Chapter 2
Bubbling Blow-Up in Critical Parabolic Problems

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Abstract These lecture notes are devoted to the analysis of blow-up of solutions for some parabolic equations that involve *bubbling phenomena*. The term *bubbling* refers to the presence of families of solutions which at main order look like scalings of a single stationary solution which in the limit become singular but at the same time have an approximately constant energy level. This arise in various problems where critical loss of compactness for the underlying energy appears. Three main equations are studied, namely: the Sobolev critical semilinear heat equation in $\mathbb{R}^n$, the harmonic map flow from $\mathbb{R}^2$ into S^2, the Patlak-Keller-Segel system in $\mathbb{R}^2$.

2.1 Introduction

These notes are devoted to the analysis of blow-up of solutions for some parabolic equations, classical in the literature, that involve so-called *bubbling phenomena*. The term *bubbling* in a variational problem refers to the presence of families of solutions which at main order look like scalings of a single stationary solution which in the limit become singular but at the same time have an approximately constant energy level. This arise in various problems where critical loss of compactness for the underlying energy appears. In time dependent versions of these problems, one expects that *blow-up by bubbling* in finite or infinite time for specific solutions appears. Those solutions are usually asymptotically not self-similar and, while not generic, their presence is among the most important features of the full dynamics since they correspond to threshold solutions between different generic regimes. In these lectures we will consider the following three problems:

1. *The Sobolev critical semilinear heat equation in* $\mathbb{R}^n$.

$$u_t = \Delta u + u^{\frac{n+2}{n-2}} \quad \text{in } \Omega \times (0, \infty), \tag{2.1}$$

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$$
\begin{aligned}
u &= 0 \quad \text{on } \partial\Omega \times (0, \infty), \\
u(\cdot, 0) &= u_0 \quad \text{in } \Omega .
\end{aligned}
$$

Here Ω designates a smooth domain in $\mathbb{R}^n$, $n \geq 3$ and u_0 is a positive, smooth initial datum.

2. *The harmonic map flow from* $\mathbb{R}^2$ into S^2

$$
\begin{aligned}
u_t &= \Delta u + |\nabla u|^2 u \quad \text{in } \Omega \times (0, T) \\
u &= \varphi \quad \text{on } \partial\Omega \times (0, T) \\
u(\cdot, 0) &= u_0 \quad \text{in } \Omega
\end{aligned}
\tag{2.2}
$$

for a function $u : \Omega \times [0, T) \to S^2$, where Ω be a bounded smooth domain in $\mathbb{R}^2$ and S^2 denotes the standard 2-sphere. Here $u_0 : \bar{\Omega} \to S^2$ is a given smooth map and $\varphi = u_0\big|_{\partial\Omega}$.

3. *The Patlak-Keller-Segel system in* $\mathbb{R}^2$.

$$
\begin{aligned}
u_t &= \Delta u - \nabla \cdot (u \nabla v) \quad \text{in } \mathbb{R}^2 \times (0, \infty) \\
v &= (-\Delta)^{-1} u := \frac{1}{2\pi} \log \frac{1}{|\cdot|} * u \\
u(\cdot, 0) &= u_0 > 0 \quad \text{in } \mathbb{R}^2
\end{aligned}
\tag{2.3}
$$

A salient common feature of these problems is the presence of *Lyapunov functionals*. In fact we let

$$
E_1(u) = \frac{1}{2} \int |\nabla u|^2 dx - \frac{1}{p+1} \int |u|^{p+1} dx, \quad p = \frac{n+2}{n-2}
$$

where the spacial domain is understood in the integral symbol. Then we compute, for sufficiently smooth solutions $u(x, t)$ of (2.1)

$$
\partial_t E_1(u(\cdot, t)) \; = \; - \int |u_t|^2 dx .
$$

Similarly, for solutions of $u(x, t)$ of (2.2) we have that

$$
\partial_t E_2(u(\cdot, t)) \; = \; - \int |u_t|^2 dx .
$$

where

$$
E_2(u) = \frac{1}{2} \int |\nabla u|^2 dx .
$$

These functionals are therefore decreasing along trajectories. In fact, we can interpret Eqs. (2.1) and (2.2) as negative L^2-gradient flows respectively for the energies E_1 and E_2.

Problem (2.3) also has a Lyapunov functional, which is less obvious. Let us write (2.3) in divergence form as

$$u_t = \nabla \cdot (u\nabla(\log u - (-\Delta)^{-1}u))$$

Then setting

$$E_3(u) := \int u(\log u - (-\Delta)^{-1}u)\, dx$$

we see that for a solution of (2.3) with sufficient regularity and space decay,

$$\partial_t E_3(u(\cdot,t)) = -\int u|\nabla(\log u - (-\Delta)^{-1}u)|^2 dx \leq 0.$$

Problem (2.3) can also be interpreted as a negative gradient flow for E_3 with respect to Wasserstein's metric, see [4].

The three problems above also have in common the presence of a *continuum of energy invariant steady states in entire space* which is what is behind the possibility of bubbling blow-up phenomena. Indeed, the steady-state equation for (2.1) in entire space is

$$\Delta u + u^{\frac{n+2}{n-2}} = 0, \quad u > 0 \quad \text{in } \mathbb{R}^n.$$

It is solved by

$$U(x) = \alpha_n \left(\frac{1}{1+|x|^2}\right)^{\frac{n-2}{2}}, \alpha_n = (n(n-2))^{\frac{1}{n-2}},$$

and so are solutions the scalings

$$U_{\lambda,x_0}(x) = \frac{1}{\lambda^{\frac{n-2}{2}}} U\left(\frac{x-x_0}{\lambda}\right) = \alpha_n \left(\frac{\lambda}{\lambda^2 + |x-x_0|^2}\right)^{\frac{n-2}{2}}. \tag{2.4}$$

We have that

$$E_1(U_{\lambda,x_0}) = E_1(U) \quad \text{for all} \quad \lambda, x_0.$$

Similarly, for (2.2) the steady state problem in $\mathbb{R}^2$ is

$$\Delta u + |\nabla u|^2 u = 0, \quad |u| = 1 \quad \text{in } \mathbb{R}^2$$

which is solved by the *1-corrotational harmonic map*

$$U(x) = \begin{pmatrix} \frac{2x}{1+|x|^2} \\ \frac{|x|^2-1}{1+|x|^2} \end{pmatrix}, \quad x \in \mathbb{R}^2.$$

We observe that the equation is also solved by

$$U_{\lambda,x_0,Q}(x) = QU\left(\frac{x-x_0}{\lambda}\right) \tag{2.5}$$

with Q a linear orthogonal transformation of $\mathbb{R}^3$. We see that

$$E_2(U_{\lambda,x_0,Q}) = E_2(U) \quad \text{for all} \quad \lambda, x_0.$$

Similarly, for (2.3) we have that the equation

$$\Delta u - \nabla \cdot (u\nabla(-\Delta)^{-1}u) = 0 \quad \text{in } \mathbb{R}^2$$

is solved by

$$U(x) = \frac{8}{(1+|x|^2)^2} \quad x \in \mathbb{R}^2,$$

and also by the scalings, singular as $\lambda \to 0$,

$$U_{\lambda,x_0}(x) = \lambda^{-2}U\left(\frac{x-x_0}{\lambda}\right) = \frac{8\lambda^2}{(\lambda^2+|x-x_0|^2)^2}. \tag{2.6}$$

We see that

$$E_3(U_{\lambda,x_0}) = E_3(U) \quad \text{for all} \quad \lambda, x_0.$$

The presence of these steady states represents loss of compactness for the respective energies, for as $\lambda \to 0^+$ they become singular, so that their limits do not belong to the natural energy space. In this way, we have the presence of non-convergent Palais-Smale sequences for the respective energies.

Our purpose in the remaining of these notes is to construct solutions of the time-dependent problems (2.1)–(2.3) that at main order look like one of the associated scalings, around one or more points, with time-dependent parameters, so that the scaling $\lambda(t)$ becomes zero in the limit. This is a *bubbling blow-up solution*. We will set up an adequate framework for each of the problems, with a common scheme. We shall do this in a rather detailed manner in Problem (2.1) and present a sketch in the case of Problems (2.2) and state the corresponding result in (2.3). In Problem (2.1) we will construct solutions with infinite time blow-up around a given arbitrary number of points of the domain. In Problem (2.2) we construct such objects but in a finite time. In (2.3) we consider the so-called critical-mass case for a fast-decay initial condition and analyze the infinite-time singularity created.

2.2 Infinite-Time Blow-Up in the Critical Heat Equation

In this section we will construct infinite time blow-up solutions to Problem (2.1).

2.2.1 Discussion and Statement of Main Result

We begin with a discussion on the blow-up topic for the more general problem

$$\begin{aligned} u_t &= \Delta u + u^p \quad \text{in } \Omega \times (0,T), \\ u &= 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) &= u_0 > 0 \quad \text{in } \Omega. \end{aligned} \tag{2.7}$$

Here Ω be a smooth domain in $\mathbb{R}^n$, $n \geq 1, p > 1$ and $0 < T \leq +\infty$.

The role of the exponent $p_S := \frac{n+2}{n-2}$ when $n \geq 3$ is fundamental in the different phenomena arising in this equation. Let us first review the steady state problem

$$\begin{aligned} \Delta u + u^p &= 0 \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.8}$$

When $1 < p < p_S$, Problem (2.8) is always solvable. In fact the best Sobolev constant

$$S_p(\Omega) = \inf_{0 \neq u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^{p+1}\right)^{\frac{2}{p+1}}}$$

is achieved by a positive function which solves (2.8) thanks to the compactness of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$. An alternative way to find a solution of Problem (2.8) is as a *mountain pass* of the energy functional

$$E_p(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1}. \tag{2.9}$$

This functional satisfies the *Palais-Smale condition*: if $u_n \in H_0^1(\Omega)$ is such that

$$E(u_n) \to c \in \mathbb{R}, \quad \nabla E(u_n) \to 0 \quad \text{in } H^{-1}(\Omega)$$

then u_n has a convergent subsequence in $H_0^1(\Omega)$. When $p = p_S$ we have that

$$S_{p_S}(\Omega) = S_{p_S}(\mathbb{R}^n) =: S_n > 0$$

and it is not attained if $\Omega \subsetneq \mathbb{R}^n$. In $\mathbb{R}^n$. The best Sobolev constant S_n is achieved precisely by (scalar multiples of) the functions

$$U_{\mu,\xi}(x) = \mu^{-\frac{n-2}{2}} U\left(\frac{x-\xi}{\mu}\right) = \alpha_n \left(\frac{\mu}{\mu^2 + |x|^2}\right)^{\frac{n-2}{2}}. \tag{2.10}$$

where

$$U(y) = \alpha_n \left(\frac{1}{1+|y|^2}\right)^{\frac{n-2}{2}}, \quad \alpha_n = (n(n-2))^{\frac{1}{n-2}}. \tag{2.11}$$

called the Aubin-Talenti bubbles, see [34]. By a result of Caffarelli-Gidas-Spruck, these functions correspond to all positive solutions of the equation

$$\Delta u + u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n.$$

namely positive critical points of the energy

$$E(u) := E_{p_S}(u) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 - \frac{n-2}{2n}\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}}.$$

The family (2.10) is energy invariant: for all μ, ξ,

$$E(U_{\mu,\xi}) = E(U) =: S_n > 0. \tag{2.12}$$

The presence of this asymptotically singular family of critical points of E in entire space precisely reflect its loss of compactness in a bounded domain Ω:

A Palais-Smale sequence in $H_0^1(\Omega)$ must asymptotically be, passing to a subsequence, of the form (called *the bubble resolution*)

$$u_n = u_\infty + \sum_{i=1}^{k} U_{\mu_n^i, \xi_n^i} + o(1), \tag{2.13}$$

for some $k \geq 0$, a critical point $u_\infty \in H_0^1(\Omega)$ of E, $\xi_n^i \in \Omega$, $\mu_n^i \to 0$ after a result by Struwe [31].

The fact that the Sobolev constant is not achieved is not just a technical obstruction for existence of solutions of (2.8): If the domain Ω is star-shaped around a point $x_0 \in \Omega$ an u solves (2.8) then Pohozaev's identity [25] yields

$$\left(\frac{2-n}{2} - \frac{n}{p+1}\right) \int_\Omega u^{p+1} = \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 (x - x_0) \cdot \nu > 0$$

hence necessarily $p < \frac{n+2}{n-2} = p_S$ and thus no solutions at all exist if $p \geq p_S$. In particular $u_\infty = 0$ in the bubble-resolution.

As for the problem in entire space,

$$\Delta u + u^p = 0, \quad u > 0 \quad \text{in } \mathbb{R}^n,$$

we have that no positive entire solution exists if $p < p_S$, while if $p = p_S$ all solutions are the Aubin-Talenti bubbles For $p > p_S$ there are positive solutions (in fact radially symmetric with $u(x) \sim |x|^{-\frac{2}{p-1}}$) but they do not have finite energy.

Coming back to the parabolic problem (2.7), let us consider a function $\varphi(x)$ positive and smooth function in Ω, with boundary value zero. Let $u_\alpha(x,t)$ be the unique (local) smooth solution of (2.7) with initial condition

$$u_\alpha(x, 0) = \alpha\varphi(x), \quad \alpha > 0.$$

We claim that for all sufficiently small ε we have that $u(x,t) \to 0$ uniformly as $t \to \infty$. To see this, let λ_1 be the first eigenvalue of $-\Delta$ in Ω under Dirichlet boundary conditions and ϕ_1 a positive first eigenfunction. Let $\varepsilon > 0$ and consider the function

$$\bar{u}(x,t) = \varepsilon e^{-\gamma t} \phi_1(x)$$

where $0 < \gamma < \lambda_1$. Then we see that

$$\bar{u}_t - \Delta\bar{u} - \bar{u}^p = \varepsilon\phi_1 e^{-\gamma t}[(\lambda_1 - \gamma) - \varepsilon^{p-1}\phi_1^{p-1}] > 0$$

and hence $\bar{u}(x,t)$ is a supersolution of (2.7) if ε is fixed sufficiently small. Then for all $0 < \alpha < \alpha_0$, we have that $u_\alpha(x,0) = \alpha\varphi(x) \leq \bar{u}(x,0)$ and hence

$$u_\alpha(x,t) \leq Ce^{-\gamma t} \to 0 \quad \text{as} \quad t \to +\infty.$$

The claim is proven. Next, we claim that for all large α, $u_\alpha(x,t)$ *blows-up in finite time*. Indeed, assume that $u_\alpha(x,t)$ is defined in $[0,T)$ and use $\phi_1(x)$ as a test function. We get

$$\frac{\partial}{\partial t} \int_\Omega \phi_1 u_\alpha(\cdot,t) = -\lambda_1 \int_\Omega \phi_1 u_\alpha(\cdot,t) + \int_\Omega \phi_1 u_\alpha(\cdot,t)^p.$$

If we let $q(t) = \int_\Omega \phi_1 u_\alpha(\cdot,t)$ then

$$q'(t) \ \geq \ -\lambda_1 q(t) + Cq(t)^p.$$

On the other hand

$$q(0) = \alpha \int_\Omega \phi_1 \varphi \, dx$$

Then for any $\alpha > \alpha_1$ we have that $-\lambda_1 q(0) + Cq(0)^p > 0$. Then we easily see that $q'(t) > 0$ for all t and hence

$$T \leq \int_{q(0)}^{\infty} \frac{dq}{-\lambda_1 q + Cq^p} \ < \ +\infty.$$

Two types of finite-time blow-up are present. Blow-up at time T for a solution $u(x,t)$ to (2.7) is said to be

- *Type I* if

$$\lim_{t\to T} (T-t)^{\frac{1}{p-1}} \|u(\cdot,t)\|_\infty \ < \ +\infty$$

- *Type II* if

$$\lim_{t\to T} (T-t)^{\frac{1}{p-1}} \|u(\cdot,t)\|_\infty \ = \ +\infty.$$

Type I means a blow-up rate that goes along with the natural scaling of (2.7)

$$\lambda \mapsto \lambda^{-\frac{1}{p-1}} u(\lambda^{-\frac{1}{2}} x, \lambda^{-1} t).$$

Said in a different way, Type I blow-up is one in which reaction overtakes diffusion effect, so that the blow-up mechanism is driven by the ODE

$$\frac{du}{dt}(t) = 0 + u(t)^p$$

whose solution with blow up at time T is precisely

$$u(t) = c_p (T-t)^{-\frac{1}{p-1}}, \quad c_p = (p-1)^{-\frac{1}{p-1}}.$$

The following facts are known:

- Type I blow-up is the only one that can arise in the subcritical case $p < p_S$ (at least for convex domains). Giga and Kohn [19].

- Type II blow-up is rare but it exists. (A radial example was found by Herrero and Velazquez [21].) One needs $p > p_{JL}$ where $p_{JL} > p_S$ is the *Joseph-Lundgren exponent*, a number which is only well-defined for dimension $n \geq 11$.
- Matano-Merle [23] proved that the set of values α for which type-II blow-up in $u_\alpha(x,t)$ may exists is just finite in the radially symmetric case.
- Still in the radial case, the condition $p > p_{JL}$ is necessary for blow-up type II to occur [17]. As seen by Galaktionov and Vázquez, type II radial blowing-up can be naturally continued beyond blow-up time for $p > p_{JL}$. In fact this radial blow-up can only take place for a given radial solution a finite number of times.

2.2.1.1 The Threshold Solution

In summary, $u_\alpha(x,t)$ blows-up in finite time for all α large and it goes to zero for small α. As a conclusion, the following number is well-defined:

$$\alpha_* = \sup\{\alpha > 0 \,/\, \lim_{t\to\infty} \|u_\alpha(\cdot,t)\|_\infty = 0\},$$

in fact $0 < \alpha_* < +\infty$. Ni et al. [24] found that $u_{\alpha_*}(x,t)$ is a well-defined L^1-weak solution of (2.7).

u_{α_*} is a type of solution which loosely speaking lies in the dynamic threshold between solutions globally defined in time and those that blow-up in finite time.

It is not clear that u_{α^*} will be smooth at all times. In fact, it will not be in general the case.

- When $1 < p < p_S$, $u_{\alpha*}(x,t)$ is smooth. Indeed it is uniformly bounded (Cazenave-Lions [7]), and up to subsequences it converges to a (positive) solution of Eq. (2.8).
- When $p > p_S$, Ω a ball and radially symmetric solutions, it turns out that $u_{\alpha^*}(x,t) \to 0$ as $t \to +\infty$, see Quittner-Souplet [28].
- The case $p = p_S$ *is special*. In fact in this case the threshold solution u_{α_*} in the radial case does have *infinite-blow up time*, namely

$$\lim_{t\to+\infty} \|u(\cdot,t)\|_\infty \;=\; +\infty.$$

We can mention in addition that by results by Du and Suzuki [13, 33]: along sequences $t_n \to +\infty$, $u_{\alpha_*}(x,t)$ does have a *bubble resolution* of the type of a Palais-Smale sequence when $p = p_S$.

$$u_{\alpha_*}(x,t_n) = u_\infty + \sum_{i=1}^{k} U_{\mu_n^i,\xi_n^i} + o(1), \tag{2.14}$$

for some $k \geq 0$, a critical point $u_\infty \in H_0^1(\Omega)$ of, $\xi_n^i \in \Omega$, $\mu_n^i \to 0$.

For any $p > 1$ the energy

$$E_p(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1}.$$

defines a Lyapunov functional for Eq. (2.7). Indeed, we readily compute, for a solution $u(x,t)$ of (2.7),

$$\frac{d}{dt}E_p(u(\cdot,t)) = -\int_\Omega |u_t|^2 dx$$

In the case $p = p_S$ the k-bubble resolution would yield

$$\lim_{n\to+\infty} E(u(\cdot,t_n)) = kS_n + E(u_\infty).$$

where we recall, necessarily $u_\infty = 0$ if Ω is star-shaped.

Galaktionov and Vázquez [17] found that in the case that $\Omega = B(0,1)$ and the threshold solution $u_{\alpha *}$ is radially symmetric, then no finite time singularities for $u_{\alpha *}(r,t)$ occur and it must become unbounded as $t \to +\infty$, thus exhibiting infinite-time blow up. Galaktionov and King discovered in [16] that this radial blow-up solution does have a bubbling asymptotic profile as $t \to +\infty$ of the form

$$u_{\alpha_*}(r,t) \;\approx\; \alpha_n \left(\frac{\mu(t)}{\mu(t)^2 + r^2}\right)^{\frac{n-2}{2}}, \quad r = |x|. \tag{2.15}$$

where for $n \geq 5$, $\mu(t) \sim t^{-\frac{1}{n-4}} \to 0$. For $\alpha > \alpha_*$ blow-up in finite time of $u_{\alpha_*}(r,t)$ occurs while, it goes to zero when $\alpha < \alpha_*$. Understanding this threshold phenomenon is central in capturing the global dynamics of Problem (2.1). These solutions are unstable, while intuitively codimension-one stable in the space of initial conditions containing $\alpha_*\varphi$.

Nothing seems to be known however on existence of infinite-time bubbling solutions in the nonradial case, or about their degree of stability. Our main goal is to build solutions with single or multiple blow-up points as $t \to +\infty$ in problem (2.1) when Ω is arbitrary and $n \geq 5$, providing precise account of their asymptotic form and investigate their stability.

Our construction unveils the interesting role played by the (elliptic) Green function of the domain Ω. In what follows we denote by $G(x,y)$ Green's function for the boundary value problem

$$-\Delta_x G(x,y) = c_n\delta(x-y) \quad \text{in } \Omega, \quad G(\cdot,y) = 0 \quad \text{on } \partial\Omega,$$

where $\delta(x)$ is the Dirac mass at the origin and c_n is the number such that

$$-\Delta_x\Gamma(x) = c_n\delta(x), \quad \Gamma(x) = \frac{\alpha_n}{|x|^{n-2}}, \tag{2.16}$$

namely $c_n = (n-2)\omega_n\alpha_n$ with ω_n the surface area of the unit sphere in $\mathbb{R}^n$ and α_n the number in (2.15). We let $H(x,y)$ be the regular part of $G(x,y)$ namely the solution of the problem

$$-\Delta_x H(x,y) = 0 \quad \text{in } \Omega, \quad H(\cdot,y) = \Gamma(\cdot - y) \quad \text{in } \partial\Omega. \tag{2.17}$$

The diagonal $H(x,x)$ is called the Robin function of Ω. It is well known that it satisfies

$$H(x,x) \to +\infty \quad \text{as dist}\,(x,\partial\Omega) \to 0. \tag{2.18}$$

Let $q = (q_1,\ldots,q_k)$ be an array of k distinct points in Ω, and define the $k\times k$ matrix

$$\mathcal{G}(q) = \begin{bmatrix} H(q_1,q_1) & -G(q_1,q_2) & \cdots & -G(q_1,q_k) \\ -G(q_1,q_2) & H(q_2,q_2) & -G(q_2,q_3)\cdots & -G(q_3,q_k) \\ \vdots & & \ddots & \vdots \\ -G(q_1,q_k) & \cdots & -G(q_{k-1},q_k) & H(q_k,q_k) \end{bmatrix} \tag{2.19}$$

Our main result states that a global solution to (2.1) which blows-up at exactly k given points q_j exists if q lies in the open region of Ω^k where the matrix $\mathcal{G}(q)$ is positive definite.

Theorem 1 ([11]) *Assume $n \geq 5$. Let $q_1,\cdots,q_k$ be distinct points in Ω such that the matrix $\mathcal{G}(q)$ is positive definite. Then there exist an initial datum u_0 and smooth functions $\xi_j(t) \to q_j$ and $0 < \mu_j(t) \to 0$, as $t \to +\infty$, $j = 1,\ldots,k$, such that the solution u_q of Problem (2.1) has the form*

$$u_q(x,t) = \sum_{j=1}^{k} \alpha_n \left(\frac{\mu_j(t)}{\mu_j(t)^2 + |x-\xi_j(t)|^2}\right)^{\frac{n-2}{2}} - \mu_j(t)^{\frac{n-2}{2}} H(x,q_j) + \mu_j(t)^{\frac{n-2}{2}}\theta(x,t), \tag{2.20}$$

where $\theta(x,t)$ is bounded, and $\theta(x,t) \to 0$ as $t \to +\infty$, uniformly away from the points q_j. In addition, for certain positive constants β_j depending on q.

$$\mu_j(t) = \beta_j t^{-\frac{1}{n-4}}(1+o(1)), \quad \xi_j(t) - q_j = O(t^{-\frac{2}{n-4}}) \quad \textit{as } t \to +\infty$$

Our construction of the solution $u_q(x,t)$ in Theorem 1 yields the codimension k-stability of its bubbling phenomenon.

Our construction of the solution $u_q(x,t)$ yields the codimension k-stability of its bubbling phenomenon in the following sense.

Corollary 2.2.1 *There exists a codimension k manifold in $C^1(\bar{\Omega})$ that contains $u_q(x,0)$ such that if $u(x,0)$ lies in that manifold and it is sufficiently close to $u_q(x,0)$, then the solution $u(x,t)$ of (2.1) has exactly k bubbling points $\tilde{q}_j$, $j = 1,\dots,k$ which lie close to the q_j.*

Positive definiteness of $\mathcal{G}(q)$ trivially holds if $k = 1$. For $k = 2$ this condition holds if and only if

$$H(q_1,q_1)H(q_2,q_2) - G(q_1,q_2)^2 > 0,$$

in particular it does not hold if both points q_1 and q_2 are too close to a given point in Ω. Given $k > 1$, using that

$$H(x,x) \to +\infty \quad \text{as dist}\,(x,\partial\Omega) \to 0.$$

we can always find k points where $\mathcal{G}(q)$ is positive definite.

The proof of the above result consists of building a first approximation to the solution, then solving for a small remainder by means of linearization and fixed point arguments. First we construct a first approximation of the desired form. We shall compute the error and see that in order to **improve the approximation** we need certain solvability conditions for the elliptic linearized operator around the bubble. These relations yield a system of ODEs for the scaling parameters, of which we find a suitable solution.

2.2.2 Construction of the Approximate Solution and Error Computations

We consider the Talenti bubbles

$$U(y) = \alpha_n \left(\frac{1}{1+|y|^2}\right)^{\frac{n-2}{2}}, \quad \alpha_n = (n(n-2))^{\frac{n-2}{4}},$$

and

$$U_{\mu,\xi}(x) = \mu^{-\frac{n-2}{2}} U\left(\frac{x-\xi}{\mu}\right), \quad \mu > 0, \quad \xi \in \mathbb{R}^n.$$

Given k points $q_1,\dots,q_k \in \mathbb{R}^n$, we want to find a solution $u(x,t)$ of equation (P) with

$$u(x,t) \approx \sum_{j=1}^{k} U_{\mu_j(t),\xi_j(t)}(x) \tag{2.21}$$

where $\xi_j(t) \to q_j$ and $\mu_j(t) \to 0$ as $t \to \infty$ for each $j = 1, \dots, k$. The functions $\xi_j(t)$ and $\mu_j(t)$ cannot of course be arbitrary.

We assume that the vanishing speed of all functions $\mu_j(t)$ is the same. More precisely that for a function $\mu_0(t) \to 0$ and positive constants $b_1, \dots, b_k$ we have

$$\mu_j(t) = b_j \mu_0(t) + O(\mu_0^2(t)) \quad \text{as} \quad t \to \infty.$$

Also, we assume

$$\xi_j(t) - q_j = O(\mu_0^2(t)) \quad \text{as} \quad t \to \infty.$$

If a solution to (2.1) satisfies $u(x,t) \approx \sum_{j=1}^k U_{\mu_j,\xi_j}(x)$ then

$$u_t \approx \Delta u + \sum_{j=1}^k U_{\mu_j,q}(x)^p$$

Besides, we see that

$$\int_\Omega U_{\mu_j,q}(x)^p dx \approx \mu_j^{\frac{n-2}{2}} a_n, \quad a_n := \int_{\mathbb{R}^n} U(y)^p dy,$$

and hence away from the points q_j

$$u_t \approx \Delta u + c_n \mu_0^{\frac{n-2}{2}} \sum_{j=1}^k b_j^{\frac{n-2}{2}} \delta_{q_j} \quad \text{in } \Omega \times (0,\infty)$$

where δ_q designates the Dirac mass at the point q.

Letting $u = \mu_0^{\frac{n-2}{2}} v(x,t)$ we get

$$v_t \approx \Delta v - \frac{n-2}{2} \mu_0^{-1} \dot{\mu}_0 v + c_n \sum_{j=1}^k b_j^{\frac{n-2}{2}} \delta_{q_j} \quad \text{in } \Omega \times (0,\infty).$$

We assume that $\mu_0^{-1} \dot{\mu}_0 \to 0$, so that

$$v_t \approx \Delta v + a_n \sum_{j=1}^k b_j^{\frac{n-2}{2}} \delta_{q_j} \quad \text{in } \Omega \times (0,\infty),$$
$$v = 0 \quad \text{on } \partial\Omega \times (0,\infty).$$

So that away from the q_j we should have

$$v(x,t) \approx a_n \sum_{j=1}^{k} b_j^{\frac{n-2}{2}} G(x,q_j),$$

$$u(x,t) \approx \sum_{j=1}^{k} \frac{\alpha_n \mu_j^{\frac{n-2}{2}}}{|x-q_j|^{n-2}} - \mu_j^{\frac{n-2}{2}} H(x,q_j).$$

Observing that for x away from the point q_j, we precisely have

$$U_{\mu_j,\xi_j}(x) \approx \frac{\alpha_n \mu_j^{\frac{n-2}{2}}}{|x-q_j|^{n-2}}$$

we see that a better global approximation to a solution $u(x,t)$ to our problem is given by the corrected k-bubble

$$u_{\xi,\mu}(x,t) := \sum_{j=1}^{k} u_j(x,t), \quad u_j(x,t) := U_{\mu_j,\xi_j}(x) - \mu_j^{\frac{n-2}{2}} H(x,q_j). \tag{2.22}$$

We have obtained this correction term out of a rough analysis to what is happening away from the blow-up points. Let us now analyze the region near them. That will allow us to identify the function $\mu_0(t)$ and the constants b_j. It is convenient to write

$$S(u) := -u_t + \Delta_x u + u^p.$$

We consider the error of approximation $S(u_0)$. We have

$$S(u_{\mu,\xi}) = -\sum_{i=1}^{k} \partial_t u_i + \left(\sum_{i=1}^{k} u_i\right)^p - \sum_{i=1}^{k} U^p_{\mu_i,\xi_i}.$$

We obtain the following estimate near a given concentration point q_j, from where the formal asymptotic derivation of the unknown parameters will be a rather direct consequence.

Given j, assuming that

$$|x-q_j| \le \frac{1}{2} \min_{i \neq l} |q_i - q_l|$$

and setting

$$y_j := \frac{x-\xi_j}{\mu_j},$$

we have

$$S(u_{\mu,\xi}) = \mu_j^{-\frac{n+2}{2}} \left(\mu_j E_{0j} + \mu_j E_{1j} \ + \ \mathcal{R}_j\right)$$

$$E_{0j} = pU(y_j)^{p-1}\big[-\mu_j^{n-3} H(q_j,q_j) \ + \ \sum_{i\neq j} \mu_j^{\frac{n-4}{2}} \mu_i^{\frac{n-2}{2}} G(q_j,q_i)\big]$$
$$+ \dot\mu_j \left[y_j \cdot \nabla U(y_j) + \frac{n-2}{2} U(y_j)\right],$$

and

$$E_{1j} = pU(y_j)^{p-1}\big[-\mu_j^{n-2} \nabla_x H(q_j,q_j)$$
$$+ \sum_{i\neq j} \mu_j^{\frac{n-2}{2}} \mu_i^{\frac{n-2}{2}} \nabla_x G(q_j,q_i)\,\big] \cdot y_j + \dot\xi_j \cdot \nabla U(y_j)$$

and $\mathcal{R}_j$ contains smaller order terms.

To see this, we write, for $y_i = \frac{x-\xi_i}{\mu_i}$,

$$u_{\mu,\xi}(x,t) = \sum_{i=1}^k \mu_i^{-\frac{n-2}{2}} U(y_i) - \mu_i^{\frac{n-2}{2}} H(x,q_i), \quad \text{and} \quad S(u_{\mu,\xi}) = S_1 + S_2$$

where

$$S_1 \ := \ \sum_{i=1}^k \mu_i^{-\frac{n}{2}} \dot\xi_i \cdot \nabla U(y_i) + \mu_i^{-\frac{n}{2}} \dot\mu_i Z_{n+1}(y_i) + \frac{n-2}{2} \mu_i^{\frac{n-4}{2}} \dot\mu_i H(x,q_i), \tag{2.23}$$

and

$$S_2 \ := \ \left(\sum_{i=1}^k \mu_i^{-\frac{n-2}{2}} U(y_i) - \mu_i^{\frac{n-2}{2}} H(x,q_i)\right)^p - \sum_{i=1}^k \mu_i^{-\frac{n+2}{2}} U(y_i)^p. \tag{2.24}$$

Then at main order we have that near q_j,

$$S_2 \approx \mu_j^{-\frac{n+2}{2}} \left[\left(U(y_j) + \Theta_j\right)^p - U(y_j)^p \right],$$

with

$$\Theta_j = -\mu_j^{n-2} H(x,q_j) + \sum_{i\neq j} \left(\mu_j \mu_i^{-1}\right)^{\frac{n-2}{2}} U(y_i) - (\mu_j \mu_i)^{\frac{n-2}{2}} H(x,q_i). \tag{2.25}$$

Taylor expanding we get

$$S_2 \approx \mu_j^{-\frac{n+2}{2}} pU(y_j)^{p-1}\Theta_j \ .$$

We make some further expansion. We have, for $i \neq j$,

$$U(y_i) \ = \ \frac{\alpha_n \mu_i^{n-2}}{(|\mu_j y_j + \xi_j - \xi_i|^2 + \mu_i^2)^{\frac{n-2}{2}}} \ \approx \ \frac{\alpha_n \mu_i^{n-2}}{|\mu_j y_j + q_j - q_i|^{n-2}}.$$

Hence we get the approximation

$$\Theta_j \approx -\mu_j^{n-2} H(q_j + \mu_j y_j, q_j) + \sum_{i\neq j} (\mu_i \mu_j)^{\frac{n-2}{2}} G(q_j + \mu_j y_j, q_i).$$

Further expanding, we get

$$\begin{aligned}\Theta_j \ \approx \ & - \mu_j^{n-2} H(q_j, q_j) + \sum_{i\neq j} (\mu_i \mu_j)^{\frac{n-2}{2}} G(q_j, q_i) \\ & + \Big[- \mu_j^{n-2} \nabla_x H(q_j, q_j) + \sum_{i\neq j} (\mu_i \mu_j)^{\frac{n-2}{2}} \nabla_x G(q_j, q_i) \Big] \cdot \mu_j y_j.\end{aligned}$$

We also approximate

$$S_1 \ \approx \ \mu_j^{-\frac{n}{2}} \dot{\xi}_j \cdot \nabla U(y_j) + \mu_j^{-\frac{n}{2}} \dot{\mu}_j \big[\frac{n-2}{2} U(y_j) + y_j \cdot \nabla U(y_j) \big],$$

2.2.3 The Choice of the Parameters at Main Order

We are looking for a solution of our equation of the form

$$u(x,t) = u_{\mu,\xi}(x,t) + \tilde{\phi}(x,t)$$

where $\tilde{\phi}$ is globally smaller. We see that

$$0 = S(u_{\mu,\xi} + \tilde{\phi}) = -\partial_t \tilde{\phi} + \Delta_x \tilde{\phi} + p u_{\mu,\xi}^{p-1} \tilde{\phi} + S(u_{\mu,\xi}) + \tilde{N}_{\mu,\xi}(\tilde{\phi})$$

where $\tilde{N}_{\mu,\xi}(\tilde{\phi}) = (u_{\mu,\xi} + \tilde{\phi})^p - u_{\mu,\xi}^p - p u_{\mu,\xi}^{p-1} \tilde{\phi}$.

We rewrite

$$\tilde{\phi}(x,t) = \mu_j^{-\frac{n-2}{2}} \phi\left(\frac{x - \xi_j}{\mu_j}, t \right)$$

so that

$$0 = \mu_j^{\frac{n+2}{2}} S(u_{\mu,\xi} + \tilde{\phi}) \approx \tag{2.26}$$

$$-\mu_j^2 \partial_t \phi + \Delta_y \phi + pU(y)^{p-1}\phi + \mu_j^{\frac{n+2}{2}} S(u_{\mu,\xi}) + A[\phi] \tag{2.27}$$

where the terms in $A(\phi)$ are all of smaller order. It is reasonable to assume that $\phi(y,t)$ decays in the y variable.

Considering the largest term E_0 in the expansion of the error $\mu_j^{\frac{n+2}{2}} S(u_{\mu,\xi})$ we find that $\phi(y,t)$ should equal at main order a solution $\phi_{0j}(y,t)$ of the elliptic equation

$$\Delta_y \phi_{0j} + pU^{p-1}\phi_{0j} = -\mu_{0j} E_{0j} \quad \text{in } \mathbb{R}^n, \quad \phi_0(y,t) \to 0 \quad \text{as} \quad |y| \to \infty. \tag{2.28}$$

where we recall

$$E_{0j} = pU(y_j)^{p-1}\Big[-\mu_j^{n-3} H(q_j,q_j) + \sum_{i\neq j} \mu_j^{\frac{n-4}{2}} \mu_i^{\frac{n-2}{2}} G(q_j,q_i)\Big]$$
$$+ \dot{\mu}_j \left[y_j \cdot \nabla U(y_j) + \frac{n-2}{2} U(y_j)\right],$$

2.2.3.1 Basic Linear Elliptic Theory

We recall some standard facts on a linear equation of the form

$$L_0(\psi) := \Delta_y \psi + pU^{p-1}\psi = h(y) \quad \text{in } \mathbb{R}^n, \quad \psi(y) \to 0 \quad \text{as} \quad |y| \to \infty.$$

It is well known that all bounded solutions of the equation $L_0(\psi) = 0$ in $\mathbb{R}^n$ consist of linear combinations of the functions $Z_1, \dots, Z_{n+1}$ defined as

$$Z_i(y) := \frac{\partial U}{\partial y_i}(y), \quad i = 1, \dots, n, \quad Z_{n+1}(y) := \frac{n-2}{2} U(y) + y \cdot \nabla U(y).$$

If $h(y) = O(|y|^{-m})$, $m > 2$, then the problem is solvable iff

$$\int_{\mathbb{R}^n} h(y) Z_i(y)\, dy = 0 \quad \text{for all} \quad i = 1, \dots, n+1.$$

Since $n \geq 5$, we can solve

$$\Delta_y \phi_{0j} + pU^{p-1}\phi_{0j} = -\mu_{0j} E_{0j} \quad \text{in } \mathbb{R}^n, \quad \phi_0(y,t) \to 0 \quad \text{as} \quad |y| \to \infty. \tag{2.29}$$

provided that

$$\int_{\mathbb{R}^n} E_{0j}(y,t)\, Z_{n+1}(y)\, dy \;=\; 0 \quad \text{for all} \quad j = 1,\dots,k \tag{2.30}$$

We compute

$$\int_{\mathbb{R}^n} E_{0j}(y,t)\, Z_{n+1}(y)\, dy \;=\; c_1\Big[\mu_j^{n-3} H(q_j,q_j) - \sum_{i\neq j} \mu_j^{\frac{n-4}{2}} \mu_i^{\frac{n-2}{2}} G(q_i,q_j)\Big] \;+\; c_2\, \dot\mu_j\,, \tag{2.31}$$

where c_1 and c_2 are the positive constants given by

$$c_1 = -p\int_{\mathbb{R}^n} U^{p-1} Z_{n+1} = \frac{n-2}{2}\int_{\mathbb{R}^n} U^p, \quad c_2 = \int_{\mathbb{R}^n} |Z_{n+1}|^2. \tag{2.32}$$

We observe that $c_2 < +\infty$ thanks to the assumed fact $n \geq 5$.

These relations define a nonlinear system of ODEs for which a solution can be found as follows: we write

$$\mu_j(t) = b_j \mu_0(t)$$

and arrive at the relations

$$b_j^{n-2} H(q_j,q_j) - \sum_{i\neq j} (b_i b_j)^{\frac{n-2}{2}} G(q_i,q_j) \;+\; c_2 c_1^{-1}\, b_j^2 \mu_0^{3-n}\, \dot\mu_0(t) = 0$$

so that $\mu_0^{3-n}\dot\mu_0(t)$ should equal a constant, which is necessarily negative since μ_0 decays to zero. This constant can be scaled out, hence it can be chosen arbitrarily to the expense of changing accordingly the values b_i. We impose

$$\dot\mu_0 = -\frac{2c_1 c_2^{-1}}{n-2}\mu_0^{n-3}, \tag{2.33}$$

which yields after a suitable translation of time,

$$\mu_0(t) \;=\; \gamma_n t^{-\frac{1}{n-4}}, \quad \gamma_n = (2^{-1}(n-4)^{-1}(n-2)c_1^{-1}c_2)^{\frac{1}{n-4}} \tag{2.34}$$

and therefore the positive constants b_j (in case they exist) must solve the nonlinear system of equations

$$b_j^{n-3} H(q_j,q_j) - \sum_{i\neq j} b_i^{\frac{n-2}{2}} b_j^{\frac{n-2}{2}-1} G(q_i,q_j) = \frac{2b_j}{n-2} \quad \text{for all} \quad j = 1,\dots,k. \tag{2.35}$$

This system has a solution (which is unique) if the matrix $\mathcal{G}(q)$ defined in is positive definite. System (2.35) can be written as a variational problem. Indeed, it is equivalent to $\nabla_b I(b) = 0$ where

$$I(b) := \frac{1}{n-2}\left[\sum_{j=1}^{k} b_j^{n-2} H(q_j, q_j) - \sum_{i\neq j} b_i^{\frac{n-2}{2}} b_j^{\frac{n-2}{2}} G(q_i, q_j) - \sum_{j=1}^{k} b_j^2\right]$$

Writing $\Lambda_j = b_j^{\frac{n-2}{2}}$ the functional becomes

$$(n-2)\, I(b) = \tilde{I}(\Lambda) = \sum_{j=1}^{k} H(q_j, q_j)\Lambda_j^2 - \sum_{i\neq j} G(q_i, q_j)\Lambda_i \Lambda_j - \sum_{j=1}^{k} \Lambda_j^{\frac{4}{n-2}}.$$

Let us assume that the matrix $\mathcal{G}(q)$ is positive definite. Then the functional $\tilde{I}(\Lambda)$ is strictly convex in the region where all $\Lambda_j > 0$. It clearly has a global minimizer with all components positive. This yields the existence of a unique critical point b of $I(b)$ with positive components which what we needed.

From the choice of the parameters μ_0, b_j we have

$$\mu_{0j} E_{0j}[\bar{\mu}_0, \dot{\mu}_{0j}] = -\gamma_j \mu_0(t)^{n-2}\, q_0(y) \tag{2.36}$$

where γ_j is a positive constant and

$$q_0(y) := p\, U^{p-1}(y) c_2 + c_1 Z_{n+1}(y),$$

so that $\int_{\mathbb{R}^n} q_0(y)\, Z_{n+1}(y)\, dy \;=\; 0$.

The problem

$$\Delta\phi_{0j} + pU(y)^{p-1}\phi_{0j} = -\gamma_j \mu_0(t)^{n-2} q_0(y), \quad \phi_{0j}(y,t) \to 0 \quad \text{as} \quad y \to \infty.$$

has a radially symmetric solution which we can describe from the variation of parameters. Let $\tilde{Z}_{n+1}(r)$ so that $L_0(\tilde{Z}_{n+1}) = 0$ with

$$\tilde{Z}_{n+1}(r) \sim r^{2-n} \quad \text{as} \quad r \to 0, \quad \tilde{Z}_{n+1}(r) \sim 1 \quad \text{as} \quad r \to \infty,$$

and the radial solution $p_0 = p_0(|y|)$ of $L_0(p_0) = q_0$ described as

$$p_0(r) = cZ_{n+1} \int_0^r \tilde{Z}_{n+1}(s) q_0(s) s^{n-1}\, ds - c\tilde{Z}_{n+1} \int_0^r Z_{n+1}(s) q_0(s) s^{n-1}\, ds.$$

p_0 satisfies

$$p_0(|y|) = O(|y|^{-2}) \quad \text{as} \quad |y| \to \infty. \tag{2.37}$$

Then a solution $\phi_{0j}(y,t)$ is simply given by the function

$$\phi_j(y,t) = \gamma_j \mu_0(t)^{n-2} p_0(y).$$

This leads us to the following corrected approximation,

$$u^*_{\mu,\xi}(x,t) := u_{\mu,\xi}(x,t) + \tilde{\Phi}(x,t), \quad \tilde{\Phi}(x,t) := \sum_{j=1}^{k} \mu_j^{-\frac{n-2}{2}} \phi_{0j}\left(\frac{x-\xi_j}{\mu_j}, t\right). \tag{2.38}$$

2.2.3.2 Total Expansion of the Error

Expansion for the error $S(u^*_{\mu,\xi})$ near each q_j. Write

$$\mu_j = b_j \mu_0 + \lambda_j, \quad |\lambda| \le \mu_0^{1+\sigma}$$

Then setting $x = \xi_j + \mu_j y_j$, we get

$$\begin{aligned} S(u^*_{\mu,\xi}) \approx \sum_{j=1}^{k} \mu_j^{-\frac{n+2}{2}} \Big\{ & \mu_{0j} \dot{\lambda}_j Z_{n+1}(y_j) \; - \; \mu_{0j} \mu_0^{n-4} p U(y_j)^{p-1} \sum_{i=1}^{k} M_{ij} \lambda_i \\ & + \mu_j \dot{\xi}_j \cdot \nabla U(y_j) + pU(y_j)^{p-1} \big[-\mu_j^{n-2} \nabla_x H(q_j, q_j) \\ & + \sum_{i \neq j} \mu_j^{\frac{n-2}{2}} \mu_i^{\frac{n-2}{2}} \nabla_x G(q_j, q_i) \big] \cdot y_j \Big\} \end{aligned}$$

where M_{ij} is a certain positive definite matrix depending on the points.

2.2.4 *The Inner-Outer Gluing Procedure*

$$\partial_t u = \Delta u + u^p \quad \text{in } \Omega \times [0,\infty), \qquad u = 0 \qquad \text{on } \partial\Omega \times [0,\infty). \tag{2.39}$$

We solve with u of the form

$$u = u^*_{\mu,\xi} + \tilde{\phi}, \tag{2.40}$$

where $\tilde{\phi}(x,t)$ is a smaller term. We construct the function $\tilde{\phi}$ by means of what we call the *inner-outer gluing* procedure.

This procedure consists in writing

$$\tilde{\phi}(x,t) = \psi(x,t) + \phi^{in}(x,t) \quad \text{where} \quad \phi^{in}(x,t) := \sum_{j=1}^{k} \eta_{j,R}(x,t)\tilde{\phi}_j(x,t)$$

with

$$\tilde{\phi}_j(x,t) := \mu_{0j}^{-\frac{n-2}{2}} \phi_j\left(\frac{x-\xi_j}{\mu_{0j}}, t\right), \quad \mu_{0j}(t) = b_j\mu_0(t)$$

and

$$\eta_{j,R}(x,t) = \eta\left(\frac{x-\xi_j}{R\mu_{0j}}\right).$$

Here $\eta(s)$ is a smooth cut-off function with $\eta(s) = 1$ for $s < 1$ and $= 0$ for $s > 2$.

In terms of $\tilde{\phi}$, Problem (2.1) reads as

$$\partial_t\tilde{\phi} = \Delta\tilde{\phi} + p(u^*_{\mu,\xi})^{p-1}\tilde{\phi} + \tilde{N}(\tilde{\phi}) + S(u^*_{\mu,\xi}) \quad \text{in } \Omega \times [0,\infty), \tag{2.41}$$

$$\tilde{\phi} = -u^*_{\mu,\xi} \quad \text{on } \partial\Omega \times [0,\infty), \tag{2.42}$$

where

$$\tilde{N}_{\mu,\xi}(\tilde{\phi}) = (u^*_{\mu,\xi} + \tilde{\phi})^p - (u^*_{\mu,\xi})^p - p(u^*_{\mu,\xi})^{p-1}\tilde{\phi},$$

$$S(u^*_{\mu,\xi}) = -\partial_t u^*_{\mu,\xi} + \Delta u^*_{\mu,\xi} + (u^*_{\mu,\xi})^p.$$

We decompose

$$S(u^*_{\mu,\xi}) \approx \sum_{j=1}^{k} S_{\mu,\xi,j} \tag{2.43}$$

where, for $y_j = \frac{x-\xi_j}{\mu_j}$,

$$S_{\mu,\xi,j} = \mu_j^{-\frac{n+2}{2}}\Big\{\mu_{0j}\left[\dot{\lambda}_j Z_{n+1}(y_j) - \mu_0^{n-4} pU(y_j)^{p-1}\sum_{i=1}^{k} M_{ij}\lambda_i\right]$$
$$+ \mu_j\Big[\dot{\xi}_j \cdot \nabla U(y_j) + pU(y_j)^{p-1}\big[-\mu_j^{n-2}\nabla_x H(q_j,q_j)$$
$$+ \sum_{i\neq j}\mu_j^{\frac{n-2}{2}}\mu_i^{\frac{n-2}{2}}\nabla_x G(q_j,q_i)\big]\cdot y_j\Big]\Big\}.$$

Let

$$V_{\mu,\xi} = p\sum_{j=1}^{k}((u^*_{\mu,\xi})^{p-1} - (\mu_j^{-\frac{n-2}{2}}U(\frac{x-\xi_j}{\mu_j}))^{p-1})\eta_{j,R} + p\big(1-\sum_{j=1}^{k}\eta_{j,R}\big)(u^*_{\mu,\xi})^{p-1}. \tag{2.44}$$

A main observation is the following: $\tilde\phi$ solves the problem if (ψ,ϕ) solves the following system:

Outer problem:

$$\begin{aligned}\partial_t\psi &= \Delta\psi + V_{\mu,\xi}\psi \\ &\quad + \sum_{j=1}^{k}[2\nabla\eta_{j,R}\nabla_x\tilde\phi_j + \tilde\phi_j(\Delta_x - \partial_t)\eta_{j,R}] \\ &\quad + \tilde N_{\mu,\xi}(\tilde\phi) + S^o_{\mu,\xi} \quad \text{in } \Omega\times[t_0,\infty), \\ \psi &= -u^*_{\mu,\xi} \quad \text{on } \partial\Omega\times[t_0,\infty),\end{aligned} \tag{2.45}$$

where

$$S^o_{\mu,\xi} = S^{(2)}_{\mu,\xi} + \sum_{j=1}^{k}(1-\eta_{j,R})S_{\mu,\xi,j},$$

and for all $j=1,\dots,k$, the *Inner problems:*

$$\partial_t\tilde\phi_j = \Delta\tilde\phi_j + pU_j^{p-1}\tilde\phi_j + pU_j^{p-1}\psi + S_{\mu,\xi,j} \quad \text{in } B_{2R\mu_{0j}}(\xi_j)\times[t_0,\infty).$$

The Inner problems in terms of $\phi_j(y,t)$, $y\in B_{2R}(0)$ become

$$\begin{aligned}\mu_{0j}^2\partial_t\phi_j &= \Delta_y\phi_j + pU(y)^{p-1}\phi_j + \mu_{0j}^{\frac{n+2}{2}}S_{\mu,\xi,j}(\xi_j+\mu_{0j}y,t) \\ &\quad + p\mu_{0j}^{\frac{n-2}{2}}\frac{\mu_{0j}^2}{\mu_j^2}U^{p-1}(y)\psi(\xi_j+\mu_{0j}y,t) + B_j[\phi_j] + B_j^0[\phi_j]\end{aligned}$$

where $B_j[\phi_j]$ is a smaller order linear operator.

We proceed as follows. For given parameters $\lambda,\xi,\dot\lambda,\dot\xi$ and functions ϕ_j fixed in a suitable range, we solve for ψ the outer problem (2.45). Indeed, in the form of a (nonlocal) operator

$$\psi = \Psi(\lambda,\xi,\dot\lambda,\dot\xi,\phi)$$

Then we replace this ψ in the inner equations and solve them by a fixed point argument involving a suitable inverse of the main part of the linear operators in ϕ_j. Let us explain how to do so.

Recall that the elliptic linear operator $L_0(\phi) := \Delta\phi + pU^{p-1}(y)\phi$ has an $n+1$ dimensional bounded kernel generated by the bounded functions

$$Z_i(y) = \frac{\partial U}{\partial y_i}, \quad i = 1, \ldots, n, \quad Z_{n+1}(y) = \frac{n-2}{2}U(y) + \nabla U(y) \cdot y.$$

If we consider the model problem for (I), in which now we do not neglect the term corresponding to time derivative, and we consider it on the whole $\mathbb{R}^n$

$$\mu_{0j}^2 \partial_t \phi = L_0(\phi) + E(y,t), \tag{2.46}$$

we observe that $\mu_{0j}^2 \partial_t \phi = L_0(\phi)$ when ϕ is any linear combination of the functions $Z_i(y)$, $i = 1, \ldots, n, n+1$. This fact suggests that solvability depends on whether the right hand side $E(y,t)$ does have component in the directions *spanned* by the $Z_i(y)$'s.

In other words, one expects solvability of (2.46) provided that some orthogonality conditions like

$$\int_{\mathbb{R}^n} E(y,t) Z_i(y)\, dy = 0, \quad i = 1, \ldots, n+1, \quad \text{for all} \quad t$$

are fulfilled. Since we have k of these conditions, for any $j = 1, \ldots, k$, this system takes the form of a nonlinear, nonlocal system of $(n+1)k$ ODEs in the $(n+1)k$ parameter functions $\lambda_1, \ldots, \lambda_k$ and $\xi_1, \ldots, \xi_k$. It is at this point that we choose the parameters λ and ξ (as functions of the given ϕ) in such a way that these orthogonality (or solvability) conditions are satisfied.

Another known fact about the elliptic $L_0(\phi)$: L_0 has a positive radially symmetric bounded eigenfunction Z_0 associated to the only negative eigenvalue λ_0 to the problem

$$L_0(\phi) + \lambda\phi = 0, \quad \phi \in L^\infty(\mathbb{R}^n).$$

Furthermore, λ_0 is simple and Z_0 decays like

$$Z_0(y) \sim |y|^{-\frac{n-1}{2}} e^{-\sqrt{|\lambda_0|}\,|y|} \quad \text{as} \quad |y| \to \infty.$$

Let $e(t) := \int_{\mathbb{R}^n} \phi(y,t) Z_0(y)\, dy$, the projection of $\phi(y,t)$ in the direction $Z_0(y)$.

Integrating equation (2.46) in $\mathbb{R}^n$, using that $\mu_{0j}(t)^2 = b_j^2 t^{-\frac{2}{n-4}}$ we get

$$b_j^2 t^{-\frac{2}{n-2}} \dot{e}(t) - \lambda_0 e(t) = f(t) := \Big(\int_{\mathbb{R}^n} Z_0(y)^2\, dy\Big)^{-1} \int_{\mathbb{R}^n} E(y,t) Z_0(y)\, dy.$$

Hence, for some $a > 0$,

$$e(t) = \exp(at^{\frac{n-2}{n-4}}) \Big(e(t_0) + \int_{t_0}^{t} s^{\frac{2}{n-4}} f(s) \exp(-as^{\frac{n-2}{n-4}}) ds \Big).$$

The only way in which $e(t)$ does not grow exponentially in time (and hence $\phi(y,t)$ does not growth exponentially in time) is for the specific value of initial condition

$$e(t_0) = \int_{\mathbb{R}^n} \phi(y,t_0) Z_0(y)\, dy = -\int_{t_0}^{\infty} s^{\frac{2}{n-4}} f(s) \exp(-as^{\frac{n-2}{n-4}}) ds.$$

This argument suggests that the (small) initial condition required for ϕ should lie on a certain manifold locally described as a translation of the hyperplane orthogonal to $Z_0(y)$. Since we have k of these hyperplanes, for any $j = 1,\ldots,k$ in (I), these constraints define a *codimension k manifold* of initial conditions which describes those for which the expected asymptotic bubbling behavior is possible.

A central point of the full proof is to design a linear theory that allows us to solve the final system by means of a contraction mapping argument. For a large number $R > 0$ we shall construct a solution to an initial value problem of the form

$$\phi_\tau = \Delta\phi + pU(y)^{p-1}\phi + h(y,\tau) \quad \text{in } B_{2R} \times (\tau_0, \infty) \tag{2.47}$$

$$\phi(y,\tau_0) = e_0 Z_0(y) \quad \text{in } B_{2R}.$$

We define

$$\|h\|_{\nu,a} := \sup_{\tau>\tau_0} \sup_{y\in B_{2R}} \tau^\nu (1+|y|^a)\, |h(y,\tau)|. \tag{2.48}$$

for a suitable number ν.

The following is a central step in the proof:

Lemma 2.2.1 *Let $0 < a < 1$, $\nu > 0$. Then, for all sufficiently large $R > 0$ and any $h = h(y,\tau)$ with $\|h\|_{\nu,2+a} < +\infty$ that satisfies for all $j = 1,\ldots,n+1$*

$$\int_{B_{2R}} h(y,\tau)\, Z_j(y)\, dy \;=\; 0 \quad \textit{for all} \quad \tau \in (\tau_0,\infty) \tag{2.49}$$

there exist $\phi = \phi[h]$ and $e_0 = e_0[h]$ which solve Problem (2.47). They define linear operators of h that satisfy the estimates

$$|\phi(y,\tau)| \;\lesssim\; \tau^{-\nu} \frac{R^{n+1-a}}{1+|y|^{n+1}} \|h\|_{\nu,2+a}. \tag{2.50}$$

and

$$|e_0[h]| \;\lesssim\; \|h\|_{\nu,2+a}. \tag{2.51}$$

After this lemma is proven, the remaining argument roughly goes as follows: We solve (this is rather straightforward) the outer problem (2.45) for given ϕ_j's in the class of the estimated in Lemma 2.2.1 and parameter functions, in the form $\psi = \Psi(\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi)$. Then replacing these into the inner problems we get for the ϕ_j's the system of equations

$$\partial_\tau \phi_j = \Delta_y \phi_j + pU(y)^{p-1}\phi_j + E_j(\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi)$$

where by definition $\partial_\tau = \mu_{0j}^2 \partial_t$. $E_j(\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi)$ is an operator with the property that the conditions

$$\int_{B_{2R}} E_j(y,\tau)\, Z_l(y)\, dy \;=\; 0 \quad \text{for all} \quad \tau \in (\tau_0, \infty)$$

for all j, l, amount to an explicit system of first order differential equations for the tuple (λ, ξ) (which involve small nonlinear, nonlocal terms). The dependence in ϕ makes the operator E_j a contraction mapping in its dependence in the operator in Lemma 2.2.1 and the result then follows from a fixed point argument. We refer the reader to [11] for the complete argument.

To establish Lemma 2.2.1 we will make use of the following basic, key lemma regarding the quadratic form associated to the linear operator $L_0 = \Delta + pU^{p-1}$,

$$Q(\phi,\phi) := \int \left[|\nabla\phi|^2 - pU^{p-1}|\phi|^2 \right]. \tag{2.52}$$

The next result provides an estimate of the associated second L^2-eigenvalue in a ball B_{2R} with large radius under zero boundary conditions.

There exists a constant $\gamma > 0$ *such that for all sufficiently large* R *and all radially symmetric function* $\phi \in H_0^1(B_{2R})$ *with* $\int_{B_{2R}} \phi Z_0 = 0$ *we have*

$$\frac{\gamma}{R^{n-2}} \int_{B_{2R}} |\phi|^2 \;\le\; Q(\phi,\phi). \tag{2.53}$$

To prove this, we let H_R be the linear space of all radial functions $\phi \in H_0^1(B_{2R})$ that satisfy the orthogonality condition $\int_{B_{2R}} \phi Z_0 = 0$, and

$$\lambda_R := \inf \left\{ Q(\phi,\phi) \;/\; \phi \in H_R,\ \int_{B_{2R}} |\phi|^2 = 1 \right\}. \tag{2.54}$$

A standard compactness argument yields that λ_R in (2.54) is achieved by a radial function $\phi_R(x) = \psi_R(r) \in H_R$ with $\int_{B_{2R}} \phi_R^2 = 1$, which satisfies the equation

$$L_0[\phi_R] + \lambda_R \phi_R = c_R Z_0 \quad \text{in } B_{2R}, \quad \phi_R = 0 \quad \text{on } \partial B_{2R}, \tag{2.55}$$

for a suitable Lagrange multiplier c_R.

We have that $\lambda_R \geq 0$. Indeed, the radial eigenvalue problem in $\mathbb{R}^n$

$$\mathcal{L}_0[\psi] + \lambda\psi = 0, \quad \psi'(0) = \psi(+\infty) = 0 \tag{2.56}$$

where

$$\mathcal{L}_0[\psi] := \psi'' + \frac{n-1}{r}\psi' + pU(r)^{p-1}\psi$$

has just one negative eigenvalue, as it follows from maximum principle, using the fact that $\mathcal{L}_0[Z] = 0$ with $Z = Z_{n+1}$, and the fact that this function changes sign just once. It follows that the associated quadratic form must be positive in $H^1(\mathbb{R}^n)$-radial, subject to the L^2-orthogonality condition with respect to Z_0. This implies $\lambda_R \geq 0$.

Thus, to establish (2.53), we assume by contradiction that

$$\lambda_R = o(R^{2-n}) \quad \text{as} \quad R \to +\infty. \tag{2.57}$$

Let χ be a smooth cut-off function with

$$\chi(s) = 1 \text{ for } s < 1 \text{ and } \chi(s) = 0 \text{ for } s > 2. \tag{2.58}$$

Testing against $Z_0(y)\eta_R(|y|)$ where $\eta_R(s) = \chi(s - \frac{R}{2})$, we get

$$c_R \int_{B_{2R}} Z_0^2 \eta_R = \int_{B_{2R}} \phi_R[Z_0 \Delta\eta_R + 2\nabla\eta_R \nabla Z_0].$$

Since $\|\phi_R\|_{L^2(B_{2R})} = 1$, it follows that, for some $\sigma > 0$, $c_R = O(e^{-\sigma R})$.

On the other hand, again using that $\|\phi_R\|_{L^2(B_{2R})} = 1$, standard elliptic estimates yield that $\|\phi_R\|_{L^\infty(B_{2R})} \lesssim 1$.

Let us represent $\phi_R(x) = \psi_R(r)$ using the variation of parameters formula. The function ψ_R satisfies the ODE

$$\mathcal{L}_0[\psi_R] = h_R(r), \quad r \in (0, R), \quad \psi_R'(0) = \psi_R(R) = 0 \tag{2.59}$$

where $h_R(r) = -\lambda_R\psi_R(r) + c_R Z_0(r)$. Furthermore, it is uniformly bounded in R.

Letting $Z = Z_{n+1}$, we consider a second, linearly independent solution $\tilde{Z}(r)$ of this problem, namely $\mathcal{L}_0[\tilde{Z}] = 0$, normalized in such a way that their Wronskian satisfies

$$\tilde{Z}'Z(r) - \tilde{Z}Z'(r) = \frac{1}{r^{n-1}}.$$

Since $Z(r) \sim 1$ near $r = 0$ and $Z(r) \sim r^{2-n}$ as $r \to \infty$, we see that $\tilde{Z}(r) \sim r^{2-n}$ near $r = 0$ and $\tilde{Z}(r) \sim 1$ as $r \to \infty$.

The formula of variation of parameters then yields the representation

$$\psi_R(r) = \tilde{Z}(r) \int_0^r h_R(s)\, Z(s)\, s^{n-1}\, ds + Z(r) \int_r^{2R} h_R(s)\, \tilde{Z}(s)\, s^{n-1}\, ds - A_R Z(r) \tag{2.60}$$

where A_R is such that $\psi_R(2R) = 0$, namely

$$A_R = Z(2R)^{-1} \tilde{Z}(2R) \int_0^{2R} h_R(s)\, Z(s)\, s^{n-1}\, ds.$$

We observe that $\|h_R\|_{L^2(B_{2R})} \lesssim \lambda_R + e^{-\sigma R}$. Then we estimate

$$\left| \int_0^r h_R(s)\, Z(s)\, s^{n-1}\, ds \right| \ \le\ \|Z\|_{L^2(B_{2R})} \|h_R\|_{L^2(B_{2R})} \ \lesssim\ (\lambda_R + e^{-\sigma R}) \|Z\|_{L^2(B_{2R})}$$

and

$$\left| \int_r^{2R} h_R(s)\, \tilde{Z}(s)\, s^{n-1}\, ds \right| \ \lesssim\ R^{\frac{n}{2}} (\lambda_R + e^{-\sigma R}).$$

Hence we have for instance,

$$\|A_R Z\|_{L^2(B_{2R})} \ \lesssim\ R^{n-2} (\lambda_R + e^{-\sigma R}) \|Z\|_{L^2(B_{2R})},$$

and estimating the other two terms we obtain at last,

$$\|\phi_R\|_{L^2(B_{2R})} \le R^{n-2} (\lambda_R + e^{-\sigma R}) \|Z\|_{L^2(B_{2R})}. \tag{2.61}$$

At this point we notice $\|Z\|_{L^2(\mathbb{R}^n)} < +\infty$. This, the fact that $\lambda_R = o(R^{2-n})$ and then $\|\phi_R\|_{L^2(B_{2R})} \to 0$. This is a contradiction and estimate (2.53) is thus proven.

How is this fact used in the complete argument? We solve first a projected problem of the form (just in the radial case for now)

$$\phi_\tau = \Delta\phi + pU(r)^{p-1}\phi + h(r,\tau) - c(\tau) Z_0 \quad \text{in } B_{2R} \times (\tau_0, \infty) \tag{2.62}$$

$$\phi = 0 \quad \text{on } \partial B_{2R} \times (\tau_0, \infty), \quad \phi(\cdot, \tau_0) = 0 \quad \text{in } B_{2R}.$$

where h decays fast and the function $c(\tau)$ is such that

$$\int_{B_{2R}} \phi(\cdot, \tau)\, Z_0 \ =\ 0 \quad \text{for all} \quad \tau \in (\tau_0, \infty). \tag{2.63}$$

We obtain the relation

$$\partial_\tau \int_{B_{2R}} \phi^2 + Q(\phi,\phi) = \int_{B_{2R}} g\phi, \quad g = h_0 - c(\tau)Z_0.$$

Using the estimate (2.53) we get that for some $\gamma > 0$,

$$\partial_\tau \int_{B_{2R}} \phi^2 + \frac{\gamma}{R^{n-2}} \int_{B_{2R}} \phi^2 \lesssim R^{n-2} \int_{B_{2R}} g^2. \tag{2.64}$$

Using that $\tilde\phi(\cdot,\tau_0) = 0$ and Gronwall's inequality, we readily get from the L^2-estimate

$$\|\phi(\cdot,\tau)\|_{L^2(B_{2R})} \lesssim \tau^{-\nu} R^{n-2} K, \quad K := \left[\|h\|_{0,\nu} + e^{-\gamma R}\|\nabla_y \tilde\phi\|_{0,\nu}\right]. \tag{2.65}$$

Using standard parabolic estimates, we obtain the desired result in Lemma 2.2.1.

2.2.5 The Cases of Dimensions $n = 4$ and $n = 3$

The construction above corresponds to dimension $n \geq 5$. Here we state the form the bubbling solutions take in lower dimensions $n = 3, 4$. For $n = 4$ the statement is similar, with different blow-up rates, but it qualitatively changes in dimension 3.

Let start with $n = 4$. In this case our result reads as follows.

Theorem 2 *Assume $n = 4$. Let $q_1, \cdots, q_k$ be distinct points in Ω such that the matrix $\mathcal{G}(q)$ is positive definite. Then there exist smooth functions $\xi_j(t) \to q_j$ and $0 < \mu_j(t) \to 0$, as $t \to +\infty$, $j = 1, \dots, k$, and a solution of Problem (2.1) of the form*

$$u(x,t) = \sum_{j=1}^{k} \alpha_4 \left(\frac{\mu_j(t)}{\mu_j(t)^2 + |x - \xi_j(t)|^2} \right) + \theta(t,x),$$

where $\|\theta(\cdot,t)\|_\infty \to 0$ as $t \to +\infty$. The functions $\mu_j(t)$ satisfy

$$\mu_j(t) = \beta_4 \Lambda_j e^{-\beta_4^{-2} t^{\frac12}} t^{\frac14} + o(e^{-\beta_4^{-2} t^{\frac12}} t^{\frac14}) \quad \text{as } t \to +\infty$$

for a certain (explicit) positive constant β_4. Here $\Lambda_j = \frac{v_j}{\|v\|} > 0$, where $v = (v_1, \dots, v_k)$ is an eigenvector associated to the first positive eigenvalue γ_1 of the matrix $\mathcal{G}(q)$.

To state the result in dimension $n = 3$, we need to introduce another Green's function. Let λ_1 be the first (positive) eigenvalue of the Laplace operator, with zero Dirichlet boundary condition on Ω and let γ be a fixed number with $\gamma \in [0, \lambda_1)$, we denote by G_γ the Green's function for the boundary value problem

$$\begin{aligned} -\Delta_x G_\gamma(x,y) - \gamma G_\gamma(x,y) &= \alpha_3 \omega_3 \delta(x-y), \quad x \in \Omega, \\ G(\cdot, y) &= 0, \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.66}$$

where ω_3 is the area of the unit sphere in $\mathbb{R}^3$ and α_3 the number in (2.15). We let $H_\gamma(x,y)$ be the regular part of $G_\gamma(x,y)$ namely the solution of the problem

$$\Delta_x H_\gamma(x,y) + \gamma H_\gamma(x,y) = \gamma \frac{\alpha_3}{|x-y|} \quad \text{in } \Omega, \quad H_\gamma(\cdot, y) = \frac{\alpha_3}{|x-y|} \quad \text{in } \partial\Omega. \tag{2.67}$$

Let $q_1, \ldots, q_k$ be given distinct points in Ω so that the matrix $\mathcal{G}(q)$ is positive definite. Define now a new matrix $\mathcal{G}_\gamma(q)$ as

$$\mathcal{G}_\gamma(q) = \begin{bmatrix} H_\gamma(q_1,q_1) & -G_\gamma(q_1,q_2) & \cdots & -G_\gamma(q_1,q_k) \\ -G_\gamma(q_1,q_2) & H_\gamma(q_2,q_2) & -G_\gamma(q_2,q_3) \cdots & -G_\gamma(q_3,q_k) \\ \vdots & & \ddots & \vdots \\ -G_\gamma(q_1,q_k) & \cdots & -G_\gamma(q_{k-1},q_k) & H_\gamma(q_k,q_k) \end{bmatrix} \tag{2.68}$$

Observe that $\mathcal{G}_0(q) = \mathcal{G}(q)$. Since, for any i, $H_\gamma(q_i,q_i) \to -\infty$, as $\gamma \uparrow \lambda_1$, the matrix $\mathcal{G}_\gamma(q)$ becomes positive definite as $\gamma \uparrow \lambda_1$. We define

$$\gamma^*(q) = \sup\{\gamma > 0 \,:\, \mathcal{G}_\gamma(q) \quad \text{is positive definite}\}. \tag{2.69}$$

Clearly $0 < \gamma^* < \lambda_1$. Furthermore, there exists a vector $b = (b_1, \ldots, b_k)$ such that

$$\mathcal{G}_{\gamma^*}(q)[b] = 0, \quad \text{and} \quad b_i > 0 \quad \text{for all} \quad i. \tag{2.70}$$

Indeed, by definition of γ^* we see that there exists $b \in \mathbb{R}^k$ with $\|b\| = 1$ such that

$$b^T \mathcal{G}_{\gamma^*}[q] b = \inf_{x \in \mathbb{R}^k,\, \|x\|=1} x^T \mathcal{G}_{\gamma^*}[q] x = 0.$$

We observe first that all components of b are positive, $b_i \geq 0$, $i = 1, \ldots, k$. If not, we consider $\tilde{b} = (|b_1|, \ldots, |b_k|)$ and observe that

$$b^T \mathcal{G}_{\gamma^*}[q] b \geq \tilde{b}^T \mathcal{G}_{\gamma^*}[q] \tilde{b}.$$

In order to show that $b_i > 0$ for all i, we assume the contrary, we call $I = \{i \,:\, b_i = 0\}$ and we define the vector $b^\varepsilon = (b_1^\varepsilon, \ldots, b_k^\varepsilon)$ with

$$b_i^\varepsilon = \varepsilon, \quad \text{if} \quad i \in I, \quad b_i^\varepsilon = b_i \quad \text{otherwise},$$

for $\varepsilon > 0$ fixed. A direct computation gives that

$$(b^\varepsilon)^T \mathcal{G}_{\gamma^*}[q] b^\varepsilon = b^T \mathcal{G}_{\gamma^*}[q] b + \varepsilon^2 \sum_{i\in I} H(q_i, q_i) - \varepsilon \sum_{i\in I, j\notin I} G(q_i, q_j) < b^T \mathcal{G}_{\gamma^*}[q] b$$

if ε is chosen small enough. We thus reach a contradiction and the claim is proven.

We can now state our result.

Theorem 3 *Assume $n = 3$. Let $q_1, \cdots, q_k$ be distinct points in Ω such that the matrix $\mathcal{G}(q)$ is positive definite. Then there exist smooth functions $\xi_j(t) \to q_j$ and $0 < \mu_j(t) \to 0$, as $t \to +\infty$, $j = 1, \ldots, k$, and a solution of Problem (2.1) of the form*

$$u(x,t) = \sum_{j=1}^{k} \alpha_3 \left(\frac{\mu_j(t)}{\mu_j(t)^2 + |x - \xi_j(t)|^2} \right)^{\frac{1}{2}} + \theta(t,x),$$

where $\|\theta(\cdot,t)\|_\infty \to 0$ as $t \to +\infty$. The functions $\mu_j(t)$ satisfy

$$\mu_j(t) = \frac{1}{2\sqrt{\gamma_*}} b_j e^{-2\gamma_* t} + o(e^{-2\gamma_* t}) \quad \text{as } t \to +\infty.$$

In the case Ω is the unit ball in $\mathbb{R}^3$, the number γ^* is explicit, it is given by $\gamma_* = \frac{\pi^2}{4}$. Recalling that, in this case, the first eigenvalue λ_1 of the Laplace operator with zero Dirichlet boundary condition is given by π^2, the previous asymptotic becomes

$$\ln \|u\|_\infty \sim \frac{\pi^2}{4} t = \frac{\lambda_1}{4} t,$$

and we recover the asymptotics in [16] obtained in the radial case.

2.3 The Harmonic Map Flow from $\mathbb{R}^2$ into S^2

The results presented in this section for Problem (2.2) correspond to joint work with Juan Dávila and Juncheng Wei.

2.3.1 Preliminaries and Statement of Main Result

We summarize some characteristics of the flow given by Eq. (2.2), some of them we already commented in the introductory section.

- Local existence and uniqueness of a classical solution of (2.2) was established in the works by Eeels-Sampson [14], Struwe [32] and Chang [8]. In fact, a solution of the equation satisfies $|u(x,t)| = 1$ at all times if initial and boundary conditions do.
- Problem (2.2) is the negative L^2-gradient flow for the Dirichlet energy $E(u) := \int_\Omega |\nabla u|^2 dx$. along smooth solutions $u(x,t)$:

$$\frac{d}{dt}E(u(\cdot,t)) \;=\; -\int_\Omega |u_t(\cdot,t)|^2 \;\leq 0\,.$$

- The problem has blowing-up families of **energy invariant steady states** in entire space (entire harmonic maps). Harmonic maps in $\mathbb{R}^2$ are solutions of

$$\Delta u + |\nabla u|^2 u = 0, \quad |u| = 1 \text{in } \mathbb{R}^2$$

for which the simplest nontrivial example is the inverse of the stereographic map,

$$U_0(x) = \begin{pmatrix} \frac{2x}{1+|x|^2} \\ \frac{|x|^2-1}{1+|x|^2} \end{pmatrix}, \quad x \in \mathbb{R}^2.$$

The *1-corrotational harmonic maps* are given by

$$U_{\lambda,x_0,Q}(x) = QU_0\left(\frac{x-x_0}{\lambda}\right)$$

with Q a linear orthogonal transformation of $\mathbb{R}^3$.

$$E_2(U_{\lambda,x_0,Q}) = E(U) = 4\pi \quad \text{for all} \quad \lambda, x_0.$$

- Struwe [32] proved the following important result: there exists a global H^1-weak solution of (2.2), where just for a finite number of points in space-time loss of regularity occurs. In fact, at those times jumps down in energy occur. This solution is unique within the class of weak solutions with degreasing energy [15].

  If $T > 0$ designates the first instant at which smoothness is lost, we must have

$$\|\nabla u(\cdot,t)\|_\infty \;\to\; +\infty$$

Several works have clarified the possible blow-up profiles as $t \uparrow T$.

The following fact follows from results in the works [12, 22, 26, 27, 32]:

Along a sequence $t_n \to T$ and points $q_1, \ldots, q_k \in \Omega$, not necessarily distinct, $u(x,t_n)$ blows-up occurs at exactly those k points in the form of *bubbling*. Precisely, we have

$$u(x,t_n) - u_*(x) - \sum_{i=1}^{k} [\, U_i\left(\frac{x-q_i^n}{\lambda_i^n}\right) - U_i(\infty)\,] \to 0 \quad \text{in } H^1(\Omega)$$

where $u_* \in H^1(\Omega)$, $q_i^n \to q_i$, $0 < \lambda_i^n \to 0$, satisfy for $i \neq j$,

$$\frac{\lambda_i^n}{\lambda_j^n} + \frac{\lambda_j^n}{\lambda_i^n} + \frac{|q_i^n - q_j^n|^2}{\lambda_i^n \lambda_j^n} \ \to\ +\infty.$$

The U_i's are entire, finite energy harmonic maps, namely solutions $U : \mathbb{R}^2 \to S^2$ of the equation

$$\Delta U + |\nabla U|^2 U = 0 \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} |\nabla U|^2 < +\infty.$$

After stereographic projection, U lifts to a conformal smooth map in S^2, so that its value $U(\infty)$ is well-defined. It is known that U is in correspondence with a complex rational function or its conjugate. Its energy corresponds to the absolute value of the degree of that map times the area of the unit sphere, and hence

$$\int_{\mathbb{R}^2} |\nabla U|^2 = 4\pi m, \quad m \in \mathbb{N}.$$

In particular, $u(\cdot, t_n) \rightharpoonup u_*$ in $H^1(\Omega)$ and for some positive integers m_i, we have

$$|\nabla u(\cdot, t_n)|^2 \ \rightharpoonup\ |\nabla u_*|^2 + \sum_{i=1}^{k} 4\pi m_i\, \delta_{q_i}$$

δ_q denotes the Dirac mass at q.

A *least energy* entire, non-trivial harmonic map is given by

$$U_0(x) \ =\ \frac{1}{1+|x|^2}\begin{pmatrix} 2x \\ |x|^2 - 1 \end{pmatrix}, \ x \in \mathbb{R}^2,$$

which satisfies

$$\int_{\mathbb{R}^2} |\nabla U_0|^2 = 4\pi, \quad U_0(\infty) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Very few examples are known of solutions which exhibit the singularity formation phenomenon, and all of them concern single-point blow-up in radially symmetric *corrotational* classes.

When Ω is a disk or the entire space, a 1-corrotational solution of (2.2) is one of the form

$$u(x,t) = \begin{pmatrix} e^{i\theta} \sin v(r,t) \\ \cos v(r,t) \end{pmatrix}, \quad x = re^{i\theta}.$$

Problem (2.2) then reduces to the simple looking scalar equation under radial symmetry,

$$v_t = v_{rr} + \frac{v_r}{r} - \frac{\sin v \cos v}{r^2}.$$

We observe that the function $w(r) = \pi - 2\arctan(r)$ is a steady state corresponding to to the harmonic map U_0:

$$U_0(x) = \begin{pmatrix} e^{i\theta} \sin w(r) \\ \cos w(r) \end{pmatrix}.$$

Chang et al. in 1991 [9] found the first example of a blow-up solution of Problem (2.2) (which was previously conjectured not to exist). It is a 1-corrotational solution in a disk with the blow-up profile $v(r,t) \sim w\left(\frac{r}{\lambda(t)}\right)$ or

$$u(x,t) \sim U_0\left(\frac{x}{\lambda(t)}\right).$$

and $0 < \lambda(t) \to 0$ as $t \to T$. No information is provided on $\lambda(t)$.

Topping [36] estimated the general blow-up rates as

$$\lambda_i = o(T-t)^{\frac{1}{2}}$$

(valid in more general targets), namely blow-up is of “type II”: it does not occur at a self-similar rate. Angenent et al. [1] estimated the blow-up rate of 1-corrotational maps as $\lambda(t) = o(T-t)$.

Using formal analysis, van den Berg et al. [38] demonstrated that this rate for 1-corrotational maps should generically be given by

$$\lambda(t) \sim \kappa \frac{T-t}{|\log(T-t)|^2}$$

for some $\kappa > 0$.

Raphael and Schweyer [29] succeeded to rigorously construct a 1-corrotational solution with this blow-up rate in entire $\mathbb{R}^2$. Their proof provides the **stability** of the blow-up phenomenon within the radially symmetric class.

A natural, important question is the nonradial case: find nonradial solutions, single and multiple blow-up in entire space or bounded domains and analyze their stability.

Our main result: For any given finite set of points of Ω and suitable initial and boundary values, then a solution with a simultaneous blow-up at those points exists, with a profile resembling a translation, scaling and rotation of U_0 around each bubbling point. Single point blow-up is **codimension-1 stable**.

The functions

$$U_{\lambda,q,Q}(x) := QU_0\left(\frac{x-q}{\lambda}\right).$$

with $\lambda > 0$, $q \in \mathbb{R}^2$ and Q an orthogonal matrix are least energy harmonic maps:

$$\int_{\mathbb{R}^2} |\nabla U_{\lambda,q,Q}|^2 = 4\pi.$$

For $\alpha \in \mathbb{R}$ we denote

$$Q_\alpha \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} e^{i\alpha}(y_1 + iy_2) \\ y_3 \end{bmatrix},$$

the α-rotation around the third axis.

Theorem 4 *Given $T > 0$, $q = (q_1, \ldots, q_k) \in \Omega^k$, there exists initial and boundary data such the solution $u_q(x,t)$ of (HMF) blows-up as $t \uparrow T$ in the form*

$$u_q(x,t) - u_*(x) - \sum_{j=1}^{k} Q_{\alpha_i^*}\left[\, U_0\left(\frac{x-q_i}{\lambda_i}\right) - U_0(\infty)\,\right] \to 0$$

in the H^1 and uniform senses where $u_* \in H^1(\Omega) \cap C(\bar{\Omega})$,

$$\lambda_i(t) = \frac{\kappa_i^*(T-t)}{|\log(T-t)|^2}.$$

$$|\nabla u(\cdot,t)|^2 \rightharpoonup |\nabla u_*|^2 + 4\pi \sum_{j=1}^{k} \delta_{q_j}$$

Raphael and Schweyer [29] proved the stability of their solution *within the 1-corrotational class*, namely perturbing slightly its initial condition in the associated radial equation the same phenomenon holds at a slightly different time. On the other hand, numerical evidence led van den Berg and Williams [37] to conjecture that this radial bubbling *loses its stability* if special perturbations off the radially symmetric class are made. Our construction shows so at a linear level.

Theorem 5 *For $k = 1$ there exists a manifold of initial data with codimension 1, that contains $u_q(x, 0)$, which leads to the solution of (2.2) to blow-up at exactly one point close to q, at a time close to T.*

A natural question is that of *Continuation after blow-up.*

Struwe defined a global H^1-weak solution of (2.2) by dropping the bubbles appearing at the blow-up time and then restarting the flow. This procedure modifies the topology of the image of $u(\cdot, t)$ across T. On the other hand, Topping [35] built a continuation of Chang-Ding-Ye solution by *attaching a bubble with opposite orientation* after blow-up (this does not change topology and makes the energy values "continuous"). This procedure is called reverse bubbling. The reverse bubble is by definition

$$\bar{U}_0(x) = \frac{1}{1+|x|^2}\begin{pmatrix} -2x \\ |x|^2 - 1 \end{pmatrix} = \begin{pmatrix} e^{i\theta}\sin\bar{w}(r) \\ \cos\bar{w}(r) \end{pmatrix}, \quad \bar{w}(r) = -w(r).$$

Our result is the following.

Theorem 6 *The solution u_q can be continued as an H^1-weak solution in $\Omega\times(0, T+\delta)$, with the property that $u_q(x, T) = u_*(x)$*

$$u_q(x,t) - u_*(x) - \sum_{j=1}^{k} Q_{\alpha_i^*}\left[\bar{U}_0\left(\frac{x-q_i}{\lambda_i}\right) - U_0(\infty)\right] \to 0 \quad as\ t \downarrow T,$$

in the H^1 and uniform senses in Ω, where

$$\lambda_i(t) = \kappa_i^* \frac{t-T}{|\log(t-T)|^2} \quad if\ t > T.$$

It is reasonable to think that the blow-up behavior obtained is generic. Is it possible to have bubbles other than those induced by U_0 or $\bar{U}_0$, and or decomposition in several bubbles at the same point? Evidence seems to indicate the opposite:

In fact, no blow-up is present in the higher corrotational class (Guan et al. [20]) and no *bubble trees* in finite time exist in the 1-corrotational class, van der Hout [39]. In infinite time they do exist and their elements have been classified (Topping [36]).

2.3.2 Sketch of the Construction of Bubbling Solution for $k = 1$

Here we present the main elements present in the construction of a first approximation. Given a $T > 0$, $q \in \Omega$, we want

$$S(u) := -u_t + \Delta u + |\nabla u|^2 u = 0 \quad \text{in } \Omega \times (0, T)$$

with

$$u(x,t) \approx U(x,t) := Q_{\alpha(t)} U_0\left(\frac{x - x_0(t)}{\lambda(t)}\right)$$

The functions $\alpha(t),\ \lambda(t),\ x_0(t)$ are continuous functions up to T and satisfy

$$\lambda(T) = 0, \quad x_0(T) = q$$

Error of approximation:

$$S(U) = -U_t = \frac{\dot\lambda}{\lambda} Q_\alpha \nabla U_0(y)\cdot y - \dot\alpha(\partial_\alpha Q_\alpha) U_0(y) + Q_\alpha \nabla U_0(y)\cdot \frac{\dot x_0}{\lambda},$$

$$y = \frac{x - x_0}{\lambda}.$$

Then $S(U) \perp U$. We recall

$$U_0(y) = \begin{pmatrix} e^{i\theta}\sin w(\rho) \\ \cos w(\rho)\end{pmatrix}, \quad w(\rho) = \pi - 2\arctan(\rho), \quad y = \rho e^{i\theta},$$

$$E_1(y) = \begin{pmatrix} -e^{i\theta}\cos w(\rho) \\ \sin w(\rho)\end{pmatrix}, \quad E_2(y) = \begin{pmatrix} ie^{i\theta} \\ 0\end{pmatrix},$$

constitute an orthonormal basis of the tangent space to S^2 at the point $U_0(y)$.

$$\begin{aligned} S(U)(x,t) = Q_\alpha[\frac{\dot\lambda}{\lambda}\rho w_\rho E_1 \ + \ \dot\alpha \rho w_\rho E_2\,] \\ + \frac{\dot x_{01}}{\lambda}\, w_\rho\, Q_\alpha[\,\cos\theta\, E_1 + \sin\theta\, E_2] \\ + \frac{\dot x_{02}}{\lambda}\, w_\rho\, Q_\alpha[\sin\theta\, E_1 - \cos\theta\, E_2\,]. \end{aligned}$$

For a small function φ, we compute

$$S(U+\varphi) = -\varphi_t + L_U(\varphi) + N_U(\varphi) + S(U).$$

$$L_U(\varphi) = \Delta\varphi + |\nabla U|^2\varphi + 2(\nabla U \nabla\varphi)U$$

$$N_U(\varphi) = |\nabla\varphi|^2 U + 2(\nabla U\nabla\varphi)\varphi + |\nabla\varphi|^2\varphi.$$

A useful observation: if φ with $|U + \varphi| = 1$ solves

$$-U_t - \partial_t \varphi + L_U(\varphi) + N_U(\varphi) + b(x,t)U = 0$$

for some scalar function $b(x,t)$ and $|\varphi| \leq \frac{1}{2}$, then $u = U + \varphi$ solves (2.2) namely $S(u) = 0$. Indeed,

$$S(u) + bU = 0$$

hence

$$-b(x,t)\, U \cdot u \;=\; S(u) \cdot u \;=\; -\frac{1}{2}\frac{d}{dt}|u|^2 + \frac{1}{2}\Delta(|u|^2) = 0.$$

which implies $b = 0$ since $U \cdot u > 0$.

We must have $|U + \varphi|^2 = 1$, namely

$$2U \cdot \varphi + |\varphi|^2 = 0.$$

If φ is small, this approximately means

$$U \cdot \varphi = 0.$$

If we neglect $N_U(\varphi)$ which is quadratic in φ, we look for canceling the linear part (up to terms along U). Thus we want:

$$-\varphi_t + L_U(\varphi) + S(U) + b(x,t)U \approx 0, \quad \varphi \cdot U = 0.$$

We describe a way of finding such a φ for suitable choices of the parameter functions.

For a function φ we write

$$\Pi_{U^\perp}\varphi := \varphi - (\varphi \cdot U)U.$$

We want to find a small function φ^* such that

$$-\partial_t \Pi_{U^\perp}\varphi^* + L_U(\Pi_{U^\perp}\varphi^*) + S(U) + b(x,t)U \approx 0.$$

φ^* will be made out of different pieces. For simplicity we fix

$$x_0 \equiv q, \quad \alpha \equiv 0.$$

Step 1 Concentrating the error. The outer problem: Far away from the concentration point the largest part of the error becomes

$$S(U)(x,t) \;\approx\; \mathcal{E}_0 \;=\; \frac{\dot\lambda}{\lambda}\rho w_\rho(\rho)\, E_1(y) \quad y = \frac{x-x_0}{\lambda} = \rho e^{i\theta}, \quad \rho = |y|.$$

So that we have

$$\mathcal{E}_0 \approx -\frac{2}{r}\begin{bmatrix} e^{i\theta}\dot\lambda \\ 0 \end{bmatrix}, \quad x = q + re^{i\theta}.$$

Set

$$\varphi^0(x,t) = \begin{bmatrix} \phi(r,t)e^{i\theta} \\ 0 \end{bmatrix}$$

so that $\Pi_{U^\perp}\varphi^0 \approx \varphi^0$ away from q.

$$-\partial_t \Pi_{U^\perp}\varphi^0 + L_U[\Pi_{U^\perp}\varphi^0] + \mathcal{E}_0 \approx -\varphi_t + \Delta_x \varphi^0 - \frac{2}{r}\begin{bmatrix} e^{i\theta}\dot\lambda \\ 0 \end{bmatrix}.$$

So we require

$$\phi_t = \phi_{rr} + \frac{\phi_r}{r} - \frac{\phi}{r^2} - \frac{2\dot\lambda}{r} \;=\; 0.$$

We solve this equations setting

$$\phi^0 = \phi_0[-2\dot\lambda]$$

where for a continuous function $p(t)$, $t \in [0,T)$, $\phi = \phi_0[p]$ is the unique solution of the Cauchy problem

$$\phi_t = \phi_{rr} + \frac{\phi_r}{r} - \frac{\phi}{r^2} + \frac{p(t)}{r} \;=\; 0, \quad (r,t) \in (0,\infty)\times(0,T),$$
$$\phi(r,0) = 0, \quad \phi(0,t) = 0 = \phi(+\infty,t).$$

With the aid of Duhamel's formula, we find

$$\phi_0[p](r,t) = \int_0^t p(s)\frac{1 - e^{-\frac{r^2}{4(t-s)}}}{2r}\, ds.$$

and modify the error as

$$\tilde{\mathcal{E}}_0 \;:=\; -\partial_t \Pi_{U^\perp}\varphi^0 + L_U(\Pi_{U^\perp}\varphi^0) + \mathcal{E}_0 \;=$$

At main order we get

$$\tilde{\mathcal{E}}_0 \approx \lambda^{-2}\left[\frac{8\phi^0}{(1+\rho^2)^2} + \frac{2\lambda\dot{\lambda}}{\rho(1+\rho^2)}\right] E_1 \ + \ \phi^0\,\lambda^{-1}\dot{\lambda}\rho w_\rho\, U.$$

Step 2 We add to $\Pi_{U^\perp}\varphi^0$ a small function $\Pi_{U^\perp}Z^*(x,t)$. We consider a small smooth function $z^*(x,t) = z_1^*(x,t) + iz_2^*(x,t)$ which solves the heat equation,

$$\begin{aligned} z_t^* &= \Delta z^*, \quad \text{in } \Omega \times (0,T), \\ z(x,t) &= z_0(x) \quad \text{in } \partial\Omega \times (0,T), \\ z(x,0) &= z_0(x) \quad \text{in } \partial\Omega\ . \end{aligned}$$

And on $z_0^*(x)$ we assume the following. For a point q_0 close to q,

$$\begin{aligned} \operatorname{div} z_0(q_0) &= \partial_{x_1} z_{01}(q_0) + \partial_{x_2} z_{02}(q_0) \ < 0 \\ \operatorname{curl} z_0(q_0) &= \partial_{x_1} z_{02}(q_0) - \partial_{x_2} z_{01}(q_0) \ = \ 0 \\ z_0(q_0) &= 0, \quad Dz_0(q_0) \text{ non-singular.} \end{aligned}$$

We write

$$Z^*(x,t) = \begin{bmatrix} z^*(x,t) \\ 0 \end{bmatrix} = \begin{bmatrix} z_1^* + iz_2^* \\ 0 \end{bmatrix}$$

and compute the linear error

$$\begin{aligned} &- \partial_t \Pi_{U^\perp} Z^* + L_U(\Pi_{U^\perp} Z^*) \\ &- \frac{1}{\lambda}\rho w_\rho^2\big[\operatorname{div} z^* E_1 + \operatorname{curl} z^*\, E_2\big] \\ &\ \frac{1}{\lambda}\rho w_\rho^2\big[\operatorname{div} \bar{z}^* \cos 2\theta + \operatorname{curl} \bar{z}^* \sin 2\theta\,\big] E_1 \\ &\ \frac{1}{\lambda}\rho w_\rho^2\big[\operatorname{div} \bar{z}^* \sin 2\theta - \operatorname{curl} \bar{z}^* \cos 2\theta\,\big] E_2 \\ &+ \ O(\rho^{-2}) \end{aligned}$$

Step 3 The improvement of approximation gets then reduced to finding φ with $\varphi \cdot U = 0$ and

$$\begin{aligned} &-\partial_t(\Pi_{U^\perp}(\varphi^0 + Z^*) + \varphi) + L_U(\Pi_{U^\perp}(\varphi^0 + Z^*) + \varphi) + \mathcal{E} + bU \\ &\quad \approx -\partial_t\varphi + L_U(\varphi) + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + bU = 0 \end{aligned}$$

where

$$\mathcal{E}_1 = \left[\lambda^{-2}\frac{4}{(1+\rho^2)^2}\Big[\,\phi_0[-2\dot\lambda] \,+\, \lambda\rho\,\mathrm{div}\, z^*\,\Big] + \frac{2\lambda^{-1}\dot\lambda}{\rho(1+\rho^2)}\right] E_1$$

$$\mathcal{E}_2 = \frac{4\lambda^{-1}\rho}{(1+\rho^2)^2}\Big\{\big[\, d_1\cos 2\theta + d_2 \sin 2\theta\,\big] E_1 + \big[\, d_1 \sin 2\theta - d_2\cos 2\theta\,\big] E_2\Big\}$$

$$\mathcal{E}_3 = \frac{4\lambda^{-1}\rho}{(1+\rho^2)^2}\,\mathrm{curl}\, z^*\, E_2 + (U\cdot \tilde z^*)\frac{2\lambda^{-1}\dot\lambda\rho}{1+\rho^2}\, E_1 + b(x,t)U + \ O(\rho^{-2})$$

We recall:

$$z^*(q,0) = 0,$$

$$\mathrm{curl}\, z^*(q,0) = 0.$$

$$\mathrm{div}\, z^*(q,0) < 0,$$

In order to find φ which cancels at main order $\mathcal{E}_1$ we consider the problem of finding φ which decays away from the concentration point and satisfies

$$L_U(\varphi) + \mathcal{E}_1 = 0 \quad \varphi\cdot U = 0.$$

the following is a necessary (and sufficient!) condition. We need the orthogonality condition

$$\int_{\mathbb{R}^2} \mathcal{E}_1\cdot Z_{01} = 0$$

where

$$Z_{01} = \rho w_\rho E_1$$

which satisfies $L_U[Z_{01}] = 0$.

After some computation the equation for $\lambda(t)$ becomes approximately

$$\int_0^{t-\lambda^2} \frac{\dot\lambda(s)}{t-s}\, ds \ = \ 4\mathrm{div}\, z^*(q,t)\ .$$

Assuming that $\log\lambda \sim \log(T-t)$ the equation is well-approximated by

$$-\dot\lambda(t)\log(T-t) + \int_0^t \frac{\dot\lambda(s)}{T-s}\, ds \ + \ 4\mathrm{div}\, z^*(q,t) \ = 0.$$

which is explicitly solved as

$$\dot{\lambda}(t) = -\frac{\kappa}{\log^2(T-t)}(1+o(1))$$

The value of κ is precisely that for which

$$\kappa \int_0^T \frac{ds}{(T-s)\log^2(T-s)} \;=\; -4\mathrm{div}\, z^*(q,T).$$

Then if T is small we get the approximation

$$\dot{\lambda}(t) \;\approx\; \dot{\lambda}_0(t) := \frac{4|\log T|}{\log^2(T-t)}\,\mathrm{div}\, z^*(q,T)$$

Since λ decreases to zero as $t \to T^-$, this is where we need the assumption

$$\mathrm{div}\, z^*(q,T) < 0.$$

With this procedure we then get a true reduction of the total error by solving $L_U[\varphi] + \mathcal{E}_j = 0, j = 1,2.$

At last we find a new approximation of the solution of the type

$$U_*(x,t) = U_0\left(\frac{x-q}{\lambda}\right) + \Pi_{U^\perp}[\phi_0[-2\dot{\lambda}] + Z^*(x,t)] + \varphi_*(x,t)$$

where $\varphi_*(x,t)$ is a decaying solution to

$$L_U[\varphi_*] = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3, \quad \varphi_* \cdot U = 0.$$

To solve the full problem we consider consider

$$\lambda(t) = \lambda_0(t) + \lambda_1(t), \quad \alpha(t) = 0 + \alpha_1(t), \quad x_0(t) = q_0 + x_1(t).$$

The true perturbations λ_1, α_1 approximately solve linear equations of the type

$$\int_0^{t-\lambda_0^2} \frac{\dot{\lambda}(s)}{t-s}\,ds \;=\; p_1(t)$$

$$\int_0^{t-\lambda_0^2} \frac{\dot{\alpha}_1(s)\lambda_0(s)}{t-s}\,ds \;=\; p_2(t)$$

which are approximated by

$$-\dot{\lambda}_1(t)\log(T-t) + \int_0^t \frac{\dot{\lambda}_1(s)}{T-s}\,ds \;= p_1(t).$$

$$\dot{\alpha}_1(t)\lambda_0 \log(T-t) \;=\; p_2(t).$$

These equations are actually a weakly coupled system. In particular the value of $\alpha_1(0)$ turns out to depend of the data, at main order in linear way. This sets a constraint in the solution which yields the codimension-one stability of the solution, a situation that confirms the non-radial instability conjecture in [37]. Actually it is so determined $\lambda(0)$, but the degree of freedom given by moving T allows to choose it a priori as an arbitrary small number. That degree of freedom is lost in α. This is what yields the codimension 1 statement. The full construction follows the same lines as that for the critical equation while it is of harder technical nature.

2.4 Infinite Time Blow-up in the Critical Mass Platak-Keller-Segel Equation

We state a result corresponding to joint work with J. Dávila, J. Dolbeault, M. Musso and J. Wei where the same general scheme of the problems in the previous two sections has been followed. For the Platak-Keller-Segel equation (2.3), our main result is existence and stability of the **critical mass solution.**

$$u_t = \Delta u - \nabla\cdot(u\nabla(-\Delta)^{-1}u), \quad u>0 \quad \text{in } \mathbb{R}^2\times(0,\infty)$$

Assuming that $u(x,0)\in L^1(\mathbb{R}^2)$, the following is known:

- If $\int_{\mathbb{R}^2} u(x,0)dx > 8\pi$ then finite-time blow-up always takes place. On the other hand, if $\int_{\mathbb{R}^2} u(x,0)dx < 8\pi$ then the solution is globally defined in time, and it goes to zero uniformly as $t\to\infty$ with a self-similar profile. See Blanchet et al. [2]. Bubbling blow-up behavior in the radial case with exact rates when mass is close from above to 8π have been built by Raphael and Schweyer [30].
- The case of *critical mass* $\int_{\mathbb{R}^2} u(x,0)dx = 8\pi$ is delicate concerning its asymptotic behavior. The solution is globally defined in time and it may or may not blow-up. If the second moment of the initial condition is finite, namely $\int_{\mathbb{R}^2} |x|^2 u(x,0)dx < +\infty$, then the solution blows-up in infinite time, with a bubbling behavior, see Carlen and Figalli [6], Blanchet et al. [3, 4]. Formal rates of bubbling when mass equals 8π have been studied by Chavanis and Sire [10] and by Campos [5]. In the very recent preprint by Ghoul and Masmoudi [18], a radial solution with exact rates has been built. Stability is proven within the radial class and the method does not yield it in general.

Theorem 7 *There exists a solution $u(x, t)$ of Problem (2.3) with fast-decay initial condition of mass 8π, which blows-up in infinite time, with a profile which at main order is*

$$u(x, t) \approx \frac{8\lambda(t)^2}{(\lambda(t)^2 + |x|^2)^2}$$

where

$$\lambda(t) \sim \frac{1}{\sqrt{\log t}}.$$

All positive initial conditions (not necessarily radial) with fast decay and mass 8π suitably close to $u(x, 0)$ lead to the same phenomenon.

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References

1. S.B. Angenent, J. Hulshof, H. Matano, The radius of vanishing bubbles in equivariant harmonic map flow from D^2 toS^2. SIAM J. Math. Anal. **41**, 1121–1137 (2009)
2. A. Blanchet, J. Dolbeault, B. Perthame, Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions. Electron. J. Differ. Equ. **44**, 32 (2006)
3. A. Blanchet, J. Carrillo, N. Masmoudi, Infinite time aggregation for the critical Patlak-Keller-Segel model in $\mathbb{R}^2$. Commun. Pure Appl. Math. **61**, 144–1481 (2008)
4. A. Blanchet, E. Carlen, J. Carrillo, Functional inequalities, thick tails and asymptotics for the critical mass Patlak-Keller-Segel model. J. Funct. Anal. **262**(5), 2142–2230 (2012)
5. J. Campos, Ph.D. Thesis U. Paris Dauphine, 2012
6. E. Carlen, A. Figalli, Stability for a GNS inequality and the log-HLS inequality, with application to the critical mass Keller-Segel equation. Duke Math. J. **162**(3), 579–625 (2013)
7. T. Cazenave, P.-L. Lions, Solutions globales d'équations de la chaleur semi linéaires. Commun. Partial Differ. Equ. **9**, 955–978 (1984)
8. K.-C. Chang, Heat flow and boundary value problem for harmonic maps. Ann. Inst. H. Poincaré Anal. Non Linéaire **6**(5), 363–395 (1989)
9. K.-C. Chang, W.Y. Ding, R. Ye, Finite-time blow-up of the heat flow of harmonic maps from surfaces. J. Differ. Geom. **36**(2), 507–515 (1992)
10. P.-H. Chavanis, C. Sire, Virial theorem and dynamical evolution of self-gravitating Brownian particles in an unbounded domain. I. Overdamped models. Phys. Rev. E (3) **73**(6), 066103, 16 pp. (2006)
11. C. Cortázar, M. del Pino, M. Musso, Green's function and infinite-time bubbling in the critical nonlinear heat equation. J. Eur. Math. Soc. (to appear). Preprint, arXiv:1604.07117
12. W. Ding, G. Tian, Energy identity for a class of approximate harmonic maps from surfaces. Commun. Anal. Geom. **3**(3–4), 543–554 (1995)
13. S.-Z. Du, On partial regularity of the borderline solution of semilinear parabolic equation with critical growth. Adv. Differ. Equ. **18**(1–2), 147–177 (2013)
14. J. Eells Jr., J.H. Sampson, Harmonic mappings of Riemannian manifolds. Am. J. Math. **86**, 109–160 (1964)

15. A. Freire, Uniqueness for the harmonic map flow in two dimensions. Calc. Var. Partial Differ. Equ. **3**(1), 95–105 (1995)
16. V.A. Galaktionov, J.R. King, Composite structure of global unbounded solutions of nonlinear heat equations with critical Sobolev exponents. J. Differ. Equ. **189**(1), 199–233 (2003)
17. V. Galaktionov, J.L. Vazquez, Continuation of blow-up solutions of nonlinear heat equations in several space dimensions. Commun. Pure Appl. Math. **50**, 1–67 (1997)
18. T. Ghoul, N. Masmoudi, Stability of infinite time blow up for the Patlak Keller Segel system. Preprint arXiv:1610.00456
19. Y. Giga, R.V. Kohn, Asymptotically self-similar blow-up of semilinear heat equations. Commun. Pure Appl. Math. **38**(3), 297–319 (1985)
20. M. Guan, S. Gustafson, T.P. Tsai, Global existence and blow-up for harmonic map heat flow. J. Differ. Equ. **246**(1), 1–20 (2009)
21. M.A. Herrero, J.J.L. Velázquez, Explosion de solutions d'equations paraboliques semilineaires supercritiques [Blowup of solutions of supercritical semilinear parabolic equations]. C. R. Acad. Sci. Paris Ser. I Math. **319**(2), 141–145 (1994)
22. F.H. Lin, C.Y. Wang, Energy identity of harmonic map flows from surfaces at finite singular time. Calc. Var. Partial Differ. Equ. **6**, 369–380 (1998)
23. H. Matano, F. Merle, Threshold and generic type I behaviors for a supercritical nonlinear heat equation. J. Funct. Anal. **261**(3), 716–748 (2011)
24. W.-M. Ni, P.E. Sacks, J. Tavantzis, On the asymptotic behavior of solutions of certain quasilinear parabolic equations. J. Differ. Equ. **54**, 97–120 (1984)
25. S.I. Pohozaev, On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. Sov. Math. Dokl. **6**, 1408–1411 (1965)
26. J. Qing, On singularities of the heat flow for harmonic maps from surfaces into spheres. Commun. Anal. Geom. **3**(1–2), 297–315 (1995)
27. J. Qing, G. Tian, Bubbling of the heat flows for harmonic maps from surfaces. Commun. Pure Appl. Math. **50**(4), 295–310 (1997)
28. P. Quittner, P. Souplet, in *Superlinear Parabolic Problems. Blow-up, Global Existence and Steady States*. Birkhauser Advanced Texts (Birkhauser, Basel, 2007)
29. P. Raphaël, R. Schweyer, Stable blowup dynamics for the 1-corotational energy critical harmonic heat flow. Commun. Pure Appl. Math. **66**(3), 414–480 (2013)
30. P. Raphaël, R. Schweyer, On the stability of critical chemotactic aggregation. Math. Ann. **359**(1–2), 267–377 (2014)
31. M. Struwe, A global compactness result for elliptic boundary value problems involving limiting nonlinearities. Math. Z. **187**(4), 511–517 (1984)
32. M. Struwe, On the evolution of harmonic mappings of Riemannian surfaces. Comment. Math. Helv. **60**(4), 558–581 (1985)
33. T. Suzuki, Semilinear parabolic equation on bounded domain with critical Sobolev exponent. Indiana Univ. Math. J. **57**(7), 3365–3396 (2008)
34. G. Talenti, Best constants in Sobolev inequality. Ann. Mat. Pura Appl. (4) **10**, 353–372 (1976)
35. P.M. Topping, Reverse bubbling and nonuniqueness in the harmonic map flow. Int. Math. Res. Not. **10**, 505–520 (2002)
36. P.M. Topping, Winding behaviour of finite-time singularities of the harmonic map heat flow. Math. Z. **247**, 279–302 (2004)
37. J.B. van den Berg, J.F. Williams, (In-)stability of singular equivariant solutions to the Landau-Lifshitz-Gilbert equation. Eur. J. Appl. Math. **24**(6), 921–948 (2013)
38. J.B. van den Berg, J. Hulshof, J.R. King, Formal asymptotics of bubbling in the harmonic map heat flow. SIAM J. Appl. Math. **63**, 1682–1717 (2003)
39. R. van der Hout, On the nonexistence of finite time bubble trees in symmetric harmonic map heat flows from the disk to the 2-sphere. J. Differ. Equ. **192**(1), 188–201 (2003)

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