Betti Diagrams with Special Shape

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Abstract We consider classes of monomial ideals whose Betti diagrams have a special shape. Monomial ideals with such a Betti diagram satisfy the subadditivity condition for the maximal shifts in the resolution by obvious reasons, and they appear quite frequently in combinatorial contexts. Examples of ideals with special shape are the edge ideal as well as the vertex cover ideal of chordal graphs, whisker graphs and triangulated d-uniform hypergraphs.

1 Introduction

In this note we consider classes of monomial ideals whose Betti diagrams have a special shape. Let K be field, $S = K[x_1, \ldots, x_n]$ the polynomial ring in n variables over K. Let $I \subset S$ be a graded ideal with $\operatorname{proj}\dim(S/I) = p$, and let $\mathbb{F}$ be the graded minimal free S-resolution of S/I. We denote by $t_i(I)$ the highest degree of a generator of F_i. The sequence $t_0(I), t_1(I), t_2(I), \ldots, t_p(I)$ is called the *t-sequence* of the ideal I. With the notation introduced, the regularity of S/I is defined to be

$$\operatorname{reg}(S/I) = \max\{t_i(I) - i\colon\ i = 1, \ldots, p\}.$$

The ideal I is said to satisfy *subadditivity* if $t_{i+j}(I) \leq t_i(I) + t_j(I)$ for all positive integers i and j with $i + j \leq p$.

The special shape of the Betti diagram we have in mind is such that the sequence of numbers $r_0(I), r_1(I), \ldots, r_p(I)$ with $r_i(I) = t_i(I) - i$ for $i = 0, \ldots, p$ form a concave sequence until they reach the regularity and from this point on are non-increasing, see Definition 1. We call the sequence of the $r_i(I)$ the *r-sequence* of the ideal.

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There exist examples which show that not all graded ideals satisfy subadditivity, see [2], but it is an open problem whether all monomial ideals satisfy subadditivity. For edge ideals of graphs the inequality $t_{i+1}(I) \leq t_i(I) + t_1(I)$ was shown by Fernández-Ramos and Gimenez [10]. The same inequality has been shown later for any monomial ideal [14] by Srinivasan and the second author of this paper. Yazdan Pour independently proved the same result and presented it in a lecture at IPM [25]. In the meantime some more notable results regarding subadditivity have been obtained by Khoury and Srinivasan [18, Theorem 2.3], Abedelfatah and Nevo [1, Theorem 1.3] and Faridi [9, Theorem 3.7].

In this note we will show (Theorem 1) that the Betti diagram of the edge ideal of any chordal graph or any whisker graph has a special shape. The condition that the r-sequence is non-increasing after it reached the regularity is guaranteed if all extremal strands of the diagram are connected and overlap properly, which for example is the case if the ideal is componentwise linear, see Proposition 1. Examples of such ideals are the vertex cover ideals of chordal graphs, as shown in the proof of [11, Theorem 3.2]. Other examples are the vertex cover ideals of simplicial trees, see [8].

Edge ideals of chordal graphs and whisker graphs are special cases of edge ideals of certain hypergraphs. The triangulated d-uniform hypergraphs $\mathscr{C}$ of Hà and Van Tuyl, see [12], are natural generalizations of chordal graphs, and indeed it can be shown that the Betti diagram of $S/I(\mathscr{C})$ has special shape, see Theorem 2. These results are obtained from the fact that their non-empty strands are connected, and that the diagonal Betti numbers $\beta_{i,di}(S/I(\mathscr{C}))$ are non-zero for $i = 0, \ldots, r$, where $rd = \text{reg}(S/I(\mathscr{C}))$. For a graph G, the maximal number i for which $\beta_{i,2i}(S/I(G)) \neq 0$ is just the induced matching number of the graph, see [17, Lemma 2.2]. A similar statement holds for d-uniform hypergraphs with a suitable definition of induced matching. It turns out, as shown in Lemma 5, that the induced matching number of d-uniform hypergraph is always less than or equal to the cardinality of a maximal set of pairwise $(d + 1)$-disjoint edges of the hypergraph, and coincides with this cardinality if the hypergraph is properly-connected.

The edge ideal of a whisker graph is the polarization of a monomial ideal $I \subset S$ containing all the squares of the variables. More generally, one may consider any monomial ideal $I \subset S$ with $\dim(S/I) = 0$. Then this ideal contains a pure power of each variable. By using a result of Mermin et al. [21] it is shown that such ideals, under additional conditions on the pure powers, satisfy the subadditivity, see Corollary 4. These assumptions are for example satisfied, if I is generated in degree 2.

2 Betti Diagrams with Special Shape

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in the indeterminates $x_1, \ldots, x_n$ over the field K, and let $I \subset S$ be a graded ideal with $p = \text{proj}\dim(S/I)$. For each $i = 0, \ldots, p$ let $r_i(I) = t_i(I) - i$. Note that the $r_i(I)$-th row in the Betti diagram of S/I

Table 1 Betti diagram of the quotient ring associated to the 5-cycle

	0	1	2	3
0:	1	-	-	-
1:	-	5	5	-
2:	-	-	-	1

is the last row with non-zero entry in the i-th column. The integer $r_i(I)$ is sometimes called the *i-regularity* of S/I, and of course $\operatorname{reg}(S/I) = \max\{r_i(I)\colon\ i = 1, \ldots, p\}$.

It is clear that I satisfies subadditivity if and only if $r_{i+j}(I) \le r_i(I) + r_j(I)$ for all positive integers i, j with $i + j \le p$.

Definition 1 Let I be a graded ideal. The Betti diagram of S/I with regularity c is said to have a *special shape*, if

(i) $r_0(I) \le r_1(I) \le \cdots \le r_g(I)$ and $r_{i+1}(I) - r_i(I) \le r_i(I) - r_{i-1}(I)$ for $1 \le i \le g$, where g is the smallest integer such that $r_g(I) = c$.
(ii) $r_{i+1}(I) \le r_i(I)$ for $g \le i \le \operatorname{proj\,dim}(S/I)$.

The first condition says that the r-sequence of the ideal is concave in the range from 0 to g, and the second condition guarantees that the r-sequence is non-increasing in the range from g to the projective dimension of S/I.

Not all Betti diagrams of monomial ideals have a special shape. This is not even the case for the edge ideal of a graph. A simplest such example is the edge ideal of the 5-cycle whose Betti diagram violates condition (i) of Definition 1 (see Table 1).

On the other hand, we do not know of any monomial ideal whose Betti diagram violates condition (ii) of Definition 1. However, we expect that such examples exist.

Lemma 1 *Any graded ideal whose Betti diagram has a special shape satisfies subadditivity.*

Proof If $i + j \le g$, then (i) implies

$$r_{i+j}(I) - r_j(I) = \sum_{k=j+1}^{i+j} (r_k(I) - r_{k-1}(I)) \le \sum_{k=1}^{i} (r_k(I) - r_{k-1}(I)) = r_i(I).$$

If $i + j \ge g$, and say $j \ge g$, then $r_i(I) + r_j(I) \ge r_i(I) + r_{i+j}(I) \ge r_{i+j}(I)$, by (ii). Finally, if $i + j > g$ and $i, j < g$, then we get

$$\begin{aligned} r_i(I) + r_j(I) = (r_i(I) + r_{g-i}(I)) + (r_j(I) - r_{g-i}(I)) &\ge r_g(I) + (r_j(I) - r_{g-i}(I)) \\ &\ge r_g(I) \ge r_{i+j}(I). \end{aligned}$$

Here we used the fact that $r_i(I) + r_{g-i}(I) \ge r_g(I)$, as we have seen before, further that $r_g(I) \ge r_{i+j}(I)$, by (ii) and that $r_j(I) - r_{g-i}(I) = \sum_{k=g-i+1}^{j} (r_k(I) - r_{k-1}(I)) \ge 0$, since each summand of this sum is ≥ 0, by (i). $\square$

It is expected that all monomial ideals satisfy subadditivity, even if their Betti diagram does not have special shape.

In the following sections we will consider classes of ideals whose Betti diagram has a special shape.

3 Strand Connectedness

A graded Betti number $\beta_{p,p+q} \neq 0$ is said to be *extremal* if $\beta_{i,i+j} = 0$ for all $(p,q) \neq (i,j)$ such that $i \geq p$ and $j \geq q$. For each graded ideal I, the ring S/I has at least one extremal Betti number which is $\beta_{p_c(I),p_c(I)+c}(S/I)$, where $c = \mathrm{reg}(S/I)$ and $p_c(I)$ is the biggest integer among all i with $\beta_{i,i+c}(S/I) \neq 0$. In particular,

$$\mathrm{reg}(S/I) = \max\{j : \ \beta_{i,i+j}(S/I) \text{ is an extremal Betti number}\}.$$

Let I be a graded ideal. For each j, the *j-strand* of I is defined to be the set

$$j\text{-strand}(I) = \{i : \beta_{i,i+j}(S/I) \neq 0\}.$$

We denote by $p_j(I)$ the greatest integer belonging to the j-strand of I. A non-empty j-strand of I is said to be *connected*, if there exists $q_j(I) \leq p_j(I)$ with the property that $\beta_{i,i+j}(S/I) \neq 0$ if and only if $q_j(I) \leq i \leq p_j(I)$. In this case we show the j-strand of I with the interval $[q_j(I), p_j(I)]$. The ideal I is called *strand connected*, if each non-empty strand of I is connected. A j-strand of I is called an *extremal strand* if $\beta_{p_j(I),p_j(I)+j}(S/I)$ is an extremal Betti number.

The edge ideal I of the 5-cycle has one extremal Betti number which is $\beta_{3,5}(S/I) = 1$. Hence, the only extremal strand of I is 2-strand(I). Moreover, all the j-strands of I are connected, see Table 1.

The following lemma makes sure that condition (ii) of Definition 1 holds for the Betti diagram of a graded ideal, if certain conditions on the extremal strands of I are satisfied.

Lemma 2 *Let I be a graded ideal, and $\{j_1, \ldots, j_s\}$ be the set of integers j for which the j-*strand *of I is extremal. Suppose that each of these strands is connected, say we have j_k-*strand*(I) $= [q_{j_k}(I), p_{j_k}(I)]$. Suppose further that for each $j_k \neq \mathrm{reg}(S/I)$ there exists j_l with $j_k < j_l$ and such that $q_{j_k}(I) \leq p_{j_l}(I)$. Then the Betti diagram of I satisfies condition (ii) of Definition 1.*

Proof We may assume that $j_1 < j_2 < \cdots < j_s$. Then $j_s = \mathrm{reg}(S/I)$, and we need to show that $r_i(I)$ is a non-increasing function on i for all $i \geq q_{j_s}(I)$. This is clear for $q_{j_s}(I) \leq i \leq p_{j_s}(I)$, because $r_i(I) = j_s$ for $i \in [q_{j_s}(I), p_{j_s}(I)]$ since the j_s-strand of I is connected.

Since the j_k-strands of I are extremal, it follows that $p_{j_1}(I) > p_{j_2}(I) > \cdots > p_{j_s}(I)$, and for any other non-empty j-strand with $j_k < j \leq j_{k+1}$ it follows that $p_j(I) \leq p_{j_{k+1}}(I)$, because j-strand(I) is not an extremal strand.

We claim that $r_i(I) = j_k$ for i with $p_{j_{k+1}}(I) < i \leq p_{j_k}(I)$. The claim implies that $r_i(I)$ is a non-increasing function for all i with $p_{j_s}(I) < i \leq p_{j_1}(I)$. Together with the fact $r_i(I) = j_s$ for i with $q_{j_s}(I) \leq i \leq p_{j_s}(I)$ and that $j_s > j_{s-1}$, the desired conclusion will follow.

For the proof of the claim we let i be an integer with $p_{j_{k+1}}(I) < i \leq p_{j_k}(I)$. We first notice that our assumptions imply that $q_{j_k}(I) \leq p_{j_{k+1}}(I)$. Therefore, since j_k-strand(I) is connected, it follows that $\beta_{i,i+j_k}(I) \neq 0$, and this implies $r_i(I) \geq j_k$. Suppose that $r_i(I) > j_k$. Then the j-strand with $j = r_i(I)$ is non-empty. Let l be the integer with $j_l < j \leq j_{l+1}$. Then $k \leq l$ and $i \leq p_j(I) \leq p_{j_{l+1}}(I) \leq p_{j_{k+1}}(I)$, a contradiction. □

For a Cohen–Macaulay ideal I the conditions described in Lemma 2 are satisfied. Indeed, in this case, I admits only one extremal strand, and this strand is connected, as follows from

Lemma 3 *Let M be a graded Cohen–Macaulay S-module of regularity c. Then the Betti number $\beta_{p_c(M),p_c(M)+c}(M)$ is the only extremal Betti number of M, and the c-strand of M is connected.*

Proof Note that $\mathrm{Ext}^i_S(M,S) \neq 0$ if and only if $i = \operatorname{proj\,dim}(M)$, see [5, Proposition 3.3.3]. We first show that the c-strand of M is connected using this fact. Suppose on the contrary that there exists $i < p_c(M)$ such that $r_i(M) = c$ and $r_{i+1}(M) < c$, i.e. $\beta_{i,i+c}(M) \neq 0$, $\beta_{i+1,(i+1)+c}(M) = 0$. Let $\mathbb{F}$ be the graded minimal free resolution of M. Since $\beta_{i,i+c}(M) \neq 0$ and since c is the regularity of M, the highest degree of the basis homogeneous elements in F_i is $i + c$. Let $e_1, \ldots, e_b$ be a homogeneous basis of F_i. We may assume that $\deg e_1 = i+c$. Let $\partial_{i+1} : F_{i+1} \to F_i$ be the $(i+1)$-differential in $\mathbb{F}$. Since $\beta_{i+1,(i+1)+c}(M) = 0$ and since $\operatorname{reg}(M) = c$, we have $\deg f \leq \deg e_1$ for all homogeneous basis elements f in F_{i+1}. Thus, since ∂_{i+1} is a graded map and since $\mathrm{Im}(\partial_{i+1}) \subset \mathfrak{m}F_i$, it follows that for all basis elements f of F_{i+1} we have

$$\partial_{i+1}(f) = \sum_{e_l \neq e_1} a_l e_l \quad \text{with } a_l \in S. \tag{1}$$

Dualizing the resolution of M with respect to S we get the acyclic complex $\mathbb{F}^*$, since $\mathrm{Ext}^i_S(M,S) = 0$ for $i < \operatorname{proj\,dim}(M)$. On the other hand (1) implies that $\partial^*_{i+1}(e^*_1) = 0$, while $e^*_1 \notin \mathrm{Im}(\partial^*_i)$ because $\mathrm{Im}(\partial^*_i) \subset \mathfrak{m}F^*_i$. This contradicts the acyclicity of $\mathbb{F}^*$. Therefore the c-strand is connected.

We know that $\beta_{p_c(M),p_c(M)+c}(M)$ is an extremal Betti number. So it suffices to show that there is no other extremal Betti number. To see this, note that with the same argument as above, $\mathrm{Ext}^{p_j(M)}_S(M,S) \neq 0$ if $\beta_{p_j(M),p_j(M)+j}(M)$ is an extremal Betti number. Since $\mathrm{Ext}^i_S(M,S) \neq 0$ if and only if $i = \operatorname{proj\,dim}(M)$, there can be only one extremal Betti number. □

A large class of strand connected ideals are the componentwise linear ideals. We first show

Lemma 4 *Let $J \subset I$ be two graded ideals equigenerated in degree d. Then*

(i) $\beta_{i,i+d}(J) \le \beta_{i,i+d}(I)$ *for all* i;
(ii) suppose in addition that J, I *have* d*-linear resolutions. If* $\beta_i(J) = \beta_i(I)$ *for some* i*, then* $\beta_k(J) = \beta_k(I)$ *for all* $k \ge i$.

Proof

(i) See [4, Lemma 2.3(a)].
(ii) We denote by $\mathrm{Gin}(I)$ the generic initial ideal of I with respect to the reverse lexicographical order induced by $x_1 > x_2 > \cdots > x_n$. It follows from a theorem of Bayer and Stillman [3] (see also [13, Corollary 4.3.18(d)]) that $\mathrm{Gin}(I)$ and $\mathrm{Gin}(J)$ have again a linear resolution. Moreover, one has $\beta_k(J) = \beta_k(\mathrm{Gin}(J))$ and $\beta_k(I) = \beta_k(\mathrm{Gin}(I))$ for all k, see [13, Problem 4.5]. Since I, J have d-linear resolutions and hence are componentwise linear, it follows from [6, Lemma 1.4] that their generic initial ideal is stable. Consequently, we may assume that $J \subset I$ are stable ideals.

Let $G(I)$ denote the set of minimal monomial generators of I. Then $G(I) = G(J) \cup \{u_1, \dots, u_r\}$ for some $u_1, \dots, u_r \in G(I) \setminus G(J)$. It follows from the Eliahou–Kervaire resolution of a stable ideal (see [7] or [13, Corollary 7.2.3]) that for all k

$$\beta_k(I) = \sum_{u \in G(I)} \binom{m(u)-1}{k} = \beta_k(J) + \sum_{t=1}^{r} \binom{m(u_t)-1}{k},$$

where $m(u)$ denotes the largest number j such that x_j divides u. Therefore, $\beta_k(J) = \beta_k(I)$ if and only if $\sum_{t=1}^{r} \binom{m(u_t)-1}{k} = 0$. This is the case if and only if $k \ge \max\{m(u_t) : 1 \le t \le r\}$. Hence, by assumption, $i \ge \max\{m(u_t) : 1 \le t \le r\}$. This implies that $\sum_{t=1}^{r} \binom{m(u_t)-1}{k} = 0$ for all $k \ge i$ which yields the desired result.

□

Let I be a graded ideal and let $I_{\langle j \rangle}$ be the ideal generated by all homogeneous polynomials of degree j belonging to I. The ideal I is called *componentwise linear* if $I_{\langle j \rangle}$ has a linear resolution for all j. It follows from [13, Lemma 8.2.10] that, in this case, the ideal $\mathfrak{m} I_{\langle j \rangle}$ has a linear resolution too, where $\mathfrak{m}$ denotes the graded maximal ideal of the ring S. Let I be a componentwise linear ideal. Then using [13, Proposition 8.2.13] one has $\beta_{i,i+j}(I) = \beta_i(I_{\langle j \rangle}) - \beta_i(\mathfrak{m} I_{\langle j-1 \rangle})$ for all i and j. We use this fact in the proof of the following proposition.

Proposition 1 *Let I be a componentwise linear ideal. Then I is strand connected with each non-empty j-strand beginning in the homological degree* 1 *for $j > 0$. In particular, $r_1(I) \ge r_2(I) \ge \cdots$, and hence the Betti diagram of S/I has a special shape and subadditivity holds for I.*

Proof By [13, Proposition 8.2.13] the graded Betti numbers of I are given by the formula

$$\beta_{i,i+j}(I) = \beta_i(I_{\langle j \rangle}) - \beta_i(\mathfrak{m}I_{\langle j-1 \rangle}).$$

Thus for $j > 0$,

$$\beta_{i,i+j}(S/I) = \beta_{i-1}(I_{\langle j+1 \rangle}) - \beta_{i-1}(\mathfrak{m}I_{\langle j \rangle}).$$

It follows from this formula that for $j > 0$, a j-strand of I is non-empty if and only if I has a generator in degree $j+1$, and this strand begins in homological degree 1. Furthermore, Lemma 4 implies that the non-empty strands are connected.

The remaining statements of the proposition follow from Lemma 2. □

Remark 1 Proposition 1, in the case that the field K is of characteristic 0, can be also obtained from the following known results. Indeed, if $\mathrm{char}(K) = 0$, then a graded ideal $I \subset S$ is componentwise linear if and only if $\beta_{i,i+j}(I) = \beta_{i,i+j}(\mathrm{Gin}(I))$ for all i and j, (see [13, Theorem 8.2.22]). It follows from [13, Theorem 4.2.1] that $\mathrm{Gin}(I)$ is Borel-fixed and [13, Proposition 4.2.4] implies that it is strongly stable. It follows from the Eliahou–Kervaire formula that for $j > 0$,

$$\beta_{i,i+j}(S/I) = \beta_{i-1,(i-1)+(j+1)}(\mathrm{Gin}(I)) = \sum_{u \in G(I)_{j+1}} \binom{m(u)-1}{i-1},$$

where $G(I)_j$ denotes the set of monomials of degree j which belong to the unique minimal set of monomial generators of I. This implies that for $j > 0$, the j-strand$(I) \neq \emptyset$ if and only if $G(I)_{j+1} \neq \emptyset$. In this case we have j-strand$(I) = \{i\colon 1 \leq i \leq p_j(I)\}$, where $p_j(I) = \max\{m(u)\colon u \in G(I)_{j+1}\}$.

As mentioned in the introduction, Faridi recently showed in [9] that the facet ideal of a simplicial tree Δ satisfies the subadditivity condition. As an application of Proposition 1 we see that the vertex cover ideal of a simplicial tree also satisfies the subadditivity condition.

Let Δ be a simplicial complex on $[n]$ with $\mathcal{F}(\Delta)$ as its facet set. The set $C \subset [n]$ is called a *vertex cover* of Δ if $C \cap F \neq \emptyset$ for all $F \in \mathcal{F}(\Delta)$. The *vertex cover ideal* of Δ is the Alexander dual of the facet ideal of Δ.

Corollary 1 *Let I be the vertex cover ideal of a simplicial forest. Then the Betti diagram of S/I has a special shape and hence I satisfies subadditivity condition.*

Proof Since the facet ideal of a simplicial forest is sequentially Cohen–Macaulay [8, Corollary 5.6], the ideal I is componentwise linear using [23, Theorem 3.8]. Thus Proposition 1 implies the desired conclusion. □

4 Edge Ideals of Graphs

A set $\mathscr{S}$ of edges of a graph G is called a *matching* of G if no two edges of $\mathscr{S}$ have a common vertex and it is called an *induced matching* if it is a matching and no two edges of $\mathscr{S}$ are connected by an edge in G. The maximal size of an induced matching of G is called the *induced matching number* of G, denoted $\mathrm{im}(G)$. It is known that $\mathrm{im}(G) \leq \mathrm{reg}(S/I(G))$, see [17, Lemma 2.2].

The induced matching number of a graph can be seen from the Betti diagram of its edge ideal. The following proposition is an immediate consequence of [17, Lemma 2.2] by Katzman. A hypergraph version of it will be shown in the next section.

Proposition 2 *Let G be a graph. Then $\beta_{i,2i}(S/I(G)) \neq 0$ if and only if $i \in \{0, 1, \ldots, \mathrm{im}(G)\}$.*

In the following we describe a situation which we will encounter throughout this section.

Corollary 2 *Let G be a graph satisfying the following properties:*

(a) $\mathrm{im}(G) = \mathrm{reg}(S/I(G))$*;*
(b) the extremal strands of $I(G)$ are connected.

Then the Betti diagram of $S/I(G)$ has a special shape. In particular, subadditivity holds for $I(G)$.

Proof Assumption (a) and Proposition 2 imply that $\beta_{i,2i}(S/I(G)) \neq 0$ for all $i = 0, \ldots, \mathrm{reg}(S/I(G))$. By [10, Corollary 1.9] we have $r_{i+1}(I) \leq r_i(I) + 1$. This implies that $r_i(I) = i$ for $i = 0, \ldots, \mathrm{reg}(S/I(G))$, and hence condition (i) of Definition 1 is satisfied. Moreover, each non-empty j-strand begins with j. It follows from Lemma 2 that condition (ii) is also satisfied. □

Example 1 In view of Corollary 2 we discuss some families of graphs.

(a) Let G be a bipartite graph. Kummini showed in [19, Theorem 1.1] that $\mathrm{im}(G) = \mathrm{reg}(S/I(G))$ if G is unmixed, and Van Tuyl showed in [22, Theorem 3.3] that $\mathrm{im}(G) = \mathrm{reg}(S/I(G))$ if G is sequentially Cohen–Macaulay. The unmixed bipartite graphs are classified by Villarreal in [24, Theorem 1.1]. In both cases, the unmixed and the sequentially Cohen–Macaulay case, condition (a) of Corollary 2 is satisfied. In particular, this happens if G is Cohen–Macaulay. In this case, by Lemma 3, condition (b) of Corollary 2 is also satisfied, and hence for Cohen–Macaulay bipartite graphs subadditivity holds.

The examples of sequentially Cohen–Macaulay bipartite graphs we considered all have the property that their Betti diagram has only one extremal strand, and this strand is connected. If this would be the case in general for bipartite graphs, then this would show that the Betti diagram of a sequentially Cohen-Macaulay bipartite graph has a special shape.

(b) Let C_n be the cycle graph of length n. Jacques in his PhD thesis [16, Chapter 7] computed the graded Betti numbers of C_n. He showed that for all i, j with $i + j < n$,

$$\beta_{i,i+j}(S/I(C_n)) = \binom{j}{i-j}\beta_{j,2j}(S/I(C_n)),$$

where

$$\beta_{i,2i}(S/I(C_n)) = \begin{cases} \frac{n}{n-2i}\binom{n-2i}{i} & \text{for } 0 \le i \le \lfloor \frac{n}{3} \rfloor, \\ 0 & \text{for } \lfloor \frac{n}{3} \rfloor < i \le \lfloor \frac{n}{2} \rfloor. \end{cases}$$

Moreover,

$$\begin{aligned} \beta_{2m+1,n}(S/I(C_n)) &= 1 \qquad \text{if } n = 3m + 1 or n = 3m + 2, \\ \beta_{2m,n}(S/I(C_n)) &= 2 \qquad \text{if } n = 3m. \end{aligned}$$

The formulas show that all non-empty strands are connected, and that the smallest element in each non-empty j-strand is j. The fact that $\beta_{i,2i}(S/I(C_n)) \neq 0$ for $i = 0, \ldots, \mathrm{reg}(S/I(C_n))$ shows that $\mathrm{im}(C_n) = \mathrm{reg}(S/I(C_n))$. Hence, subadditivity holds for $I(C_n)$.

The following theorem describes two other prominent classes of graphs which satisfy the conditions of Corollary 2.

Theorem 1 *Let G be a chordal or a whisker graph. Then G satisfies the conditions of Corollary 2. In particular, the Betti diagram of $S/I(G)$ has a special shape, and hence subadditivity holds for $I(G)$.*

The theorem is a special case of a more general result which will be presented in the next section.

5 Generalizations

The results saying that chordal and whisker graphs satisfy subadditivity condition can be generalized to monomial ideals which are not only generated by monomials of degree 2.

First we will prove the analogue of Proposition 2 for hypergraphs. Let $V = \{v_1, \ldots, v_n\}$ be a finite set, and let $\mathscr{C} = \{F_1, \ldots, F_t\}$ be a family of distinct non-empty subsets of V. This family is called a *hypergraph* on the vertex set V. The elements of $\mathscr{C}$ are called the *edges* of $\mathscr{C}$. A hypergraph is called *d-uniform* if all of its edges have the same cardinality d. Let $\mathscr{C}$ be a d-uniform hypergraph on the vertex set $[n]$. The set of edges of $\mathscr{C}$ will be denoted by $E(\mathscr{C})$. With each edge $F = \{i_1, \ldots, i_d\} \in E(\mathscr{C})$ we associate the monomial $\mathbf{x}_F = \prod_{j=1,\ldots,d} x_{i_j}$. The ideal

$I(\mathscr{C})$, generated by the monomials $\mathbf{x}_F$ with F an edge of $\mathscr{C}$, is called the *edge ideal* of $\mathscr{C}$.

Let $V \subset [n]$. The hypergraph $\mathscr{C}_V$ induced by V is the hypergraph with $E(\mathscr{C}_V) = \{F \in \mathscr{C}\colon\ F \subset V\}$. Let $\{F_1, \ldots, F_i\} \subset E(\mathscr{C})$ and set $V = \bigcup_{j=1}^{i} F_j$. The set $\{F_1, \ldots, F_i\}$ is called an *induced matching* of $\mathscr{C}$, if $E(\mathscr{C}_V) = \{F_1, \ldots, F_i\}$ and the edges of $\mathscr{C}_V$ are pairwise disjoint. The maximal number of edges of $\mathscr{C}$ which form an induced matching of $\mathscr{C}$ is called the *induced matching number* of $\mathscr{C}$, denoted $\mathrm{im}(\mathscr{C})$. Note that in the case of graphs this is the classical definition of the induced matching number of a graph.

In view of Lemma 5, the following proposition may be considered as a generalization of [12, Theorem 6.5].

Proposition 3 *Let $\mathscr{C}$ be a d-uniform hypergraph. Then $\beta_{i,di}(S/I(\mathscr{C})) \neq 0$ if and only if $i \in \{0, 1, \ldots, \mathrm{im}(\mathscr{C})\}$.*

Proof Let $E(\mathscr{C}) = \{F_1, \ldots, F_r\}$ and let $\{F_{i_1}, \ldots, F_{i_c}\}$ be an induced matching of $\mathscr{C}$. Since the F_{i_j} are pairwise disjoint, it follows that the edge ideal of $\mathscr{C}_V$ for $V = \bigcup_{j=1}^{c} F_{i_j}$ is generated by the regular sequence $\mathbf{x}_{F_{i_1}}, \ldots, \mathbf{x}_{F_{i_c}}$. Thus the Taylor complex on this sequence provides a minimal free resolution of $S/I(\mathscr{C}_V)$. It follows that $\beta_{i,di}(S/I(\mathscr{C}_V)) \neq 0$ for $i = 0, \ldots, c$. By the restriction lemma [15, Lemma 4.4] we have $\beta_{i,j}(S/I(\mathscr{C}_V)) \leq \beta_{i,j}(S/I(\mathscr{C}))$ for all i and j. This implies that $\beta_{i,di}(S/I(\mathscr{C})) \neq 0$ if $i \in \{0, 1, \ldots, c\}$.

Now let $i > c$. We must show that $\beta_{i,di}(S/I(\mathscr{C})) = 0$. To prove this, we apply the strategy of Katzman [17] and use the Taylor complex to compute $\beta_{i,id}(S/I(\mathscr{C}))$. The Taylor complex $\mathbb{T}$ provides a graded free S-resolution of $S/I(\mathscr{C})$ which is rarely minimal. Nevertheless it can be used to compute the (multi)graded Betti numbers of $S/I(\mathscr{C})$. The module T_1 is a free S-module with basis $e_1, \ldots, e_r$, and $T_i = \bigwedge^i T_1$ for $i = 0, \ldots, r$. The differential $\partial_i : T_i \to T_{i-1}$ is given by

$$\partial_i(e_{k_1,\ldots,k_i}) = \sum_{j=1}^{i} (-1)^{j+1} \mu_j e_{k_1,\ldots,\widehat{k_j},\ldots,k_i},$$

where for $k_1 < k_2 < \cdots < k_i$ the element $e_{k_1,\ldots,k_i}$ stands for $e_{k_1} \wedge \cdots \wedge e_{k_i}$ and $\mu_j = \mathbf{x}_{G_j}$, where $G_j = \bigcup_{l=1}^{i} F_{k_l} \setminus \bigcup_{\substack{l=1 \\ l\neq j}}^{i} F_{k_l}$.

Suppose now that $\beta_{i,di}(S/I(\mathscr{C})) \neq 0$. It follows from the Taylor complex that the multigraded shifts (monomial notation) of the resolution of $S/I(\mathscr{C})$ are squarefree and are least common multiples of the monomial generators of $I(\mathscr{C})$. Therefore there exist edges $F_{k_1}, \ldots, F_{k_i}$ of $\mathscr{C}$ which are pairwise disjoint and there is a a non-zero Betti number $\beta_{i,\mathbf{a}_F}(S/I(\mathscr{C}))$, where $a_F(l) = 1$ if $l \in F := \bigcup_{l=1}^{i} F_{k_l}$ and 0 otherwise. Let $e = e_{k_1,\ldots,k_i}$. Then $\partial_i(e) = \sum_{j=1}^{i} (-1)^{j+1} \mathbf{x}_{F_{k_j}} e_{k_1,\ldots,\widehat{k_j},\ldots,k_i}$, and hence $\partial_i(e) = 0$ in $\mathbb{T}/\mathfrak{m}\mathbb{T}$, where $\mathfrak{m}$ is the graded maximal ideal of S. We denote by $\bar{e}$ the residue class of e in $\mathbb{T}/\mathfrak{m}\mathbb{T}$. Then $\bar{e}$ is a cycle in $\mathbb{T}/\mathfrak{m}\mathbb{T}$. Any other cycle in $\mathbb{T}/\mathfrak{m}\mathbb{T}$ of multidegree $\mathbf{a}_F$ arises like $\bar{e}$ from a decomposition of F as a union of pairwise disjoint edges. Thus we may as well assume that the homology class of $\bar{e}$ gives the non-zero contribution to $\mathrm{Tor}_i^S(S/I(\mathscr{C}), K)_{id} \cong H_i(\mathbb{T}/\mathfrak{m}\mathbb{T})_{id}$. We will obtain a contradiction by showing next that $\bar{e}$ is a boundary in $\mathbb{T}/\mathfrak{m}\mathbb{T}$. Indeed, since $i > \mathrm{im}(\mathscr{C})$, $\{F_{k_1}, \ldots, F_{k_i}\}$ is not an induced matching. In other words there exists an edge F_{k_t} with $F_{k_t} \neq F_{k_l}$ for $l = 1, \ldots, i$ and $F_{k_t} \subset \bigcup_{l=1}^{i} F_{k_l}$. Let $f = e_{k_t} \wedge e$. It follows from the definition of F_{k_t} that $\partial_{i+1}(f) = e - e_{k_t} \wedge \partial_i(e)$. Hence $\partial_{i+1}(\bar{f}) = \bar{e}$. Thus $\bar{e}$ is a boundary. □

Let, as before, $\mathscr{C}$ be a d-uniform hypergraph on $[n]$. A subset $D \subset [n]$ is called an *independent set* of $\mathscr{C}$, if D does not contain any edge of $\mathscr{C}$. We call the maximal cardinality of an independent set of $\mathscr{C}$ the *independence number* of $\mathscr{C}$ and denote it by $\mathrm{ind}(\mathscr{C})$.

Proposition 4 *Let $\mathscr{C}$ be a d-uniform hypergraph. Then*

$$\mathrm{im}(\mathscr{C})(d-1) \leq \mathrm{reg}(S/I(\mathscr{C})) \leq \mathrm{ind}(\mathscr{C}).$$

Proof The lower bound for $\mathrm{reg}(S/I(\mathscr{C}))$ follows from Proposition 3. To see the upper bound, let $c = \mathrm{reg}(S/I(\mathscr{C}))$. Then there exists an integer i and a non-zero homology class $[z] \in H_i(\mathbf{x}; S/I(\mathscr{C}))_{i+c}$, where $H_i(\mathbf{x}; S/I(\mathscr{C}))$ is the i-th Koszul homology module of $S/I(\mathscr{C})$ with respect to $x_1, \ldots, x_n$. Since for a squarefree monomial ideal all shifts in the resolution are squarefree, the cycle z is of the form $\sum_I \lambda_I u_I e_I$, where each I is a subset of $[n]$ of cardinality i, $\lambda_I \in K$, u_I is the residue class of a squarefree monomial v_I modulo $I(\mathscr{C})$ and $e_I = e_{j_1} \wedge \cdots \wedge e_{j_i}$ for $I = \{j_1 < \cdots < j_i\}$. It follows that $c = \deg v_I$ for all I with $u_I \neq 0$. Since $z \neq 0$, there exists at least one $u_I \neq 0$. This set I must be an independent set, otherwise $u_I = 0$. □

In Proposition 5 we give an example of a class of hypergraphs for which the Betti diagram has a special shape. The proof is based on a recursive formula, Formula (2) below, for the graded Betti numbers of hypergraphs, given by Há and Van Tuyl [12, Theorem 4.16] under certain conditions on the hypergraphs. First we recall this formula. See [12] for more details.

Let $\mathscr{C}$ be a d-uniform hypergraph and F, F' two edges of $\mathscr{C}$. The*distance* between F and F', denoted by $\mathrm{dist}_{\mathscr{C}}(F, F')$, is the smallest integer t such that there exists a chain $L_0, L_1, \ldots, L_t \in \mathscr{C}$ with $F = L_0$, $F' = L_t$ and $|L_i \cap L_{i+1}| = d - 1$ for all $0 \leq i \leq t - 1$. If no such chain exists, we set $\mathrm{dist}_{\mathscr{C}}(F, F') = \infty$.

A d-uniform hypergraph $\mathscr{C}$ is said to be *properly-connected* if for any two edges F and F' of $\mathscr{C}$ with the property that $F \cap F' \neq \emptyset$, we have $\mathrm{dist}_{\mathscr{C}}(F, F') = d - |F \cap F'|$. The class of simple graphs is an obvious class of properly-connected hypergraphs. The 3-uniform hypergraph $\mathscr{C}$ shown in Fig. 1 is properly-connected while $\mathscr{D}$ is not, because for $F = \{2, 4, 5\}$ and $F' = \{3, 4, 6\}$, one has $\mathrm{dist}_{\mathscr{D}}(F, F') = 3$.

By a *splitting edge* F of a hypergraph $\mathscr{C}$ we mean an edge such that $I(\mathscr{C}) = (\mathbf{x}_F) + I(\mathscr{C} \setminus F)$ is a splitting of $I(\mathscr{C})$ in the sense of [7].

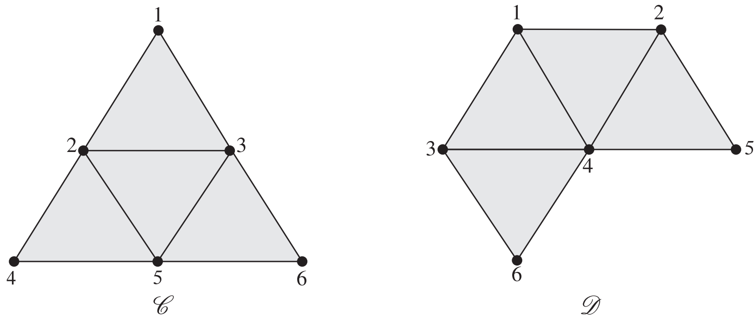

Fig. 1 $\mathscr{C}$ is properly connected, while $\mathscr{D}$ is not

Now let $\mathscr{C}$ be a d-uniform properly-connected hypergraph with a splitting edge F. Then the recursive formula for the graded Betti numbers of $S/I(\mathscr{C})$, given by Há and Van Tuyl, is as follows:

$$\beta_{i,i+j}(S/I(\mathscr{C}))=\beta_{i,i+j}(S/I(\mathscr{C}\setminus F))+\alpha_{i,j}(\mathscr{C})+\sum_{l=0}^{i-2}\binom{t}{l}\beta_{i-1-l,(i-1-l)+(j-d+1)}(S/I(\mathscr{C}')), \quad (2)$$

where

$$\alpha_{i,j}(\mathscr{C})=\begin{cases}\binom{t}{i-1} & \text{if } j=d-1,\\ 0 & \text{if } j\neq d-1.\end{cases}$$

Here, $\mathscr{C}\setminus F$ is a hypergraph obtained from $\mathscr{C}$ by removing the edge F, t is the cardinality of the set

$$\bigcup_{\substack{F'\in\mathscr{C}\\ F'\cap F\neq\emptyset}} F'\setminus F,$$

and $\mathscr{C}'$ is the hypergraph

$$\mathscr{C}'=\{F'\in\mathscr{C}:\ \mathrm{dist}_{\mathscr{C}}(F,F')\geq d+1\}.$$

A d-uniform properly-connected hypergraph $\mathscr{C}$ with the vertex set $[n]$ is said to be *triangulated* if for every non-empty subset $V\subset[n]$, the induced hypergraph $\mathscr{C}_V$ contains a vertex $v\in V$ such that the induced hypergraph of $\mathscr{C}_V$ on the set $\mathrm{N}_{\mathscr{C}_V}[v]$ is a d-complete hypergraph of order $|\mathrm{N}_{\mathscr{C}_V}[v]|$. Here, $\mathrm{N}_{\mathscr{C}_V}[v]=\{v\}\cup\{u\in[n]:\{u,v\}\subset F \text{ for some } F\in\mathscr{C}_V\}$. Há and Van Tuyl showed in [12, Theorem 5.6] that properly-connected f-forests are examples of triangulated hypergraphs. The hypergraph $\mathscr{C}$ in Fig. 1 is an f-forest, so triangulated. Figure 2 illustrates a 3-uniform triangulated hypergraph which is not an f-forest.

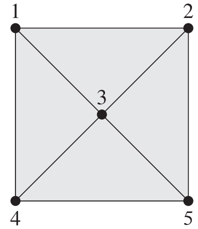

Fig. 2 A triangulated 3-uniform hypergraph $\mathscr{C}$

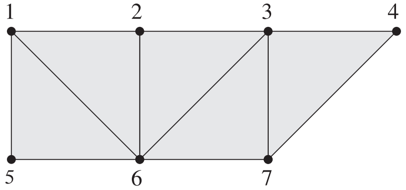

Fig. 3 A 3-uniform hypergraph which is not properly-connected

Há and Van Tuyl also showed in [12, Theorem 5.8] that for triangulated hypergraphs one can compute the graded Betti numbers recursively by using Formula (2), that is, if $\mathscr{C}$ is a triangulated hypergraph, then there exists a splitting edge F in $\mathscr{C}$ such that $\mathscr{C} \setminus F$ and $\mathscr{C}'$ are both triangulated, [12, Lemma 5.7]. We make use of this fact to prove Proposition 5. Before we state this proposition, we first compute the integers $r_i(I)$, where I is the edge ideal of a triangulated d-uniform hypergraph.

Let $\mathscr{C}$ be a d-uniform properly-connected hypergraph. Two edges F, F' of $\mathscr{C}$ are called *t-disjoint* if $\operatorname{dist}_{\mathscr{C}}(F, F') \geq t$. A set of edges E of $\mathscr{C}$ is *pairwise t-disjoint* if every pair of edges of E is t-disjoint. We denote by $\mathfrak{p}(\mathscr{C})$ the cardinality of the biggest set of pairwise $(d+1)$-disjoint edges of $\mathscr{C}$. In Lemma 5 we show that for a d-uniform hypergraph $\mathscr{C}$ we have $\operatorname{im}(\mathscr{C}) \leq \mathfrak{p}(\mathscr{C})$. The equality does not hold in general. However, in the same lemma, we prove the equality for properly-connected d-uniform hypergraphs. Figure 3 illustrates a 3-uniform hypergraph which is not properly-connected. The set $\{\{1,5,6\},\{3,4,7\}\}$ is pairwise 4-disjoint, while all the induced matchings of this clutter are of size 1.

Lemma 5 *Let $\mathscr{C}$ be a d-uniform hypergraph. Then* $\operatorname{im}(\mathscr{C}) \leq \mathfrak{p}(\mathscr{C})$. *Suppose in addition that $\mathscr{C}$ is properly-connected. Then* $\operatorname{im}(\mathscr{C}) = \mathfrak{p}(\mathscr{C})$.

Proof Suppose that $\mathscr{D} \subset \mathscr{C}$ is an induced matching of $\mathscr{C}$. We show that $\mathscr{D}$ is a $(d+1)$-pairwise disjoint set. It is enough to show that for any pair $F, F' \in \mathscr{D}$, $\operatorname{dist}_{\mathscr{C}}(F, F') \geq d+1$. Let $\operatorname{dist}_{\mathscr{C}}(F, F') = t$. Thus there exists a chain $L_0, \ldots, L_t$ in $\mathscr{C}$ such that $F = L_0$, $F' = L_t$, and $|L_i \cap L_{i+1}| = d-1$ for all $0 \leq i \leq t-1$. Since $\mathscr{D}$ is an induced matching, we have $F \cap F' = \emptyset$. It follows that $|F \cap L_{t-1}| \leq 1$ because $|L_{t-1} \cap F'| = d-1$. If $|F \cap L_{t-1}| = 1$, then $L_{t-1} \subset F \cup F'$. This implies that $L_{t-1} \in \mathscr{C}_{F \cup F'}$, a contradiction. Thus $|F \cap L_{t-1}| = 0$. On the other hand by using induction we have $|F \cap L_i| \geq d - i$ for all i. In particular, $|F \cap L_{t-1}| \geq d - (t-1)$. Therefore $t \geq d+1$.

Now suppose that $\mathscr{C}$ is properly-connected. It follows from [12, Theorem 6.5] that $\beta_{i,di}(S/I(\mathscr{C})) \neq \emptyset$ if and only if $i \leq \mathfrak{p}(\mathscr{C})$. Thus $\mathrm{im}(\mathscr{C}) = \mathfrak{p}(\mathscr{C})$ using Proposition 3. □

Lemma 6 *Let $\mathscr{C}$ be a triangulated d-uniform hypergraph and let $I := I(\mathscr{C})$ be its edge ideal. Then* $\mathrm{reg}(S/I) = \mathrm{im}(\mathscr{C})(d-1)$ *and $r_i(I) = i(d-1)$ for all $i \leq \mathrm{im}(\mathscr{C})$.*

Proof Since $\mathscr{C}$ is properly-connected, by Lemma 5, $\mathrm{im}(\mathscr{C}) = \mathfrak{p}(\mathscr{C})$. Thus $\mathrm{reg}(S/I) = \mathrm{im}(\mathscr{C})(d-1)$ using [12, Theorem 6.8]. Proposition 3 implies that $\beta_{i,id}(S/I) \neq 0$ for all $i \leq \mathrm{im}(\mathscr{C})$. Thus $r_i(I) \geq i(d-1)$ for all $i \leq \mathrm{im}(\mathscr{C})$. By [14, Corollary 4] we have $r_i(I) \leq r_{i-1}(I) + d - 1$ for all $i \geq 2$. Since $r_1(I) = d-1$ we conclude that $r_i(I) = i(d-1)$ for all $i \leq \mathrm{im}(\mathscr{C})$. □

Proposition 5 *Let $\mathscr{C}$ be a triangulated d-uniform hypergraph and let $I := I(\mathscr{C})$. Let j be an integer such that* $0 \leq j \leq \mathrm{reg}(S/I)$. *Then*

(a) j-strand$(I) \neq \emptyset$ if and only if $j = k(d-1)$ for some positive integer k;
(b) if $j = k(d-1)$ for some positive integer k, then k is the smallest integer belonging to the j-strand of I, and
(c) if the j-strand of I is non-empty, then it is connected.

Proof Since $\mathscr{C}$ is triangulated [12, Lemma 5.7] implies that it has a splitting edge, say F, and both $\mathscr{C} \setminus F$ and $\mathscr{C}'$ are triangulated. Using induction on $|\mathscr{C}|$ we may assume that the statements (a), (b) and (c) hold for $I(\mathscr{C} \setminus F)$ and $I(\mathscr{C}')$.

We first prove the statements for $j = d-1$. It is clear that (a) and (b) hold if $j = d-1$. By the definition of $p_{d-1}(I)$ we have $(d-1)$-strand$(I) \subset [1, p_{d-1}(I)]$. Let i be an integer with $i \in [1, p_{d-1}(I)]$. If $i \in [1, p_{d-1}(I(\mathscr{C} \setminus F))]$, then since (c) holds for $I(\mathscr{C} \setminus F)$, $\beta_{i,i+(d-1)}(S/I(\mathscr{C} \setminus F)) \neq 0$. Thus (2) implies that $i \in (d-1)$-strand(I). Suppose $i \notin [1, p_{d-1}(I(\mathscr{C} \setminus F))]$. Hence $p_{d-1}(I(\mathscr{C} \setminus F)) < p_{d-1}(I)$. Since $\beta_{p_{d-1}(I), p_{d-1}(I)+(d-1)}(S/I) \neq 0$, using (2), we have $\alpha_{p_{d-1}(I),d-1}(\mathscr{C}) \neq 0$ or $\sum_{l=0}^{p_{d-1}(I)-2} \binom{t}{l} \beta_{p_{d-1}(I)-1-l, p_{d-1}(I)-1-l}(S/I(\mathscr{C}')) \neq 0$. The latter can not happen because otherwise there exists an integer l with $0 \leq l \leq p_{d-1}(I) - 2$ such that $\binom{t}{l} \beta_{p_{d-1}(I)-1-l, p_{d-1}(I)-1-l}(S/I(\mathscr{C}')) \neq 0$. It follows that $p_{d-1}(I) = l+1 \leq p_{d-1}(I) - 1$, a contradiction. So $\alpha_{p_{d-1}(I),d-1}(\mathscr{C}) \neq 0$. Then $0 \neq \binom{t}{p_{d-1}(I)-1} \leq \binom{t}{i-1} = \alpha_{i,d-1}(\mathscr{C})$ which implies that $\beta_{i,i+(d-1)}(S/I) \neq 0$ and so $i \in (d-1)$-strand(I). Therefore, $(d-1)$-strand$(I) = [1, p_{d-1}(I)]$. In particular, the $(d-1)$-strand of I is connected. So (c) holds for $j = d-1$.

Now let $j > d-1$. Then $\alpha_{i,j}(\mathscr{C}) = 0$ for all i in Formula (2). First let $j = k(d-1)$ for some $k > 1$. Since by Lemma 6 we have $r_k(I) = j$, it follows that $\beta_{k,k+j}(S/I) \neq 0$. Thus the j-strand is non-empty in this case. Let $(k-1)(d-1) < j < k(d-1)$ for some $k > 1$. Since by induction hypothesis (a) holds for $I(\mathscr{C} \setminus F)$ and $I(\mathscr{C}')$, the j-strand of $I(\mathscr{C} \setminus F)$ and $(j-d+1)$-strand of $I(\mathscr{C}')$ are empty. Hence the Há-Vantuyl formula (2) implies that the j-strand of I is also empty. This proves (a).

Note that if $j = k(d-1)$ and $\beta_{i,i+j}(S/I) \neq 0$ for some $i < k$, then $r_i(I) = i(d-1) \leq (k-1)(d-1) < j$ which is impossible. Therefore k is the smallest integer such that $\beta_{k,k+j}(S/I) \neq 0$. So (b) holds for I.

Now we prove (c). Suppose j-strand$(I) \neq \emptyset$ for some $j > d-1$. Then $j = k(d-1)$ and j-strand$(I) \subset [k, p_j(I)]$ for some $k > 1$. Suppose on contrary that j-strand$(I) \neq [k, p_j(I)]$. So there exists $i \in [k, p_j(I)]$ such that $\beta_{i,i+j}(S/I) = 0$ while $\beta_{i+1,(i+1)+j}(S/I) \neq 0$. Using Formula (2) we have $\beta_{i,i+j}(S/I(\mathscr{C} \setminus F)) = 0$. So induction hypothesis implies that $i > p_j(I(\mathscr{C} \setminus F))$. Therefore,

$$\beta_{i+1,(i+1)+j}(S/I) = \sum_{l=0}^{i-1} \binom{t}{l} \beta_{i-l,(i-l)+(j-d+1)}(S/I(\mathscr{C}')) \neq 0. \tag{3}$$

Thus there exists $0 \leq l' \leq i-1$ such that

$$\binom{t}{l'} \beta_{i-l',(i-l')+(j-d+1)}(S/I(\mathscr{C}')) \neq 0. \tag{4}$$

Hence $l' \leq t$ and $i - l' \in (j-d+1)$-strand$(I(\mathscr{C}'))$. Using induction hypothesis we have $i - l' \in [k-1, p_{j-d+1}(I(\mathscr{C}'))]$.

On the other hand $\beta_{i,i+j}(S/I) = 0$ implies that for all $0 \leq l \leq i-2$,

$$\binom{t}{l} \beta_{i-1-l,(i-1-l)+(j-d+1)}(S/I(\mathscr{C}')) = 0. \tag{5}$$

It follows that for all $0 \leq l \leq i-2$ either $l > t$ or $i-1-l \notin [k-1, p_{j-d+1}(I(\mathscr{C}'))]$. If $l' = i-1$, then $i-l' \in [k-1, p_{j-d+1}(I(\mathscr{C}'))]$ implies that $k = 2$. Thus $i-l' = k-1$. Now suppose $l' < i-1$. Since $l' \leq t$ we have $i-1-l' \notin [k-1, p_{j-d+1}(I(\mathscr{C}'))]$. Consequently, $i - l' = k-1$ also in this case. Since $i \geq k$ we have $l' > 0$. Now putting $l = l'-1$ in (5) we have either $l'-1 > t$ or $i-l' \notin [k-1, p_{j-d+1}(I(\mathscr{C}'))]$. Both are impossible because $l' \leq t$ and $i - l' = k-1$. Therefore there does not exist such l', which by (3) implies that $\beta_{i+1,(i+1)+j}(S/I) = 0$, a contradiction. So the j-strand of I is connected. □

Theorem 2 *Let $\mathscr{C}$ be a triangulated d-uniform hypergraph and let I be its edge ideal. Then the Betti diagram of S/I has a special shape, and hence I satisfies subadditivity.*

Proof It follows from Lemma 6 that condition (i) of Definition 1 is satisfied. We show that (ii) is also satisfied. Let g be the smallest integer with $\beta_{g,g+\mathrm{reg}(S/I)}(S/I) \neq 0$. By the same Lemma we have $\mathrm{reg}(S/I) = r_g(I) = g(d-1)$. Since $\mathrm{reg}(S/I) = \mathrm{im}(\mathscr{C})(d-1)$, we conclude that $g = \mathrm{im}(\mathscr{C})$. Let $i \geq g$ with $r_i(I) < r_{i+1}(I)$. It follows that $i > g$, because $r_j(I) \leq \mathrm{reg}(S/I)$ for all j. Since r_i-strand$(I) \neq \emptyset$ by Proposition 5(a) we have $r_i(I) = k(d-1)$ for some k. Moreover, since $r_i(I) \leq \mathrm{reg}(S/I) = g(d-1)$, we have $k \leq g$. Therefore $i > k$.

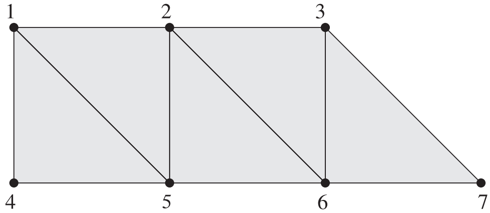

Fig. 4 A triangulated 3-uniform hypergraph $\mathscr{C}$

Table 2 Betti diagram of $S/I(\mathscr{C})$ with $\mathscr{C}$ as in Fig. 4

	0	1	2	3
0:	1	-	-	-
1:	-	-	-	-
2:	-	5	4	-
3:	-	-	-	-
4:	-	-	1	1

Proposition 5 implies that there exists an integer k' such that $r_{i+1}(I) = k'(d-1)$ and r_{i+1}-strand$(I) = [k', p_{r_{i+1}(I)}(I)]$. Since, by [14, Corollary 3], $r_{i+1}(I) \leq r_i(I) + d - 1$, we have $k' \leq k+1$. It follows that $k' = k+1$, because $r_i(I) < r_{i+1}(I)$. Therefore $i \geq k'$. Since $i+1 \in r_{i+1}$-strand(I) we have $i < p_{r_{i+1}(I)}(I)$. Therefore $i \in r_{i+1}$-strand(I), i.e. $\beta_{i,i+r_{i+1}(I)}(I) \neq 0$, a contradiction to the definition of $r_i(I)$. Hence, $r_i(I) \geq r_{i+1}(I)$ for all $i \geq g$. □

Figure 4 shows a triangulated 3-uniform hypergraph $\mathscr{C}$. The Betti diagram of the quotient ring $S/I(\mathscr{C})$ is given in Table 2.

As seen in Theorem 2, this Betti diagram has a special shape.

A *simplicial vertex* in a graph G is a vertex v such that $G_{N_G[v]}$ is a complete graph. It is a well known fact that any chordal graph admits a simplicial vertex. As mentioned in Theorem 1, chordal graphs have a special shape:

Corollary 3 *Let G be a chordal graph. Then $I(G)$ has a special shape, and hence it satisfies subadditivity.*

Proof Each simple graph is properly-connected. Since any induced subgraph of a chordal graph is again chordal and since any chordal graph has a simplicial vertex, we conclude that any chordal graph is triangulated. Thus the assertion follows from Theorem 2. □

Next we consider generalizations of whisker graphs. Let G be a graph on $[n]$ and G^* its whisker graph. The graph G^* is obtained from G by attaching to each vertex i of G a new edge $\{i, i'\}$. Then $I(G^*) = I(G) + (x_1y_1, \ldots, x_ny_n)$, and hence $I(G^*)$ is the polarization of the 0-dimensional ideal $(I(G), x_1^2, \ldots, x_n^2)$.

We now consider more generally a monomial ideal $I \subset S$ with $\dim(S/I) = 0$. Then I contains pure powers of all the variables. Thus we can write $I = (J, x_1^{a_1}, \dots, x_n^{a_n})$, where J is the monomial ideal generated by monomials $u = x_1^{b_1} \cdots x_n^{b_n}$ with $b_i < a_i$ for $i = 1, \dots, n$. The graded Betti numbers of I are determined by those of J and certain colon ideals, see Formula (6).

For a standard graded K-algebra $R = S/I$, we denote by

$$P_R = \sum_{i,j} \dim_K \operatorname{Tor}_i^S(R, K)_j t^i s^j$$

the graded Poincaré series of R. Under our assumptions on I and J we will use a result of Mermin, Peeva and Stillman [21, Theorem 2.1] in the more general version as quoted by Mermin and Murai [20, Equation (1.3)], according to which

$$P_{S/I} = \sum_{T \subset [n]} t^{|T|} s^{a_T} P_{S/J_T}, \tag{6}$$

where $a_T = \sum_{i \in T} a_i$ and where $J_T = J : \prod_{i \in T} x_i^{a_i}$.

Let Δ be a simplicial complex on the vertex set $[n]$. To each vertex i of Δ we attach a number a_i and for $F \in \Delta$ we set $a_F = \sum_{i \in F} a_i$. Finally, we let

$$t_j(\mathbf{a}, \Delta) = \max\{a_F\colon\ F \in \Delta, |F| = j\}.$$

Here $\mathbf{a}$ stands for the sequence $a_1, \dots, a_n$.

Proposition 6 *Let $J \subset S$ be a monomial ideal whose minimal monomial generating set does not contain any pure power of the variables. and let $I = (J, x_1^{a_1}, \dots, x_n^{a_n})$. Assume that the highest degree of a minimal monomial generator of J is less than or equal to* $\min\{a_1, \dots, a_n\}$. *Furthermore, let Δ be the simplicial complex with $I_\Delta = \sqrt{J}$, and let* $\dim \Delta = d - 1$. *Then*

(a) $t_j(I) \geq t_j(\mathbf{a}, \Delta)$ *for $j = 1, \dots, d$. If Δ is unmixed or if $a_1 = a_2 = \cdots = a_n$, then equality holds.*
(b) $\operatorname{reg}(S/I) = t_d(\mathbf{a}, \Delta)$, *if J is squarefree.*

Proof

(a) It follows from formula (6) that

$$t_j(I) = \max_T\{t_{j-|T|}(J_T) + a_T\}, \tag{7}$$

where the maximum is taken over all $T \subset [n]$ with $|T| \leq j$ and such that $J_T \neq S$. We claim that $J_T \neq S$ if and only if $T \in \Delta$. Indeed, let $P_1, \dots, P_m$ be the minimal prime ideals of J. Then these prime ideals are also the minimal prime ideals of I_Δ. Thus, if $F_1, \dots, F_m$ are the facets of Δ, then $P_i = P_{[n] \setminus F_i}$ for $i = 1, \dots, m$. It follows that $J_T \neq S$ if and only if $T \subset F_i$ for some i, in other words, if $T \in \Delta$.

It follows that

$$t_j(I) = \max\{t_{j-|T|}(J_T) + a_T : \ T \in \Delta, \ |T| \leq j\} \tag{8}$$

Since we assume that $j \leq d$, there exists $T \in \Delta$ with $|T| = j$. Thus

$$t_j(I) \geq \max\{t_0(J_T)+a_T : \ T \in \Delta, \ |T| = j\} = \max\{a_T : \ T \in \Delta, \ |T| = j\} = t_j(\mathbf{a}, \Delta).$$

Assume now that Δ is unmixed, and let $T \in \Delta$ with $|T| \leq j$. Since Δ is unmixed there exists $F \in \Delta$ with $|F| = j$ and $T \subset F$. Let g be the maximal degree of a generator of J. Then [14, Corollary 4] implies that

$$t_{j-|T|}(J_T) + a_T \leq (j - |T|)g + a_T \leq a_{F\setminus T} + a_T = a_F \leq t_j(\mathbf{a}, \Delta).$$

Finally assume that $a = a_i$ for all i. Then

$$t_{j-|T|}(J_T) + a_T \leq (j - |T|)g + a|T| \leq ja = t_j(\mathbf{a}, \Delta).$$

Thus in both cases $t_j(I) \leq t_j(\mathbf{a}, \Delta)$.

(b) A K-basis of S/I consists of the monomials $\prod_{i\in F} x_i^{b_i}$ with $F \in \Delta$ and $b_i < a_i$ for $i = 1, \ldots, n$. Since $\dim(S/I) = 0$, the regularity of S/I is given by the highest degree of one of these basis elements. Basis elements of largest degree are of the form $\prod_{i\in F} x_i^{a_i-1}$, where F is a face with $|F| = d$. This yields the desired result. □

Theorem 1 for whisker graphs is contained as special case of the following result. We maintain the notation of Proposition 6. After relabeling we may assume in addition that $a_1 \leq a_2 \leq \cdots \leq a_n$.

Corollary 4

(a) *Let Δ be a simplicial complex and let $J = I_\Delta$. If Δ is unmixed or if $a_1 = a_2 = \cdots = a_n$, then I satisfies subadditivity.*
(b) *Suppose in addition that there exists a facet F of Δ with $a_F = \sum_{j=1}^{d} a_{n-j+1}$. Then the Betti diagram of S/I has a special shape.*

Proof

(a) It is clear from the definition of the numbers $t_i(\mathbf{a}, \Delta)$ that $t_{i+j}(\mathbf{a}, \Delta) \leq t_i(\mathbf{a}, \Delta) + t_j(\mathbf{a}, \Delta)$ for all i and j with $i + j \leq \dim \Delta + 1$. Thus as long as $i + j \leq d$ subadditivity holds, and for i with $d \leq i \leq \operatorname{proj}\dim(S/I)$, we have that $t_i(I) = t_d(\mathbf{a}, \Delta)$ by Proposition 6(b). Thus subadditivity holds for S/I.

(b) The assumption implies that $t_i(\mathbf{a}, \Delta) = \sum_{j=1}^{i} a_{n-j+1}$. This yields the desired conclusion. □

The following example demonstrates Corollary 4. Here $J = (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4) \cap (x_4, x_5)$ and $a_1 = 2, a_2 = 3, a_3 = 4, a_4 = 5$. The dimension of S/J is 3.

In this example, as seen in Table 3, $r_1(I) - r_0(I) = 4 < r_2(I) - r_1(I) = 3 < r_3(I) - r_2(I) = 2$.

Table 3 Betti diagram of S/J

	0	1	2	3	4	5
0:	1	-	-	-	-	-
1:	-	2	-	-	-	-
2:	-	7	14	7	1	
3:	-	1	9	15	8	1
4:	-	1	2	4	3	1
5:	-	-	6	7	2	-
6:	-	-	1	4	4	1
7:	-	-	1	1	-	-
8:	-	-	-	2	3	1
9:	-	-	-	1	2	1

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