Chapter 2
Quantum Mechanics and Central Extensions

In this short chapter we discuss the implementation of symmetries in a quantum-mechanical context. For definiteness and simplicity we assume throughout that these symmetries span a Lie group. We start in Sect. 2.1 with a brief review of the symmetry representation theorem of Wigner and show how quantum mechanics gives rise to projective unitary representations. The problem of classifying such representations then leads to Sects. 2.2 and 2.3, respectively devoted to Lie algebra cohomology and group cohomology. The presentation is inspired by [1–4]; see also [5].

2.1 Symmetries and Projective Representations

In this section we review the interplay between quantum mechanics and symmetries. After a brief general reminder on the formalism of quantum theory, we state the symmetry representation theorem which justifies the study of unitary representations of groups and Lie algebras. We also show how the fact that quantum states are rays in a Hilbert space (rather than individual vectors) leads to projective representations, hence to central extensions. We end with a discussion of topological central extensions, while algebraic central extensions are postponed to Sect. 2.2.

2.1.1 Quantum Mechanics

Definition A (complex) *Hilbert space* $\mathscr{H}$ is a vector space over $\mathbb{C}$ endowed with a Hermitian form

$$\langle\cdot|\cdot\rangle : \mathscr{H}\times\mathscr{H}\to\mathbb{C} : (\Phi,\Psi)\mapsto\langle\Phi|\Psi\rangle, \tag{2.1}$$

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such that the norm of a vector Ψ be $\sqrt{\langle\Psi|\Psi\rangle}$, and such that the resulting normed vector space be complete.[1] We take the scalar product (2.1) to be linear in its second argument and antilinear in the first one.

Note that our notation is *not* the standard Dirac notation of bras and kets: a vector in $\mathscr{H}$ is denoted as Ψ (not $|\Psi\rangle$), and its dual is the linear form $\langle\Psi|\cdot\rangle$ on $\mathscr{H}$. Accordingly, the Hermitian conjugate $\hat{A}^\dagger$ of a linear operator $\hat{A}$ is defined by

$$\langle\Phi|\hat{A}^\dagger\Psi\rangle \equiv \langle\hat{A}\Phi|\Psi\rangle \quad \text{for all } \Phi, \Psi \in \mathscr{H}. \tag{2.2}$$

An operator $\hat{A}$ is Hermitian (or self-adjoint)[2] if $\hat{A}^\dagger = \hat{A}$.

Now consider a quantum system whose space of states is a Hilbert space $\mathscr{H}$. A pure *quantum state* of the system is a ray in $\mathscr{H}$, that is, a one-dimensional subspace

$$[\Psi] = \{z\Psi | z \in \mathbb{C}\} \tag{2.3}$$

where Ψ is some non-zero state vector. The vanishing vector does not represent a quantum state, so the set of mutually inequivalent pure states is the projective space $\mathbb{P}\mathscr{H} = (\mathscr{H}\backslash 0)/\mathbb{C}$. It is the set of one-dimensional subspaces of $\mathscr{H}$. Stated differently, the set of distinct states in $\mathscr{H}$ is the quotient of the unit sphere in $\mathscr{H}$ by the equivalence relation

$$\Psi \sim e^{i\theta}\Psi \quad \text{for all } \theta \in \mathbb{R}. \tag{2.4}$$

We shall denote by $[\Psi]$ the resulting equivalence class of Ψ. For example, in a two-level system where $\mathscr{H} = \mathbb{C}^2$, the set of inequivalent states is $\mathbb{C}P^1 \cong S^2$.

Now let the system be in a state $[\Psi]$. If $\hat{A}$ is an observable and if λ is one of its eigenvalues with eigenvector Φ say, the probability of finding the value λ is

$$\text{Prob}(\lambda, \hat{A}, [\Psi]) = \frac{|\langle\Phi|\Psi\rangle|^2}{\langle\Phi|\Phi\rangle\langle\Psi|\Psi\rangle}. \tag{2.5}$$

(We are assuming for simplicity that the eigenvalue λ is not degenerate.) Note that this expression is independent, as it should, of the choice of both the representative Ψ of the state $[\Psi]$, and the eigenvector Φ.

Remark In quantum mechanics, one generally assumes that the Hilbert space is *separable*, i.e. that it admits a countable basis. Any such space is isometric to the space $\ell^2(\mathbb{N})$ of square-integrable sequences of complex numbers — so there really exists only *one* infinite-dimensional separable Hilbert space. This is not to say that all separable Hilbert spaces describe the same quantum system, because the definition of a system also involves the set of observables that act on it — and identical Hilbert spaces may well come with very different operator algebras.

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[1] Recall that a metric space is *complete* if any Cauchy sequence converges.

[2] We will not take into account issues related to the domains of operators.

2.1.2 *Symmetry Representation Theorem*

Symmetry Groups

A *symmetry* is a transformation of a system that leaves it invariant. In particular, the set of symmetries of a system always contains the identity transformation, and any symmetry transformation is invertible. In addition the composition of any two symmetry transformations is itself a symmetry, and composition is associative. Put together, these properties imply that

the set of symmetries of any system forms a group.

Accordingly, the framework suited for the study of symmetries is *group theory*.

In this thesis we will be concerned with *Lie groups*, consisting of symmetry transformations that depend smoothly on a certain number of real parameters. This number is the *dimension* of the group. In part I of the thesis, all Lie groups are finite-dimensional.

Remark The notion of symmetry can be relaxed in such a way that not all pairs of symmetry transformations are allowed to be composed together. The resulting set of symmetry transformations then spans a *groupoid* rather than a group (see e.g. [6, 7]). This relaxed notion of symmetry is relevant to gauge theories [8], and in particular to BMS symmetry in four dimensions [9]. However, standard group theory suffices for all symmetry considerations in three-dimensional gravity (and in particular for BMS_3), so we will not deal with groupoids in this thesis.

Symmetries in Quantum Mechanics

Consider a quantum Hilbert space of states $\mathscr{H}$. In these terms a symmetry is a bijection $\mathbb{P}\mathscr{H} \to \mathbb{P}\mathscr{H} : [\Psi] \mapsto \mathcal{S}([\Psi])$ that preserves the probabilities (2.5). Equivalently, if we represent rays in $\mathscr{H}$ by normalized vectors subject to the identification (2.4), a symmetry transformation $\mathcal{S}$ must be such that

$$|\langle\Phi|\Psi\rangle| = \left|\langle\Phi'|\Psi'\rangle\right| \tag{2.6}$$

for all normalized vectors Φ, Ψ, Φ', Ψ' such that $\Phi' \in \mathcal{S}([\Phi])$ and $\Psi' \in \mathcal{S}([\Psi])$. The key result on symmetries in quantum mechanics is the following [10]:

Symmetry Representation Theorem Let $\mathcal{S} : \mathbb{P}\mathscr{H} \to \mathbb{P}\mathscr{H}$ be an invertible transformation satisfying property (2.6). Then it takes the form $\mathcal{S}([\Psi]) = [\hat{U} \cdot \Psi]$, where $\hat{U}$ is either a linear, unitary operator so that

$$\hat{U} \cdot (\lambda\Phi + \mu\Psi) = \lambda\,\hat{U} \cdot \Phi + \mu\,\hat{U} \cdot \Psi \qquad \text{and} \qquad \langle\hat{U} \cdot \Phi|\hat{U} \cdot \Psi\rangle = \langle\Phi|\Psi\rangle\,,$$

or an antilinear, antiunitary operator so that

$$\hat{U} \cdot (\lambda\Phi + \mu\Psi) = \bar{\lambda}\,\hat{U} \cdot \Phi + \bar{\mu}\,\hat{U} \cdot \Psi \qquad \text{and} \qquad \langle\hat{U} \cdot \Phi|\hat{U} \cdot \Psi\rangle = \langle\Psi|\Phi\rangle$$

for all $\lambda, \mu \in \mathbb{C}$ and all $\Phi, \Psi \in \mathscr{H}$. A proof of this theorem can be found in Chap. 2 (Appendix A) of [1].

Note that symmetries represented by antiunitary operators only arise when the symmetry group is disconnected. For example, in Lorentz-invariant theories, time-reversal is always represented in an antiunitary way (see e.g. [11]). In this work we will restrict attention to connected symmetry groups, in which case all symmetry operators are linear and unitary. In particular they satisfy $\hat{U}^\dagger = \hat{U}^{-1}$, where Hermitian conjugation is defined by (2.2).

2.1.3 *Projective Representations*

The symmetry representation theorem implies that all (connected) symmetry groups are represented unitarily in a quantum-mechanical system, and thus motivates the study of unitary representations in general. Let us first recall the basics:

Definition A *representation* of a group G in a vector space $\mathscr{H}$ is a homomorphism[3]

$$\mathcal{T} : G \to \mathrm{GL}(\mathscr{H}) : g \mapsto \mathcal{T}[g]$$

where $\mathrm{GL}(\mathscr{H})$ is the group of invertible linear transformations of $\mathscr{H}$. When $\mathscr{H}$ is a Hilbert space, the representation is *unitary* if $\mathcal{T}[g]$ is a unitary operator for each $g \in G$.

In quantum mechanics the notion of symmetry as a transformation that satisfies (2.6) leads to a key subtlety. Let us call $\mathcal{T}[f]$ the unitary operator that represents a symmetry transformation f belonging to some group G. Then, because a quantum state is really an equivalence class (2.3) of vectors in $\mathscr{H}$, there is no need to require $\mathcal{T}$ to be a homomorphism; rather, all we need is that the ray of $\mathcal{T}[f] \cdot \mathcal{T}[g] \cdot \Phi$ coincides with that of $\mathcal{T}[f \cdot g] \cdot \Phi$ (for all $f, g \in G$ and any $\Phi \in \mathscr{H}$). Accordingly, $\mathcal{T}$ must really be a unitary representation *up to a phase*,

$$\mathcal{T}[f] \cdot \mathcal{T}[g] = e^{i\mathsf{C}(f,g)}\, \mathcal{T}[f \cdot g] \qquad \text{for } f, g \in G, \tag{2.7}$$

where C is some real function on $G \times G$. In more abstract terms, $\mathcal{T}$ must define a group action on the projective space $\mathbb{P}\mathscr{H}$, which is to say that the map

$$[\mathcal{T}] : G \to \mathrm{GL}(\mathscr{H})/\mathbb{C}^* : f \mapsto \big[\mathcal{T}[f]\big] \tag{2.8}$$

is a homomorphism. Here $\mathrm{GL}(\mathscr{H})/\mathbb{C}^*$ is the projective group of $\mathscr{H}$, i.e. the quotient of the linear group of $\mathscr{H}$ by its normal subgroup consisting of multiples of the identity. For any operator $\mathcal{O}$ in $\mathrm{GL}(\mathscr{H})$, the symbol $[\mathcal{O}]$ denotes its class in the projective group. Throughout this thesis, any map $\mathcal{T}$ satisfying this property will be called

[3] Throughout this thesis representations of groups are denoted by the letters $\mathcal{R}, \mathcal{S}, \mathcal{T}$, etc. The letter G denotes a group whose elements are written f, g, h, etc. The identity in G is denoted e.

a *projective representation*. In quantum mechanics, symmetries are represented by *unitary* projective representations, i.e. projective representations whose operators are unitary.

From now on, if we wish to stress that a representation is *not* projective, we will call it *exact*. Quantum mechanics tells us that exact representations are overrated: the truly important ones are generally projective. This seemingly anecdotal observation is at the core of the richest aspects of the representation theory of the Virasoro algebra, and it will also play a key role for BMS_3 particles. For instance, all interesting two-dimensional conformal field theories are such that the conformal group is represented projectively in their Hilbert space, and how exactly this phenomenon takes place is measured by the central charge. For this reason, this whole chapter is devoted to the various ways in which projective effects occur; they are accounted for by group and Lie algebra cohomology.

Remark Since we are focussing on Lie groups, the representations of interest are *continuous* in the sense that the map $G \times \mathscr{H} \to \mathscr{H} : (f, \Psi) \mapsto \mathcal{T}[f] \cdot \Psi$ is continuous. From now on it is understood that all representations are continuous.

2.1.4 Central Extensions

The function C appearing in (2.7) is not completely arbitrary. Indeed, the product (2.7) must be associative in the sense that $\mathcal{T}[f] \cdot (\mathcal{T}[g] \cdot \mathcal{T}[h]) = (\mathcal{T}[f] \cdot \mathcal{T}[g]) \cdot \mathcal{T}[h]$ for all group elements f, g, h, so that

$$\mathsf{C}(f, gh) + \mathsf{C}(g, h) = \mathsf{C}(fg, h) + \mathsf{C}(f, g) \quad \text{for all } f, g, h \in G. \tag{2.9}$$

Any function $\mathsf{C} : G \times G \to \mathbb{R}$ satisfying this requirement is known as a (real) *two-cocycle*, and the condition itself is known as the *cocycle condition*. Given any such function one can define a new group

$$\widehat{G} \equiv G \times \mathbb{R} \tag{2.10}$$

whose elements are pairs (f, λ), endowed with a group operation

$$\boxed{(f, \lambda) \cdot (g, \mu) = \big(f \cdot g, \lambda + \mu + \mathsf{C}(f, g)\big).} \tag{2.11}$$

The group (2.10) is called a *central extension* of the group G. We will study this notion in much greater detail in Sect. 2.3. For now let us only work out the basic consequences of this structure and its relation to representation theory.

Projective Versus Exact Representations

Property (2.7) says that $\mathcal{T}$ is an exact unitary representation of the centrally extended group (2.10), provided one represents the pair (f, λ) by $e^{i\lambda}\mathcal{T}[f]$. In other words, exact representations are not overrated after all: we may view any projective representation of G as an exact (i.e. non-projective) representation of a central extension $\widehat{G}$ of G, and the problem of classifying projective unitary representations of G boils down to that of classifying *exact* unitary representations of its central extensions.

The question then is whether G admits central extensions to begin with. For any group, an obvious type of central extension always exists. Namely, suppose K is a real function on G and define $\mathsf{C} : G \times G \to \mathbb{R}$ by

$$\mathsf{C}(f, g) \equiv \mathsf{K}(fg) - \mathsf{K}(f) - \mathsf{K}(g). \tag{2.12}$$

This automatically satisfies condition (2.9). A two-cocycle of that form is said to be *trivial*. In particular, if the cocycle in (2.7) is trivial, it can be absorbed by defining $\tilde{\mathcal{T}}[f] \equiv e^{i\mathsf{K}(f)}\mathcal{T}[f]$, which is an exact representation of G. Thus, what we wish to know is not quite whether G admits two-cocycles at all (since trivial ones are always available), but rather whether it admits *non-trivial* two-cocycles. If yes, it admits genuine projective representations, whose phases cannot be absorbed by a mere redefinition.

This question leads to group (and Lie algebra) cohomology, studied in detail in Sects. 2.2 and 2.3. For now we simply point out that central extensions may arise via two distinct mechanisms. The first is *algebraic* in that it follows from the local group structure of G, or equivalently from the commutation relations of its Lie algebra. In short, in some cases, the Lie algebra $\mathfrak{g}$ of G can be enlarged into a bigger algebra $\widehat{\mathfrak{g}}$ which contains extra generators commuting with those of $\mathfrak{g}$ (see Eq. (2.27) below). The group corresponding to this enlarged algebra then is a central extension of G. The second mechanism is *topological* in the sense that it is due to the global structure of G. We now describe this topological mechanism in some more detail.

2.1.5 Topological Central Extensions

If the group G is not simply connected (i.e. its fundamental group is non-trivial), there exist closed paths in G that cannot be continuously deformed into a point. Let $\gamma : [0, 1] \to G$ be such a path, starting and ending at some group element f so that $\gamma(0) = \gamma(1) = f$. Suppose we are given a (continuous) projective unitary representation $\mathcal{T}$ of G, and consider the path

$$\mathcal{T} \circ \gamma : [0, 1] \to \mathrm{GL}(\mathscr{H}) : t \mapsto \mathcal{T}[\gamma(t)]$$

in the space of unitary operators on $\mathscr{H}$. Since $\mathcal{T}$ is projective, the fact that γ is a closed path does *not* imply that $\mathcal{T} \circ \gamma$ is closed: in general $\mathcal{T}[\gamma(0)]$ and $\mathcal{T}[\gamma(1)]$ differ by a γ-dependent phase, $\mathcal{T}[\gamma(1)] = e^{i\phi(\gamma)}\mathcal{T}[\gamma(0)]$.

Owing to the fact that the map $\mathcal{T}$ is continuous, the phase $\phi(\gamma)$ only depends on the homotopy class of γ. In addition, if γ_1 and γ_2 are two closed paths starting at f, we can concatenate them into a single path $\gamma_1 \cdot \gamma_2$ (which is γ_1 at double speed followed by γ_2 at double speed); the phase ϕ must be compatible with this operation in the sense that $e^{i\phi(\gamma_1)} \cdot e^{i\phi(\gamma_2)} = e^{i\phi(\gamma_1 \cdot \gamma_2)}$. Thus, any one-dimensional unitary representation of the fundamental group of G, multiplying an exact unitary representation of G, produces a projective unitary representation of G.

This is the topological notion of central extensions that we wanted to exhibit: if G is multiply connected, it admits genuine projective representations (whose phases cannot be removed by redefinitions) due to one-dimensional unitary representations of its fundamental group.[4] Projective representations of that type may equivalently be seen as exact representations of the universal cover $\widetilde{G}$ of G, which is the unique connected and simply connected group locally isomorphic to G.

Remark One might be worried by the fact that only *one-dimensional* unitary representations of the fundamental group are allowed to appear in this construction. Indeed, if the fundamental group was non-Abelian, it would generally admit no non-trivial one-dimensional unitary representation. Fortunately, it turns out that the fundamental group of any finite-dimensional Lie group is a discrete commutative group, whose irreducible unitary representations are necessarily one-dimensional.

Rotations and Anyons

The simplest example of topological projective representations arises with the group $\mathrm{U}(1)$. The latter is diffeomorphic to a circle and has a fundamental group isomorphic to $\mathbb{Z}$ (see Fig. 2.1). Any exact irreducible, unitary representation of $\mathrm{U}(1)$ takes the form

$$\mathcal{T} : \mathrm{U}(1) \to \mathbb{C}^* : \theta \mapsto e^{is\theta} \tag{2.13}$$

where θ is identified with $\theta + 2\pi$, as a consequence of which the "spin" s is an integer. For example, when $s = 2$, a rotation by $\theta = \pi$ is represented by the identity. (We will see in Sect. 4.3 that the label s actually *is* the spin of a particle in certain representations of the Poincaré groups.) But there is a subtlety: $\mathrm{U}(1)$ is multiply connected and admits topological projective representations, which from the viewpoint of quantum mechanics are just as acceptable as exact ones. For example, the map (2.13) with $s = 1/2$ definitely isn't an exact representation because a full rotation by 2π is now represented by an inversion, $\mathcal{T}[2\pi] = e^{i\pi} = -1$. Nevertheless, in quantum mechanics, the vectors Ψ and $\mathcal{T}[2\pi] \cdot \Psi$ define the same state by virtue of the identification (2.4), so in this sense $\mathcal{T}[2\pi]$ acts as an "almost-identity" operator. More generally, formula (2.13) is a projective representation of $\mathrm{U}(1)$ for *any* real value of the spin s.

The example just described occurs in Nature. Indeed, fermions provide a well-known example of projective representations, as already suggested above by the case $s = 1/2$. By the spin-statistics theorem, all fermions have half-integer spins, and therefore transform according to a projective representation of the Lorentz group. The latter is multiply connected (its fundamental group is $\mathbb{Z}_2$), which is why it admits projective representations in the first place. We will return to the representation theory

[4]Beware: a manifold being *multiply connected* means that it has a non-trivial fundamental group, and *not* that it has several connected components.

Fig. 2.1 The group U(1) is diffeomorphic to a circle S^1, whose universal cover is the real line $\mathbb{R}$. The projection $\mathbb{R} \to S^1 \cong \mathbb{R}/\mathbb{Z}$ is obtained by identifying points of $\mathbb{R}$ that differ by some periodicity, typically $\theta \sim \theta + 2\pi$. In particular, paths in $\mathbb{R}$ which are not closed may be projected on closed paths in S^1. As an application we can picture topological projective representations: if $\mathcal{T}$ is projective and if γ is a closed path in the circle, the sequence $\mathcal{T}[\gamma(t)]$ may not be a closed path in the space of operators

of the Lorentz group (as a subgroup of Poincaré) in much greater detail in Sect. 4.2. In the cases where arbitrary real values of spin are allowed by quantum mechanics, as for example in three space-time dimensions, the particles whose spin is neither an integer nor a half-integer are known as *anyons*. We will encounter this phenomenon in Sect. 10.1 when dealing with BMS_3 particles.

2.1.6 Classifying Projective Representations

Given a group G, suppose we wish to find all its projective unitary representations. The above considerations provide an algorithm that allows us, in principle, to solve that problem:

- First find the universal cover $\widetilde{G}$ of G to take care of topological central extensions.
- Then find the most general central extension $\widehat{\widetilde{G}}$ of $\widetilde{G}$ in order to take care of differentiable central extensions. (We will deal with the actual definition of these extensions in the next section.)
- Finally, consider an *exact* unitary representation of $\widehat{\widetilde{G}}$; any projective unitary representation of G may be seen as a representation of that type.

Thus we now have a systematic procedure allowing us to build arbitrary projective unitary representations of symmetry groups in quantum mechanics. We will apply it later to the Virasoro algebra (Sect. 8.4) and the BMS_3 group (Sect. 10.1), where central extensions play a crucial role.

2.2 Lie Algebra Cohomology

This section is devoted to a thorough investigation of the concept of central extensions at the Lie-algebraic level. In fact, we shall describe the more general framework of Lie algebra cohomology and we will show how statements on algebraic central extensions can be recast in that language. The group-theoretic analogue of this construction is relegated to Sect. 2.3.

2.2.1 Cohomology

Let $\mathfrak{g}$ be a Lie algebra with Lie bracket $[\cdot, \cdot]$. We recall that a *representation* of $\mathfrak{g}$ in a vector space $\mathbb{V}$ is a linear map $\mathscr{T} : \mathfrak{g} \to \mathrm{End}(\mathbb{V})$ such that $\mathscr{T}[X] \circ \mathscr{T}[Y] - \mathscr{T}[Y] \circ \mathscr{T}[X] = \mathscr{T}\big[[X, Y]\big]$ for all Lie algebra elements X, Y.[5]

Definition Let k be a non-negative integer, $\mathscr{T}$ a representation of $\mathfrak{g}$ in $\mathbb{V}$. Then a $\mathbb{V}$-valued *k-cochain* on $\mathfrak{g}$ is a continuous, multilinear, completely antisymmetric map[6]

$$\mathsf{c} : \underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}_{k \text{ times}} \to \mathbb{V} : (X_1, ..., X_k) \mapsto \mathsf{c}(X_1, ..., X_k). \tag{2.14}$$

In other words, a $\mathbb{V}$-valued k-cochain on $\mathfrak{g}$ is a k-form on $\mathfrak{g}$ with values in $\mathbb{V}$; note that $0 \leq k \leq \dim(\mathfrak{g})$. A zero-cochain on $\mathfrak{g}$ is a vector in $\mathbb{V}$ while a $\dim(\mathfrak{g})$-cochain is a volume form on $\mathfrak{g}$. We denote the space of $\mathbb{V}$-valued k-cochains on $\mathfrak{g}$ by $\mathcal{C}^k(\mathfrak{g}, \mathbb{V})$ and we define the associated cochain complex $\mathcal{C}^*(\mathfrak{g}, \mathbb{V}) \equiv \oplus_{k=0}^{\dim(\mathfrak{g})} \mathcal{C}^k(\mathfrak{g}, \mathbb{V})$. The latter is sometimes called the *Chevalley-Eilenberg complex*.

Definition The *Chevalley-Eilenberg differential* $\mathsf{d} : \mathcal{C}^*(\mathfrak{g}, \mathbb{V}) \to \mathcal{C}^*(\mathfrak{g}, \mathbb{V})$ is defined by $\dim(\mathfrak{g})$ linear maps

$$\mathsf{d}_k : \mathcal{C}^k(\mathfrak{g}, \mathbb{V}) \to \mathcal{C}^{k+1}(\mathfrak{g}, \mathbb{V}) : \mathsf{c} \mapsto \mathsf{d}_k\mathsf{c}$$

where k runs from 0 to $\dim(\mathfrak{g}) - 1$ and the $(k+1)$-cochain $\mathsf{d}_k\mathsf{c}$ is given by

$$\begin{aligned}(\mathsf{d}_k\mathsf{c})(X_1, ..., X_{k+1}) \equiv & \sum_{1 \leq i < j \leq k+1} (-1)^{i+j-1} \mathsf{c}\big([X_i, X_j], X_1, ..., \widehat{X_1}, ..., \widehat{X_j}, ..., X_{k+1}\big) \\ & + \sum_{1 \leq i \leq k+1} (-1)^i \mathscr{T}[X_i] \cdot \mathsf{c}\big(X_1, ..., \widehat{X_i}, ..., X_{k+1}\big)\end{aligned} \tag{2.15}$$

for all $X_1, ..., X_{k+1}$ in $\mathfrak{g}$; the hat denotes omission. Note that the representation $\mathscr{T}$ of $\mathfrak{g}$ in $\mathbb{V}$ appears explicitly in this definition. In particular, when $\mathscr{T}$ is trivial, formula (2.15) simplifies since its last line disappears.

Cocycles and Coboundaries

Using the fact that $\mathscr{T}$ is a representation, one can verify that the Chevalley-Eilenberg differential (2.15) is nilpotent:

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[5] Throughout this thesis the elements of a Lie algebra $\mathfrak{g}$ will be denoted as X, Y, etc. Representations of Lie algebras will be denoted by script capital letters such as $\mathscr{R}, \mathscr{S}, \mathscr{T}$.

[6] Cochains on Lie algebras will be denoted by lowercase sans serif letters such as c, s, etc.

$$\mathsf{d}_k \circ \mathsf{d}_{k-1} = 0 \qquad \forall\, k = 0, ..., \dim(\mathfrak{g}) \tag{2.16}$$

where it is understood that the "extreme differentials" are $\mathsf{d}_{-1} : 0 \to \mathbb{V} : 0 \mapsto 0$ and $\mathsf{d}_{\dim \mathfrak{g}} : \mathcal{C}^{\dim \mathfrak{g}}(\mathfrak{g}, \mathbb{V}) \to 0 : \mathsf{c} \mapsto 0$. Accordingly, one adapts the standard terminology of differential forms to cochains on a Lie algebra: a *k-cocycle* is a k-cochain c such that $\mathsf{d}_k\mathsf{c} = 0$; a *k-coboundary* is a k-cochain c of the form $\mathsf{c} = \mathsf{d}_{k-1}\mathsf{b}$, where b is some $(k-1)$-cochain. By virtue of property (2.16), one has $\mathrm{Im}(\mathsf{d}_{k-1}) \subseteq \mathrm{Ker}(\mathsf{d}_k)$ for each k (any coboundary is a cocycle). One can therefore define the k^{th} *cohomology space* of $\mathfrak{g}$ with coefficients in $\mathbb{V}$ as the quotient of the space of k-cocycles by the space of k-coboundaries:

$$\mathcal{H}^k(\mathfrak{g}, \mathbb{V}) \equiv \mathrm{Ker}(\mathsf{d}_k)/\mathrm{Im}(\mathsf{d}_{k-1}). \tag{2.17}$$

A k-cocycle is said to be *trivial* if its equivalence class vanishes in $\mathcal{H}^k$, i.e. if the cocycle is a coboundary; the cocycle is *non-trivial* otherwise. When $\mathbb{V} = \mathbb{R}$ with $\mathscr{T}$ the trivial representation of $\mathfrak{g}$, we write $\mathcal{H}^k(\mathfrak{g}, \mathbb{R}) \equiv \mathcal{H}^k(\mathfrak{g})$.

Isomorphic Lie algebras have the same cohomology for any choice of the representation $\mathscr{T}$. Thus, cohomology is a way to associate invariants with Lie algebras: if two algebras have different cohomology spaces, then they cannot be isomorphic. This is analogous to, say, de Rham cohomology in differential geometry, as manifolds with different de Rham cohomologies cannot be diffeomorphic.

Low Degree Cohomologies

There is a simple interpretation for the lowest cohomology spaces. For example, zero-cocycles are vectors $v \in \mathbb{V}$ that are invariant under $\mathfrak{g}$ in the sense that

$$\mathscr{T}[X] \cdot v = 0 \qquad \text{for all } X \in \mathfrak{g}\,, \tag{2.18}$$

so the zeroth cohomology space of $\mathfrak{g}$ classifies the invariants of the representation $\mathscr{T}$. Similarly, one-cocycles are known as *derivations* of $\mathfrak{g}$ and are classified by the first cohomology space $\mathcal{H}^1(\mathfrak{g}, \mathbb{V})$. In the particular case where $\mathscr{T}$ is trivial and $\mathbb{V} = \mathbb{R}$, a one-cocycle is a linear map $\mathsf{c} : \mathfrak{g} \to \mathbb{R}$ such that $\mathsf{c}([X, Y]) = 0$ for all Lie algebra elements X, Y. Hence the first real cohomology space of $\mathfrak{g}$ can be written as

$$\mathcal{H}^1(\mathfrak{g}) \cong \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]\,, \tag{2.19}$$

which motivates the following definition:

Definition A Lie algebra $\mathfrak{g}$ is *perfect* if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, i.e. if any Lie algebra element can be written as the bracket of two other elements.

It follows from (2.19) that $\mathfrak{g}$ is perfect if and only if $\mathcal{H}^1(\mathfrak{g})$ vanishes. We will use this property in Sect. 2.2.2 when defining central extensions.

By the definitions above, a two-cochain is an antisymmetric map $\mathsf{c} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{V}$. It is a coboundary if

$$\mathsf{c}(X, Y) \overset{(2.15)}{=} (\mathsf{d}_1\mathsf{k})(X, Y) = \mathsf{k}([X, Y]) - \mathscr{T}[X] \cdot \mathsf{k}(Y) + \mathscr{T}[Y] \cdot \mathsf{k}(X) \quad (2.20)$$

for some one-cochain k; and it is a cocycle if

$$\begin{aligned} &\mathsf{c}([X, Y], Z) + \mathsf{c}([Y, Z], X) + \mathsf{c}([Z, X], Y) = \\ &= \mathscr{T}[X] \cdot \mathsf{c}(Y, Z) + \mathscr{T}[Y] \cdot \mathsf{c}(Z, X) + \mathscr{T}[Z] \cdot \mathsf{c}(X, Y). \end{aligned} \quad (2.21)$$

As we shall see shortly, when $\mathscr{T}$ is trivial, a two-cocycle defines a *central extension* of $\mathfrak{g}$. Thus the second cohomology of $\mathfrak{g}$ classifies its extensions. More generally, cohomology may be seen as a measure of flexibility: Lie algebras with high-dimensional cohomology groups can be "deformed" in many inequivalent ways; by contrast, Lie algebras with trivial cohomology are "rigid" in the sense that any deformation is equivalent to no deformation at all.

Remark Here we have been using the word "deformation" in a vague way, but there is an exact definition of the notion of deformations. Namely, a (true) *deformation* of a Lie algebra $\mathfrak{g}$ is a Lie algebra $\tilde{\mathfrak{g}}$ that coincides with $\mathfrak{g}$ as a vector space, but whose brackets are

$$\tilde{[}X, Y\tilde{]} = [X, Y] + \mathsf{c}(X, Y) \quad (2.22)$$

where $[\cdot, \cdot]$ is the bracket in $\mathfrak{g}$ while c is a $\mathfrak{g}$-valued two-cocycle on $\mathfrak{g}$,[7] such that the image of c belongs to its kernel. The latter condition means that $\mathsf{c}\big(X, \mathsf{c}(Y, Z)\big) = 0$ for all Lie algebra elements X, Y, Z; together with the fact that c is a cocycle, this ensures that (2.22) is a Lie bracket.

Examples

For finite-dimensional semi-simple Lie algebras, cohomology is trivial:

Whitehead's Lemma Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra, $\mathscr{T}$ an irreducible, finite-dimensional representation of $\mathfrak{g}$ in a space $\mathbb{V}$. Then

$$\mathcal{H}^k(\mathfrak{g}, \mathbb{V}) = 0 \quad \text{for all } k > 0. \quad (2.23)$$

Despite this result, examples of non-trivial cohomologies do exist in physics. For instance, let c be an arbitrary non-vanishing antisymmetric bilinear form on $\mathbb{R}^2$, and view the latter as an Abelian Lie algebra. Then c defines a non-trivial, real-valued two-cocycle on $\mathbb{R}^2$, so the real-valued second cohomology of $\mathbb{R}^2$ is non-trivial; in fact one can prove that

$$\mathcal{H}^2(\mathbb{R}^2) \cong \mathbb{R}. \quad (2.24)$$

We shall see below that this property is related to the (three-dimensional) Heisenberg algebra, which is crucial for quantum mechanics. Other important examples of algebras with non-trivial cohomology spaces include the Galilei algebra (Sect. 4.4), the Virasoro algebra (Chap. 6) and the $\mathfrak{bms}_3$ algebra (Chap. 9).

[7]It is understood that the relevant representation of $\mathfrak{g}$ in this case is the adjoint, $\mathscr{T}[X] \cdot Y \equiv [X, Y]$.

2.2.2 Central Extensions

Definition Let $\mathfrak{g}$ be a (real) Lie algebra and let $\mathsf{c} \in \mathcal{C}^2(\mathfrak{g}, \mathbb{R})$ be a real two-cocycle on $\mathfrak{g}$. Then c defines a *central extension* $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$, which is a Lie algebra whose underlying vector space

$$\widehat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R} \qquad \text{(as vector spaces)} \tag{2.25}$$

is endowed with the centrally extended Lie bracket

$$\big[(X, \lambda), (Y, \mu)\big] \equiv \big([X, Y], \mathsf{c}(X, Y)\big). \tag{2.26}$$

In particular, elements of $\widehat{\mathfrak{g}}$ are pairs (X, λ) where $X \in \mathfrak{g}$ and $\lambda \in \mathbb{R}$, so that $\mathbb{R}$ is an Abelian subalgebra of $\widehat{\mathfrak{g}}$. The bracket (2.26) satisfies the Jacobi identity on account of the fact that c is a two-cocycle with respect to a trivial representation of $\mathfrak{g}$ (so that the right-hand side of Eq. (2.21) vanishes).

In (2.26) we displayed the definition of central extensions in intrinsic terms thanks to the two-cocycle c. The same definition can be written in terms of Lie algebra generators: let $\{t_a | a = 1, ..., n\}$ be a basis of $\mathfrak{g}$ with brackets $[t_a, t_b] = f_{ab}{}^c\, t_c$. Then a central extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ is a Lie algebra generated by the basis elements $T_a \equiv (t_a, 0)$ together with a central element $\mathcal{Z} = (0, 1)$, whose Lie brackets read

$$[T_a, T_b] = f_{ab}{}^c\, T_c + c_{ab}\, \mathcal{Z} \tag{2.27}$$

where $c_{ab} = \mathsf{c}(t_a, t_b)$, while all brackets with $\mathcal{Z}$ vanish. The cocycle condition on c then becomes the requirement

$$f_{ab}{}^d c_{dc} + f_{bc}{}^d c_{da} + f_{ca}{}^d c_{db} = 0$$

for the coefficients c_{ab}. Note that this construction can be readily generalized to multiple central extensions $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}^N$, in which case there are N central generators $\mathcal{Z}_1, ..., \mathcal{Z}_N$.

Non-Trivial Central Extensions

When the two-cocycle c is trivial in the sense of Lie algebra cohomology, it takes the form (2.20) in terms of some one-cocycle k and the map

$$\mathfrak{g} \to \widehat{\mathfrak{g}} : X \mapsto \big(X, \mathsf{k}(X)\big) \tag{2.28}$$

is an injective homomorphism of Lie algebras. The central extension is then said to be *trivial*: the cocycle c can be absorbed by the "redefinition" (2.28), and $\widehat{\mathfrak{g}}$ is isomorphic to the direct sum $\mathfrak{g} \oplus \mathbb{R}$ as a Lie algebra. By contrast, when c is non-trivial, it defines a non-zero element in the second cohomology space $\mathcal{H}^2(\mathfrak{g})$; such a two-cocycle cannot be removed by a mere redefinition, and the central extension is *non-trivial*.

For example, as on p.xx, consider the Abelian Lie algebra $\mathbb{R}^2$ and let c be a non-zero antisymmetric bilinear form on $\mathbb{R}^2$. We then define the three-dimensional *Heisenberg algebra* as the algebra $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$ whose elements are pairs (X, λ), endowed with the Lie bracket (2.26). Since c is non-trivial, so is the central extension. If we choose a basis $\{Q, P\}$ of $\mathbb{R}^2$ such that $\mathsf{c}(Q, P) = 1$ and if we call Z the central element $(0, 1)$, the commutation relations of the Heisenberg algebra take the form

$$[Q, P] = Z. \tag{2.29}$$

Property (2.24) says that there is only one linearly independent central extension of $\mathbb{R}^2$, i.e. that Heisenberg algebras built using different (non-zero) two-cocycles c are mutually isomorphic. This can be generalized to higher dimensions: by seeing $\mathbb{R}^{2n}$ as an Abelian Lie algebra and taking c an arbitrary non-zero $2n$-form on $\mathbb{R}^{2n}$, the Lie algebra defined by the bracket (2.26) is the $(2n + 1)$-dimensional Heisenberg algebra.

Universal Central Extensions

It is important to know how many inequivalent central extensions an algebra may possess. This leads to the following notion:

Definition A central extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ is *universal* if, for any other central extension $\widehat{\mathfrak{g}}'$ of $\mathfrak{g}$, there exists a unique isomorphism of Lie algebras $\widehat{\mathfrak{g}}' \cong \widehat{\mathfrak{g}}$.

As it turns out, any perfect Lie algebra admits a universal central extension. By virtue of (2.19), this is to say that any algebra such that $\mathcal{H}^1(\mathfrak{g}) = 0$ admits a universal central extension. For example we will see in Chap. 6 that the Virasoro algebra is the universal central extension of the Lie algebra of vector fields on the circle.

2.3 Group Cohomology

This section is devoted to the group-theoretic analogue of the considerations of the previous pages. We start by discussing generalities on group cohomology before focussing on central extensions of groups.

2.3.1 Cohomology

Let G be a Lie group, $\mathcal{T} : G \to \mathrm{GL}(\mathbb{V})$ a representation of G in a vector space $\mathbb{V}$.

Definition Let $k \geq 0$ be an integer. A $\mathbb{V}$-valued *k-cochain* on G is a smooth map[8]

[8]Cochains on a group will be denoted by capital sans serif symbols such as C, S, etc.

$$\mathsf{C} : \underbrace{G \times \cdots \times G}_{k \text{ times}} \to \mathbb{V} : (g_1, ..., g_k) \mapsto \mathsf{C}(g_1, ..., g_k). \tag{2.30}$$

Note that, in contrast to the Lie-algebraic definition (2.14), there is no restriction on k. The new ingredient in the group-theoretic context is the requirement that the map (2.30) be smooth. As in the case of Lie algebras, we denote by $\mathcal{C}^k(G, \mathbb{V})$ the vector space of $\mathbb{V}$-valued k-cochains on G and we let $\mathcal{C}^*(G, \mathbb{V}) = \oplus_{k=0}^{+\infty}\mathcal{C}^k(G, \mathbb{V})$ be the associated cochain complex. The space of zero-cochains is just $\mathbb{V}$.

Definition The *differential* $\mathsf{d} : \mathcal{C}^*(G, \mathbb{V}) \to \mathcal{C}^*(G, \mathbb{V})$ is defined by the maps

$$\mathsf{d}_k : \mathcal{C}^k(G, \mathbb{V}) \to \mathcal{C}^{k+1}(G, \mathbb{V}) : \mathsf{C} \mapsto \mathsf{d}_k\mathsf{C}$$

where $k \in \mathbb{N}$ and the $(k+1)$-cochain $\mathsf{d}_k\mathsf{C}$ is given by

$$\begin{aligned}(\mathsf{d}_k\mathsf{C})(g_1, ..., g_{k+1}) \equiv \mathcal{T}[g_1] \cdot \mathsf{C}(g_2, ..., g_{k+1}) + (-1)^{k+1}\,\mathsf{C}(g_1, ..., g_k)\\ + \sum_{i=1}^{k}(-1)^i\,\mathsf{C}(g_1, ..., g_i g_{i+1}, ..., g_{k+1})\end{aligned} \tag{2.31}$$

for all $g_1, ..., g_{k+1}$ in G.

The differential (2.31) satisfies the key property (2.16), so the usual machinery of homological algebra applies: one defines a *k-cocycle* as a closed k-cochain, that is, a cochain C such that $\mathsf{d}_k\mathsf{C} = 0$. One also defines a *k-coboundary* to be an exact k-cochain, i.e. one that can be written as the differential of a $(k-1)$-cochain. As before any coboundary is trivially a cocycle, so one defines the k^{th} *cohomology space* of G with values in $\mathbb{V}$ as the quotient of the space of k-cocycles by the space of k-coboundaries:

$$\mathcal{H}^k(G, \mathbb{V}) \equiv \mathrm{Ker}(\mathsf{d}_k)/\mathrm{Im}(\mathsf{d}_{k-1}).$$

A k-cocycle is *trivial* if its class in $\mathcal{H}^k(G, \mathbb{V})$ vanishes; it is non-trivial otherwise. When $\mathbb{V} = \mathbb{R}$ with $\mathcal{T}$ the trivial representation, we write $\mathcal{H}^k(G, \mathbb{R}) \equiv \mathcal{H}^k(G)$.

Interpretation

As in the case of Lie algebras, cohomology spaces are invariants that measure the flexibility of a group structure; isomorphic Lie groups have the same cohomology. This interpretation is simplest to illustrate with the cohomology spaces of lowest degree.

A $\mathbb{V}$-valued zero-cocycle on G is a vector $v \in \mathbb{V}$ such that $(\mathsf{d}_0 v)(f) = \mathcal{T}[f] \cdot v - v = 0$ for any group element f. Accordingly, the zeroth cohomology space of G classifies vectors $v \in \mathbb{V}$ that are left invariant by G. This is the group-theoretic analogue of (2.18).

A $\mathbb{V}$-valued one-cocycle is a (smooth) map $\mathsf{S} : G \to \mathbb{V}$ satisfying the property

$$\mathsf{S}(fg) = \mathcal{T}[f] \cdot \mathsf{S}(g) + \mathsf{S}(f) \qquad \forall\, f, g \in G. \tag{2.32}$$

Given a one-cocycle S, one defines the associated *affine module* as the space $\mathbb{V} \oplus \mathbb{R}$ acted upon by the following representation $\widehat{\mathcal{T}}$ of G:

$$\widehat{\mathcal{T}}[f] \cdot (v, \lambda) \equiv \big(\mathcal{T}[f] \cdot v + \lambda \, \mathsf{S}(f), \lambda\big). \tag{2.33}$$

The cocycle condition (2.32) ensures that $\widehat{\mathcal{T}}$ is indeed a representation. In addition one can show that affine modules defined using different one-cocycles are equivalent if (and only if) their cocycles differ by a coboundary. Thus $\mathcal{H}^1(G, \mathbb{V})$ classifies affine G-modules based on $\mathbb{V}$. For example, in Sect. 6.3 we will see that the Schwarzian derivative is a one-cocycle on the group of diffeomorphisms of the circle; this is why we denote the cocycle in (2.33) by S. The corresponding affine module will be the coadjoint representation of the Virasoro group and the parameter λ left invariant by (2.33) will be a Virasoro central charge. More generally one can think of the term $\lambda \mathsf{S}[f]$ in (2.33) as an anomaly that adds an inhomogeneous term to the otherwise homogeneous transformation law of v under G.

Two-cocycles lead to the notion of group extensions; in particular, when $\mathbb{V} = \mathbb{R}$ with $\mathcal{T}$ the trivial representation, $\mathcal{H}^2(G)$ classifies central extensions of G. Indeed, when C is a real two-cocycle on G, the requirement $\mathsf{d}_2\mathsf{C} = 0$ becomes the cocycle condition (2.9); the central extension is trivial when C is a coboundary, i.e. if it takes the form (2.12) for some one-cochain K. We will return to central extensions of groups in Sect. 2.3.2.

Relation to Lie Algebra Cohomology

One may ask how group and Lie algebra cohomology are related. The following result provides a first answer:

Proposition Let G be a Lie group, $\mathfrak{g}$ its Lie algebra. Let $\mathbb{V}$ be a vector space, $\mathcal{T}$ a smooth representation of G in $\mathbb{V}$, and $\mathscr{T}$ the representation of $\mathfrak{g}$ corresponding to $\mathcal{T}$ by differentiation. Then, for any non-negative integer k, there is a homomorphism

$$\mathcal{H}^k(G, \mathbb{V}) \to \mathcal{H}^k(\mathfrak{g}, \mathbb{V}) : [\mathsf{C}] \mapsto \big[\delta \mathsf{C}\big] \tag{2.34}$$

given by

$$\delta\mathsf{C}(X_1, ..., X_k) \equiv \frac{\partial^k}{\partial t_1 ... \partial t_k} \left[\sum_{1 \leq i_1 < ... < i_k \leq k} \epsilon_{i_1 ... i_k} \mathsf{C}\big(e^{t_{i_1} X_{i_1}}, ..., e^{t_{i_k} X_{i_k}}\big) \right] \Bigg|_{t_1=0, ..., t_k=0}$$

for all $X_1, ..., X_k$ in $\mathfrak{g}$, with e^X the exponential of $X \in \mathfrak{g}$ and $\epsilon_{i_1 ... i_k}$ the Levi-Civita symbol with k indices (and $\epsilon_{12...k} \equiv +1$). For $k = 2$ this can be rewritten as

$$\delta\mathsf{C}(X, Y) = \frac{\partial^2}{\partial t \, \partial s} \left[\mathsf{C}\left(e^{tX}, e^{sY}\right) - \mathsf{C}\left(e^{sY}, e^{tX}\right) \right] \Big|_{t=0, \, s=0}. \tag{2.35}$$

The fact that (2.34) is a homomorphism ensures that, if $\delta\mathsf{C}$ is a non-trivial cocycle, then C itself is non-trivial. The converse is not true since the map need not be injective: a non-trivial cocycle C may well be such that $\delta\mathsf{C}$ is trivial.

We will use formula (2.35) in Sect. 6.2 to relate the Virasoro algebra to the Virasoro group. The key point here is that any differentiable group cocycle C admits an algebraic analogue $\delta\mathsf{C}$. The converse problem is to start from a Lie algebra cocycle, say c, and ask whether there exists a group cocycle whose differential is c. This is the problem of integrating Lie algebra cocycles to group cocycles, and it is generally much more complicated than differentiation. However, for "sufficiently connected" Lie groups, the Van Est theorem states that integration is trivial because group and Lie algebra cohomologies coincide (see e.g. [4]). In particular, when the universal cover of a group is homotopic to a point, the cohomology of the universal cover coincides with that of the Lie algebra.

2.3.2 Central Extensions

Here we return in more detail to the notion of centrally extended groups, already outlined around (2.11). For simplicity we deal only with simply connected groups, so as to avoid the topological complications of Sect. 2.1.5. Including these subtleties would lead to a definition of central extensions somewhat more general (see e.g. [4]) than the one given here:

Definition Let G be a Lie group, C a real two-cocycle on G. Then the associated *centrally extended group* $\widehat{G}$ is topologically a product $G \times \mathbb{R}$ whose elements are pairs (f, λ) with $f \in G$ and $\lambda \in \mathbb{R}$, endowed with a group operation (2.11).

It is straightforward to generalize this definition to the case where $\mathbb{R}$ is replaced by an arbitrary (additive) Abelian group such as $\mathbb{R}^N$.

Non-Trivial Central Extensions

As in the Lie-algebraic case, a central extension of G is *trivial* if the two-cocycle C defining the group operation (2.11) is a coboundary (2.12) for some one-cochain K. Then the map $G \to \widehat{G} : f \mapsto \big(f, \mathsf{K}(f)\big)$ is an injective homomorphism whose Lie-algebraic analogue is (2.28), and $\widehat{G}$ is isomorphic, as a group, to the direct product $G \times \mathbb{R}$. Thus any trivial central extension can be absorbed by a redefinition of the group, and is irrelevant as regards projective representations. By contrast, when the cohomology class of C is a non-zero vector in $\mathcal{H}^2(G)$, the central extension cannot be removed by a redefinition and is said to be *non-trivial*.

Example Let us find the group corresponding to the $(2n+1)$-dimensional Heisenberg algebra. Consider the Abelian additive group $G = \mathbb{R}^n \times \mathbb{R}^n$ (whose elements are pairs of column vectors (α, β)) and define the *Heisenberg group* as

$$\widehat{G} \equiv \left\{ \begin{pmatrix} 1 & \alpha^t & \lambda \\ 0 & \mathbb{I}_n & \beta \\ 0 & 0 & 1 \end{pmatrix} \middle| \alpha, \beta \in \mathbb{R}^n,\ \lambda \in \mathbb{R} \right\} \tag{2.36}$$

where $\mathbb{I}_n$ denotes the $n \times n$ identity matrix and α^t is the transpose of α. The group operation in $\widehat{G}$ is given by matrix multiplication and can be written as

$$(\alpha, \beta, \lambda) \cdot (\alpha', \beta', \lambda') = \big(\alpha + \alpha', \beta + \beta', \lambda + \lambda' + \alpha^t \cdot \beta'\big) \tag{2.37}$$

where $\alpha^t \cdot \beta' \equiv \alpha^i \beta'^i$ is the Euclidean scalar product of α and β'. Thus the Heisenberg group is a central extension of $\mathbb{R}^{2n}$ defined by the two-cocycle

$$\mathsf{C}\big((\alpha, \beta), (\alpha', \beta')\big) = \alpha^t \cdot \beta'. \tag{2.38}$$

By differentiation, one can associate with C a Lie algebra cocycle given by (2.35). For example, when $n = 1$ (and writing elements of the Lie algebra $\mathbb{R}^2$ as pairs $X = (x, y)$),

$$\delta\mathsf{C}\big((x, y), (x', y')\big) \overset{(2.35)}{=} \frac{\partial^2}{\partial t\, \partial s} \big(tx \cdot sy' - sx' \cdot ty\big)\big|_{t=0,\, s=0} = xy' - yx'.$$

This is a non-zero antisymmetric bilinear form on $\mathbb{R}^2$, hence defining the Heisenberg algebra of (2.29). Note that this is an example of "cocycle integration": we have found the explicit group two-cocycle whose differential defines the Heisenberg Lie algebra.

Universal Central Extensions

Universal central extensions of groups can be defined exactly as for Lie algebras. A central extension $\widehat{G}$ of G is *universal* if, for any other central extension $\widehat{G}'$ of G by A, there exists a unique isomorphism $\widehat{G} \to \widehat{G}'$.

As in the algebraic case, there is a simple criterion for knowing when a group admits a universal central extension. A group is said to be *perfect* if it coincides with the group of its commutators, i.e. if any $f \in G$ can be written as $f = ghg^{-1}h^{-1}$ for some $g, h \in G$. It turns out that any perfect group admits a universal central extension. In Chaps. 6 and 9 we will see that both $\mathrm{Diff}(S^1)$ and BMS_3 are perfect groups, so that their central extensions are universal.

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