Preface

This book focuses on percolation on high-dimensional lattices. We give a general introduction to percolation, stating the main results and defining the central objects. We assume no prior knowledge about percolation. This text is aimed at graduate students and researchers who wish to enter the wondrous world of high-dimensional percolation, with the aim to demystify the lace-expansion methodology that has been the key technique in high dimensions. This text can be used for reading seminars or advanced courses as well as for reference and individual study. The exposition is complemented with many exercises, and we invite readers to try them out and gain deeper understanding of the techniques presented here. Let us now summarize the content in more detail.

We describe mean-field results in high-dimensional percolation that make the intuition that "faraway critical percolation clusters are close to being independent" precise. We have two main purposes. The first main purpose is to give a self-contained proof of mean-field behavior for high-dimensional percolation, by proving that percolation in high dimensions has mean-field critical exponents $\beta = \gamma = 1$, $\delta = 2$ and $\eta = 0$, as for percolation on the tree. This proof is obtained by combining the Aizenman–Newman and Barsky–Aizenman differential inequalities, that rely on the *triangle condition*, with the lace-expansion proof of Hara and Slade of the infrared bound that, in turn, verifies the triangle condition. While there are expository texts discussing lace-expansion methodology, such as Slade's excellent Saint-Flour lecture notes, an introduction to high-dimensional percolation did not yet exist.

Aside from these classical results, that are now over 25 years old, our second main purpose is to discuss recent extensions and additions. We focus on (1) the recent proof that mean-field critical behavior holds for percolation in $d \geq 11$; (2) the proof of existence of arm exponents; (3) results on finite-size scaling and percolation on high-dimensional tori and their relationship to the Erdős–Rényi random graph; (4) extensions of these finite-size scaling results to hypercube percolation; (5) the existence of the *incipient infinite cluster* and its scaling properties, as well as the proof of the Alexander–Orbach conjecture for random walks on the high-dimensional incipient infinite cluster; (6) the novel lace expansion for the two-point function with a fixed number of pivotals; and (7) super-process limits of critical percolation clusters. The text is enriched with numerous open problems, which, we hope, will stimulate further research in the field.

This text is organized as follows. In Part I, consisting of Chaps. 1–3, we introduce percolation and prove its main properties such as the sharpness of the phase transition. In Part II, consisting of Chaps. 4–9, we discuss mean-field critical behavior by describing the two main techniques used, namely, differential inequalities and the lace expansion. In Parts I and Part II, all results are proved, making this the first self-contained text discussing high-dimensional percolation. In Part III, consisting of Chaps. 10–13, we describe recent progress in high-dimensional percolation. We provide partial proofs and give substantial overview and heuristics about how the proofs are obtained. In many of these results, the lace expansion and differential inequalities or their discrete analogues are central. In Part IV, consisting of Chaps. 14–16, we discuss related models and further open problems. Here we only provide heuristics and few details of the proofs, thus focussing on the overview and big picture.

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