

PREFACE

The aim of the book is to give a rigorous and at the same time accessible presentation of the theory of stochastic processes. This imposed hard restrictions on the selection of material, which reflects of course the preferences of the author.

As befits a theory, this presentation is largely self-contained. The text includes a very limited number of references to assertions, whose proofs should be looked for in other books. As a rule, this does not affect on the main style of the presentation. Accordingly, the different parts of the book can serve a source of ready materials for lecture courses. The text contains many examples showing how to apply theoretical results to solving concrete problems.

We do not dwell on the history of the creation of the theory of stochastic processes. This was covered brilliantly in many monographs that are much closer in time to the milestones of the theory than this book.

A significant part of the book is devoted to the classical theory of stochastic processes. At the same time there are new topics not presented previously in books. As to well-known results, we try to clarify the main ideas of the proofs, sometimes providing them with new approaches.

The first chapter contains basic facts of Probability Theory that will be useful for more detailed treatment in the subsequent chapters. Considerable attention is paid to conditional probabilities and conditional expectations, which are effective tools in the theory of stochastic processes. The foundations of the theory of martingales created by J. L. Doob are explained. The value of this theory is difficult to overestimate. Martingales were effectively applied outside probability theory, for example, in mathematical analysis. We consider only discrete-time martingales. The basic ideas of the theory of martingales are well illustrated for the case of a discrete-time parameter. Furthermore, the results for this case form a basis for the continuous-time theory. For our purposes the discrete time is sufficient and each time we need the continuous case, a corresponding justification will be given. We introduce a fairly detailed description of Markov processes. The Markov property provides a basic way of random changes, which is closest to what happens in reality. The chapter is completed with a consideration of a Brownian motion process. It is no exaggeration to assert that the Brownian motion is the fundamental stochastic process. Far ahead of its time many brilliant conjectures of P. Lévy were confirmed just for this process.

The second chapter is devoted to stochastic calculus, the foundations of which were laid down by K. Itô. At the initial stage of the creation of this theory, it was almost impossible to foresee how fruitful it will become. Its role in the theory of stochastic processes can be compared with the role of differential calculus in mathematical analysis or other disciplines. The theory of stochastic differential equations is the natural development of the theory of ordinary differential equations. Stochastic integrals with respect to Brownian motion, whose sample paths have unbounded variation, differ fundamentally from the classical integrals. This

difference leads to the fact that the stochastic differentials of superpositions of smooth functions with the solutions of the stochastic differential equations depend on the second derivatives of functions under differentiation, while this is absolutely impossible in the classical analysis. This difference also shows that descriptions of some physical phenomena, having a constructive nature and including stochastic interactions, are based on second-order differential equations. This gives a possible answer to the following fundamental question: why many phenomena in the real world are described by second-order differential equations? The role of the stochastic analysis in the description of the real world is not completely understood at the present time. At the end of the book, in Appendix 1, the problem of heat transfer is treated by means of the rigorous mathematical arguments. The approach is based on the energy exchange between individual molecules. This description is close to the real physical process.

The third chapter presents the theory of distributions of functionals of Brownian motion. The foundations of this theory were laid by A. N. Kolmogorov and M. Kac. In 1931 the famous forward and backward Kolmogorov's equations were introduced. An impetus to the emergence of probabilistic representations for the solutions of parabolic equations with potential was given in the doctoral thesis of the Nobel prizewinner in physics R. Feynman. He described the solutions of the Schrödinger equation in terms of path integrals. M. Kac saw in this result an analogy with the theory of distributions of integral functionals of Brownian motion and established the basis for this theory. The third chapter is devoted to the investigation of distributions of functionals of Brownian motion stopped at various random times. A sufficiently rich collection of stopping random times is considered. For all of them we derive effective results that enable us to compute distributions of various functionals of Brownian motion stopped at these moments. These results have already been used in Mathematical Statistics, Insurance Theory and Financial Mathematics.

The fourth chapter is devoted to a class of diffusion processes, generalizing the Brownian motion in a natural way. The necessity of studying the diffusion processes was probably realized by physicists earlier than by mathematicians. A striking example of this is the Einstein–Smoluchowski equation, describing the motion of a light particle in a viscous fluid. On the one hand, the random motion of fluid molecules interacting via collisions makes the particle move randomly, and, on the other hand, the viscosity restricts the speed of the movement. These two factors had a significant role in the discovery of the Einstein–Smoluchowski stochastic differential equation. A rigorous mathematical definition of diffusion processes was given by A. N. Kolmogorov. After the appearance of Itô's stochastic calculus, it was shown that under certain assumptions the diffusion processes defined by Kolmogorov are the solutions of the corresponding stochastic differential equations.

The fifth chapter is entirely devoted to a detailed study of the properties of Brownian local time, the cornerstone in the structure of additive functionals of a Brownian motion. The Brownian local time is the simplest positive continuous additive functional of a Brownian motion, in the sense that any such functional can be represented by the Stieltjes integral of the Brownian local time with respect to a non-decreasing function. This shows that the Brownian local time is

extremely useful for studying functionals of Brownian motion. Even from a purely mathematical point of view, the Brownian local time is a very interesting object to investigate. This is manifested in a number of deep properties, discovered through careful consideration, and in the beauty of analytical methods used to prove them. The concept of the Brownian local time and its most important properties emerged thanks to the intuition of P. Lévy. A special landmark in the study of the Brownian local time is F. Knight and D. Ray's description of the local time as a Markov process with respect to the space parameter. Methods enabling to compute the distributions of functionals of the Brownian local time play an important role in the study of properties of the Brownian local time. The main progress in this sphere was made possible thanks to the fact that, in contrast to the real potential, which corresponds to the integral functional in the theory of distributions, the Dirac δ -function corresponds to the local time. Involvement of the Dirac δ -function makes differential problems much simpler than those for real potentials.

The subject of the sixth chapter is a class of diffusions with jumps. The appearance of this topic is mainly dictated by needs of Financial Mathematics. In this theory, the continuous variation of the price of some assets is interrupted occasionally by abrupt collapses or, conversely, by growth of their ratings. An important feature here is the presence of random factors affecting the price of the asset. Between moments of jumps the process evolves as a classical diffusion. The most natural way of the appearance of jumps is the following. On disjoint intervals of arbitrarily small length the jumps occur independently with the probability proportional to the interval length. In fact, this probability may also depend on the value of the process at a moment of jump. In this way one comes to the processes having the Markov property. This, in turn, allows us to develop a sufficiently rich theory of such processes, which to some extent is comparable with the theory of classical diffusions.

The final, seventh chapter is devoted to the invariance principle. The Brownian motion and, more generally, diffusions are idealized limiting objects for random walks or more general processes with discrete time that have a recurrent structure. The random walk describes objectively some phenomena occurring in reality. A good example is given by the model of heat transfer (for details, see Appendix 1). This model is a perfect illustration of the following paradigm: the random walk is a simple and intuitively understandable process as regards the structure of sample paths, whereas its finite-dimensional distributions are rather complicated and the distributions of various functionals of the process are in fact extremely complicated. But the sample paths of a Brownian motion process are difficult to imagine at all. They are continuous, but non-differentiable almost everywhere. The level sets of a Brownian motion are Cantor sets (closed, uncountable, with zero topological dimension, and without isolated points), whereas there are only finitely many points in the level sets of a random walk. The finite-dimensional distributions of a Brownian motion are rather simple. Moreover, one can find explicit formulas for distributions of many functionals of a Brownian motion. In this situation, it is important to know how the Brownian motion process and the correspondingly transformed random walk can be close to each other. It is natural to use the specifics of both the random walk and the Brownian motion, in a certain extent,

identifying these processes.

In Appendix 1 the advantages of random walk and Brownian motion modelling are clearly demonstrated in the study of heat transfer. Using the random walk, we visually describe how heat is transferred, whereas using a Brownian motion approximation of it we derive a differential equation describing the change of the temperature in time and in space.

Appendix 2 contains a summary of the main properties of the special functions that are relevant to Probability Theory.

The theory of distributions of nonnegative functionals of processes is largely based on the Laplace transform of the distributions of these functionals. Appendix 3 contains tables of the inverse Laplace transforms that are often used in this theory.

In Appendix 4, we gather certain second-order differential equations and their nonnegative linearly independent solutions that are used to express Laplace transforms of transition functions of various diffusion processes.

Appendix 5 contains examples of transformations of measures generated by some well-known diffusions.

Finally, in Appendix 6 formulas that can be used for computing the moments of the distribution of a functional by its Laplace transform are given.

A. Yu. Zaitsev carefully read the manuscript and made many useful comments, which contributed to a significant improvement of the text. I am truly grateful to him for this great work. I am thankful to I. Ponomarenko for the help with the translation of some parts of the book. My thanks also go to the reviewers for their valuable suggestions.

Finally, I want to thank the staff of Birkhäuser for their excellent work when preparing the book. It is also my pleasure to thank the copy-editor whose careful work made the text more readable.

St. Petersburg, Russia, July 2017

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<http://www.springer.com/978-3-319-62309-2>

Stochastic Processes

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2017, XIV, 626 p. 1 illus., Hardcover

ISBN: 978-3-319-62309-2

A product of Birkhäuser Basel