Chapter 3
Boolean Inverse Semigroups and Additive Semigroup Homomorphisms

Tarski investigates in [109] partial commutative monoids constructed from *partial bijections* on a given set. In Kudryavtseva et al. [71], this study is conveniently formalized in the language of *inverse semigroups*. Further connections can be found in works on K-theory of rings, such as Ara and Exel [7].

By definition, a partial bijection on a set Ω is a bijection from a subset of Ω onto another subset of Ω. Partial bijections can be composed, by letting $(g \circ f)(x)$ be defined if $f(x)$ is defined and belongs to the domain of g. Instead of forming a group, the partial bijections on Ω form an *inverse semigroup*. Moreover, two partial bijections f and g, with disjoint domains and ranges, can be added, by defining their orthogonal join $f \oplus g$ as the smallest common extension of f and g. This brings us naturally to the widely studied concept of a *Boolean inverse semigroup*, in particular Lawson [74, 75], Lawson and Lenz [76] (the original definition of a Boolean inverse semigroup got subsequently extended by no longer assuming the existence of finite meets; see Sect. 3.1.2 for more detail). While the literature contains a number of interesting weakenings of the concept of Boolean inverse semigroup, most notably the one of *distributive inverse semigroup* (cf. Lawson [74], Lawson and Scott [77]), Boolean inverse semigroups will take up most of our discussion, mostly due to our ring-theoretical emphasis and the results of Sect. 6.1.

In Sect. 3.1 we recall a few basic results on inverse semigroups and Boolean inverse semigroups, in particular emphasizing with Proposition 3.1.9 that they are distributive, and beginning the discussion of additivity in Sect. 3.1.3.

In Sect. 3.2, we prove that the category of all Boolean inverse semigroups, with additive semigroup homomorphisms, is identical to a variety of algebras (in the sense of universal algebra) that we call *biases*, with their homomorphisms.

In Sect. 3.3, we discuss the faithfulness of Exel's regular representation, defined for any inverse semigroup, with emphasis on the class of all Boolean inverse semigroups. We also present a variant of this representation which is valid for all distributive inverse semigroups, as a specialization of a duality theorem due to Lawson and Lenz [76].

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In Sect. 3.4 we study the bias congruences of a given Boolean inverse semigroup, in terms of the semigroup operations and the orthogonal join. We also describe the congruence associated with an additive ideal.

Section 3.5 is of preparatory nature, and it introduces a minor extension of the concept of *generalized rook matrices* introduced in Kudryavtseva et al. [71]. The results of that section are applied in Sect. 3.6 to an extension, to the class of all Boolean inverse semigroups, of the ring-theoretical concept of *crossed product*.

Section 3.7 introduces some material on two subclasses of Boolean inverse semigroups, called *fundamental Boolean inverse semigroups* and *Boolean inverse meet-semigroups*.

Section 3.8 is devoted to a brief study of *inner automorphisms* of a Boolean inverse semigroup, which can be defined even in the non-unital case.

Our main textbook references on inverse semigroups will be Howie [60] and Lawson [73].

3.1 Boolean Inverse Semigroups

3.1.1 Arbitrary Inverse Semigroups

We first recall a few classical definitions. Let S be a semigroup (i.e., a set endowed with an associative binary operation). For $x, y \in S$, we say that y is a *quasi-inverse* (resp., an *inverse*) of x if $x = xyx$ (resp., $x = xyx$ and $y = yxy$).

Recall (cf. Howie [60]) that S is

- a *regular semigroup* if every element of S has a quasi-inverse (this is consistent with Definition 1.5.1),
- an *inverse semigroup* if every $x \in S$ has a unique inverse, then denoted by x^{-1}. The assignment $x \mapsto x^{-1}$ is the *inversion map* of S.

Every semigroup homomorphism between inverse semigroups is also a homomorphism of inverse semigroups. We denote by $\operatorname{Idp} S$ the set of all idempotent elements of S. A regular semigroup S is an inverse semigroup iff all the idempotent elements of S commute (cf. Howie [60, Theorem V.1.2]). In that case, $(xy)^{-1} = y^{-1}x^{-1}$ for all $x, y \in S$, and e idempotent implies that xex^{-1} is also idempotent (cf. Howie [60, Proposition V.1.4]).

For the remainder of this section we shall fix an inverse semigroup S. We set $XY = \{xy \mid (x, y) \in X \times Y\}$, $aX = \{a\} X$, $Xa = X \{a\}$, and $X^{-1} = \left\{x^{-1} \mid x \in X\right\}$, for all $a \in S$ and all $X, Y \subseteq S$.

We set $\mathbf{d}(x) = x^{-1}x$ (the *domain* of x) and $\mathbf{r}(x) = xx^{-1}$ (the *range* of x), for any $x \in S$. Both $\mathbf{d}(x)$ and $\mathbf{r}(x)$ are idempotent.

Recall that *Green's relations* $\mathscr{L}$, $\mathscr{R}$, $\mathscr{D}$, $\mathscr{H}$, and $\mathscr{J}$ can be defined on S by

$$x \mathscr{L} y \quad \text{if} \quad \mathbf{d}(x) = \mathbf{d}(y)\,;$$
$$x \mathscr{R} y \quad \text{if} \quad \mathbf{r}(x) = \mathbf{r}(y)\,,$$

$\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$ (cf. Howie [60, Proposition II.1.3]), $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$, and

$$x \mathscr{J} y \quad \text{if} \quad SxS = SyS\,.$$

Every congruence of S with respect to the semigroup structure is also a congruence with respect to the inverse semigroup structure.

The following very useful lemma, contained in Schein [101], yields an alternate characterization of inverse semigroups. We include a proof for convenience.

Lemma 3.1.1 *Let* $(S, \cdot)$ *be a semigroup and let* $i\colon S \to S$ *be a map satisfying the following conditions:*

(I1) $x = x \cdot i(x) \cdot x$ *for all* $x \in S$.
(I2) $i(x \cdot y) = i(y) \cdot i(x)$ *for all* $x, y \in S$.
(I3) $i(i(x)) = x$ *for all* $x \in S$.
(I4) $i(x) \cdot x \cdot x \cdot i(x) = x \cdot i(x) \cdot i(x) \cdot x$ *for all* $x \in S$.

Then S *is an inverse semigroup, with inversion map* i.

Proof By applying (I1) to $i(x)$, we obtain, by virtue of (I3),

$$i(x) \cdot x \cdot i(x) = i(x)\,, \quad \text{for all } x \in S\,. \tag{3.1.1}$$

By (I1) and (3.1.1), $i(x)$ is an inverse of x.

We claim that $i(e) = e$, for every idempotent element e of S. Indeed, by applying (I4), we obtain, by virtue of (I3),

$$i(e) \cdot e \cdot e \cdot i(e) = e \cdot i(e) \cdot i(e) \cdot e\,,$$

hence, as e and $i(e)$ are both idempotent (use (I2)),

$$i(e) \cdot e \cdot i(e) = e \cdot i(e) \cdot e\,.$$

By (I1) and (3.1.1), this means that $i(e) = e$, thus proving our claim.

Since every element of S has an inverse [use (I1) and (3.1.1)], it suffices, in order to reach the desired conclusion, to prove that any idempotent elements a and b of S commute. By (I2) together with the claim above,

$$i(a \cdot b) = b \cdot a\,. \tag{3.1.2}$$

By applying (I1) to $x = a \cdot b$, we thus obtain that $a \cdot b = a \cdot b \cdot b \cdot a \cdot a \cdot b$, that is, $a \cdot b = a \cdot b \cdot a \cdot b$, which means that $a \cdot b$ is idempotent. By the claim above, $i(a \cdot b) = a \cdot b$. By (3.1.2), it thus follows that $a \cdot b = b \cdot a$. □

Let $x \leq y$ hold if $x = y\,\mathbf{d}(x)$, for all elements x and y in an inverse semigroup S. This relation is a partial ordering on S, called the *natural ordering of S*. It is compatible with the multiplication and the inversion operation on S (cf. Howie [60, Proposition V.2.4], or Lemma 1.4.6 and Proposition 1.4.7 in Lawson [73]). Various statements equivalent to $x \leq y$ can be found in Howie [60, Proposition V.2.2] or Lawson [73, Proposition 1.4.6]: $x = \mathbf{r}(x)y$; $x = ye$ for some idempotent e; $x = ey$ for some idempotent e; $\mathbf{r}(x) = yx^{-1}$; $\mathbf{d}(x) = y^{-1}x$; $x = xy^{-1}x$. The set $\operatorname{Idp} S$ of all idempotents of S is a lower subset of $(S, \leq)$.

For $x, y \in S$, let $x \sim y$ hold (we say that x and y are *compatible*) if $x^{-1}y$ and xy^{-1} are both idempotent. Equivalently (cf. Lawson [73, Lemma 1.4.11]), the meet $x \wedge y$ exists in S, $\mathbf{d}(x \wedge y) = \mathbf{d}(x)\,\mathbf{d}(y)$, and $\mathbf{r}(x \wedge y) = \mathbf{r}(x)\,\mathbf{r}(y)$. In that case (see, for example, Lawson [73, Lemma 1.4.12]),

$$x \wedge y = \mathbf{r}(x)y = y\,\mathbf{d}(x) = xy^{-1}x\,. \tag{3.1.3}$$

If $\{x, y\}$ is bounded above, then $x \sim y$; the converse fails for easy examples. A subset A of S is compatible if any two elements of A are compatible.

Definition 3.1.2 Let S be an inverse semigroup with zero and let $x, y \in S$.

(1) We say that x and y are *left orthogonal*, in notation $x \perp_{\mathrm{lt}} y$, if $xy^{-1} = 0$; equivalently, $\mathbf{d}(x)\,\mathbf{d}(y) = 0$.
(2) We say that x and y are *right orthogonal*, in notation $x \perp_{\mathrm{rt}} y$, if $x^{-1}y = 0$; equivalently, $\mathbf{r}(x)\,\mathbf{r}(y) = 0$.
(3) We say that x and y are *orthogonal*, in notation $x \perp y$, if $x \perp_{\mathrm{lt}} y$ and $x \perp_{\mathrm{rt}} y$.

A subset A of S is orthogonal if any two distinct elements of A are orthogonal.

In particular, $x \perp y$ (orthogonality) implies that $x \sim y$ (compatibility).

For a congruence relation $\boldsymbol{\theta}$ on S and elements $x, y \in S$, let $x \leq_{\boldsymbol{\theta}} y$ hold if $x \equiv_{\boldsymbol{\theta}} y\,\mathbf{d}(x)$. Equivalently, $\theta(x) \leq \theta(y)$, where $\theta\colon S \twoheadrightarrow S/\boldsymbol{\theta}$ denotes the canonical projection. Observe, in particular, that since $S/\boldsymbol{\theta}$ is an inverse semigroup, $x \leq_{\boldsymbol{\theta}} y$ and $y \leq_{\boldsymbol{\theta}} x$ iff $x \equiv_{\boldsymbol{\theta}} y$, for all $x, y \in S$.

For any $a \in S$, let $\lambda_a\colon S \to S$, $x \mapsto ax$ and $\rho_a\colon S \to S$, $x \mapsto xa$. The following lemma is well known but we could not trace it back to any particular source. It enables us to reduce order properties of an inverse semigroup to its semilattice of idempotents. We include a proof for convenience.

Lemma 3.1.3 (Folklore) *The following statements hold, for any $a \in S$.*

(1) *The maps λ_a and $\lambda_{a^{-1}}$ restrict to mutually inverse, domain-preserving order-isomorphisms, from $S \downarrow \mathbf{d}(a)$ onto $S \downarrow a$ and from $S \downarrow a$ onto $S \downarrow \mathbf{d}(a)$, respectively. The graphs of those maps are all contained in $\mathscr{L}$.*
(2) *The maps ρ_a and $\rho_{a^{-1}}$ restrict to mutually inverse, range-preserving order-isomorphisms, from $S \downarrow \mathbf{r}(a)$ onto $S \downarrow a$ and from $S \downarrow a$ onto $S \downarrow \mathbf{r}(a)$, respectively. The graphs of those maps are all contained in $\mathscr{R}$.*

Furthermore, all the isomorphisms above preserve orthogonality, and also all existing meets and joins, evaluated in S.

Proof It is clear that λ_a, $\lambda_{a^{-1}}$, ρ_a, $\rho_{a^{-1}}$ are all isotone.

(1) Any $x \in S \downarrow \mathbf{d}(a)$ satisfies $x = \mathbf{d}(a)\,\mathbf{d}(x)$ (so x is idempotent), thus $\lambda_a(x) = a\,\mathbf{d}(x) \leq a$. Furthermore, $\lambda_{a^{-1}}\lambda_a(x) = \mathbf{d}(a)x = x$, and further, by using the idempotence of x, $\mathbf{d}(\lambda_a(x)) = x^{-1}a^{-1}ax = x$, so $(x, \lambda_a(x)) \in \mathscr{L}$. This proves that $\lambda_a[S \downarrow \mathbf{d}(a)] \subseteq S \downarrow a$, $\lambda_{a^{-1}}\lambda_a \restriction_{S\downarrow \mathbf{d}(a)} = \mathrm{id}_{S\downarrow \mathbf{d}(a)}$, and the graph of λ_a is contained in $\mathscr{L}$.

Any $y \in S \downarrow a$ satisfies $y = a\,\mathbf{d}(y)$, thus $\lambda_{a^{-1}}(y) = \mathbf{d}(a)\,\mathbf{d}(y) \leq \mathbf{d}(a)$. Furthermore, $\lambda_a\lambda_{a^{-1}}(y) = aa^{-1}a\,\mathbf{d}(y) = a\,\mathbf{d}(y) = y$, and $\lambda_{a^{-1}}(y) = a^{-1}a\,\mathbf{d}(y) = \mathbf{d}(y)$, so $(y, \lambda_{a^{-1}}(y)) \in \mathscr{L}$. This proves that $\lambda_{a^{-1}}[S \downarrow a] \subseteq S \downarrow \mathbf{d}(a)$, $\lambda_a\lambda_{a^{-1}} \restriction_{S\downarrow a} = \mathrm{id}_{S\downarrow a}$, and the graph of $\lambda_{a^{-1}}$ is contained in $\mathscr{L}$. This completes the proof of (1).

The proof of (2) is symmetric.

For all $x, y, z \in S$, if $x \perp y$, then $(xz)^{-1}yz = z^{-1}(x^{-1}y)z = 0$ and $xz(yz)^{-1} = xzz^{-1}y^{-1} \leq xy^{-1} = 0$, thus $xz \perp yz$. Symmetrically, $zx \perp zy$. In particular, all maps λ_a, $\lambda_{a^{-1}}$, ρ_a, $\rho_{a^{-1}}$ preserve orthogonality. By virtue of Lawson [73, Proposition 1.4.19], all those maps preserve all existing meets evaluated in S.

Let $u \in S$ and let $X \subseteq S \downarrow \mathbf{d}(a)$ such that $u = \bigvee X$ within S. In particular, $u \in S \downarrow \mathbf{d}(a)$. Let $y \in S$ such that $aX \leq y$. Then $x = a^{-1}ax \leq a^{-1}y$, for each $x \in X$. It follows that $X \leq a^{-1}y$, so $u \leq a^{-1}y$, and so $au \leq aa^{-1}y \leq y$. This proves that $au = \bigvee(aX)$ within S.

Let $u \in S$ and let $X \subseteq S \downarrow a$ such that $u = \bigvee X$. In particular, $u \in S \downarrow a$. Let $y \in S$ such that $a^{-1}X \leq y$. Then $x = aa^{-1}x \leq ay$, for each $x \in X$. It follows that $u \leq ay$, so $a^{-1}u \leq a^{-1}ay \leq y$. This proves that $a^{-1}u = \bigvee(a^{-1}X)$ within S.

Therefore, both λ_a and $\lambda_{a^{-1}}$ preserve all existing joins from S. The proofs for ρ_a and $\rho_{a^{-1}}$ are symmetric. □

In particular, since $\mathbf{d}(x) = a^{-1}x$ and $\mathbf{r}(x) = xa^{-1}$ whenever $x \in S \downarrow a$, we obtain the following result, contained in Schein [102, Lemma 1.12]; see also Lawson [73, Proposition 1.4.17].

Lemma 3.1.4 *The maps* $\mathbf{d}$ *and* $\mathbf{r}$ *both preserve all existing meets and joins in S.*

Since the map $\mathbf{d}$ preserves all existing meets and joins, it follows that $\mathrm{Idp}\, S$ *is closed under all existing meets and joins in S.*

3.1.2 Boolean Inverse Semigroups

Definition 3.1.5 (Orthogonal Join in an Inverse Semigroup with Zero) For elements x, y, z in an inverse semigroup S with zero, let $z = x \oplus y$ hold if $z = x \vee y$ in S and $x \perp y$.

Definition 3.1.6 An inverse semigroup S is

– *distributive* if $\mathrm{Idp}\, S$ is a distributive lattice and $x \vee y$ exists for all compatible $x, y \in \mathrm{Idp}\, S$;

– *Boolean* if $\operatorname{Idp} S$ is a generalized Boolean lattice and $x \vee y$ exists for all compatible $x, y \in \operatorname{Idp} S$.

Although distributive inverse semigroups will be met on an occasional basis throughout the present work, Boolean inverse semigroups will be given the lion's share.

It is well known that an inverse semigroup S is Boolean iff $\operatorname{Idp} S$ is a generalized Boolean lattice and $x \vee y$ exists for all *orthogonal* $x, y \in \operatorname{Idp} S$ (thus $x \vee y = x \oplus y$). The original definition of a Boolean inverse semigroup (cf. Lawson [74]) assumed that the natural ordering is a meet-semilattice. This definition got subsequently relaxed, by dropping the meet-semilattice assumption (cf. Lawson [75]). In the latter paper, inverse semigroups for which the natural ordering is a meet-semilattice are called *inverse* $\wedge$*-semigroups* (cf. Definition 3.7.7). Proposition 3.1.9 shows, in particular, that our Boolean inverse semigroups are identical to Lawson's Boolean inverse semigroups from [75].

As the following example shows, those concepts are stronger than the eponymous one introduced in Exel [41]. For further discussion about this, see Sect. 3.2.

Example 3.1.7 Table 3.1 describes a finite commutative inverse monoid S with zero, such that $\operatorname{Idp} S$ is the Boolean semilattice with two atoms, but the atoms of S have no join.

The atoms of S are 1 and 2. They are both idempotent, and they join to 4 in $\operatorname{Idp} S$. However, they do not have a join in S.

Another inverse monoid, which is Boolean in Exel's sense but not in ours, is the final example in Exel [41]. This will be discussed further in Sect. 3.2.

Example 3.1.8 The semigroup $\mathfrak{I}_X$, of all partial one-to-one functions between subsets of a set X, is a Boolean inverse monoid, the so-called *symmetric inverse monoid on* X. It has a zero element, namely the function with empty domain (and range). Its unit element is the identity function on X. For $u, v \in \mathfrak{I}_X$, the inequality $u \leq v$ holds iff v extends u. Furthermore, $u \sim v$ iff u and v agree on the intersection of the domains of u and v, and $u \perp v$ iff $\operatorname{dom}(u) \cap \operatorname{dom}(v) = \operatorname{rng}(u) \cap \operatorname{rng}(v) = \varnothing$. If $X = [n] = \{1, \dots, n\}$, for a nonnegative integer n, then we shall write $\mathfrak{I}_n$ instead of $\mathfrak{I}_{[n]}$.

The inverse monoid $\mathfrak{I}_X$ has the additional property that every collection $\mathcal{F}$ of elements of $\mathfrak{I}_X$ has a meet (with respect to the natural ordering), whose domain is the set of all elements of X on which all members of $\mathcal{F}$ agree. In particular, $\mathfrak{I}_X$ is an *inverse meet-semigroup* as introduced further (cf. Definition 3.7.7).

Table 3.1 A non-Boolean inverse monoid with zero, with Boolean semilattice of idempotents

S	0	1	2	3	4
0	0	0	0	0	0
1	0	1	0	1	1
2	0	0	2	2	2
3	0	1	2	4	3
4	0	1	2	3	4

The monoid $\mathfrak{I}_X$ can be viewed as a "skeleton" of $X \times X$ matrix rings. In particular, for $i, j \in X$, the unique function $e_{i,j}$ with domain $\{j\}$ and range $\{i\}$ belongs to $\mathfrak{I}_X$. Denoting by $\delta_{x,y}$ the Kronecker symbol and interpreting $0 \cdot f$ as the empty function (which is the zero element of $\mathfrak{I}_X$), the $e_{i,j}$ satisfy the following relations:

$$e_{i,j}e_{k,l} = \delta_{j,k}e_{i,l}\,, \tag{3.1.4}$$

$$e_{i,j}^{-1} = e_{j,i}\,, \tag{3.1.5}$$

for all $i, j, k, l \in X$. We call the $e_{i,j}$ the *matrix units* of $\mathfrak{I}_X$.

The following result is at the basis of many calculations in distributive and Boolean inverse semigroups.

Proposition 3.1.9 *The following statements hold for any distributive inverse semigroup S, with a zero element required in (2)–(4):*

(1) *For every nonempty finite compatible subset $\{b_1, \dots, b_n\}$ of S, the join $\bigvee_{i=1}^{n} b_i$ exists and the following statements hold:*
   (i) *$a \cdot \bigvee_{i=1}^{n} b_i = \bigvee_{i=1}^{n}(a \cdot b_i)$ and $\left(\bigvee_{i=1}^{n} b_i\right) \cdot a = \bigvee_{i=1}^{n}(b_i \cdot a)$, for every $a \in S$.*
   (ii) *For every $a \in S$, $a \wedge \bigvee_{i=1}^{n} b_i$ exists iff each $a \wedge b_i$ exists, and then $a \wedge \bigvee_{i=1}^{n} b_i = \bigvee_{i=1}^{n}(a \wedge b_i)$.*

(2) *The partial operation $(x, y) \mapsto x \oplus y$ endows S with a structure of a conical partial refinement monoid.*
(3) *The results of (1) above extend to the orthogonal join $\oplus$ in place of the join $\vee$.*
(4) *If $a \oplus c = b \oplus c$ in S, then $a = b$.*

Proof (1i) follows from the finite version of Lawson [73, Proposition 1.4.20].

(1ii) follows from the finite version of Resende [98].

(2) Let $u = (x \oplus y) \oplus z$ in S. From $x \oplus y \perp z$ it follows that $x \perp z$ and $y \perp z$. Since S is distributive, it follows that $y \oplus z$ is defined. By (1ii), it follows that $x \perp y \oplus z$. Since S is distributive, $x \oplus (y \oplus z)$ is defined, with value $x \vee (y \vee z) = (x \vee y) \vee z = u$. Hence, $\oplus$ is associative, so $(S, \oplus, 0)$ is a partial commutative monoid.

Let $x \oplus x' = y \oplus y'$ in S. Applying (1ii), we get the following refinement matrix:

	y	y'
x	$x \wedge y$	$x \wedge y'$
x'	$x' \wedge y$	$x' \wedge y'$

within $(S, \oplus)$. (3.1.6)

Item (2) follows. If $a \oplus c = b \oplus c$, then, taking $x = y = c$, $x' = a$, and $y' = b$ yields $x \wedge y' = x' \wedge y = 0$, thus $x' = y'$. Item (4) follows.

Finally, it is straightforward to obtain (3) from (1). □

As an immediate application of Proposition 3.1.9(1ii), we record the following.

Corollary 3.1.10 *Let S be a distributive inverse semigroup, let m and n be positive integers, and let $a_1, \dots, a_m, b_1, \dots, b_n \in S$. Then $\left(\bigvee_{i=1}^{m} a_i\right) \wedge \left(\bigvee_{j=1}^{n} b_j\right)$ exists iff each $a_i \wedge b_j$ exists, and then*

$$\Bigl(\bigvee_{i=1}^{m} a_i\Bigr) \wedge \Bigl(\bigvee_{j=1}^{n} b_j\Bigr) = \bigvee_{1 \leq i \leq m,\ 1 \leq j \leq n} (a_i \wedge b_j)\,. \tag{3.1.7}$$

Furthermore, if S has a zero element, then the analogue of (3.1.7), with $\vee$ replaced by $\oplus$, also holds.

It is well known (see, for example, Lawson and Lenz [76, Lemma 3.27]) that for any elements x and y in a Boolean inverse semigroup S, if $x \leq y$, then there exists a unique $z \in S$ such that $y = x \oplus z$. Consequently, the natural ordering $\leq$ of S is also the algebraic ordering $\leq^{\oplus}$ of the partial commutative monoid $(S, \oplus, 0)$.

Notation 3.1.11 For any elements x and y in a Boolean inverse semigroup S such that $x \leq y$, we denote by $y \smallsetminus x$ the unique $z \in S$ such that $y = x \oplus z$. The range of this symbol is extended to all pairs (x, y) such that $x \wedge y$ exists, by defining $x \smallsetminus y = x \smallsetminus (x \wedge y)$.

A direct application of Proposition 3.1.9 yields the following result, whose easy proof we omit.

Lemma 3.1.12 *The following statements hold, for every Boolean inverse semigroup S and all $x, y, z \in S$ such that $x \leq y$:*

(1) $z(y \smallsetminus x) = (zy) \smallsetminus (zx)$ *and* $(y \smallsetminus x)z = (yz) \smallsetminus (xz)$.
(2) *If $y \wedge z$ exists, then $x \wedge z$ exists and* $(y \smallsetminus x) \wedge z = (y \wedge z) \smallsetminus (x \wedge z)$.

We will repeatedly use the following easy fact.

Lemma 3.1.13 *Let S be an inverse subsemigroup of a Boolean inverse semigroup T. If S is closed under finite orthogonal joins and $a \smallsetminus b \in S$ whenever $a, b \in \operatorname{Idp} S$ with $b \leq a$, then S is a Boolean inverse semigroup, and $x \smallsetminus y \in S$ whenever $x, y \in S$ with $y \leq x$.*

Proof It follows from our assumptions, together with the identity $\mathsf{x} \smallsetminus \mathsf{y} = \mathsf{x} \smallsetminus \mathsf{xy}$ (in generalized Boolean algebras), that $\operatorname{Idp} S$ is a subsemigroup of $\operatorname{Idp} T$, closed under the operation $(x, y) \mapsto x \smallsetminus y$ and under finite orthogonal joins. Since the latter is a Boolean ring, so is the former.

By assumption, S contains the empty sum 0. Hence, the orthogonality relation on S is the restriction to S of the orthogonality relation on T. By our assumption, $x \oplus y$ exists in S whenever x and y are orthogonal elements of S. Therefore, S is Boolean.

Let $x, y \in S$ with $y \leq x$. From $y \leq x$ it follows that $y = x\,\mathbf{d}(y)$, whence $x \smallsetminus y = x(\mathbf{d}(x) \smallsetminus \mathbf{d}(y))$. By assumption, $\mathbf{d}(x) \smallsetminus \mathbf{d}(y) \in S$; whence $x \smallsetminus y \in S$. $\square$

The following example shows that the additional assumption, in Lemma 3.1.13, that $\operatorname{Idp} S$ be closed under $(x, y) \mapsto x \smallsetminus y$, cannot be dropped.

Example 3.1.14 The powerset algebra T of $\{0, 1\}$, endowed with set intersection, is a Boolean inverse semigroup. The subset $S = \{\varnothing, \{0\}, \{0, 1\}\}$ is an inverse subsemigroup of T, closed under finite orthogonal joins. However, S is not a Boolean inverse semigroup.

3.1.3 Additivity in Boolean Inverse Semigroups

The definition of additivity given below is the restriction, to Boolean inverse semigroups, of a definition by Lawson and Lenz [76].

Definition 3.1.15 Let S and T be Boolean inverse semigroups.

- A semigroup homomorphism $f\colon S \to T$ is *additive* if the equality $f(x \oplus y) = f(x) \oplus f(y)$ holds whenever x and y are orthogonal elements in S.
- A one-to-one map $f\colon S \hookrightarrow T$ is a *lower semigroup embedding* if it is an additive semigroup homomorphism and $f[S]$ is a lower subset of T with respect to the natural ordering.

In particular, every lower semigroup embedding is also a V-embedding (cf. Definition 2.1.2). Further, it is well known, and an easy exercise, that any additive semigroup homomorphism between Boolean inverse semigroups preserves finite compatible joins. We postpone a more complete characterization of additive semigroup homomorphisms until Theorem 3.2.5.

Definition 3.1.16 An inverse subsemigroup S of an inverse semigroup T is an *ideal of* T (resp., a *quasi-ideal of* T) if $TST \subseteq S$ (resp., $STS \subseteq S$).

It is easy to verify that every ideal is a quasi-ideal. Our concept of quasi-ideal is the restriction, to inverse semigroups, of the classical one (cf. Lawson [72]), defined for arbitrary semigroups. The five-element inverse semigroup T represented in Table 3.2, with the subsemigroup $S = \{0, 1\}$, shows that the assumption that S is an inverse subsemigroup of T is not redundant in Definition 3.1.16.

Definition 3.1.17 A subset S in a Boolean inverse semigroup T is

- an *additive inverse subsemigroup of* T if S is an inverse subsemigroup of T, $\mathrm{Idp}\, S$ is closed under the operation $(x, y) \mapsto x \smallsetminus y$, and S is closed under finite orthogonal joins in T;

Table 3.2 The subsemigroup $S = \{0, 1\}$ of T has $STS = S$ and $S^{-1} \neq S$

T	0	1	2	3	4
0	0	0	0	0	0
1	0	0	3	0	1
2	0	4	0	2	0
3	0	1	0	3	0
4	0	0	2	0	4

- a *lower inverse subsemigroup* of T if S is an additive inverse subsemigroup of T and S is a lower subset of T with respect to the natural ordering;
- an *additive ideal* of T if S is an ideal of T and S is closed under finite orthogonal joins in T.

Our additive ideals are called $\vee$-*ideals* in Kudryavtseva et al. [71]. The following result shows that the concepts introduced above occur only between Boolean inverse semigroups.

Proposition 3.1.18 *The following implications hold, for any subset S in a Boolean inverse semigroup T:*

$$S \text{ additive ideal of } T \Rightarrow S \text{ additive quasi-ideal of } T \Rightarrow$$

$$S \text{ lower inverse subsemigroup of } T \Rightarrow S \text{ additive inverse subsemigroup of } T \Rightarrow$$

$$S \text{ Boolean inverse semigroup}\,.$$

Proof It is trivial that any ideal of T is also a quasi-ideal of T. Now suppose that S is a quasi-ideal of T and let $x \in T$ and $y \in S$ such that $x \leq y$. Then $x = \mathbf{r}(y)x\,\mathbf{d}(y) \in STS \subseteq S$. Hence, S is a lower subset of T. Now if S is a lower inverse subsemigroup of T, then $\operatorname{Idp} S$ is a lower subset of $\operatorname{Idp} T$, thus it is closed under the operation $(x, y) \mapsto x \smallsetminus y$. The final implication follows immediately from Lemma 3.1.13. □

Proposition 3.1.19 *Every additive inverse subsemigroup S of a Boolean inverse semigroup T is closed under finite compatible joins.*

Proof Let $x, y \in S$ with $x \sim y$. Using (3.1.3), we get $x \wedge y = y\,\mathbf{d}(x) \in S$. Further, by Lemma 3.1.13, $x \smallsetminus y = x \smallsetminus (x \wedge y)$ belongs to S. It follows that $x \vee y = (x \smallsetminus y) \oplus y \in S$. □

Proposition 3.1.20 *Let X be a subset in a Boolean inverse semigroup S. Then $(SXS)^{\oplus}$ is the smallest additive ideal of S containing X.*

Proof For each $x \in X$, $x = \mathbf{r}(x)x\,\mathbf{d}(x) \in SXS \subseteq (SXS)^{\oplus}$; thus $X \subseteq (SXS)^{\oplus}$. Moreover, SXS is an ideal of S, thus, using Proposition 3.1.9, it follows that $(SXS)^{\oplus}$ is also an ideal of S. This ideal is obviously additive in S. □

Definition 3.1.21 A Boolean inverse semigroup T is an *additive enlargement* of a quasi-ideal S if $T = (TST)^{\oplus}$.

Our concept of additive enlargement is obtained, from the one of enlargement introduced in Lawson [72], by replacing the condition $T = TST$ by the weaker condition $T = (TST)^{\oplus}$. For an interpretation of additive enlargements in terms of the type monoid introduced in Definition 4.1.3, see Theorem 4.2.2. An important class of additive enlargements is given by the following result.

Proposition 3.1.22 *Let e be an idempotent element in a Boolean inverse semigroup S. Then eSe is an additive quasi-ideal of S, and $(SeS)^{\oplus}$ is an additive enlargement of eSe.*

Proof Set $B = \operatorname{Idp} S$. It is trivial that eSe is an inverse subsemigroup of S. Clearly, $\operatorname{Idp}(eSe) = B \downarrow e$ is Boolean. Furthermore, it follows from Proposition 3.1.9 that eSe is closed under finite orthogonal joins. Hence, eSe is an additive inverse subsemigroup of S. It is trivial that $(eSe)S(eSe) \subseteq eSe$. Thus, eSe is an additive quasi-ideal of S.

By Proposition 3.1.20, $(SeS)^{\oplus}$ is the additive ideal of S generated by e. Setting $S' = eSe$ and $T' = (SeS)^{\oplus}$, it is straightforward to verify that $(T'S'T')^{\oplus} = T'$. □

The two following examples show that none of the converse implications in Proposition 3.1.18 holds.

Example 3.1.23 Let $T = \mathfrak{I}_2$ (cf. Example 3.1.8). Then $S = \{\varnothing, \mathrm{id}_{\{1\}}\}$ is an additive quasi-ideal of T, but not an ideal of T.

The following result gives a convenient characterization of lower inverse subsemigroups.

Proposition 3.1.24 *An inverse subsemigroup S of a Boolean inverse semigroup T is a lower inverse subsemigroup of T iff S is closed under finite orthogonal joins and* $\operatorname{Idp} S$ *is a lower subset of* $\operatorname{Idp} T$.

Proof It is sufficient to prove that if $\operatorname{Idp} S$ is a lower subset of $\operatorname{Idp} T$, then S is a lower subset of T. Let $t \leq s$ where $t \in T$ and $s \in S$. Since S is an inverse subsemigroup of T, $\mathbf{d}(s) \in S$. Since $\mathbf{d}(t) \leq \mathbf{d}(s)$ and by assumption, it follows that $\mathbf{d}(t) \in S$. Therefore, $t = s\,\mathbf{d}(t) \in S$. □

In particular, $\operatorname{Idp} S$ is a lower inverse subsemigroup of S, for every Boolean inverse semigroup S.

Example 3.1.25 Let $T = \mathfrak{I}_2$. Then $S = \operatorname{Idp} T = \{\varnothing, \mathrm{id}_{\{1\}}, \mathrm{id}_{\{2\}}, \mathrm{id}_{\{1,2\}}\}$ is a lower inverse subsemigroup of T, but not a quasi-ideal. In fact, $STS = T$.

3.2 The Concept of Bias: An Equational Definition of Boolean Inverse Semigroups

Due to their formulation in terms of the partial operation $\oplus$, the original defining axioms of the class of all Boolean inverse semigroups are not identities in the usual sense of universal algebra. For example, the formula $\mathsf{x}(\mathsf{y} \oplus \mathsf{z}) = (\mathsf{xy}) \oplus (\mathsf{xz})$ makes sense only in case $\mathsf{y} \oplus \mathsf{z}$ and $(\mathsf{xy}) \oplus (\mathsf{xz})$ are both defined. This causes confusion when it comes to handling standard concepts of universal algebra, such as homomorphisms, colimits, or free algebras.

In general, a *similarity type* is a set of "operation symbols" (or just, using a standard abuse of language, "operations"), each one given a nonnegative natural number called the *arity*. Operations with arity zero are usually called *constants*. Formal compositions of operations, starting with variables, are called *terms*. An *identity* is a formal expression of the form $\mathsf{p} = \mathsf{q}$, where p and q are terms. A *variety* is the class of all algebras that satisfy a given set of identities. For more detail, see McKenzie et al. [82].

We shall provide in this section an alternative characterization of Boolean inverse semigroups by a finite set of identities. This characterization will be formulated in the language of inverse semigroups (i.e., a binary operation for the product, a unary operation for the inversion, and a constant for the zero), enriched by two additional binary operations $\otimes$ and $\triangledown$ (cf. Definition 3.2.1). This will enable us to define natural concepts such as homomorphisms, congruences, or free objects, within Boolean inverse semigroups, and more generally, study that class from the vantage point of universal algebra.

Our definition of the operations $\otimes$ and $\triangledown$ is inspired by Leech [78, Example 1.7(c)].

Definition 3.2.1 Let S be a Boolean inverse semigroup. We set

$$x \otimes y = (\mathbf{r}(x) \smallsetminus \mathbf{r}(y))x(\mathbf{d}(x) \smallsetminus \mathbf{d}(y)) \text{ and } x \triangledown y = (x \otimes y) \oplus y\,, \quad \text{for all } x, y \in S\,. \tag{3.2.1}$$

We shall call $x \otimes y$ the *skew difference* and $x \triangledown y$ the *left skew join*—from now on *skew join*[1] —of x and y.

Since B is a Boolean inverse semigroup and $\mathbf{d}(x)$, $\mathbf{d}(y)$, $\mathbf{r}(x)$, $\mathbf{r}(y)$ are all idempotent, both differences $\mathbf{d}(x) \smallsetminus \mathbf{d}(y)$ and $\mathbf{r}(x) \smallsetminus \mathbf{r}(y)$ are always defined, thus $x \otimes y$ is always defined. Furthermore, $\mathbf{r}(y)(x \otimes y) = (x \otimes y)\,\mathbf{d}(y) = 0$, thus $x \otimes y \perp y$, and thus $x \triangledown y$ is also always defined.

Notation 3.2.2 We denote by $\mathcal{L}_{\mathrm{IS}}$ the similarity type of inverse semigroups. It is thus defined as $\mathcal{L}_{\mathrm{IS}} = (0, {}^{-1}, \cdot)$, where 0 is a symbol of constant, ${}^{-1}$ is a symbol of unary operation, and $\cdot$ is a symbol of binary operation.[2]

We also denote by $\mathcal{L}_{\mathrm{BIS}}$ the similarity type obtained by enriching $\mathcal{L}_{\mathrm{IS}}$ with two binary operation symbols $\otimes$ and $\triangledown$.

As the sequel of the present section will involve relatively complicated identities, we shall use a number of abbreviations, such as $\mathbf{d}(\mathsf{x}) = \mathsf{x}^{-1}\mathsf{x}$, $\mathbf{r}(\mathsf{x}) = \mathsf{x}\mathsf{x}^{-1}$, $\mathsf{x}^2 = \mathsf{x} \cdot \mathsf{x}$, $\mathsf{x} \leq \mathsf{y}$ instead of $\mathsf{x} = \mathsf{y}\,\mathbf{d}(\mathsf{x})$, $\mathsf{x} \perp \mathsf{y}$ instead of $\mathsf{x}^{-1}\mathsf{y} = \mathsf{x}\mathsf{y}^{-1} = 0$, and so on. For instance, the identity $\mathsf{x}^{-1}\mathsf{x}\mathsf{y}^{-1}\mathsf{y} = (\mathsf{x}\mathsf{y})^{-1}\mathsf{x}\mathsf{y}$ (which is not valid in all inverse semigroups!) will be abbreviated by $\mathbf{d}(\mathsf{x}\mathsf{y}) = \mathbf{d}(\mathsf{x})\,\mathbf{d}(\mathsf{y})$.

The characterization of the class of all Boolean inverse semigroups by a set of identities will be performed via the following concept.

Definition 3.2.3 A *bias* is a $\mathcal{L}_{\mathrm{BIS}}$-structure $(S, 0, {}^{-1}, \cdot, \otimes, \triangledown)$, that is, a set S together with a distinguished element $0 \in S$, a unary operation $x \mapsto x^{-1}$ on S, and binary operations $(x, y) \mapsto x \cdot y$, $(x, y) \mapsto x \otimes y$, $(x, y) \mapsto x \triangledown y$ on S, subject to the following

[1]The right skew join of x and y could of course be defined as $x \oplus (y \otimes x)$, that is, $y \triangledown x$.

[2]Although strictly speaking, the operation symbols should not be denoted the same way as their interpretations (in a given structure), that confusion is widespread and harmless.

(finite) collection of identities:

(InvSem) Any set of identities defining inverse semigroups with zero. For example, state that $\cdot$ is associative, 0 is a zero element with respect to $\cdot$, $\mathsf{x} = \mathsf{x}\mathsf{x}^{-1}\mathsf{x}$, $(\mathsf{x}^{-1})^{-1} = \mathsf{x}$, and $\mathbf{d}(\mathsf{x})\,\mathbf{d}(\mathsf{y}) = \mathbf{d}(\mathsf{y})\,\mathbf{d}(\mathsf{x})$.

(GBa$_{\oslash,\triangledown}$) All defining identities (1.4.3) of generalized Boolean algebras, with $\wedge$ changed to the product operation $\cdot$, $\smallsetminus$ changed to $\oslash$, $\vee$ changed to $\triangledown$, and x, y, z respectively replaced by $\mathbf{d}(\mathsf{x})$, $\mathbf{d}(\mathsf{y})$, $\mathbf{d}(\mathsf{z})$. For example, the identity $\mathbf{d}(\mathsf{x}) = (\mathbf{d}(\mathsf{x}) \oslash \mathbf{d}(\mathsf{y})) \triangledown (\mathbf{d}(\mathsf{x})\,\mathbf{d}(\mathsf{y}))$, which is the translation of the identity $\mathsf{x} = (\mathsf{x} \smallsetminus \mathsf{y}) \vee (\mathsf{x} \wedge \mathsf{y})$, belongs to the list.

(Idp$_{\oslash,\triangledown}$) $(\mathbf{d}(\mathsf{x}) \oslash \mathbf{d}(\mathsf{y}))^2 = \mathbf{d}(\mathsf{x}) \oslash \mathbf{d}(\mathsf{y})$ and $(\mathbf{d}(\mathsf{x}) \triangledown \mathbf{d}(\mathsf{y}))^2 = \mathbf{d}(\mathsf{x}) \triangledown \mathbf{d}(\mathsf{y})$. This says that the set of all idempotents is closed under both operations $\oslash$ and $\triangledown$.

(Distr$_{\oslash,\triangledown}$) $\mathsf{z}\Big(\big(\mathbf{d}(\mathsf{x}) \oslash \mathbf{d}(\mathsf{y})\big) \triangledown \mathbf{d}(\mathsf{y})\Big) = \mathsf{z}\big(\mathbf{d}(\mathsf{x}) \oslash \mathbf{d}(\mathsf{y})\big) \triangledown \mathsf{z}\,\mathbf{d}(\mathsf{y})$. This states a certain restricted distributivity of the product $\cdot$ on the skew join $\triangledown$.

(Maj$_{\oslash,\triangledown}$) $\mathsf{x} \oslash \mathsf{y} \leq \mathsf{x} \triangledown \mathsf{y}$ and $\mathsf{y} \leq \mathsf{x} \triangledown \mathsf{y}$.

(Dom$_{\triangledown}$) $\mathbf{d}(\mathsf{x} \triangledown \mathsf{y}) = \mathbf{d}(\mathsf{x} \oslash \mathsf{y}) \triangledown \mathbf{d}(\mathsf{y})$.

(Def$_{\oslash}$) $\mathsf{x} \oslash \mathsf{y} = \big(\mathbf{r}(\mathsf{x}) \oslash \mathbf{r}(\mathsf{y})\big)\mathsf{x}\big(\mathbf{d}(\mathsf{x}) \oslash \mathbf{d}(\mathsf{y})\big)$.

The equivalence between the concept of bias on the one hand, and the one of Boolean inverse semigroup on the other hand, is achieved by the following result.

Theorem 3.2.4

(1) *Every Boolean inverse semigroup* $(S, 0, {}^{-1}, \cdot)$ *expands, via the operations* $\oslash$ *and* $\triangledown$ *defined in* (3.2.1)*, to a bias.*

(2) *For every bias* $(S, 0, {}^{-1}, \oslash, \triangledown)$*, the reduct* $(S, 0, {}^{-1}, \cdot)$ *to the similarity type* $\mathcal{L}_{\mathrm{IS}}$ *is a Boolean inverse semigroup.*

(3) *Any two biases on* S *with the same inverse semigroup reduct are equal. In particular, the two operations of expansion and reduction, defined in* (1) *and* (2) *above, are mutually inverse.*

Proof (1) The identities **(InvSem)** and **(Maj$_{\oslash,\triangledown}$)** are both satisfied by definition. For all idempotents $a, b \in S$, $a \cdot b = a \wedge b$, $a \oslash b = a \smallsetminus b$, and $a \triangledown b = a \vee b$, thus, since $\operatorname{Idp} S$ is a generalized Boolean algebra, the identities **(GBa$_{\oslash,\triangledown}$)** and **(Idp$_{\oslash,\triangledown}$)** are satisfied. It follows that **(Def$_{\oslash}$)** holds as well.

In order to verify **(Distr$_{\oslash,\triangledown}$)**, we just need to observe that $z(a \oplus b) = (za) \oplus (zb)$, for all $z \in S$ and all orthogonal $a, b \in \operatorname{Idp} S$. (Indeed, whenever $x, y \in S$, the elements $a = \mathbf{d}(x) \oslash \mathbf{d}(y) = \mathbf{d}(x) \smallsetminus \mathbf{d}(y)$ and $b = \mathbf{d}(y)$ are orthogonal idempotents, thus $a \triangledown b = a \oplus b$.)

Now we verify **(Dom$_{\triangledown}$)**. Observe first that whenever $x, y \in S$, the elements $x' = x \oslash y$ and y are orthogonal, thus $x' \triangledown y = x' \oplus y$. Further, $\mathbf{d}(x')$ and $\mathbf{d}(y)$ are orthogonal, and $\mathbf{d}(x' \oplus y) = \mathbf{d}(x') \oplus \mathbf{d}(y)$.

(2) It follows from **(InvSem)** that $(S, 0, {}^{-1}, \cdot)$ is an inverse semigroup. Further, it follows from **(GBa$_{\oslash,\triangledown}$)** and **(Idp$_{\oslash,\triangledown}$)** that $\operatorname{Idp} S$, endowed with the restriction $\wedge$ of $\cdot$, the restriction $\smallsetminus$ of $\oslash$, and the restriction $\vee$ of $\triangledown$, is a generalized Boolean algebra.

Now let $x, y \in S$ be orthogonal elements. Since $\mathbf{d}(x)$ and $\mathbf{d}(y)$ are orthogonal idempotents, $\mathbf{d}(x) \oslash \mathbf{d}(y) = \mathbf{d}(x) \smallsetminus \mathbf{d}(y) = \mathbf{d}(x)$, and similarly, $\mathbf{r}(x) \oslash \mathbf{r}(y) = \mathbf{r}(x)$.

Further, it follows from $(\mathbf{Def}_{\obslash})$ that $x \obslash y = \mathbf{r}(x)x\,\mathbf{d}(x) = x$. By $(\mathbf{Maj}_{\obslash,\triangledown})$, this implies that $\begin{matrix} x \\ y \end{matrix} \leq x \triangledown y$. By using $(\mathbf{Dom}_{\triangledown})$, we get $\mathbf{d}(x \triangledown y) = \mathbf{d}(x) \triangledown \mathbf{d}(y)$.

Now let $z \in S$ such that $\begin{matrix} x \\ y \end{matrix} \leq z$. We compute:

$$
\begin{aligned}
z\,\mathbf{d}(x \triangledown y) &= z\big(\mathbf{d}(x) \triangledown \mathbf{d}(y)\big) && \text{(by the above)}\\
&= z\big((\mathbf{d}(x) \obslash \mathbf{d}(y)) \triangledown \mathbf{d}(y)\big) && \text{(because } \mathbf{d}(x) \obslash \mathbf{d}(y) = \mathbf{d}(x))\\
&= z(\mathbf{d}(x) \obslash \mathbf{d}(y)) \triangledown z\,\mathbf{d}(y) && \text{(by } (\mathbf{Distr}_{\obslash,\triangledown}))\\
&= z\,\mathbf{d}(x) \triangledown z\,\mathbf{d}(y) && \text{(because } \mathbf{d}(x) \obslash \mathbf{d}(y) = \mathbf{d}(x))\\
&= x \triangledown y && \text{(because } x \leq z \text{ and } y \leq z)\,,
\end{aligned}
$$

so $x \triangledown y \leq z$. This completes the proof that $x \triangledown y$ is the orthogonal join of $\{x, y\}$ in S. Therefore, S is a Boolean inverse semigroup.

(3) We need to prove that in the presence of the bias identities, the operations $\obslash$ and $\triangledown$ are necessarily given by (3.2.1). Observe from the start that by (2) above, S is a Boolean inverse semigroup.

Due to $(\mathbf{GBa}_{\obslash,\triangledown})$ and $(\mathbf{Idp}_{\obslash,\triangledown})$, this certainly holds on $\operatorname{Idp} S$: that is, $a \obslash b = a \smallsetminus b$ and $a \triangledown b = a \vee b$ (within $\operatorname{Idp} S$), for any $a, b \in \operatorname{Idp} S$. Due to $(\mathbf{Def}_{\obslash})$, it follows that the operation $\obslash$ is given by (3.2.1); thus it is uniquely determined.

Now let $x, y \in S$. We must prove that $x \triangledown y = (x \obslash y) \oplus y$. Since S is a Boolean inverse semigroup and by $(\mathbf{Maj}_{\obslash,\triangledown})$, $(x \obslash y) \oplus y \leq x \triangledown y$. Further, it follows from $(\mathbf{Dom}_{\triangledown})$ that $\mathbf{d}(x \triangledown y) = \mathbf{d}(x \obslash y) \triangledown \mathbf{d}(y)$. Since $x \obslash y \perp y$ and since the restriction of $\triangledown$ to the idempotents is the join within $\operatorname{Idp} S$, it follows that $\mathbf{d}(x \triangledown y) = \mathbf{d}(x \obslash y) \oplus \mathbf{d}(y)$. Therefore, we get

$$
\begin{aligned}
(x \obslash y) \oplus y &= (x \triangledown y)\,\mathbf{d}\big((x \obslash y) \oplus y\big) && \text{(because } (x \obslash y) \oplus y \leq x \triangledown y)\\
&= (x \triangledown y)\,\mathbf{d}(x \triangledown y)\\
&= x \triangledown y\,,
\end{aligned}
$$

so the operation $\triangledown$ is given by (3.2.1).

The second statement of (3) follows immediately. □

In particular, given a Boolean inverse semigroup S, Theorem 3.2.4 enables us to define the *Boolean inverse subsemigroup of S generated by a subset X*, as the sub-bias of S generated by X.

The following result, crucial despite the easiness of its proof, identifies the homomorphisms on Boolean inverse semigroups, with respect to the structure of bias.

Theorem 3.2.5 *Let S and T be Boolean inverse semigroups and let $f: S \to T$ be a semigroup homomorphism. The following are equivalent:*

(i) *f is a bias homomorphism.*
(ii) *The domain-range restriction of f from* $\mathrm{Idp}\, S$ *to* $\mathrm{Idp}\, T$ *is a homomorphism of Boolean rings.*
(iii) *$c = a \oplus b$ implies that $f(c) = f(a) \oplus f(b)$, for all $a, b, c \in \mathrm{Idp}\, S$.*
(iv) *f is additive.*

Proof (i)⇒(ii). Since f is a semigroup homomorphism, it sends $\mathrm{Idp}\, S$ into $\mathrm{Idp}\, T$. Since the bias operations $\oslash$ and $\triangledown$ restrict, on the idempotents, to the difference $(x, y) \mapsto x \smallsetminus y$ and the join $(x, y) \mapsto x \vee y$, (ii) follows.

(ii)⇒(iii) is trivial.

(iii)⇒(iv) Let $z = x \oplus y$ in S, we must prove that $f(z) = f(x) \oplus f(y)$ in T. Since f is a homomorphism of inverse semigroups with zero, $f(x) \perp f(y)$; whence $f(x) \oplus f(y) \leq f(z)$. It follows from Lemma 3.1.4 that $\mathbf{d}(z) = \mathbf{d}(x) \oplus \mathbf{d}(y)$, thus, by assumption and since f is a homomorphism of inverse semigroups,

$$\mathbf{d}(f(z)) = f(\mathbf{d}(z)) = f(\mathbf{d}(x)) \oplus f(\mathbf{d}(y)) = \mathbf{d}(f(x)) \oplus \mathbf{d}(f(y)) = \mathbf{d}(f(x) \oplus f(y))\,.$$

Since $f(x) \oplus f(y) \leq f(z)$, it follows that $f(x) \oplus f(y) = f(z)$.

(iv)⇒(i) Suppose that f is an additive semigroup homomorphism from S to T. For all $a, b \in \mathrm{Idp}\, S$, it follows from the additivity of f together with the equation $a = (a \smallsetminus b) \oplus ab$ that $f(a) = f(a \smallsetminus b) \oplus f(ab) = f(a \smallsetminus b) \oplus f(a)f(b)$. It follows that $f(a \smallsetminus b) = f(a) \smallsetminus f(b)$. Since f is an inverse semigroup homomorphism, it follows that $f(x \oslash y) = f(x) \oslash f(y)$, for all $x, y \in S$. Since f is additive, it follows that $f(x \triangledown y) = f(x) \triangledown f(y)$, for all $x, y \in S$. □

Corollary 3.2.6 *Let S and T be Boolean inverse semigroups and let $f: S \to T$ be an additive semigroup homomorphism. Then $f[S]$ is a sub-bias of T. In particular, it is a Boolean inverse semigroup.*

The following result relates Theorem 3.2.5 with the concept of additive inverse subsemigroup introduced in Definition 3.1.17.

Corollary 3.2.7 *An inverse subsemigroup S of a Boolean inverse semigroup T is a sub-bias of T iff it is an additive inverse subsemigroup of T.*

Proof It is trivial that every sub-bias is an additive inverse subsemigroup. Suppose, conversely, that S is an additive inverse subsemigroup of T. By Lemma 3.1.13, S is a Boolean inverse semigroup. The desired conclusion follows then from Theorem 3.2.5. □

In particular, a Boolean inverse subsemigroup S of a Boolean inverse semigroup T is an additive inverse subsemigroup iff S is a sub-bias of T. Even more particularly, an ideal I of S is a sub-bias of S iff it is an additive ideal of S.

By Theorems 3.2.4 and 3.2.5, the category of all Boolean inverse semigroups and additive semigroup homomorphisms is identical to the category of all biases and bias homomorphisms. In particular, this category is a variety of algebras (in the sense of universal algebra).

3.3 The Prime Spectrum Representation of a Distributive Inverse Semigroup

Cayley's Theorem states that every group embeds into some symmetric group, and the Wagner[3]-Preston Theorem (cf. Lawson [73, Theorem 1.5.4]) states that every inverse semigroup embeds into some symmetric inverse semigroup. As observed in Exel [41], the implied embedding does not preserve finite joins as a rule, even starting with a Boolean inverse semigroup.

The following theorem is an analogue of those results for distributive inverse semigroups and embeddings preserving finite joins and meets. Although it is not explicitly stated there, most of it can, in principle, deduced from results of Lawson and Lenz [76] via elementary arguments: ε being one-to-one is essentially contained in the combination of Lemma 3.6, Propositions 3.12, and 3.19 in [76], and ε preserving existing meets can be deduced from Lemma 2.16 and Corollary 2.18 in [76]. Since the required translations involve the digestion of a fair number of nontrivial definitions, we provide direct proofs for convenience.

Theorem 3.3.1 *Let S be a distributive inverse semigroup with zero. Then there are a set Ω and a zero-preserving semigroup embedding $\varepsilon\colon S \hookrightarrow \mathfrak{I}_\Omega$ such that the following conditions hold for every positive integer n and all $x_1, \dots, x_n \in S$:*

(i) *$\bigvee_{i=1}^n x_i$ exists in S iff $\bigvee_{i=1}^n \varepsilon(x_i)$ exists in $\mathfrak{I}_\Omega$, and then*

$$\bigvee_{i=1}^{n} \varepsilon(x_i) = \varepsilon\Big(\bigvee_{i=1}^{n} x_i\Big)\,. \tag{3.3.1}$$

(ii) *If $\bigwedge_{i=1}^n x_i$ exists in S, then*

$$\bigwedge_{i=1}^{n} \varepsilon(x_i) = \varepsilon\Big(\bigwedge_{i=1}^{n} x_i\Big)\,. \tag{3.3.2}$$

Note Remember that every subset of $\mathfrak{I}_\Omega$ has a meet (cf. Example 3.1.8).

Proof Let us recall the definition of the *prime spectrum* $\mathsf{G}_{\mathrm{P}}(S)$ of S, as considered in Lawson and Lenz [76]. By definition, a nonempty subset $\mathfrak{p}$ of S is a *filter* of S if it is a downward directed, upper subset of S, with respect to the natural ordering of S. In addition, we say that $\mathfrak{p}$ is *prime* if $x \vee y \in \mathfrak{p}$ implies that either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$, for all $x, y \in S$ such that $x \vee y$ is defined. By definition, $\Omega = \mathsf{G}_{\mathrm{P}}(S)$ is[4] the set of all prime filters of S. Set $D = \operatorname{Idp} S$ and $\Omega_e = \{\mathfrak{p} \in \Omega \mid e\mathfrak{p} \subseteq \mathfrak{p}\}$, for every $e \in D$.

[3] Often transliterated as "Vagner".

[4] This set can be endowed with a well studied structure of topological groupoid, which will however not be of concern in the present work.

For all $x \in S$ and all $\mathfrak{p} \in \Omega_{\mathbf{d}(x)}$, we define $\varepsilon(x)(\mathfrak{p}) = \uparrow x\mathfrak{p}$ (where $\uparrow X$ is shorthand for $S \uparrow X$). If $\mathfrak{p} \notin \Omega_{\mathbf{d}(x)}$, let $\varepsilon(x)(\mathfrak{p})$ be undefined.

Claim 1 Let $x \in S$ and let $\mathfrak{p} \in \Omega_{\mathbf{d}(x)}$. Then $\varepsilon(x)(\mathfrak{p})$ is a prime filter of S. Moreover, $\varepsilon(x)(\mathfrak{p}) \in \Omega_{\mathbf{r}(x)}$, and $\varepsilon(x^{-1})\big(\varepsilon(x)(\mathfrak{p})\big) = \mathfrak{p}$.

Proof of Claim. If is obvious that $\varepsilon(x)(\mathfrak{p})$ is a proper filter of S. Let $y_0, y_1 \in S$ be compatible such that $y_0 \vee y_1 \in \varepsilon(x)(\mathfrak{p})$, so $xp \leq y_0 \vee y_1$ for some $p \in \mathfrak{p}$. Since $\mathbf{d}(x)\mathfrak{p} \subseteq \mathfrak{p}$, we may assume that $p = \mathbf{d}(x)p$. Since S is distributive, it follows from Proposition 3.1.9 that

$$xp = xp \wedge (y_0 \vee y_1) = (xp \wedge y_0) \vee (xp \wedge y_1)\,,$$

thus, by applying again Proposition 3.1.9, $p = \mathbf{d}(x)p = p_0 \vee p_1$ where each $p_i = x^{-1}(xp \wedge y_i)$. Since $p \in \mathfrak{p}$ and $\mathfrak{p}$ is prime, there is $i \in \{0, 1\}$ such that $p_i \in \mathfrak{p}$. Since $xp_i = \mathbf{r}(x)(xp \wedge y_i) \leq y_i$, it follows that $y_i \in \uparrow x\mathfrak{p}$, thus completing the proof that $\varepsilon(x)(\mathfrak{p})$ is prime.

The proofs of the relations $\varepsilon(x)(\mathfrak{p}) \in \Omega_{\mathbf{r}(x)}$ and $\varepsilon(x^{-1})\big(\varepsilon(x)(\mathfrak{p})\big) = \mathfrak{p}$ are routine and we omit them. □ Claim 1.

It follows from Claim 1 that ε takes its values in $\mathcal{I}_\Omega$.

Denote by $\overline{\Omega}$ the prime spectrum of D. An argument, similar to the one of the proof of Claim 1, yields the following claim.

Claim 2 Let $\mathfrak{p} \in \overline{\Omega}$. Then $\uparrow\mathfrak{p} \in \Omega$, and $\mathfrak{p} = D \cap \uparrow\mathfrak{p}$.

Claim 3 Let $a, b \in D$. Then $\Omega_{a \wedge b} = \Omega_a \cap \Omega_b$ and $\Omega_{a \vee b} = \Omega_a \cup \Omega_b$. Furthermore, $\Omega_a = \Omega_b$ implies that $a = b$.

Proof of Claim. The relation $\Omega_{a \wedge b} = \Omega_a \cap \Omega_b$ follows immediately from the distributivity of the multiplication on the meet in S, while the relation $\Omega_{a \vee b} = \Omega_a \cup \Omega_b$ follows immediately from Proposition 3.1.9.

Now suppose that $\Omega_a = \Omega_b$. By Claim 2, it follows that $a \cdot (\uparrow\mathfrak{p}) \subseteq \uparrow\mathfrak{p}$ iff $b \cdot (\uparrow\mathfrak{p}) \subseteq \uparrow\mathfrak{p}$, for every $\mathfrak{p} \in \overline{\Omega}$; that is, $a \in \mathfrak{p}$ iff $b \in \mathfrak{p}$, for every $\mathfrak{p} \in \overline{\Omega}$. By Proposition 1.4.1, this implies that $a = b$. □ Claim 3.

The proof of the following claim is routine (and it does not require distributivity), and we omit it.

Claim 4 The map ε is a semigroup homomorphism from S to $\mathcal{I}_X$.

Claim 5 The map ε is one-to-one.

Proof of Claim. Let $x, y \in S$ such that $\varepsilon(x) = \varepsilon(y)$. By equating the domains of the two sides, we get $\Omega_{\mathbf{d}(x)} = \Omega_{\mathbf{d}(y)}$, thus, by Claim 3, $\mathbf{d}(x) = \mathbf{d}(y)$. Set $e = \mathbf{d}(x)$. The filter $\uparrow\mathfrak{p}$ belongs to Ω_e, for every $\mathfrak{p} \in \overline{\Omega}(e)$ (cf. Claim 2). Hence, it follows from our assumption $\varepsilon(x) = \varepsilon(y)$ that $\uparrow x\mathfrak{p} = \uparrow y\mathfrak{p}$. This implies easily that for every $\mathfrak{p} \in \overline{\Omega}(e)$, there exists $p \in \mathfrak{p} \downarrow e$ such that $xp = yp$. Setting $\Delta = \{p \in D \downarrow e \mid xp = yp\}$, this means that

$$\overline{\Omega}(e) \subseteq \bigcup \big(\overline{\Omega}(p) \mid p \in \Delta\big)\ .$$

Since $\overline{\Omega}(e)$ is compact and all $\overline{\Omega}(p)$ are open within $\overline{\Omega}$ (cf. Theorem 1.4.2), there is a finite subset X of Δ such that

$$\overline{\Omega}(e) \subseteq \bigcup \left(\overline{\Omega}(p) \mid p \in X\right) .$$

By Proposition 1.4.1, this means that $e \leq \bigvee X$. Since $xp = yp$ for every $p \in X$, it follows from the distributivity of S that $xe = ye$, that is, $x = y$. □ Claim 5.

Now we know that ε is a semigroup embedding. Trivially, this embedding maps 0 to the empty function.

Let us prove (i). Since S and $\mathfrak{I}_\Omega$ are both distributive inverse semigroups and since compatibility can be expressed equationally, $\bigvee_{i=1}^{n} \varepsilon(x_i)$ is defined iff $\bigvee_{i=1}^{n} x_i$ is defined. Suppose that this holds and set $y = \bigvee_{i=1}^{n} x_i$. Obviously,

$$\bigvee_{i=1}^{n} \varepsilon(x_i) \leq \varepsilon(y) . \tag{3.3.3}$$

Furthermore, by using Lemma 3.1.4 together with Claim 3, we obtain the relations

$$\operatorname{dom} \varepsilon(y) = \Omega_{\mathbf{d}(y)} = \bigcup_{i=1}^{n} \Omega_{\mathbf{d}(x_i)} = \bigcup_{i=1}^{n} \operatorname{dom} \varepsilon(x_i) = \operatorname{dom}\Big(\bigvee_{i=1}^{n} \varepsilon(x_i)\Big) .$$

By (3.3.3), it follows that $\bigvee_{i=1}^{n} \varepsilon(x_i) = \varepsilon(y)$, thus completing the proof of (i).

Finally, suppose that $z = \bigwedge_{i=1}^{n} x_i$ exists in S. Obviously,

$$\varepsilon(z) \leq \bigwedge_{i=1}^{n} \varepsilon(x_i) . \tag{3.3.4}$$

Thus, in order to complete the proof of (ii), it suffices to prove that the domain of the right hand side of (3.3.4) is contained in the domain of its left hand side. That is, for every element $\mathfrak{p}$ of the domain of $\bigwedge_{i=1}^{n} \varepsilon(x_i)$, we must prove that $\mathbf{d}(z)\mathfrak{p} \subseteq \mathfrak{p}$. Let $p \in \mathfrak{p}$. For all $i, j \in [n]$, $\varepsilon(x_i)(\mathfrak{p}) = \varepsilon(x_j)(\mathfrak{p})$, thus there is $q_{i,j} \in \mathfrak{p} \downarrow p$ such that $x_i q_{i,j} \leq x_j q_{i,j}$. Pick $q \in \mathfrak{p}$ such that $q \leq q_{i,j}$ for all $i, j \in [n]$; since $\mathbf{d}(x_1)\mathfrak{p} \subseteq \mathfrak{p}$, we may assume that $q = \mathbf{d}(x_1)q$. Then $x_i q = x_j q$ for all $i, j \in [n]$, whence $x_1 q = \bigwedge_{i=1}^{n} x_i q = zq$. From $z \leq x_1$ it follows that $x_1^{-1} z = \mathbf{d}(z)$. Therefore, $q = \mathbf{d}(x_1)q = x_1^{-1} x_1 q = x_1^{-1} zq = \mathbf{d}(z)q \leq \mathbf{d}(z)p$, so $\mathbf{d}(z)p \in \mathfrak{p}$, as desired. □

Specializing Theorem 3.3.1 to Boolean inverse semigroups, we obtain immediately the following result.

Corollary 3.3.2 *Every Boolean inverse semigroup S has an additive semigroup embedding into some symmetric inverse semigroup $\mathfrak{I}_\Omega$, preserving all existing nonempty finite meets. In particular, S is a sub-bias of $\mathfrak{I}_\Omega$.*

Remark 3.3.3 The set Ω of Corollary 3.3.2 is identical to the one of Theorem 3.3.1, that is, it is the prime spectrum of S. In the context of Corollary 3.3.2 (i.e., S is

Boolean), more can be said: the prime filters of S are exactly the *ultrafilters* of S, that is, the maximal elements of the set of all filters of S with respect to set inclusion (cf. Lawson and Lenz [76, Lemma 3.20]).

For an arbitrary inverse semigroup S, the canonical semigroup homomorphism $\lambda\colon S \to \mathfrak{I}_{\Omega'}$ introduced in Exel [41], where Ω' is the set of all ultrafilters of S (denoted by $\mathsf{G}_{\mathrm{M}}(S)$ in Lawson and Lenz [76]), is tight in Exel's sense. As in [41], λ will be called the *regular representation of* S. Although Exel's concept of a Boolean inverse semigroup is less restrictive than ours, it follows from Exel [41, Proposition 6.2], together with Theorem 3.2.5, that his concept of a tight homomorphism extends our concept of an additive semigroup homomorphism. Moreover, for a Boolean inverse semigroup S, $\mathsf{G}_{\mathrm{M}}(S) = \mathsf{G}_{\mathrm{P}}(S)$ and the canonical embedding $\varepsilon\colon S \hookrightarrow \mathfrak{I}_{\mathsf{G}_{\mathrm{P}}(S)}$ of Theorem 3.3.1 is identical to Exel's regular representation λ.

On the other hand, $\mathsf{G}_{\mathrm{M}}(S) \ncong \mathsf{G}_{\mathrm{P}}(S)$ for most distributive inverse semigroups S (consider the three-element chain), so there are examples where $\varepsilon \neq \lambda$.

Remark 3.3.4 Say that an inverse semigroup is *Exel-Boolean* if its semilattice of idempotents is Boolean (not necessarily unital). The final example of Exel [41] is an Exel-Boolean inverse semigroup with no additive semigroup embedding into any symmetric inverse semigroup. Of course, by Corollary 3.3.2, such an inverse semigroup cannot be Boolean. A much easier example, serving the same purpose, is the one of Example 3.1.7: in that example, the ultrafilters of S are $\mathfrak{p}_i = \{i, 3, 4\}$, for $i \in \{1, 2\}$; and $3\mathfrak{p}_i = 4\mathfrak{p}_i = \mathfrak{p}_i$, whenever $i \in \{1, 2\}$. In particular, $\lambda(3) = \lambda(4)$, with $3 \neq 4$.

We say that two elements x and y in an inverse semigroup S with zero *essentially coincide*, in notation $x \equiv y$, if $\mathbf{d}(x) = \mathbf{d}(y)$ and for every nonzero idempotent $e \leq \mathbf{d}(x)$ there exists a nonzero idempotent $a \leq e$ such that $xa = ya$. We say that S is *continuous* if $x \equiv y$ implies that $x = y$, for all $x, y \in S$. Exel proved in [41, Theorem 7.5] that every continuous Exel-Boolean inverse semigroup embeds tightly (in his sense) into some symmetric inverse monoid. He also asks, just before the statement of [41, Theorem 7.5], whether $x \equiv y$ implies $\lambda(x) = \lambda(y)$. The following example, whose construction is inspired by the final counterexample of Exel [41], shows that this is not the case as a rule. This example turns out to be Boolean.

Example 3.3.5 A Boolean inverse monoid S, with unit element 1_S and an element x such that $1_S \equiv x$ and $\lambda(1_S) \neq \lambda(x)$. In particular, S is not continuous.

Proof We denote by $\mathcal{B}$ the Boolean algebra of all subsets of $\mathbb{Z}^+$ that are either finite or cofinite, and we fix a nontrivial group G. For each $x \in \mathcal{B}$, we set $N_x = G$ if x is finite, and $N_x = \{1\}$ if x is cofinite. The semigroup $\mathcal{B} \times G$ is an inverse monoid, and the binary relation $\sim$ on S defined by the rule

$$(x, g) \sim (y, h) \quad \text{if} \quad x = y \text{ and } g \equiv h \pmod{N_x}, \qquad \text{for any } (x, g), (y, h) \in \mathcal{B}\times G,$$

is a monoid congruence on $S \times G$. The quotient monoid $S = (\mathcal{B} \times G)/{\sim}$ is an inverse monoid with zero. Denoting by $[x, g]$ the equivalence class of (x, g) modulo $\sim$, the zero element of S is $[\varnothing, 1] = [\varnothing, g]$ (for all $g \in G$) and the unit element of S

is $1_S = [\mathbb{Z}^+, 1]$. Easy calculations show that $[x, g]^{-1} = [x, g^{-1}]$ and $\mathbf{d}([x, g]) = \mathbf{r}([x, g]) = [x, 1]$ whenever $(x, g) \in \mathcal{B} \times G$. Two elements $[x_0, g_0]$ and $[x_1, g_1]$ of S are orthogonal if $x_0 \cap x_1 = \varnothing$ (thus one of x_0 and x_1 needs to be finite), and then their orthogonal sum is $[x_0 \cup x_1, g_{1-i}]$ if x_i is finite. The semilattice of all idempotents of S is $B = \{[x, 1] \mid x \in \mathcal{B}\}$, which is isomorphic to $\mathcal{B}$. Therefore, S is a Boolean inverse monoid.

Pick $g \in G \setminus \{1\}$ and set $x = [\mathbb{Z}^+, g]$. Every nonzero idempotent of S contains an idempotent of the form $e_n = [\{n\}, 1]$, where $n \in \mathbb{Z}^+$; and $1_S e_n = x e_n = e_n$. This proves that $1_S \equiv x$.

However, since S is Boolean, it follows from Corollary 3.3.2 that Exel's regular representation λ of S (cf. Remark 3.3.3) is one-to-one; whence $\lambda(1_S) \neq \lambda(x)$. □

3.4 Additive Congruences of Boolean Inverse Semigroups

In this section we shall investigate in our context the crucial universal-algebraic concept of a *congruence*, in particular by describing bias congruences in terms of the semigroup operations and the orthogonal join operation $\oplus$.

Proposition 3.4.1 *Let S be a Boolean inverse semigroup. An equivalence relation $\boldsymbol{\theta}$ on S is a bias congruence iff $\boldsymbol{\theta}$ is a semigroup congruence and the following condition holds:*

For all $x \in S$ and all $a, b \in \operatorname{Idp} S$ orthogonal,

$$(xa \equiv_{\boldsymbol{\theta}} a \text{ and } xb \equiv_{\boldsymbol{\theta}} b) \Rightarrow x(a \oplus b) \equiv_{\boldsymbol{\theta}} a \oplus b\,. \qquad (3.4.1)$$

Proof We prove the non-trivial direction. Let $\boldsymbol{\theta}$ be a semigroup congruence of S (thus also an inverse semigroup congruence) satisfying (3.4.1).

The assumption (3.4.1) means that for all orthogonal idempotents a and b, from $\begin{matrix} a \\ b \end{matrix} \leq_{\boldsymbol{\theta}} x$ it follows that $a \oplus b \leq_{\boldsymbol{\theta}} x$, for each $x \in S$. (Recall that $x \leq_{\boldsymbol{\theta}} y$ is shorthand for $x \equiv_{\boldsymbol{\theta}} y\,\mathbf{d}(x)$.) Denoting by $\theta\colon S \twoheadrightarrow S/\boldsymbol{\theta}$ the canonical projection, this means that $\theta(a \oplus b) = \theta(a) \oplus \theta(b)$ within $S/\boldsymbol{\theta}$.

Claim 1 $\theta(a \vee b)$ is the join of $\{\theta(a), \theta(b)\}$ within $S/\boldsymbol{\theta}$, for any idempotents a and b of S. Hence, $\boldsymbol{\theta}$ is compatible with the operations $\wedge$, $\vee$, and $\smallsetminus$ on idempotents.

Proof of Claim. Any upper bound, within $S/\boldsymbol{\theta}$, of $\{\theta(a), \theta(b)\}$ is also an upper bound of the set $\{\theta(a \smallsetminus b), \theta(b)\}$, thus, by (3.4.1), it is an upper bound of $\theta((a \smallsetminus b) \oplus b) = \theta(a \vee b)$. Hence, $\theta(a \vee b)$ is the join of $\{\theta(a), \theta(b)\}$ within $S/\boldsymbol{\theta}$, and hence $\boldsymbol{\theta}$ is compatible with the $\vee$ operation on $\operatorname{Idp} S$. Since $\boldsymbol{\theta}$ is also a congruence with respect to the product operation, its restriction to the generalized Boolean algebra $\operatorname{Idp} S$ is a congruence with respect to join and meet, thus it is also a congruence with respect to the difference operation $\smallsetminus$. □ Claim 1.

Claim 2 The equivalence relation $\boldsymbol{\theta}$ is compatible with the operation $\oslash$ on S.

Proof of Claim. Since $\boldsymbol{\theta}$ is compatible with the product operation, it is also compatible with the operations $\mathbf{d}$ and $\mathbf{r}$, thus, by Claim 1, it is also compatible with the operations $(x, y) \mapsto \mathbf{r}(x) \smallsetminus \mathbf{r}(y)$ and $(x, y) \mapsto \mathbf{d}(x) \smallsetminus \mathbf{d}(y)$. Since it is compatible with the product operation, the desired conclusion follows. □ Claim 2.

Claim 3 Let $x_0, y_0, x_1, y_1 \in S$ such that $x_0 \equiv_{\boldsymbol{\theta}} y_0$, $x_1 \equiv_{\boldsymbol{\theta}} y_1$, $x_0 \perp x_1$, and $y_0 \perp y_1$. Then $x_0 \oplus x_1 \equiv_{\boldsymbol{\theta}} y_0 \oplus y_1$.

Proof of Claim. Set $x = x_0 \oplus x_1$ and $y = y_0 \oplus y_1$. Then

$$\mathbf{d}(x_i) = x_i^{-1} x_i = x^{-1} x_i \equiv_{\boldsymbol{\theta}} x^{-1} y_i \leq_{\boldsymbol{\theta}} x^{-1} y\,, \quad \text{for each } i \in \{0, 1\}\ .$$

Thus, by our assumption (3.4.1), $\mathbf{d}(x_0) \oplus \mathbf{d}(x_1) \leq_{\boldsymbol{\theta}} x^{-1}y$, that is, $\mathbf{d}(x) \leq_{\boldsymbol{\theta}} x^{-1}y$, and thus $x = x\,\mathbf{d}(x) \leq_{\boldsymbol{\theta}} xx^{-1}y$, and so $x \leq_{\boldsymbol{\theta}} y$. Symmetrically, $y \leq_{\boldsymbol{\theta}} x$, and therefore, since $\boldsymbol{\theta}$ is an inverse semigroup congruence, $x \equiv_{\boldsymbol{\theta}} y$. □ Claim 3.

Let $x_0, x_1, y_0, y_1 \in S$ such that $x_0 \equiv_{\boldsymbol{\theta}} y_0$ and $x_1 \equiv_{\boldsymbol{\theta}} y_1$. It follows from Claim 2 that $x_0 \oslash x_1 \equiv_{\boldsymbol{\theta}} y_0 \oslash y_1$. Since $x_0 \oslash x_1 \perp x_1$, $y_0 \oslash y_1 \perp y_1$, and $y_0 \equiv_{\boldsymbol{\theta}} y_1$, it follows from Claim 3 that $(x_0 \oslash x_1) \oplus x_1 \equiv_{\boldsymbol{\theta}} (y_0 \oslash y_1) \oplus y_1$, that is, $x_0 \triangledown x_1 \equiv_{\boldsymbol{\theta}} y_0 \triangledown y_1$. Therefore, $\boldsymbol{\theta}$ is compatible with the operation $\triangledown$. □

Define an *additive congruence* of a Boolean inverse semigroup S as a semigroup congruence satisfying (3.4.1). Proposition 3.4.1 says that the concepts of additive congruence and bias congruence are equivalent.

It would be nicer if, within the statement of Proposition 3.4.1, the assumption (3.4.1) could be replaced by the weaker assumption that the restriction of $\boldsymbol{\theta}$ to $\operatorname{Idp} S$ is a ring congruence. The following example shows that this cannot be done, even for idempotent-separating congruences. (A congruence $\boldsymbol{\theta}$ of S is *idempotent-separating* if $a \equiv_{\boldsymbol{\theta}} b$ implies that $a = b$, for all $a, b \in \operatorname{Idp} S$. By Howie [60, Proposition II.4.8], this is equivalent to saying that $\boldsymbol{\theta} \subseteq \mathscr{H}$.)

Example 3.4.2 Denote the two-element group $G = \mathbb{Z}/2\mathbb{Z}$ multiplicatively, so $G = \{1, u\}$ with $u^2 = 1$. The inverse semigroup $S = G^{\sqcup 0} \times \{0, 1\}$ is a Boolean inverse monoid. The equivalence relation $\boldsymbol{\theta}$ on S, defined as the union of the diagonal of S with the set $\{((u, 0), (1, 0)), ((1, 0), (u, 0))\}$, is an idempotent-separating semigroup congruence of S.

This congruence is not additive, for $(u, 0) \equiv_{\boldsymbol{\theta}} (1, 0)$ while $(u, 1) \not\equiv_{\boldsymbol{\theta}} (1, 1)$, the latter meaning that $(u, 0) \oplus (0, 1) \not\equiv_{\boldsymbol{\theta}} (1, 0) \oplus (0, 1)$.

Observe the contrast between Example 3.4.2 and Theorem 3.2.5. The point is that the quotient inverse semigroup $S/\boldsymbol{\theta}$, in Example 3.4.2, is not Boolean.

Notation 3.4.3 We set

$$x\,\langle y \rangle = xyx^{-1}\,, \text{ for all } x\,,\ y \text{ in any inverse semigroup}\,. \tag{3.4.2}$$

Recall that if y is idempotent, then so is $x\langle y\rangle$. The following observation will be used repeatedly without mentioning, throughout this work.

Lemma 3.4.4 *Let x, u, v be elements in an inverse semigroup, with either u or v idempotent. Then $x\langle uv\rangle = x\langle u\rangle \cdot x\langle v\rangle$.*

Proof If, for example, u is idempotent, then u and $x^{-1}x$ commute (they are both idempotent), thus $x\langle uv\rangle = xx^{-1}xuvx^{-1} = xux^{-1}xvx^{-1} = x\langle u\rangle \cdot x\langle v\rangle$. □

As our next result shows, the somewhat irregular-looking behavior witnessed by Example 3.4.2 does not occur for the largest idempotent-separating congruence of a Boolean inverse semigroup.

Proposition 3.4.5 *Let S be a Boolean inverse semigroup. Then the largest idempotent-separating congruence $\boldsymbol{\mu}$ of S is an additive congruence of S. In particular, $S/\boldsymbol{\mu}$ is a Boolean inverse semigroup and the canonical projection $S \twoheadrightarrow S/\boldsymbol{\mu}$ is an additive semigroup homomorphism.*

Proof Recall (cf. Howie [60, Theorem V.3.2]) that $\boldsymbol{\mu}$ can be described by

$$\boldsymbol{\mu} = \{(x,y) \in S \times S \mid (\forall e \in \operatorname{Idp} S)(x\langle e\rangle = y\langle e\rangle)\} \tag{3.4.3}$$

(cf. Notation 3.4.3). Now let $a, b \in \operatorname{Idp} S$ be orthogonal and let $x \in S$ with $xa \equiv_{\mu} a$ and $xb \equiv_{\mu} b$. By (3.4.3), this means that $x\langle ae\rangle = ae$ and $x\langle be\rangle = be$ for every $e \in \operatorname{Idp} S$. Now for every $e \in \operatorname{Idp} S$,

$$x\langle (a \oplus b)e\rangle = x\langle ae \oplus be\rangle = x\langle ae\rangle \oplus x\langle be\rangle = ae \oplus be = (a \oplus b)e\,,$$

so $x(a \oplus b) \equiv_{\mu} a \oplus b$. By Proposition 3.4.1, it follows that $\boldsymbol{\mu}$ is a bias congruence. The last part of Proposition 3.4.5 follows immediately. □

Proposition 3.4.6 *Let I be an additive ideal in a Boolean inverse semigroup S. Then the binary relation $\equiv_I$ on S, defined by the rule*

$$x \equiv_I y \Leftrightarrow (\exists z)\bigl(z \leq x \text{ and } z \leq y \text{ and } \{x \smallsetminus z, y \smallsetminus z\} \subseteq I\bigr)\,, \quad \text{for all } x, y \in S\,, \tag{3.4.4}$$

is the least additive congruence of S for which the equivalence class of 0 contains I. Moreover, $0/{\equiv_I} = I$.

Proof It is obvious that $0/{\equiv_I} = I$ and that every additive congruence of S, for which the equivalence class of 0 contains I, contains $\equiv_I$. Hence, it suffices to prove that $\equiv_I$ is an additive congruence of S. It is trivial that $\equiv_I$ is both reflexive and symmetric. Let $x, y, z \in S$ such that $x \equiv_I y$ and $y \equiv_I z$. There are $u, v \in S$ with $u \leq x$, $v \leq z$, and $\begin{smallmatrix} u \\ v \end{smallmatrix} \leq y$, such that $x \smallsetminus u$, $y \smallsetminus u$, $y \smallsetminus v$, and $z \smallsetminus v$ all belong to I. From $\begin{smallmatrix} u \\ v \end{smallmatrix} \leq y$ it follows that $u \sim v$ and $y \smallsetminus u \sim y \smallsetminus v$. The latter relation implies that $(y \smallsetminus u) \vee (y \smallsetminus v)$ exists in S. Since $\{y \smallsetminus u, y \smallsetminus v\} \in I$, it follows that $(y \smallsetminus u) \vee (y \smallsetminus v) \in I$ (cf. Proposition 3.1.19), that is, $y \smallsetminus (u \wedge v) \in I$. (All statements, such as $y \smallsetminus (u \wedge v) = (y \smallsetminus u) \vee (y \smallsetminus v)$, can easily be proved by reduction to the idempotent case, via Lemma 3.1.3.) By meeting that relation with u, we get $u \smallsetminus (u \wedge v) \in I$ and $v \smallsetminus (u \wedge v) \in I$. Therefore, $x \smallsetminus (u \wedge v) = (x \smallsetminus u) \oplus (u \smallsetminus (u \wedge v)) \in I$, and, similarly, $z \smallsetminus (u \wedge v) \in I$, so $x \equiv_I z$.

Let $x, y, z \in S$ with $x \equiv_I y$. There exists $u \in S$ such that $u \leq \begin{matrix} x \\ y \end{matrix}$ and $\{x \smallsetminus u, y \smallsetminus u\} \subseteq I$. By Lemma 3.1.12 and since I is an ideal of S, it follows that $\{xz \smallsetminus uz, yz \smallsetminus uz\} \subseteq I$ and $\{zx \smallsetminus zu, zy \smallsetminus zu\} \subseteq I$, thus $xz \equiv_I yz$ and $zx \equiv_I zy$. Therefore, $\equiv_I$ is a semigroup congruence of S.

In order to verify that $\equiv_I$ is an additive congruence, it suffices to verify (3.4.1). Let $a, b \in \operatorname{Idp} S$ be orthogonal idempotents and let $x \in S$ such that $xa \equiv_I a$ and $xb \equiv_I b$. There are $u \leq \begin{matrix} xa \\ a \end{matrix}$ and $v \leq \begin{matrix} xb \\ b \end{matrix}$ such that $xa \smallsetminus u$, $a \smallsetminus u$, $xb \smallsetminus v$, and $b \smallsetminus v$ all belong to I. From $u \leq a$ and $v \leq b$ it follows that u and v are both idempotent. Further, $u \oplus v \leq \begin{matrix} x(a \oplus b) \\ a \oplus b \end{matrix}$, and

$$(a \oplus b) \smallsetminus (u \oplus v) = (a \smallsetminus u) \oplus (b \smallsetminus v) \in I\,,$$

$$x(a \oplus b) \smallsetminus (u \oplus v) = (xa \oplus xb) \smallsetminus (u \oplus v) = (xa \smallsetminus u) \oplus (xb \smallsetminus v) \in I\,,$$

so $x(a \oplus b) \equiv_I a \oplus b$. Therefore, $\equiv_I$ is an additive congruence of S. By virtue of Theorem 3.2.4 and Proposition 3.4.1, the final statement of Proposition 3.4.6 follows immediately. □

For an additive ideal I in a Boolean inverse semigroup S, we will denote by x/I the equivalence class of x with respect to $\equiv_I$, for each $x \in S$. Observe that $0/I = I$.

In the context of Proposition 3.4.6, $\equiv_I$ is a bias congruence of S (cf. Proposition 3.4.1), thus the quotient structure $S/I = S/{\equiv_I}$ is a Boolean inverse semigroup.

Our next group of results introduces an alternate way to view additive ideals of S, by focusing attention on the idempotents of S.

Definition 3.4.7 Let S be a Boolean inverse semigroup. An ideal I of the Boolean ring $\operatorname{Idp} S$ is *$\mathscr{D}$-closed* if for all $a, b \in \operatorname{Idp} S$, $a \mathscr{D}_S b$ and $a \in I$ implies that $b \in I$.

Our next result shows that additive ideals (of Boolean inverse semigroups) are essentially the same concept as $\mathscr{D}$-closed ideals (in Boolean rings of idempotents).

Proposition 3.4.8 *Let S be a Boolean inverse semigroup and set $B = \operatorname{Idp} S$. The following statements hold:*

(1) *For any additive ideal J of S, the intersection $J \cap B$ is a $\mathscr{D}$-closed ideal of B.*
(2) *For any $\mathscr{D}$-closed ideal I of the Boolean ring B, the equality $\mathbf{d}^{-1}[I] = \mathbf{r}^{-1}[I]$ holds. Furthermore, this set is the ideal of S generated by I, and it is also an additive ideal.*
(3) *The two transformations described in* (1) *and* (2) *above are mutually inverse.*

Proof (1) From $0 \in J$ it follows that $0 \in J \cap B$. Let $a \in B$ and $b \in J \cap B$ with $a \leq b$. Then $a = ab \in SJ \subseteq J$, so $a \in J \cap B$, and so $J \cap B$ is a lower subset of B. Since J is closed under finite orthogonal joins, so is $J \cap B$. Hence $J \cap B$ is an ideal of

B. Now let $a, b \in B$ with $a \mathscr{D} b$. Pick $x \in S$ with $\mathbf{d}(x) = a$ and $\mathbf{r}(x) = b$. In particular, $b = xax^{-1}$, hence $a \in I$ implies that $b \in I$.

(2) Since $\mathbf{d}(x) \mathscr{D} \mathbf{r}(x)$ for all $x \in S$, the equality $\mathbf{d}^{-1}[I] = \mathbf{r}^{-1}[I]$ is obvious. It is then easy to verify, in particular by using Lemma 3.1.4, that this set is an additive ideal of S. It obviously contains I. Let J be an ideal of S containing I. For any $x \in \mathbf{d}^{-1}[I]$, the element $\mathbf{d}(x)$ belongs to I, thus to J, thus $x = x\mathbf{d}(x) \in J$; whence $\mathbf{d}^{-1}[I] \subseteq J$.

(3) Let J be an additive ideal of S and set $I = J \cap B$. We claim that $J = \mathbf{d}^{-1}[I]$. For each $x \in J$, $\mathbf{d}(x) = x^{-1}x \in J$, thus $\mathbf{d}(x) \in I$, that is, $x \in \mathbf{d}^{-1}[I]$. Conversely, let $x \in \mathbf{d}^{-1}[I]$. Then $\mathbf{d}(x) \in J$ as well, so $x = x\mathbf{d}(x) \in J$, thus completing the proof of our claim.

Finally, it is trivial that $I = \mathbf{d}^{-1}[I] \cap B$, for any $\mathscr{D}$-closed ideal I of B. □

Proposition 3.4.9 *Let S and T be Boolean inverse semigroups and let $f: S \to T$ be an additive map. Then the set $\ker f = f^{-1}\{0\}$ is an additive ideal of S. Furthermore, denoting by $p: S \twoheadrightarrow S/\ker f$ the canonical projection, there is a unique additive semigroup homomorphism $\overline{f}: S/\ker f \to T$ such that $f = \overline{f} \circ p$.*

Note The set $\ker f = f^{-1}\{0\}$, which is a subset of the domain of f, should not be confused with the kernel $\operatorname{Ker} f$ of f, which is an equivalence relation on the domain of f (cf. Sect. 1.3).

Proof It is straightforward to verify that the subset $I = \ker f$ is an additive ideal of S.

Since biases form a variety of algebras, the standard concepts of universal algebra apply to the category of biases and bias homomorphisms. This is the case, in particular, for the First Isomorphism Theorem. Since bias homomorphisms and additive semigroup homomorphisms are the same concept (cf. Theorem 3.2.5), in order to prove the final statement of Proposition 3.4.9, it suffices to prove that $p(x) = p(y)$ (i.e., $x \equiv_I y$) implies that $f(x) = f(y)$, for all $x, y \in S$. Let $z \in S$ witness $x \equiv_I y$, that is, $z \leq \begin{matrix} x \\ y \end{matrix}$ and $\{x \smallsetminus z, y \smallsetminus z\} \subseteq I$. Since $x = (x \smallsetminus z) \oplus z$ and f is additive, we get $f(x) = f(x \smallsetminus z) \oplus f(z) = f(z)$. Similarly, $f(y) = f(z)$, so $f(x) = f(y)$. □

Say that a congruence $\boldsymbol{\theta}$ of S is *ideal-induced* if $\boldsymbol{\theta}$ is equal to $\equiv_I$ for some additive ideal I of S. As the following example shows, a Boolean inverse semigroup may have many additive congruences that are not ideal-induced. This example also shows that the map $\overline{f}$ of the statement of Proposition 3.4.9 may not be one-to-one.

Example 3.4.10 For any group G, the inverse semigroup $G^{\sqcup 0}$ is a Boolean inverse semigroup, where $x \perp y$ iff either $x = 0$ or $y = 0$. If an additive congruence $\boldsymbol{\theta}$ of $G^{\sqcup 0}$ identifies 0 with some nonzero element, then $\boldsymbol{\theta} = G^{\sqcup 0} \times G^{\sqcup 0}$ is the largest congruence. If $\boldsymbol{\theta}$ does not identify 0 with any nonzero element, then $\boldsymbol{\theta}$ is the congruence $\boldsymbol{\theta}_H$ associated with a normal subgroup H of G, in the sense that $x \equiv_{\boldsymbol{\theta}} y$ iff either $x = y = 0$ or $x, y \neq 0$ and $x^{-1}y \in H$. Observe that $\boldsymbol{\theta}_H$ is ideal-induced iff $H = \{1\}$.

It follows, in particular, that the lattice of all additive congruences of $G^{\sqcup 0}$ is isomorphic to the normal subgroup lattice NSub G of G, with a top element added. In particular, taking for G the Klein group $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, the lattice of all additive congruences of $G^{\sqcup 0}$ is the five-element modular non distributive lattice M_3, with a top element added. Thus we get the following observation: *The lattice of all additive congruences of a Boolean inverse semigroup may not be distributive*.

On the other hand, it is well know that the lattice NSub G is modular, for any group G. Hence, the lattice of all additive congruences of $G^{\sqcup 0}$ is modular. We shall now see that this observation can be extended to any Boolean inverse semigroup.

To this end, let us introduce the following ternary term m, in the similarity type $\mathcal{L}_{\mathrm{BIS}}$ of all biases (cf. Notation 3.2.2; recall that $\mathbf{d}(\mathsf{x})$ and $\mathbf{r}(\mathsf{z})$ are shorthand for $\mathsf{x}^{-1}\mathsf{x}$ and $\mathsf{x}\mathsf{x}^{-1}$, respectively):

$$\mathsf{m}(\mathsf{x},\mathsf{y},\mathsf{z}) = \Big(\mathsf{x}\big(\mathbf{d}(\mathsf{x}) \oslash \mathbf{d}(\mathsf{y})\big) \mathbin{\triangledown} \mathsf{x}\mathsf{y}^{-1}\mathsf{z}\Big) \mathbin{\triangledown} \big(\mathbf{r}(\mathsf{z}) \oslash \mathbf{r}(\mathsf{y})\big)\mathsf{z}\,. \tag{3.4.5}$$

It is worthwhile noticing that the right hand side of (3.4.5) contains, as a subterm, the group-theoretical term $\mathsf{x}\mathsf{y}^{-1}\mathsf{z}$, which is the standard Mal'cev term for groups.

Recall that a variety $\mathbf{V}$ of algebras is *congruence-permutable* if $\alpha \circ \beta = \beta \circ \alpha$ for any congruences α and β of any algebra in $\mathbf{V}$. We also say that $\mathbf{V}$ is *congruence-modular* if the lattice of all congruences of any algebra $A \in \mathbf{V}$ is modular, that is, $\alpha \cap (\beta \vee (\alpha \cap \gamma)) = (\alpha \cap \beta) \vee (\alpha \cap \gamma)$ for any congruences α, β, γ of A. It is well known that every congruence-permutable variety is congruence-modular (for every lattice of pairwise commuting equivalence relations is modular, and even Arguesian; this originates in Jónsson [63], see also Grätzer [53, Theorem 410]).

Theorem 3.4.11 *The term* m *is a Mal'cev term for the variety of all biases; that is, the equations* $\mathsf{m}(\mathsf{x},\mathsf{x},\mathsf{y}) = \mathsf{m}(\mathsf{y},\mathsf{x},\mathsf{x}) = \mathsf{y}$ *hold identically in every bias. Therefore, the variety of all biases is congruence-permutable, thus also congruence-modular.*

Proof Let S be a bias. It is straightforward to verify that $x \mathbin{\triangledown} 0 = 0 \mathbin{\triangledown} x = x$, for every $x \in S$. Since the operations $\oslash$ and $\smallsetminus$ agree on the idempotents of S, while $\triangledown$ and $\oplus$ agree on orthogonal pairs, we can compute

$$\begin{aligned}
\mathsf{m}(x,x,y) &= \Big(x\big(\mathbf{d}(x) \smallsetminus \mathbf{d}(x)\big) \mathbin{\triangledown} xx^{-1}y\Big) \mathbin{\triangledown} \big(\mathbf{r}(y) \smallsetminus \mathbf{r}(x)\big)y \\
&= \mathbf{r}(x)y \oplus \big(\mathbf{r}(y) \smallsetminus \mathbf{r}(x)\big)y \\
&= \big(\mathbf{r}(x) \vee \mathbf{r}(y)\big)y \\
&= y\,,
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{m}(y,x,x) &= \Big(y\big(\mathbf{d}(y) \smallsetminus \mathbf{d}(x)\big) \mathbin{\triangledown} yx^{-1}x\Big) \mathbin{\triangledown} \big(\mathbf{r}(x) \smallsetminus \mathbf{r}(x)\big)x \\
&= y\big(\mathbf{d}(y) \smallsetminus \mathbf{d}(x)\big) \mathbin{\triangledown} y\,\mathbf{d}(x)
\end{aligned}$$

$$= y\bigl(\mathbf{d}(y) \vee \mathbf{d}(x)\bigr)$$
$$= y\,.$$

Hence m is a Mal'cev term for biases. It is well known since Mal'cev [80] (cf. McKenzie et al. [82, Theorem 4.141]) that this implies the congruence-permutability result, whence the congruence-modularity result. □

Theorem 3.4.11 marks a crucial difference between Boolean inverse semigroups on the one hand, and inverse semigroups on the other hand. Indeed, it is well known that there is no lattice identity satisfied by the congruence lattices of all semilattices (cf. Freese and Nation [48]), thus, a fortiori, by the congruence lattices of all inverse semigroups.

3.5 Generalized Rook Matrices over Boolean Inverse Semigroups

The following concept is taken from Wallis [116, § 4.5], see also Kudryavtseva et al. [71]. It extends the one of a *rook matrix* introduced in Solomon [103]. Solomon's rook matrices are identical to generalized rook matrices taken over the two-element inverse semigroup.

Recall that left and right orthogonality are both introduced in Definition 3.1.2.

Definition 3.5.1 Let S be an inverse semigroup with zero and let Ω be a (possibly infinite) set. A square matrix $a = \bigl(a_{i,j} \mid (i,j) \in \Omega \times \Omega\bigr)$, with all $a_{i,j} \in S$, is a $\Omega \times \Omega$ *generalized rook matrix over* S if any two distinct rows (resp., columns) of S are left orthogonal (resp., right orthogonal). In formula,

$$a_{i,j} \perp_{\mathrm{rt}} a_{i,k} \text{ and } a_{j,i} \perp_{\mathrm{lt}} a_{k,i}\,, \quad \text{for all } i,j,k \in \Omega \text{ with } j \neq k\,,$$

or, equivalently,

$$a_{i,j}^{-1} a_{i,k} = a_{j,i} a_{k,i}^{-1} = 0\,, \quad \text{for all } i,j,k \in \Omega \text{ with } j \neq k\,.$$

We denote by $\mathrm{R}_{\Omega}^{\oplus}(S)$ the set of all $\Omega \times \Omega$ generalized rook matrices over S. We also consider the following subsets of $\mathrm{R}_{\Omega}^{\oplus}(S)$:

- the set $\mathrm{B}_{\Omega}^{\oplus}(S)$ of all generalized rook matrices a that are both *row-finite* (i.e., for each $i \in \Omega$, $a_{i,j} = 0$ for all but finitely many $j \in \Omega$) and *column-finite* (i.e., for each $j \in \Omega$, $a_{i,j} = 0$ for all but finitely many $i \in \Omega$);
- the set $\mathrm{M}_{\Omega}^{\oplus}(S)$ of all generalized rook matrices a such that $a_{i,j} = 0$ for all but finitely many $(i,j) \in \Omega \times \Omega$.

If $\Omega = [n]$, for $n \in \mathbb{N}$, we will write $\mathrm{M}_{n}^{\oplus}(S) = \mathrm{B}_{n}^{\oplus}(S) = \mathrm{R}_{n}^{\oplus}(S) = \mathrm{R}_{[n]}^{\oplus}(S)$.

The basic properties of generalized rook matrices over S are summed up in Wallis [116, § 4.5], Kudryavtseva et al. [71, Proposition 3.5]. Since we are dealing with a slightly more general context (due to the possibility that Ω be infinite), we include proofs for convenience.

In what follows, for any family $(a_i \mid i \in I)$ of elements in a Boolean inverse semigroup S, we say that the orthogonal join $\bigoplus_{i\in I} a_i$ is *defined* if the a_i are pairwise orthogonal and $a_i = 0$ for all but finitely many $i \in I$.

Lemma 3.5.2 *Let S be an inverse semigroup with zero and let Ω be a set. The following statements hold, for any generalized rook matrices $a = \bigl(a_{i,j} \mid (i,j) \in \Omega \times \Omega\bigr)$ and $b = \bigl(b_{i,j} \mid (i,j) \in \Omega \times \Omega\bigr)$ over S:*

(1) *For any $i, j \in \Omega$, the elements $a_{i,k}b_{k,j}$, where $k \in \Omega$, are pairwise orthogonal.*
(2) *If S is Boolean inverse and all elements $c_{i,j} = \bigoplus_{k\in\Omega} a_{i,k}b_{k,j}$, for $i, j \in \Omega$, are defined* (in which case we say that the matrix ab is defined)*, then $c = \bigl(c_{i,j} \mid (i,j) \in \Omega \times \Omega\bigr)$ is a generalized rook matrix over S.*
(3) *If S is Boolean inverse, $a, b \in \mathrm{R}^{\oplus}_{\Omega}(S)$, and either a is row-finite or b is column-finite, then ab is defined. Furthermore, if a is row-finite and b is column-finite, then ab is both row-finite and column-finite.*
(4) *If S is Boolean inverse, then $\mathrm{M}^{\oplus}_{\Omega}(S)$ is an ideal of $\mathrm{B}^{\oplus}_{\Omega}(S)$.*

In the context of Lemma 3.5.2(2), we say that c is the product of a and b, and we write $c = ab$.

Proof (1) For any distinct $k, l \in \Omega$, from $b_{k,j}b_{l,j}^{-1} = 0$ it follows that

$$a_{i,k}b_{k,j}(a_{i,l}b_{l,j})^{-1} = a_{i,k}b_{k,j}b_{l,j}^{-1}a_{i,l}^{-1} = 0\,,$$

so $a_{i,k}b_{k,j} \perp_{\mathrm{lt}} a_{i,l}b_{l,j}$. Similarly, from $a_{i,k}^{-1}a_{i,l} = 0$ it follows that

$$(a_{i,k}b_{k,j})^{-1}a_{i,l}b_{l,j} = b_{k,j}^{-1}a_{i,k}^{-1}a_{i,l}b_{l,j} = 0\,.$$

so $a_{i,k}b_{k,j} \perp_{\mathrm{rt}} a_{i,l}b_{l,j}$. Hence, $a_{i,k}b_{k,j} \perp a_{i,l}b_{l,j}$.

(2) Suppose that the matrix $c = ab$ is defined. Let $i, j, k \in \Omega$ with $j \neq k$. We claim that $c_{i,j} \perp_{\mathrm{rt}} c_{i,k}$ and $c_{j,i} \perp_{\mathrm{lt}} c_{k,i}$. In order to verify the first statement, it suffices to verify that $a_{i,p}b_{p,j} \perp_{\mathrm{rt}} a_{i,q}b_{q,k}$, that is, $b_{p,j}^{-1}a_{i,p}^{-1}a_{i,q}b_{q,k} = 0$, for all $p, q \in \Omega$. If $p \neq q$, then this follows from $a_{i,p}^{-1}a_{i,q} = 0$. If $p = q$, then $a_{i,p}^{-1}a_{i,q} = \mathbf{d}(a_{i,p})$ is idempotent, thus, since $b_{p,j}^{-1}b_{p,k} = 0$, we get

$$b_{p,j}^{-1}a_{i,p}^{-1}a_{i,q}b_{q,k} \leq b_{p,j}^{-1}b_{p,k} = 0\,,$$

thus $b_{p,j}^{-1}a_{i,p}^{-1}a_{i,q}b_{q,k} = 0$, as desired. The proof of the relation $c_{j,i} \perp_{\mathrm{lt}} c_{k,i}$ is similar.

(3) Suppose first that a is row-finite and let $i, j \in \Omega$. By assumption, the set $X = \{k \in \Omega \mid a_{i,k} \neq 0\}$ is finite. It follows that $\bigoplus_{k\in\Omega} a_{i,k}b_{k,j} = \bigoplus_{k\in X} a_{i,k}b_{k,j}$ is defined. Hence ab is defined. The argument is similar in case b is column-finite.

Now suppose that a is row-finite and b is column-finite. By the paragraph above, $c = ab$ is defined. Let $i \in \Omega$. Since a is row-finite, the set $X = \{k \in \Omega \mid a_{i,k} \neq 0\}$ is finite. Since b is column-finite, the set $Y = \bigl\{j \in \Omega \mid (\exists k \in X)(b_{k,j} \neq 0)\bigr\}$ is finite. We shall prove that $a_{i,k}b_{k,j} = 0$, for any $j \in \Omega \setminus Y$ and any $k \in \Omega$. If $k \notin X$, then $a_{i,k} = 0$ and we are done. If $k \in X$, then, since $j \notin Y$, we get $b_{k,j} = 0$. In any case, we are done. This proves that $c_{i,j} = 0$ whenever $j \in \Omega \setminus Y$, thus completing the proof that c is row-finite. The proof that c is column-finite is symmetric.

A similar type of argument yields (4). □

Proposition 3.5.3 *The following statements hold, for any Boolean inverse semigroup S and every set Ω:*

(1) *The multiplication, $(a, b) \mapsto ab$, defined in the statement of Lemma 3.5.2, endows $\mathrm{B}^{\oplus}_{\Omega}(S)$ with a structure of an inverse semigroup, for which the inverse of a matrix $a = \bigl(a_{i,j} \mid (i,j) \in \Omega \times \Omega\bigr)$ is given by $a^{-1} = \Bigl(a_{j,i}^{-1} \mid (i,j) \in \Omega \times \Omega\Bigr)$. The idempotent elements of $\mathrm{B}^{\oplus}_{\Omega}(S)$ are the diagonal matrices with idempotent entries.*
(2) *Let $a, b \in \mathrm{B}^{\oplus}_{\Omega}(S)$. Then $a \leq b$ iff $a_{i,j} \leq b_{i,j}$ for all $i, j \in \Omega$.*
(3) *Two matrices $a, b \in \mathrm{B}^{\oplus}_{\Omega}(S)$ are left orthogonal (resp., right orthogonal) iff any row of a is left orthogonal to any row of b (resp., any column of a is right orthogonal to any column of b). Furthermore, if a and b are orthogonal, then their orthogonal join $a \oplus b$ is defined, and*

$$a \oplus b = \bigl(a_{i,j} \oplus b_{i,j} \mid (i,j) \in \Omega \times \Omega\bigr)\ .$$

(4) *$\mathrm{B}^{\oplus}_{\Omega}(S)$ is a Boolean inverse semigroup, in which $\mathrm{M}^{\oplus}_{\Omega}(S)$ is an additive ideal.*

Proof (1) The proof of the associativity of the matrix multiplication on $\mathrm{B}^{\oplus}_{\Omega}(S)$, given in the statement of Lemma 3.5.2, is identical, *mutatis mutandis* (and using Proposition 3.1.9), to the one of the associativity of the matrix multiplication over any ring, so we omit it.

Now set $\iota(a) = \Bigl(a_{j,i}^{-1} \mid (i,j) \in \Omega \times \Omega\Bigr)$, for any generalized rook matrix a over S. A straightforward calculation yields that $a \cdot \iota(a)$ is the diagonal matrix with diagonal entries $\bigoplus_{j \in \Omega} \mathbf{r}(a_{i,j})$, for $i \in \Omega$. A further easy calculation yields $a \cdot \iota(a) \cdot a = a$. In particular, any matrix of the form $a \cdot \iota(a)$ is diagonal with idempotent diagonal entries. Hence, any two such matrices commute. Since the map ι is obviously involutive, it follows from Lemma 3.1.1 that $\mathrm{B}^{\oplus}_{\Omega}(S)$ is an inverse semigroup, with inversion map ι. Further, the zero matrix is the zero element of $\mathrm{B}^{\oplus}_{\Omega}(S)$.

(2) As observed in the proof of (1), $\mathbf{r}(a)$ is the diagonal matrix with entries $e_i = \bigoplus_{j \in \Omega} \mathbf{r}(a_{i,j})$, for $i \in \Omega$. Hence, $\mathbf{r}(a)b = \Bigl(b'_{i,j} \mid (i,j) \in \Omega \times \Omega\Bigr)$ where we set $b'_{i,j} = e_i b_{i,j}$ whenever $i, j \in \Omega$. In particular, if $a \leq b$, that is, $a = \mathbf{r}(a)b$, then $a_{i,j} \leq b_{i,j}$ for all $i, j \in \Omega$. Suppose, conversely, that $a_{i,j} \leq b_{i,j}$ for all $i, j \in \Omega$. Let $k \in \Omega \setminus \{j\}$. From $\mathbf{r}(b_{i,k})b_{i,j} = 0$ and $a_{i,k} \leq b_{i,k}$ it follows that $\mathbf{r}(a_{i,k})b_{i,j} = 0$. Since $\mathbf{r}(a_{i,j})b_{i,j} = a_{i,j}$, a direct application of Proposition 3.1.9 yields that $e_i b_{i,j} = a_{i,j}$. Hence, $a = \mathbf{r}(a)b$, that is, $a \leq b$.

(3) For all $i, j \in \Omega$, the (i, j)-th entry of ab^{-1} is $\bigoplus_{k \in \Omega} a_{i,k} b_{j,k}^{-1}$. Hence, $ab^{-1} = 0$ iff $a_{i,k} b_{j,k}^{-1} = 0$ for each $k \in \Omega$, that is, any row of a is left orthogonal to any row of b. The proof of the statement about right orthogonality is similar.

Now suppose that $a \perp b$. Let $i, j, k \in \Omega$ with $j \neq k$. Since a and b are both generalized rook matrices over S, $a_{i,j} \perp_{\mathrm{rt}} a_{i,k}$ and $b_{i,j} \perp_{\mathrm{rt}} b_{i,k}$. Moreover, by the paragraph above, $a_{i,j} \perp_{\mathrm{rt}} b_{i,k}$ and $b_{i,j} \perp_{\mathrm{rt}} a_{i,k}$. Therefore, $a_{i,j} \oplus b_{i,j} \perp_{\mathrm{rt}} a_{i,k} \oplus b_{i,k}$. The proof of the relation $a_{j,i} \oplus b_{j,i} \perp_{\mathrm{lt}} a_{k,i} \oplus b_{k,i}$ is similar. It follows that the matrix $c = \big(a_{i,j} \oplus b_{i,j} \mid (i, j) \in \Omega \times \Omega\big)$ is a generalized rook matrix over S. An easy application of (2) yields then that c is the orthogonal join of $\{a, b\}$.

(4) By (1) above, $\operatorname{Idp} \mathrm{B}_{\Omega}^{\oplus}(S)$ is isomorphic to $(\operatorname{Idp} S)^{\Omega}$ endowed with the componentwise ordering. By (3) above, it follows that $\operatorname{Idp} \mathrm{B}_{\Omega}^{\oplus}(S)$ is Boolean. Hence, $\mathrm{B}_{\Omega}^{\oplus}(S)$ is a Boolean inverse semigroup. The subset $\mathrm{M}_{\Omega}^{\oplus}(S)$ is an ideal (cf. Lemma 3.5.2), closed under finite orthogonal sum by (3) above, so it is an additive ideal. □

For a Boolean inverse semigroup S and a set Ω, denote by $x_{(i,j)}$ the matrix with (i, j)-th entry x and all other entries 0, for all $x \in S$ and all $(i, j) \in \Omega \times \Omega$. It follows from Proposition 3.5.3 that every element of $\mathrm{M}_{\Omega}^{\oplus}(S)$ is a finite orthogonal join of elements of the form $x_{(i,j)}$. The $x_{(i,j)}$ behave essentially like matrix units:

$$x_{(i,j)} \cdot y_{(k,l)} = \delta_{j,k} \cdot (xy)_{(i,l)}\,, \qquad \text{for all } x, y \in S \text{ and all } i, j, k, l \in \Omega\,, \tag{3.5.1}$$

$$(x_{(i,j)})^{-1} = (x^{-1})_{(j,i)}\,, \qquad \text{for all } x \in S \text{ and all } i, j \in \Omega\,, \tag{3.5.2}$$

where $\delta_{j,k}$ denotes the Kronecker symbol. In particular,

$$e_{(i,i)} = e_{i,j}(e_{(i,j)})^{-1} \text{ and } e_{(j,j)} = (e_{(i,j)})^{-1} e_{(i,j)}\,, \quad \text{for all } e \in \operatorname{Idp} S \text{ and all } i, j \in \Omega\,, \tag{3.5.3}$$

so $e_{(i,i)} \mathrel{\mathscr{D}} e_{(j,j)}$ within $\mathrm{M}_{\Omega}^{\oplus}(S)$.

Corollary 3.5.4 *Let S be a Boolean inverse semigroup, let Ω be a set, and let $o \in \Omega$. Then the map $\eta \colon S \hookrightarrow \mathrm{M}_{\Omega}^{\oplus}(S)$, $x \mapsto x_{(o,o)}$ is a lower semigroup embedding and $\mathrm{M}_{\Omega}^{\oplus}(S)$ is an additive enlargement of $\eta[S]$.*

Proof It is straightforward to verify from Proposition 3.5.3 that η is an additive semigroup embedding. Set $\overline{S} = \eta[S]$ and $\overline{T} = \mathrm{M}_{\Omega}^{\oplus}(S)$. Then $\overline{S}$ consists of all matrices with all entries, with the possible exception of the (o, o)-th, zero. By the definition of the multiplication in $\overline{T}$, we obtain easily that $\overline{S}\,\overline{T}\,\overline{S} = \overline{S}$. Since $\overline{S}^{-1} = \overline{S}$, it follows that $\overline{S}$ is an additive quasi-ideal of $\overline{T}$.

Finally, it follows from Proposition 3.5.3(3) that the orthogonal joins in $\overline{T}$ are evaluated componentwise, thus every element of $\overline{T}$ is a finite orthogonal join of elements of the form $x_{(i,j)}$, where $x \in S$ and $(i, j) \in \Omega \times \Omega$. From $x_{(i,j)} = x_{(i,o)} x_{(o,o)} x_{(o,j)}$ it follows that $x_{(i,j)} \in \overline{T}\,\overline{S}\,\overline{T}$. Therefore, $\overline{T} = (\overline{T}\,\overline{S}\,\overline{T})^{\oplus}$. □

It is interesting to compare the results of this section, especially Proposition 3.5.3, to the corresponding results in ring theory. A unital ring R is an *exchange ring* if for every $x \in R$, there is an idempotent $e \in R$ such that $eR \subseteq xR$ and $(1-e)R \subseteq (1-x)R$.

Every von Neumann regular ring is an exchange ring, but the converse fails. A C*-algebra is an exchange ring iff it has real rank zero (cf. Ara et al. [13, Theorem 7.2]). O'Meara proves in [91] that the ring $\mathrm{B}(R)$, of all countably infinite, row-finite, and column-finite matrices over a regular ring R, is an exchange ring. He also observes there that for an arbitrary exchange ring R, $\mathrm{B}(R)$ may not be an exchange ring. On the other hand, it is well known that the ring $\mathrm{B}(R)$ is not regular unless R is trivial (if s is the matrix of the shift operator, then $1 - s$ has no quasi-inverse in $\mathrm{B}(R)$).

3.6 Crossed Product of a Boolean Inverse Semigroup by a Group Action

The goal of this section is to extend, to Boolean inverse semigroups, the classical construction of the crossed product of a ring by a group action (cf. Sect. 2.8).

Let a group G act by automorphisms on a Boolean inverse semigroup S. We denote the group action by $(g, x) \mapsto g(x)$. We set $x \cdot g = \bigl(\delta_{p,gq} p^{-1}(x) \mid (p, q) \in G \times G\bigr)$, for any $(x, g) \in S \times G$, where $\delta_{p,q}$ denotes the Kronecker symbol. The set $S \cdot G = \{x \cdot g \mid (x, g) \in S \times G\}$ is a subset of the Boolean inverse semigroup $\mathrm{B}_G^{\oplus}(S)$ of row-finite and column-finite $G \times G$ generalized rook matrices over S (cf. Proposition 3.5.3). The following lemma records a few elementary properties of the elements $x \cdot g$. Its proof is straightforward and we leave it to the reader.

Lemma 3.6.1 *The following statements hold, for any $x, y \in S$ and $g, h \in G$:*

(1) $x \cdot g = y \cdot h$ *iff* $x = y$ *and either* $g = h$ *or* $x = 0$*;*
(2) $(x \cdot g)(y \cdot h) = \bigl(xg(y)\bigr) \cdot gh$*;*
(3) $(x \cdot g)^{-1} = g^{-1}(x^{-1}) \cdot g^{-1}$*;*
(4) $\mathbf{d}(x \cdot g) = g^{-1}(\mathbf{d}(x)) \cdot 1$*;*
(5) $\mathbf{r}(x \cdot g) = \mathbf{r}(x) \cdot 1$.

In particular, $S \cdot G$ is an inverse subsemigroup of $\mathrm{B}_G^{\oplus}(S)$*, and the idempotent elements of $S \cdot G$ are the $e \cdot 1$, where $e \in \operatorname{Idp} S$.*

Definition 3.6.2 The *crossed product of S by* (*the action of*) G, denoted by $S \rtimes G$, is the closure of $S \cdot G$ under finite orthogonal joins, within $\mathrm{B}_G^{\oplus}(S)$.

Hence the elements of $S \rtimes G$ are the orthogonal joins of the form

$$x = \bigoplus_{i=1}^{n} (x_i \cdot g_i)\,, \quad \text{where } n \in \mathbb{Z}^+ \text{ and each } (x_i, g_i) \in S \times G\,. \tag{3.6.1}$$

The orthogonality, within $\mathrm{B}_G^{\oplus}(S)$, of the finite sequence $(x_i \cdot g_i \mid i \in [n])$ is, by Lemma 3.6.1, equivalent to the orthogonality, within $\operatorname{Idp} S$, of both finite sequences $\bigl(g_i^{-1}(\mathbf{d}(x_i)) \mid i \in [n]\bigr)$ and $(\mathbf{r}(x_i) \mid i \in [n])$.

Proposition 3.6.3 *Let a group G act by automorphisms on a Boolean inverse semigroup S. Then $S \rtimes G$ is an additive inverse subsemigroup of* $\mathrm{B}_G^{\oplus}(S)$. *In particular, it is a Boolean inverse semigroup. Furthermore,* $\mathrm{Idp}(S \rtimes G) = (\mathrm{Idp}\, S) \cdot 1$, *and the canonical map* $\varepsilon \colon S \hookrightarrow S \rtimes G$, $x \mapsto x \cdot 1$ *is a lower semigroup embedding.*

Proof It follows from the definition of $S \rtimes G$, together with Lemma 3.6.1 and Proposition 3.1.9, that $S \rtimes G$ is an inverse subsemigroup of $\mathrm{B}_G^{\oplus}(S)$, closed under finite orthogonal sums. Any element $x \in S \rtimes G$ can be written in the form (3.6.1), and then, using Lemma 3.6.1, we get $\mathbf{r}(x) = e \cdot 1$ where $e = \bigoplus_{i=1}^{n} \mathbf{r}(x_i)$. It follows that $\mathrm{Idp}(S \rtimes G) = (\mathrm{Idp}\, S) \cdot 1$. Since $\mathrm{Idp}\, S$ is Boolean, so is $\mathrm{Idp}(S \rtimes 1)$. In particular, $\mathrm{Idp}(S \rtimes 1)$ is closed under the operation $(x, y) \mapsto x \smallsetminus y$. By Lemma 3.1.13, $S \rtimes G$ is a Boolean inverse semigroup.

Let $x \in S \rtimes G$, written as in (3.6.1), and let $y \in S$ such that $x \leq y \cdot 1$ within $S \rtimes G$. For each $i \in [n]$, $x_i \cdot g_i = \mathbf{r}(x_i \cdot g_i)(y \cdot 1) = \bigl(\mathbf{r}(x_i) \cdot 1\bigr)(y \cdot 1) = \mathbf{r}(x_i)y \cdot 1$, thus $x_i = \mathbf{r}(x_i)y$ (i.e., $x_i \leq y$) and either $g_i = 1$ or $x_i = 0$. In any case, $x_i \cdot g_i = x_i \cdot 1$. Then it follows from Lemma 3.6.1 that the x_i are pairwise orthogonal in S; whence $x = \bigl(\bigoplus_{i=1}^{n} x_i\bigr) \cdot 1$ belongs to the range of ε. Therefore, ε is a lower semigroup embedding. □

Our next encounter with crossed products of Boolean inverse semigroups, involving type monoids, will occur in Theorem 4.1.10.

3.7 Fundamental Boolean Inverse Semigroups and Boolean Inverse Meet-Semigroups

Two important subclasses of the class of all Boolean inverse semigroups will come up repeatedly in our work, namely *fundamental Boolean inverse semigroups* and *Boolean inverse meet-semigroups*. This will also motivate the introduction of a definition of *semisimplicity* for Boolean inverse semigroups, close in spirit to the eponymous ring-theoretical concept.

We start by recalling the following definition.

Definition 3.7.1 An inverse semigroup S is *fundamental* (cf. Howie [60], Munn [87])[5] if the identity is the only idempotent-separating congruence of S. Equivalently, every element of S, which commutes with all idempotent elements of S, is idempotent.

Denote by $\boldsymbol{\mu}$ the largest idempotent-separating congruence of an inverse semigroup S. By Howie [60, Theorem V.3.4], the quotient $S/\boldsymbol{\mu}$ is then fundamental and $\mathrm{Idp}(S/\boldsymbol{\mu}) \cong \mathrm{Idp}\, S$.

The following lemma records a useful basic property of fundamental inverse semigroups.

[5] In Wagner [112, 113], Zhitomirskiy [126, 127], such semigroups are called *antigroups*.

Lemma 3.7.2 *Let S be a fundamental inverse semigroup and let p and q be atoms of* $\operatorname{Idp} S$. *Then there is at most one element $x \in S$ such that* $\mathbf{d}(x) = p$ *and* $\mathbf{r}(x) = q$.

Proof We first deal with the case where $p = q$. Let $x \in S$ such that $\mathbf{d}(x) = \mathbf{r}(x) = p$. In particular, $x = pxp$. Since p is an atom, every $e \in \operatorname{Idp} S$ satisfies either $p \leq e$ or $pe = 0$. In the first case, $xe = ex = x$. In the second case, $xe = ex = 0$. In either case, $xe = ex$, so x commutes with every idempotent of S. Since S is fundamental, x is idempotent, so $x = p$.

Now we deal with the general case. Let $x, y \in S$ such that $\mathbf{d}(x) = \mathbf{d}(y) = p$ and $\mathbf{r}(x) = \mathbf{r}(y) = q$. It follows that $\mathbf{d}(x^{-1}y) = \mathbf{r}(x^{-1}y) = p$, thus, by the paragraph above, $x^{-1}y = p$. It follows that $y = qy = xx^{-1}y = xp = x$. □

Example 3.7.3 For any set X, the symmetric inverse monoid $\mathfrak{I}_X$ (cf. Example 3.1.8) is a fundamental Boolean inverse semigroup.

Example 3.7.4 For a group G, the monoid $G^{\sqcup 0}$ (cf. Definition 1.5.1) is a Boolean inverse semigroup, with the same unit as G. It is fundamental iff G is trivial.

Let $\tau_g : G \to G$, $x \mapsto gx$, for all $g \in G$. Then the assignment $\tau : G^{\sqcup 0} \to \mathfrak{I}_G$ (cf. Example 3.1.8), defined by $0 \mapsto \varnothing$, $g \mapsto \tau_g$, is a semigroup embedding from $G^{\sqcup 0}$ into $\mathfrak{I}_G$. The orthogonality relation on the range of τ is trivial, hence the range of τ is closed under finite orthogonal joins. Therefore, $G^{\sqcup 0}$ is isomorphic to an inverse subsemigroup of $\mathfrak{I}_G$, closed under finite orthogonal joins. More generally, recall from Corollary 3.3.2 that every Boolean inverse semigroup has an additive semigroup embedding into some symmetric inverse monoid, thus into some fundamental Boolean inverse semigroup. This shows that Lemma 3.1.13 does not extend to fundamental Boolean inverse semigroups.

Definition 3.7.5 The *pedestal* of a Boolean inverse semigroup S is defined as the set $\operatorname{Ped} S = \{x \in S \mid S \downarrow x \text{ is finite}\}$. We say that S is *semisimple* if $\operatorname{Ped} S = S$.

The terminology in Definition 3.7.5 is consistent with the one, introduced in Ara and Goodearl [10, Definition 2.3], for conical refinement monoids. With an eye on ring theory, it would seem reasonable to call the subset defined above the *socle* of S. However, the concept of the (left or right) socle, of a semigroup with zero (cf. Clifford and Preston [30, § 6.4]), is related, but not equivalent, to our concept of a pedestal, even in the particular case of Boolean inverse semigroups.

It is not hard to verify that an element x, in a Boolean inverse semigroup S, belongs to $\operatorname{Ped} S$ iff $\mathbf{d}(x)$ (equivalently, $\mathbf{r}(x)$) is a finite join of atoms of the Boolean ring $\operatorname{Idp} S$, iff x is a finite orthogonal join of atoms of S. Further, $\operatorname{Ped} S$ is an additive ideal of S. Observe also that every finite Boolean inverse semigroup is semisimple.

Proposition 3.7.6 *Every additive congruence $\boldsymbol{\theta}$ of a fundamental semisimple Boolean inverse semigroup S is ideal-induced, and $S/\boldsymbol{\theta}$ is a fundamental semisimple Boolean inverse semigroup.*

Proof By applying Proposition 3.4.9 to the canonical projection $\theta : S \twoheadrightarrow S/\boldsymbol{\theta}$, we obtain that the subset $I = 0/\boldsymbol{\theta}$ is an additive ideal of S. In order to prove that $\boldsymbol{\theta}$ is induced by that ideal, it suffices to prove that the additive semigroup homomorphism

$\overline{\theta}: S/I \twoheadrightarrow S/\boldsymbol{\theta}$ given by Proposition 3.4.9 is one-to-one, that is, $\theta(x) = \theta(y)$ implies that $x \equiv_I y$, for all $x, y \in S$.

We first settle the case where $x, y \in qSp$, for atoms p and q of B. Since S is fundamental and by Lemma 3.7.2, either $x = y$ or $0 \in \{x, y\}$. In the first case, $x \equiv_I y$ trivially. In the second case, say $x = 0$, then $\theta(y) = 0$, that is, $y \in I$, so $x \equiv_I y$.

Now we settle the general case. Since S is semisimple, there is a finite set P of atoms of $\operatorname{Idp} S$ whose (orthogonal) join contains $\mathbf{d}(x)$, $\mathbf{r}(x)$, $\mathbf{d}(y)$, $\mathbf{r}(y)$. For any $p, q \in P$, $\theta(qxp) = \theta(qyp)$, thus, by the paragraph above, $qxp \equiv_I qyp$. By evaluating the orthogonal join, over $p \in P$, of both sides of that equation, we obtain, since $\equiv_I$ is an additive congruence (cf. Proposition 3.4.6), the relation

$$\bigoplus_{p \in P} qxp \equiv_I \bigoplus_{p \in P} qyp\,,$$

thus, using Proposition 3.1.9, $qx \equiv_I qy$. By the same token, now summing up over q instead of p, we obtain $x \equiv_I y$. This completes the proof that $\overline{\theta}$ is one-to-one.

Observe that $x/\boldsymbol{\theta}$ is either zero or an atom, for every atom x of S, according to whether $x \in I$ or $x \notin I$, respectively. Since every element of S is a finite join of atoms, it follows that every element of $S/\boldsymbol{\theta}$ is a finite join of atoms, that is, $S/\boldsymbol{\theta}$ is semisimple.

Finally we prove that $S/\boldsymbol{\theta}$ is fundamental. This amounts to proving that for every $x \in S$, if $xe \equiv_{\boldsymbol{\theta}} ex$ for all $e \in \operatorname{Idp} S$, then $x/\boldsymbol{\theta}$ is idempotent in $S/\boldsymbol{\theta}$. Since $\overline{\theta}$ is one-to-one, we get $xe \equiv_I ex$, for all $e \in \operatorname{Idp} S$. The latter relation means that there is $z_e \in S$ such that $z_e \leq \begin{matrix} xe \\ ex \end{matrix}$ and $\{xe \smallsetminus z_e, ex \smallsetminus z_e\} \subseteq I$. Set $v = \mathbf{d}(x) \vee \mathbf{r}(x)$ (any larger idempotent would do). Since S is semisimple, the set $P = (\operatorname{Idp} S) \downarrow v$ is finite. Since I is an additive ideal of S, the idempotent element

$$u = \bigvee_{e \in P} \bigl(\mathbf{d}(xe \smallsetminus z_e) \vee \mathbf{d}(ex \smallsetminus z_e)\bigr)$$

belongs to I; moreover, $u \leq v$. Observe that $xe \smallsetminus z_e = (xe \smallsetminus z_e)u$ for each $e \in P$. Hence, from $z_e \leq xe$ it follows that $xe(v \smallsetminus u) = z_e(v \smallsetminus u)$. Likewise, $ex(v \smallsetminus u) = z_e(v \smallsetminus u)$, so $xe(v \smallsetminus u) = ex(v \smallsetminus u)$. It follows that $x(v \smallsetminus u)$ commutes with every element of P, thus with every idempotent below v. Since it also commutes with every idempotent e orthogonal to v (for in that case, $xe = ex = 0$), it follows that $x(v \smallsetminus u)$ commutes with every idempotent of S. Since S is fundamental, $x \smallsetminus xu = x(v \smallsetminus u)$ is idempotent in S, thus $x/\boldsymbol{\theta} = x(v \smallsetminus u)/\boldsymbol{\theta}$ is idempotent in $S/\boldsymbol{\theta}$. □

Definition 3.7.7 An inverse semigroup S is an *inverse meet-semigroup* (cf. Leech [79], also Lawson [75]) if it is a meet-semilattice under $\leq$, that is, the meet $x \wedge y$ exists for all $x, y \in S$.

As witnessed by Example 3.1.7, not every finite inverse monoid with zero is an inverse meet-semigroup: in that example, $\frac{1}{2} \leq \frac{3}{4}$, but there is no x such that $\frac{1}{2} \leq x \leq \frac{3}{4}$. For Boolean inverse semigroups, this strange behavior does not occur.

Proposition 3.7.8 *Let S be a Boolean inverse semigroup, let $x \in \operatorname{Ped} S$, and let $y \in S$. Then $x \wedge y$ exists in S. In particular, every semisimple Boolean inverse semigroup S is an inverse meet-semigroup.*

Proof The set X, of all common lower bounds of x and y, is a compatible subset of the finite set $S \downarrow x$. Since S is Boolean, X has a join in S, which is necessarily the meet of $\{a, b\}$. □

The following example shows that not every fundamental Boolean inverse semigroup is an inverse meet-semigroup. By Proposition 3.7.8, any such example is infinite.

Example 3.7.9 Define S as the inverse subsemigroup of the symmetric inverse semigroup $\mathfrak{I}_{\mathbb{Z}^+}$ (cf. Example 3.1.8) consisting of all functions whose domain is either finite or cofinite. Then S is a fundamental Boolean inverse semigroup. However, for any permutation α of $\mathbb{Z}^+$ whose fixed point set consists of all even numbers, $\alpha \wedge \mathrm{id}_{\mathbb{Z}^+}$ does not exist in S. Hence S is not an inverse meet-semigroup.

It is well known that any compatible elements x and y in an inverse semigroup S have a meet, given (among many other expressions) by (3.1.3). In particular, *every semigroup homomorphism preserves compatible meets*. On the other hand, we will see shortly that additive semigroup homomorphisms between Boolean inverse meet-semigroups may not preserve meets (cf. Example 3.7.12). Nevertheless, the following result shows that under certain conditions, additive semigroup homomorphisms may preserve all meets.

Proposition 3.7.10 *Let S be a fundamental Boolean inverse semigroup, let T be a Boolean inverse semigroup, and let $f : S \to T$ be an additive semigroup homomorphism. Then $f(x \wedge y) = f(x) \wedge f(y)$, for all $x \in \operatorname{Ped} S$ and all $y \in S$.*

Note Although, by Proposition 3.7.8, the meet $x \wedge y$ exists in S, we are not assuming that T is an inverse meet-semigroup.

Proof Set $B = \operatorname{Idp} S$. The set P, of all atoms of B below $\mathbf{d}(x) \vee \mathbf{r}(x)$, is finite. Let $z \in T$ such that $z \leq \frac{f(x)}{f(y)}$. By multiplying those inequalities by $f(p)$ on the right side and $f(q)$ on the left side, we obtain

$$f(q)zf(p) \leq \frac{f(qxp)}{f(qyp)}, \quad \text{for any } p, q \in P. \tag{3.7.1}$$

It follows from Lawson [73, Proposition 1.4.19] that $(qxp) \wedge (qyp) = q(x \wedge y)p$. Further, by Lemma 3.7.2, either $qxp = qyp$ or $0 \in \{qxp, qyp\}$. Hence, in any case,

$$f(qxp) \wedge f(qyp) = f(q(x \wedge y)p) = f(q)f(x \wedge y)f(p),$$

and hence, by (3.7.1), we get $f(q)zf(p) \leq f(q)f(x \wedge y)f(p)$. This holds for all $p, q \in P$, thus, since $z \leq \bigoplus_{p \in P} f(p)$ and by using the additivity of f, we get $z \leq f(x \wedge y)$. □

The following two examples show that the assumption in Proposition 3.7.6, that $x \in \operatorname{Ped} S$, cannot be dropped. Moreover, Example 3.7.11 witnesses that Proposition 3.7.10 cannot be extended to arbitrary finite Boolean inverse semigroups S, and Example 3.7.12 witnesses that the finiteness assumption is necessary in Proposition 3.7.10, even for inverse meet-semigroups S.

Example 3.7.11 Finite Boolean inverse monoids S and T, together with a surjective, non one-to-one additive semigroup homomorphism $f: S \twoheadrightarrow T$ such that $\ker f = \{0\}$.

Proof Let G be any non-trivial group. Set $S = G^{\sqcup 0}$ and $T = \{0, \infty\}$ (the two-element join-semilattice), and let $f: S \twoheadrightarrow T$ the map that sends 0 to 0 and any element of G to ∞. Then f is an additive semigroup homomorphism and $\ker f = \{0\}$. Since G is non-trivial, f is not one-to-one. □

Example 3.7.12 Fundamental, unital, Boolean inverse meet-semigroups S and T, together with a surjective additive semigroup homomorphism $f: S \twoheadrightarrow T$, with an invertible element $\alpha \in S \setminus \{1\}$ such that $f(\alpha \wedge 1) < f(\alpha) = f(1)$ and $\alpha \not\equiv_{\ker f} 1$. In particular, f is not ideal-induced.

Proof Define S as the inverse submonoid of the symmetric inverse monoid $\mathfrak{I}_{\mathbb{Z}^+}$ (cf. Example 3.1.8) consisting of all bijections $x: A \to B$, where A and B are both either finite or cofinite subsets of $\mathbb{Z}^+$, and such that if A is cofinite, then there exists $n \in \mathbb{Z}$ such that $x(k) = n + k$ for all large enough $k \in A$ (*this condition is put there in order to ensure that S is an inverse meet-semigroup*). Further, define T as the two-element join-semilattice $\{0, \infty\}$, and define $f: S \twoheadrightarrow T$ by letting $f(x) = \infty$ iff the domain of x is cofinite, whenever $x \in S$. Then S and T are both fundamental unital Boolean inverse meet-semigroups and f is an additive semigroup homomorphism from S to T.

Now let α be any permutation of $\mathbb{Z}^+$ without fixed points (e.g., let α interchange $2n$ and $2n + 1$, for any $n \in \mathbb{Z}^+$). Then $f(\alpha) = f(\mathrm{id}) = \infty$ and $f(\alpha \wedge \mathrm{id}) = f(0) = 0$. If $\alpha \equiv_{\ker f} \mathrm{id}$, then α and id would need to agree on some cofinite subset of $\mathbb{Z}^+$, which is not the case. □

The following example shows that an additive homomorphic image of a fundamental unital Boolean inverse meet-semigroup may not be fundamental.

Example 3.7.13 A fundamental unital Boolean inverse meet-semigroup S, a unital Boolean inverse meet-semigroup T, and a surjective additive semigroup homomorphism $f: S \twoheadrightarrow T$, such that T is not fundamental.

Proof We use the same monoid S as in Example 3.7.12 together with the larger $T = \mathbb{Z}^{\sqcup 0}$. For every $x \in S$, we set $f(x) = 0$ in case the domain of x is finite. If x is infinite, we define $f(x)$ as the unique $n \in \mathbb{Z}$ such that $x(k) = n + k$ for all large enough k. Observe that T is not fundamental (cf. Example 3.7.4). □

We will need later the following preservation result for fundamental Boolean inverse semigroups and Boolean inverse meet-semigroups.

Proposition 3.7.14 *Let T be a Boolean inverse semigroup. If T is fundamental (resp., a Boolean inverse meet-semigroup), then so is any additive quasi-ideal of T, and so is $\mathrm{M}_{\Omega}^{\oplus}(T)$, for any set Ω.*

Proof Let S be any additive quasi-ideal of T. It follows from Proposition 3.1.18 that S is a lower inverse subsemigroup of T.

Set $\overline{T} = \mathrm{M}_{\Omega}^{\oplus}(T)$.

Suppose first that T is fundamental and let $x \in S$ commute with all idempotents of S. Since x commutes with both $\mathbf{d}(x)$ and $\mathbf{r}(x)$, we get $\mathbf{d}(x) = \mathbf{r}(x)$. Denote by a this element. Since S is a lower subset of T, $T \downarrow a$ is contained in $\operatorname{Idp} S$. By assumption, it follows that x commutes with all elements of $T \downarrow a$. On the other hand, $xe = ex = 0$ for any $e \in \operatorname{Idp} T$ orthogonal to a. Since $e = ea \oplus (e \smallsetminus a)$ for any $e \in \operatorname{Idp} T$ and by Proposition 3.1.9, it follows that x commutes with all idempotent elements of T. Since T is fundamental, x is idempotent. Therefore, S is fundamental.

Any element $x \in \overline{T}$ that commutes with all idempotents must commute with all $e_{(i,i)}$, where $e \in \operatorname{Idp} S$ and $i \in [n]$. It follows easily that x must be a diagonal matrix, each of whose diagonal entries commutes with all idempotents of T. Since T is fundamental, it follows that x is a diagonal matrix with idempotent entries; hence x is idempotent. Therefore, $\overline{T}$ is fundamental.

Finally, we only assume that T is an inverse meet-semigroup. Since S is a lower subset of T, it is a meet-subsemilattice of T, thus it is also a fundamental unital Boolean inverse meet-semigroup. Furthermore, since T is an inverse meet-semigroup and by Proposition 3.5.3(2), $\overline{T}$ is an inverse meet-semigroup and the meets in $\overline{T}$ are evaluated componentwise. □

3.8 Inner Endomorphisms and Automorphisms of a Boolean Inverse Semigroup

We set $\mathrm{ad}_g(x) = g\langle x\rangle = gxg^{-1}$, for all elements g and x in an inverse semigroup S. We call ad_g the *inner endomorphism* determined by g. If S is unital, then inner endomorphisms with respect to invertible elements are automorphisms, called *inner automorphisms* of S.

In order to extend this definition to the case where S is not unital, we add the assumption that S is Boolean, then we need to drop the assumption that g be invertible but we keep the assumption that $\mathbf{d}(g) = \mathbf{r}(g)$. Then we replace g

by $g \oplus e$, for large enough idempotents e ranging through the ideal $g^{\perp} = \{e \in \operatorname{Idp} S \mid e \perp g\} = \{e \in \operatorname{Idp} S \mid ge = eg = 0\}$ of the Boolean ring $\operatorname{Idp} S$. As the following lemma shows, for large enough $e \in g^{\perp}$, the value of $(g \oplus e)\langle x \rangle$ depends only on g and x.

Lemma 3.8.1 *Let S be a Boolean inverse semigroup and let $g, x \in S$. Then the value of $(g \oplus e)\langle x \rangle$, where $e \in g^{\perp}$ and $\mathbf{d}(x) \vee \mathbf{r}(x) \leq (\mathbf{d}(g) \vee \mathbf{r}(g)) \oplus e$, depends only on g and x.*

Proof Both elements $\overline{x} = \mathbf{d}(x) \vee \mathbf{r}(x)$ and $\overline{g} = \mathbf{d}(g) \vee \mathbf{r}(g)$ are idempotent. Let $e_i \in \operatorname{Idp} S$ such that $e_i \perp g$ and $\overline{x} \leq \overline{g} \oplus e_i$, for $i \in \{0, 1\}$. From $e_i = (\overline{g} \oplus e_i) \smallsetminus \overline{g}$ it follows, by multiplying on the left by x, that $xe_i = x \smallsetminus x\overline{g}$ is independent of i. Symmetrically, $e_i x = x \smallsetminus \overline{g}x$ is also independent of i. It follows that $e_i x e_i = xe_i \smallsetminus \overline{g}xe_i = (x \smallsetminus x\overline{g}) \smallsetminus \overline{g}(x \smallsetminus x\overline{g})$ is also independent of i. Therefore,

$$(g \oplus e_i)\langle x \rangle = (g \oplus e_i)x(g^{-1} \oplus e_i) = gxg^{-1} \oplus gxe_i \oplus e_i xg^{-1} \oplus e_i xe_i$$

is independent of i. □

Notation 3.8.2 We shall denote by $\operatorname{inn}_g(x)$ the constant value of $(g \oplus e)\langle x \rangle$, for large enough $e \in g^{\perp}$.

Hence, inn_g is the directed union, over all $e \in g^{\perp}$, of all maps $\operatorname{ad}_{g \oplus e}$.

We will be interested in situations where inn_g is an automorphism of S. We wish to identify those $g \in S$ such that inn_g defines an automorphism of aSa for any large idempotent a. Accordingly, we define a subset of S as follows.

Notation 3.8.3 We set $\operatorname{Self} S = \{g \in S \mid \mathbf{d}(g) = \mathbf{r}(g)\}$, for any Boolean inverse semigroup S.

Observe that $\operatorname{Self} S$ is usually not a subsemigroup of S.

Lemma 3.8.4 *The following statements hold, for any Boolean inverse semigroup S:*

(1) $\operatorname{inn}_{g \oplus e} = \operatorname{inn}_g$, *for any $g \in S$ and any $e \in g^{\perp}$. In particular,* $\operatorname{inn}_e = \operatorname{id}_S$ *whenever e is idempotent.*
(2) $\operatorname{inn}_{fg} = \operatorname{inn}_f \circ \operatorname{inn}_g$, *for any $f, g \in \operatorname{Self} S$ with $\mathbf{d}(f) = \mathbf{d}(g)$.*
(3) inn_g *is an automorphism of S, for any $g \in \operatorname{Self} S$. We call the automorphisms of that form the* inner automorphisms *of S.*
(4) $x \mathscr{D} \operatorname{inn}_g(x)$, *for any $x \in S$ and any $g \in \operatorname{Self} S$.*
(5) *The inner automorphisms of S form a subgroup of the automorphism group of S.*

Proof (1) is trivial.

(2) Set $a = \mathbf{d}(f) = \mathbf{d}(g)$. For all $x \in S$ and all large enough $e \in a^{\perp}$,

$$\begin{aligned}(\operatorname{inn}_f \circ \operatorname{inn}_g)(x) = (f \oplus e)\langle (g \oplus e)\langle x \rangle \rangle &= ((f \oplus e)(g \oplus e))\langle x \rangle \\ &= (fg \oplus e)\langle x \rangle \\ &= \operatorname{inn}_{fg}(x)\,.\end{aligned}$$

(3) follows trivially from (1) and (2).

(4) Since $x \mathscr{D} \mathbf{d}(x)$ for every x, a direct application of (3) reduces the problem to the case where x is idempotent. Set $a = \mathbf{d}(g) = \mathbf{r}(g)$ and let $e \in a^{\perp}$ such that $\mathbf{d}(x) \vee \mathbf{r}(x) \leq a \oplus e$. Setting $h = g \oplus e$, we get $\mathbf{d}(h) = \mathbf{r}(h) = a \oplus e$. It follows from Lemma 3.8.1 that $\mathrm{inn}_g(x) = h\,\langle x \rangle$; thus $\mathrm{inn}_g(x) = (hx)(hx)^{-1}$. Moreover,

$$(hx)^{-1}hx = x^{-1}(h^{-1}h)x = x^{-1}(a \oplus e)x = x^{-1}x = x\,.$$

Hence $x \mathscr{D} \mathrm{inn}_g(x)$.

(5) Let $f, g \in \mathrm{Self}\, S$, with respective domains a and b. We must prove that $\mathrm{inn}_f \circ \mathrm{inn}_g$ is an inner automorphism of S. By (1), we may replace f by $f \oplus (b \smallsetminus a)$ and g by $g \oplus (a \smallsetminus b)$, and thus suppose that $\mathbf{d}(f) = \mathbf{d}(g)$. The conclusion follows then immediately from (2). □

Observe that every inner automorphism of S fixes all elements in some $a^{\perp}$, where $a \in S$: if $g \in \mathrm{Self}\, S$ and $\mathbf{d}(g) = \mathbf{r}(g) = a$, then $\mathrm{inn}_g(x) = x$ for every $x \in a^{\perp}$.

Notation 3.8.5 We denote by $\mathrm{Inn}\, S$ the group of all inner automorphisms of S.

Of course, if S has a unit, then $\mathrm{Inn}\, S = \{\mathrm{ad}_g \mid g \text{ invertible element of } S\}$. However, $\mathrm{Inn}\, S$ is also defined if S has no unit. In fact, it can be proved that $\mathrm{Inn}\, S \cong \mathrm{Inn}\, \widetilde{S}$, where $\widetilde{S}$ is the Boolean inverse monoid, introduced in Sect. 6.6, which we will call the *Boolean unitization* of S.

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