Chapter 2
Fixing Ideas: Percolation on a Tree and Branching Random Walk

This text discusses percolation in high dimensions. When the dimension is high, space is so vast that faraway pieces of percolation clusters are close to being independent. The main purpose of this text is to make this imprecise statement precise. One reflection of this is that critical percolation clusters in high dimensions have relatively few cycles. On a tree, they are *precisely* independent, so that the above heuristic suggests that percolation on the high-dimensional hypercubic lattice is close to percolation on a tree. However, a tree does not have a Euclidean structure, and after discussing percolation on a tree, we discuss branching random walk, which we consider to be the proper mean-field model for percolation in high dimensions.

2.1 Percolation on a Tree

We start by studying percolation on the regular tree. In particular, we identify the critical exponents for percolation on a tree. We follow Grimmett [122, Sect. 10.1] or the second author's lecture notes [153, Sect. 1.2.2]. Let $\mathbb{T}_r$ denote the r-regular tree of degree r. The advantage of trees is that they do not contain cycles, which makes explicit computations possible. We first prove that the critical exponents for percolation on a regular tree exist and identify their values in the following theorem:

Theorem 2.1 (Critical behavior on the *r*-regular tree). *On the r-regular tree $\mathbb{T}_r$, $p_{\mathrm{c}}(\mathbb{T}_r) = p_T(\mathbb{T}_r) = 1/(r-1)$, and $\beta = \gamma = \gamma' = \rho_{\mathrm{in}} = 1$ and $\delta = \Delta = \Delta' = 2$ in the asymptotic sense.*

Proof. We make substantial use of the fact that percolation on a tree can be described in terms of *branching processes*. Let o denote a distinguished vertex that we call the *root* of the tree. For vertices $x, y \in \mathbb{T}_r$, we write $x \sim y$ whenever x and y are linked by an edge in the tree $\mathbb{T}_r$ and denote by $(I_{x,y})_{x\sim y}$ an i.i.d. family of Bernoulli random variables with parameter p indicating

M. Heydenreich and R. van der Hofstad, *Progress in High-Dimensional Percolation and Random Graphs*, CRM Short Courses, DOI 10.1007/978-3-319-62473-0_2

whether the edge $\{x, y\}$ is occupied or not. For $x \neq o$, we write $\mathscr{C}C_{\mathrm{BP}}(x)$ for the forward cluster of x in $\mathbb{T}_r$, i.e., those vertices $y \in \mathbb{T}_r$ that are connected to x and for which the unique path from x to y only moves away from the root o. Then, clearly,

$$|\mathscr{C}(o)| = 1 + \sum_{e \sim o} I_{o,e} |\mathscr{C}_{\mathrm{BP}}(e)| \,, \tag{2.1.1}$$

where the sum is over all neighbors e of o, and $(|\mathscr{C}_{\mathrm{BP}}(e)|)_{e\sim o}$ is an i.i.d. sequence independent of $(I_{o,e})_{e\sim o}$. Equation (2.1.1) allows us to deduce all information concerning $|\mathscr{C}(o)|$ from the information of $|\mathscr{C}_{\mathrm{BP}}(e)|$. Also, for each $x \neq o$, $|\mathscr{C}_{\mathrm{BP}}(x)|$ satisfies the formula

$$|\mathscr{C}_{\mathrm{BP}}(x)| = 1 + \sum_{v \sim x : h(v) > h(x)} I_{x,v} |\mathscr{C}_{\mathrm{BP}}(v)| \,, \tag{2.1.2}$$

where $h(x)$ is the distance to the root (or *height*) of x in $\mathbb{T}_r$, and $(|\mathscr{C}_{\mathrm{BP}}(v)|)_{v\sim x:h(v)>h(x)}$ is a set of $r-1$ independent copies of $|\mathscr{C}_{\mathrm{BP}}(x)|$. Thus, $|\mathscr{C}_{\mathrm{BP}}(x)|$ is the *total population size* of a branching process, also known as its *total progeny*. We now derive the critical exponents one by one, in the order γ, Δ, β, γ', Δ', δ, and ρ_{in}.

Proof that $(\gamma = 1)$ *on the Tree*. We use (2.1.2) to conclude that

$$\begin{aligned} \chi_{\mathrm{BP}}(p) &:= \mathbb{E}_p |\mathscr{C}_{\mathrm{BP}}(x)| = 1 + (r-1)p\, \mathbb{E}_p |\mathscr{C}_{\mathrm{BP}}(x)| \\ &= 1 + (r-1)p\chi_{\mathrm{BP}}(p) \,, \end{aligned} \tag{2.1.3}$$

so that

$$\chi_{\mathrm{BP}}(p) = \mathbb{E}_p |\mathscr{C}_{\mathrm{BP}}(x)| = \frac{1}{1-(r-1)p} \,. \tag{2.1.4}$$

From (2.1.1), we then obtain that, for $p < 1/(r-1)$,

$$\chi(p) = 1 + rp\chi_{\mathrm{BP}}(p) = 1 + \frac{rp}{1-(r-1)p} = \frac{1+p}{1-(r-1)p} \,, \tag{2.1.5}$$

while, for $p \geq 1/(r-1)$, $\chi(p) = \infty$. In particular, $p_{\mathrm{T}} = 1/(r-1)$ and $\gamma = 1$ in the asymptotic sense. The computation of $\chi(p)$ can also be performed without the use of (2.1.2), by noting that, for $p \in [0, 1]$,

$$\tau_p(x) = p^{h(x)} \,, \tag{2.1.6}$$

and the fact that, for $n \geq 1$, there are $r(r-1)^{n-1}$ vertices in $\mathbb{T}_r$ at height n, so that, for $p < 1/(r-1)$,

$$\chi(p) = 1 + \sum_{n=1}^{\infty} r(r-1)^{n-1} p^n = 1 + \frac{rp}{1-(r-1)p} = \frac{1+p}{1-(r-1)p} \,. \tag{2.1.7}$$

However, for related results for percolation on a tree, the connection to branching processes in (2.1.2) is vital.

Proof that $(\Delta = 2)$ *on the Tree. You* do it:

Exercise 2.1 $(\boldsymbol{\Delta} = \mathbf{2}$ **on the tree).** Prove that $\Delta = 2$ on the tree.

Proof that $(\beta = 1)$ *on the Tree*. We continue to investigate the critical exponent β for the percolation function on the tree. Let $\theta_{\mathrm{BP}}(p) = \mathbb{P}_p(|\mathscr{C}_{\mathrm{BP}}(x)| = \infty)$. Then $\theta_{\mathrm{BP}}(p)$ is the survival probability of a branching process with a binomial offspring distribution with parameters $r-1$ and p. Thus, $\theta_{\mathrm{BP}}(p)$ satisfies the equation

$$\theta_{\mathrm{BP}}(p) = 1 - \big(1 - p + p(1-\theta_{\mathrm{BP}}(p))\big)^{r-1} = 1 - \big(1 - p\theta_{\mathrm{BP}}(p)\big)^{r-1}\,. \tag{2.1.8}$$

To compute $\theta_{\mathrm{BP}}(p)$, it is more convenient to work with the extinction probability $\zeta_{\mathrm{BP}}(p) = 1 - \theta_{\mathrm{BP}}(p)$, which is the probability that the branching process dies out. The extinction probability $\zeta_{\mathrm{BP}}(p)$ satisfies

$$\zeta_{\mathrm{BP}}(p) = \big(1 - p + p\zeta_{\mathrm{BP}}(p)\big)^{r-1}\,. \tag{2.1.9}$$

This equation can be seen by noting that each of the $r-1$ possible children of the root needs to die out for the process to go extinct. By the absence of cycles, these events are independent and have the same probability, which explains the power $r-1$ in (2.1.9). Further, the probability that a child of the root dies out equals $1 - p + p\zeta_{\mathrm{BP}}(p)$, since either the edge leading to it is vacant, or the edge leading to it is occupied and then the branching process generated from this child needs to die out as well.

Equation (2.1.9) can be solved explicitly when $r = 2$ (the 'line graph'), where the unique solution is $\zeta_{\mathrm{BP}}(p) = 1$ for $p \in [0, 1)$ and $\zeta_{\mathrm{BP}}(1) = 0$. As a result, $\theta_{\mathrm{BP}}(p) = 0$ for $p \in [0, 1)$ and $\theta_{\mathrm{BP}}(1) = 1$, so that $p_{\mathrm{c}}(\mathbb{T}_2) = 1$. Having dealt with $r = 2$, we henceforth assume $r \geq 3$.

When $r = 3$, (2.1.9) reduces to

$$p^2\zeta_{\mathrm{BP}}(p)^2 + (2p(1-p) - 1)\zeta_{\mathrm{BP}}(p) + (1-p)^2 = 0\,, \tag{2.1.10}$$

so that

$$\zeta_{\mathrm{BP}}(p) = \frac{1 - 2p(1-p) \pm |2p-1|}{2p^2}\,. \tag{2.1.11}$$

Since $\zeta_{\mathrm{BP}}(1) = 0$, we must have that

$$\zeta_{\mathrm{BP}}(p) = \frac{1 - 2p(1-p) - |2p-1|}{2p^2}\,, \tag{2.1.12}$$

so that $\zeta_{\mathrm{BP}}(p) = 1$ for $p \in [0, \frac{1}{2}]$, while, for $p \in [\frac{1}{2}, 1]$,

$$\zeta_{\mathrm{BP}}(p) = \frac{1-2p(1-p)+(1-2p)}{2p^2} = \frac{2-4p+2p^2}{2p^2} = \left(\frac{1-p}{p}\right)^2 . \tag{2.1.13}$$

As a result, we have the explicit form $\theta_{\mathrm{BP}}(p) = 0$ for $p \in [0, \frac{1}{2}]$ and

$$\theta_{\mathrm{BP}}(p) = 1 - \left(\frac{1-p}{p}\right)^2 = \frac{2p-1}{p^2} , \tag{2.1.14}$$

for $p \in [\frac{1}{2}, 1]$, so that $p_{\mathrm{c}}(\mathbb{T}_3) = \frac{1}{2}$. In particular, $p \mapsto \theta_{\mathrm{BP}}(p)$ is continuous, and

$$\theta_{\mathrm{BP}}(p) = 8(p-p_{\mathrm{c}})\big(1+o(1)\big) \quad \text{as } p \searrow p_{\mathrm{c}} . \tag{2.1.15}$$

It is not hard to see that (2.1.15) together with (2.1.1) implies that

$$\theta(p) = 12(p-p_{\mathrm{c}})\big(1+o(1)\big) \quad \text{as } p \searrow p_{\mathrm{c}} . \tag{2.1.16}$$

Thus, for $r = 3$, the percolation function is continuous and $\beta = 1$ in the asymptotic sense.

One can easily extend the asymptotic analysis in (2.1.15)–(2.1.16) to $r \geq 4$, for which $p_{\mathrm{c}}(\mathbb{T}_r) = p_{\mathrm{T}}(\mathbb{T}_r) = 1/(r-1)$. We leave this as an exercise:

Exercise 2.2 (Asymptotics $p \mapsto \theta(p)$ for $\mathbb{T}_r$ with $r \geq 4$). Prove that, on $\mathbb{T}_r$ with $r \geq 4$,

$$\theta_{\mathrm{BP}}(p) = \frac{2(r-1)^2}{r-2}(p-p_{\mathrm{c}})\big(1+o(1)\big) , \tag{2.1.17}$$

and

$$\theta(p) = \frac{2r(r-1)}{r-2}(p-p_{\mathrm{c}})\big(1+o(1)\big) . \tag{2.1.18}$$

Proof that $\gamma' = 1$ on the Tree. In order to study $\chi^{\mathrm{f}}_{\mathrm{BP}}(p) = \mathbb{E}_p[|\mathscr{C}_{\mathrm{BP}}(x)|\mathbb{1}_{\{|\mathscr{C}_{\mathrm{BP}}(x)|<\infty\}}]$ for $p > p_{\mathrm{c}} = 1/(r-1)$, we make use of the fact that

$$\chi^{\mathrm{f}}_{\mathrm{BP}}(p) = \big(1-\theta_{\mathrm{BP}}(p)\big)\,\mathbb{E}_p[|\mathscr{C}_{\mathrm{BP}}(x)| \mid |\mathscr{C}_{\mathrm{BP}}(x)| < \infty] , \tag{2.1.19}$$

and the conditional law of percolation on the tree given that $|\mathscr{C}_{\mathrm{BP}}(x)| < \infty$ is percolation on a tree with p replaced by the *dual* percolation probability p_{d} given by

$$p_{\mathrm{d}} = p\big(1-\theta_{\mathrm{BP}}(p)\big) . \tag{2.1.20}$$

Indeed, each of the edges incident to the root that is occupied needs to be leading to a vertex that dies out itself, and all these events are independent. This explains that the offspring distribution of the root is binomial with parameter p_{d} as in (2.1.20) and $r-1$. But then each of the children of the root is again conditioned to go extinct, so that also their offspring distribution is binomial with parameters p_{d} and $r-1$.

The crucial fact is that $p_{\rm d} < p_{\rm c}(\mathbb{T}_r)$, which follows from the equality $1 - \theta_{\rm BP}(p) = \zeta_{\rm BP}(p)$, (2.1.9) and the fact that

$$\begin{aligned} (r-1)p\zeta_{\rm BP}(p) &= (r-1)p\big(1-p+p\zeta_{\rm BP}(p)\big)^{r-1} \\ &< (r-1)p\big(1-p+p\zeta_{\rm BP}(p)\big)^{r-2} \\ &= \frac{\mathrm{d}}{\mathrm{d}s}(1-p+ps)^{r-1}\Big|_{s=\zeta_{\rm BP}(p)} . \end{aligned} \tag{2.1.21}$$

Since $\zeta_{\rm BP}(p)$ is the smallest solution of $(1-p+ps)^{r-1} = s$, this implies that the derivative of $(1-p+ps)^{r-1}$ at $s = \zeta_{\rm BP}(p)$ is strictly bounded above by 1 for $p > p_{\rm c}(\mathbb{T}_r)$. Thus, by conditioning a supercritical cluster in percolation on a tree to die out, we obtain a subcritical cluster at an appropriate subcritical $p_{\rm d}$ which is related to the original percolation parameter. This fact is sometimes called the *discrete duality principle*.

We use (2.1.7) to conclude that

$$\begin{aligned} \chi^{\rm f}_{\rm BP}(p) &= \big(1-\theta_{\rm BP}(p)\big)\, \mathbb{E}_p[|\mathscr{C}_{\rm BP}(x)| \mid |\mathscr{C}_{\rm BP}(x)| < \infty] \\ &= \big(1-\theta_{\rm BP}(p)\big)\frac{1}{1-(r-1)p\big(1-\theta_{\rm BP}(p)\big)} . \end{aligned} \tag{2.1.22}$$

Using that $\theta_{\rm BP}(p) = C(p-p_{\rm c})\big(1+o(1)\big)$ for $C > 1$, cf. (2.1.17), in the asymptotic sense then gives that

$$\chi^{\rm f}_{\rm BP}(p) = \frac{C_{\gamma'} + o(1)}{p - p_{\rm c}} . \tag{2.1.23}$$

By (2.1.1), this can easily be transferred to $\chi^{\rm f}(p)$, so that also $\gamma' = 1$ in the asymptotic sense.

Proof that ($\Delta' = 2$) *on the Tree*. The above analysis can be extended to $\Delta' = 2$, by looking at higher moments of the cluster size conditioned to be finite. By the duality described above, this follows from the fact that $\Delta = 2$ and $\beta = 1$.

Proof that ($\delta = 2$) *on the Tree*. We can compute δ by using the *random walk hitting time theorem*, see Grimmett's percolation book [122, Prop. 10.22] or the more recent proof by the second author and Keane in [162], where a simple proof is given for general branching processes. This result yields that

$$\mathbb{P}_p(|\mathscr{C}_{\rm BP}(x)| = k) = \frac{1}{k}\,\mathbb{P}(X_1 + \cdots + X_k = k-1), \tag{2.1.24}$$

where $(X_i)_{i\geq 1}$ is an i.i.d. sequence of binomial random variables with parameter $r-1$ and success probability p.

Exercise 2.3 (Branching processes and random walks). Prove that

$$\mathbb{P}_p(|\mathscr{C}_{\mathrm{BP}}(x)| = k) = \mathbb{P}_1(S_k = 0 \text{ for the first time}) , \tag{2.1.25}$$

where $S_k = 1 + \sum_{i=1}^{k}(X_i - 1)$.

Exercise 2.4 (Random walk hitting time theorem). Prove (2.1.24) using induction on the equality

$$\mathbb{P}_m(S_k = 0 \text{ for first time}) = \frac{m}{k}\,\mathbb{P}_1(X_1 + \cdots + X_k = k-1) , \tag{2.1.26}$$

where $S_k = m + \sum_{i=1}^{k}(X_i - 1)$, and the previous exercise.

From (2.1.24), we conclude that

$$\mathbb{P}_p(|\mathscr{C}_{\mathrm{BP}}(x)| = k) = \frac{1}{k}\binom{k(r-1)}{k-1} p^{k-1}(1-p)^{k(r-1)-(k-1)} . \tag{2.1.27}$$

To prove that $\delta = 2$, we note that for $p = p_{\mathrm{c}}(\mathbb{T}_r) = 1/(r-1)$, by a local limit theorem and for some C_δ,

$$\mathbb{P}_{p_{\mathrm{c}}}(|\mathscr{C}_{\mathrm{BP}}(x)| = k) = \big(C_\delta/2 + o(1)\big)\frac{1}{\sqrt{k^3}} . \tag{2.1.28}$$

Summing over $k \geq n$, we obtain

$$\mathbb{P}_{p_{\mathrm{c}}}(|\mathscr{C}_{\mathrm{BP}}(x)| \geq n) = \sum_{k \geq n}\big(C_\delta/2 + o(1)\big)\frac{1}{\sqrt{k^3}} = \frac{C_\delta + o(1)}{\sqrt{n}} . \tag{2.1.29}$$

This proves that $\delta = 2$ in an asymptotic sense on the tree.

Proof that $(\rho_{in} = 1)$ *on the Tree*. We can compute ρ_{in} by noting that

$$\theta_n = \mathbb{P}_{p_{\mathrm{c}}}(\exists v \in \mathscr{C}_{\mathrm{BP}}(o) \text{ such that } h(v) = n) \tag{2.1.30}$$

satisfies the recursion relation

$$1 - \theta_n = (1 - p_{\mathrm{c}}\theta_{n-1})^{r-1} . \tag{2.1.31}$$

It is not hard to see that (2.1.31) together with $p_{\mathrm{c}}(\mathbb{T}_r) = 1/(r-1)$ implies that $\theta_n = (C_{\mathrm{in}} + o(1))/n$, so that $\rho_{\mathrm{in}} = 1$. This is left as Exerc. 2.5 below. □

Exercise 2.5 (Proof of (2.1.31) and its asymptotics). (a) Prove (2.1.31).

(b) Prove that (2.1.31) implies that $\theta_n = 2/\big(\sigma^2 n\big)(1 + O(1/n)\big)$, where $\sigma^2 = (r-2)/(r-1)$ is the variance of the offspring distribution. *Hint*: Perform induction on n for $v_n = 1/\theta_n$.

Exercise 2.6 (A lower bound on p_c for general graphs). (a) Use the percolation critical values $p_c(\mathbb{T}_r) = p_T(\mathbb{T}_r) = 1/(r-1)$ on the r-regular tree $\mathbb{T}_r$ in Thm. 2.1 to show that $p_c(\mathscr{G}) \geq 1/r$ when $\mathscr{G}$ is a transitive graph with degree r.

(b) Improve the bound in part (a) to $p_c(\mathscr{G}) \geq 1/(r-1)$ when $\mathscr{G}$ is a transitive graph with degree r.

The computation of the key objects for percolation on a tree is feasible due to the close relationship to branching processes, a topic which has attracted substantial interest in the probability community. See the books by Athreya and Ney [23], Harris [142] and Jagers [183] for detailed discussions about branching processes.

2.2 Branching Random Walk as the Percolation Mean-Field Model

We next argue that *branching random walk,* henceforth abbreviated by BRW, can be viewed as the mean-field model for percolation, and we shall see that the critical behavior of percolation in high dimensions is closely related to the critical behavior of BRW. Of course, percolation on the tree lacks a geometric embedding into (Euclidean) space. Therefore, the critical exponents ρ_{ex}, ν, ν' ,and η are not so easily defined on the tree.

BRW is a *random embedding* of a branching process with $\mathsf{Bin}(2d, p)$-offspring distribution into $\mathbb{Z}^d$. This can be intuitively understood as follows. Every vertex x in percolation on $\mathbb{Z}^d$ has a binomial number of neighbors with parameters p and $2d$ for which the edge leading to it is occupied. Thus, one could imagine *exploring* a cluster vertex by vertex. In high dimensions, space is quite vast, so that it is *relatively rare* to close a cycle. Cycles form the difference between percolation on a tree and percolation in $\mathbb{Z}^d$. BRW is precisely the process in which we *ignore cycles*. Thus, one might hope that BRW is closely related to percolation in sufficiently high dimensions. One of the main aims of this text is to make this intuition precise. Before starting with that, though, let us first investigate BRW in more detail, so as to obtain insight in the kind of results that we might be able to show for high-dimensional percolation.

We may think of branching random walk as percolation on the $2d$-ary tree that is embedded into the Euclidean lattice $\mathbb{Z}^d$, and we explain this *BRW embedding* now. For every $v \in \mathbb{T}_r$, we associate a spatial location $\phi(v) \in \mathbb{Z}^d$ in the following (random) way. We let $\phi(o) = 0$, so that the root in $\mathbb{T}_r$ is mapped to the origin in $\mathbb{Z}^d$. Further, for $v \in \mathbb{T}_r$, we let $p(v) \in \mathbb{T}_r$ denote the unique parent of v, i.e., the neighbor of v that is on the unique path to the root o in the tree $\mathbb{T}_r$. Then, for every $v \in \mathbb{T}_r$ having parent $p(v) \in \mathbb{T}_r$, we let $\phi(v) = \phi\big(p(v)\big) + Y_v$, where $Y_v \in \mathbb{Z}^d$ is a random neighbor of the origin, i.e., for every e with $|e| = 1$,

$$\mathbb{P}(Y_v = e) = 1/(2d) . \tag{2.2.1}$$

The random variables $(Y_v)_{v\in\mathbb{T}_r\setminus\{o\}}$ form a collection of i.i.d. random variables.

We denote the *BRW two-point function* $G_p(x)$ by

$$G_p(x) = \mathbb{E}_p\Bigg[\sum_{v\in\mathscr{C}(o)} \mathbb{1}_{\{\phi(v)=x\}}\Bigg], \quad x\in\mathbb{Z}^d , \tag{2.2.2}$$

and its truncated version by

$$G_p^{\mathrm{f}}(x) = \mathbb{E}_p\Bigg[\sum_{v\in\mathscr{C}(o)} \mathbb{1}_{\{\phi(v)=x\}}\mathbb{1}_{\{|\mathscr{C}(o)|<\infty\}}\Bigg], \quad x\in\mathbb{Z}^d . \tag{2.2.3}$$

The BRW two-point functions $G_p(x)$ and $G_p^{\mathrm{f}}(x)$ have similar interpretations as the percolation two-point functions $\tau_p(x)$ and $\tau_p^{\mathrm{f}}(x)$ in (1.2.6)–(1.2.7).

Alternatively, denoting by $N(x)$ the total number of particles in $\mathscr{C}(o)$ that are mapped to $x\in\mathbb{Z}^d$ by ϕ,

$$\begin{aligned} G_p^{\mathrm{f}}(x) &= \mathbb{E}_p[N(x)\mathbb{1}_{\{|\mathscr{C}(o)|<\infty\}}] \\ &= \mathbb{E}_p\Bigg[\sum_{v\in\mathbb{T}_r} \mathbb{1}_{\{\phi(v)=x\}}\mathbb{1}_{\{v\in\mathscr{C}(o)\}}\mathbb{1}_{\{|\mathscr{C}(o)|<\infty\}}\Bigg] \\ &= \sum_{v\in\mathbb{T}_r} \mathbb{P}_p\big(\phi(v)=x, v\in\mathscr{C}(o), |\mathscr{C}(o)|<\infty\big) \\ &= \sum_{v\in\mathbb{T}_r} \mathbb{P}(\phi(v)=x)\,\mathbb{P}_p(v\in\mathscr{C}(o), |\mathscr{C}(o)|<\infty) , \end{aligned} \tag{2.2.4}$$

the latter by the independence of the embedding of the tree and the occupation statuses of the bonds.

We now turn to $p\le p_{\mathrm{c}}(\mathbb{T}_r)=1/(r-1)$, in which case we can remove the condition that $|\mathscr{C}(o)|<\infty$. When $h(v)=n$, in order for $\{v\in\mathscr{C}(o)\}$ to occur, all the n edges on the path between o and v have to be occupied, so that $\mathbb{P}_p\big(v\in\mathscr{C}(o)\big)=p^n$. Further,

$$\mathbb{P}(\phi(v)=x) = \mathbb{P}\Bigg(\sum_{u\in\pi_v} Y_u = x\Bigg) , \tag{2.2.5}$$

where π_v contains all the vertices on the unique path between o and v. Again, when $h(v)=n$, and by the independence of the random variables $(Y_v)_{v\in\mathbb{T}_r\setminus\{o\}}$,

$$\mathbb{P}(\phi(v)=x) = \mathbb{P}\Bigg(\sum_{i=1}^n Y_i = x\Bigg) = D^{\star n}(x) , \tag{2.2.6}$$

where $D^{\star n}$ denotes the n-fold convolution of D with itself, and we recall that D denotes the simple random walk transition probability defined in (1.2.18). As a result, we obtain that

$$G_p(x) = \sum_{n\geq 0} r(r-1)^{n-1} p^n D^{\star n}(x) = \frac{r}{r-1} C_{(r-1)p}(x) , \tag{2.2.7}$$

where $C_\mu(x)$ denotes the random walk Green's function given by

$$C_\mu(x) = \sum_{n\geq 0} \mu^n D^{\star n}(x) . \tag{2.2.8}$$

It is well known that, for any $d \geq 3$, $\mu = 1$ serves as a critical value for the simple random walk Green's function $C_\mu(x)$ and that there exists a constant $A > 0$ such that

$$C_1(x) = \frac{A}{|x|^{d-2}}\big(1 + o(1)\big), \tag{2.2.9}$$

cf. Uchiyama [256]. Probabilistically, $C_1(x)$ describes the expected number of visits to the site x of a random walk starting at the origin. We take $p = p_c(\mathbb{T}_r) = 1/(r-1)$, so that

$$G_{p_c}(x) = \frac{r}{r-1} C_1(x) . \tag{2.2.10}$$

Thus, (2.2.9) implies that $\eta = 0$ in x-space for BRW.

The connection to BRW yields a powerful intuitive way to predict properties of percolation in high dimensions. Further, it yields a powerful relation between BRW and the random walk Green's function that will prove to be extremely useful later on.

In Fourier language, the fact that $\eta = 0$ is much simpler. Indeed, taking the Fourier transform of (2.2.8) leads to

$$\widehat{C}_\mu(k) = \sum_{n\geq 0} \mu^n \widehat{D}^n(k) = \frac{1}{1 - \mu \widehat{D}(k)} . \tag{2.2.11}$$

Since $|\widehat{D}(k)| \leq 1$ with $\widehat{D}(0) = 1$, we again see that $\mu = 1$ serves as a critical value. For $\mu = 1$, we obtain

$$\widehat{C}_1(k) = \sum_{n\geq 0} \widehat{D}^n(k) = \frac{1}{1 - \widehat{D}(k)} . \tag{2.2.12}$$

Using a series expansion of cosine, we obtain for $k \to 0$ that

$$1 - \widehat{D}(k) = \frac{1}{d}\sum_{i=1}^{d}[1-\cos(k_i)]$$
$$= \frac{1}{2d}\sum_{i=1}^{d} k_i^2\big(1+o(1)\big) = \frac{1}{2d}|k|^2\big(1+o(1)\big)\,, \tag{2.2.13}$$

and arrive at

$$\widehat{C}_1(k) = \frac{2d}{|k|^2}\big(1+o(1)\big) \quad \text{as } k \to 0\,. \tag{2.2.14}$$

This proves that $\eta = 0$ in k-space.

BRW also allows us to define ρ_{ex} and ν_2, ν_2', ν, ν'. For ρ_{ex}, we write

$$\mathbb{P}_{p_c}(\exists v \in \mathscr{C}(o)\colon \phi(v) \in \partial\Lambda_n) \sim n^{-1/\rho_{\mathrm{ex}}}\,, \tag{2.2.15}$$

recalling that $\partial\Lambda_n$ consists of lattice sites at ℓ^∞-distance n from the origin. The exponents ν and ν' are the critical exponents of the BRW correlation length

$$\xi^{\mathrm{BRW}}(p) = -\lim_{n\to\infty}\left(\frac{\log\big(G_p^{\mathrm{f}}(ne)\big)}{n}\right)^{-1}\,, \tag{2.2.16}$$

where we recall (1.2.7) and (2.2.3). Similarly, the exponents ν_2 and ν_2' are the critical exponents of

$$\xi_2^{\mathrm{BRW}}(p) = \sqrt{\frac{1}{\chi^{\mathrm{f}}(p)}\sum_{x\in\mathbb{Z}^d}|x|^2 G_p^{\mathrm{f}}(x)}\,, \tag{2.2.17}$$

where we recall (1.2.8) and (2.2.3). The following theorem collects results for the critical behavior of BRW:

Theorem 2.2 (Critical behavior branching random walk). *For BRW with a binomial offspring distribution with parameters $r-1$ and p, the critical value equals $p_c = p_T = 1/(r-1)$, and $\beta = \gamma = \gamma' = 1$, $\delta = \Delta = \Delta' = 2$, $\eta = 0$, and $\nu_2 = \nu_2' = \nu = \nu' = \rho_{\mathrm{ex}} = \frac{1}{2}$ in the asymptotic sense.*

Proof. All these critical exponents follow from Thm. 2.1, except $\eta, \nu_2, \nu_2', \nu, \nu'$, and ρ_{ex}. The fact that $\eta = 0$ follows from (2.2.9) and (2.2.10).

For ν_2 and ν_2', we note that

$$G_p^{\mathrm{f}}(x) = \sum_{v\in\mathbb{T}_r} D^{\star h(v)}(x)\tau_p^{\mathrm{f}}(v)\,. \tag{2.2.18}$$

Fix $p < p_c = 1/(r-1)$, so that $\tau_p^{\mathrm{f}}(v) = \tau_p(v) = p^{h(v)}$. Then, using the simple random walk variance given by $\sum_{x\in\mathbb{Z}^d}|x|^2 D^{\star n}(x) = n$, we compute

$$\sum_{x\in\mathbb{Z}^d} |x|^2 G_p^{\mathrm{f}}(x) = \sum_{v} \sum_{x\in\mathbb{Z}^d} |x|^2 D^{\star h(v)}(x)\tau_p^{\mathrm{f}}(v) = \sum_{v} h(v)\tau_p^{\mathrm{f}}(v)$$
$$= \sum_{n\geq 1} nr(r-1)^{n-1} p^n = \frac{rp}{[1-(r-1)p]^2}, \tag{2.2.19}$$

so that $\nu_2 = \frac{1}{2}$ for BRW. This can be extended to $\nu_2' = \frac{1}{2}$ by using the duality between supercritical BRW conditioned to go extinct and subcritical BRW as discussed around (2.1.20). The fact that $\nu = \frac{1}{2}$ for BRW follows from a careful analysis of $C_\mu(ne_1)$ when $n \to \infty$, using large deviations for random walks. Again this can be extended to $\nu' = \frac{1}{2}$ by using the duality between supercritical BRW conditioned to go extinct and subcritical BRW.

The proof that $\rho_{\mathrm{ex}} = \frac{1}{2}$ is more involved and is therefore omitted here. □

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Progress in High-Dimensional Percolation and Random Graphs
Heydenreich, M.; van der Hofstad, R.
2017, XII, 285 p. 10 illus., 1 illus. in color., Hardcover
ISBN: 978-3-319-62472-3