Chapter 2
Superstring Actions in $AdS_5 \times S^5$ and $AdS_4 \times \mathbb{CP}^3$ Spaces

Following the earlier construction of the covariant action for supersymmetric particles [1], the Green-Schwarz (GS) superstring action in flat space was proposed in [2] and interpreted as a coset sigma-model of Wess-Zumino type by Henneaux and Mezincescu in [3]. The action displays a local fermionic symmetry, called *κ-symmetry*, which generalizes the one exhibited by massive and massless superparticles [4, 5]. Superstrings of the type IIB can be formulated on a generic supergravity background with preservation of this gauge invariance [6].

Along the lines of the approach in [3], R.R. Metsaev and A.A. Tseytlin constructed the covariant κ-symmetric action for type IIB superstring on $AdS_5 \times S^5$ in [7, 8]. Given its central relevance in the main instance (1.1) of AdS/CFT, we devote Sect. 2.1 to review the sigma-model based on the supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$.

The importance of constructing superstring theory for the lower-dimensional duality (1.5) prompted the formulation of the $\frac{OSp(4|6)}{SO(1,3)\times U(3)}$ supercoset action on the $AdS_4 \times \mathbb{CP}^3$ background [9, 10]. However, some subtleties related to the κ-symmetry transformations of this sigma-model will lead us in Sect. 2.2 to consider the alternative formulation of the action proposed by Uvarov [11, 12].

2.1 Supercoset Construction of the String Action in $AdS_5 \times S^5$

This $AdS_5 \times S^5$ background is a maximally supersymmetric solution [13] of the type IIB supergravity equations of motion in ten-dimensions, together with the flat Minkowski space $\mathbb{R}^{1,9}$ and the "plane-wave" background [14]. The presence of the self-dual Ramond-Ramond (RR) five-form flux supporting this "vacuum" geometry precludes the use of the Neveu-Schwarz-Ramond (NSR) approach [15, 16] to construct the action. The Green-Schwarz formalism [2, 17] has proven to be a viable

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method when the RR fields are not vanishing, with the additional advantage of realizing supersymmetry manifestly in the ten-dimensional ambient space. For any type II supergravity background, the formal expression of the superstring action exists [6], but it is not very practical to explicitly find the action in terms of the coordinate fields.[1] For this reason, one is led to devise an alternative approach to write the complete action on this space.

A more advantageous route to the superstring action in the $AdS_5 \times S^5$ background is tailored to the peculiar structure of the superisometry group $PSU(2, 2|4)$, namely the supersymmetric extension of the $SU(2, 2) \times SU(4)$ group which is locally isomorphic to the bosonic isometries $SO(2, 4) \times SO(6)$ of this product manifold.

The generalization to the curved $AdS_5 \times S^5$ background has been developed in [7, 8], prompted by the then-recent conjecture of the AdS/CFT correspondence [18], and it is also conceptually very close to the construction of the GS action for strings moving in $\mathbb{R}^{1,9}$ [3]. Taking inspiration from the flat-space counterpart, the superstring action can be formulated as a type of Wess-Zumino-Witten (WZW) non-linear sigma-model in two-dimensions with the supercoset $PSU(2, 2|4)/(SO(1, 4) \times SO(5))$ as target space. We recall that the bosonic reduction of this coset is precisely a representation of the $AdS_5 \times S^5$ space, as the quotient of $SO(2, 4) \times SO(6) \subset PSU(2, 2|4)$ over $SO(1, 4) \times SO(5)$, where the 10 bosonic degrees of freedom of the superstring propagate. The additional 32 fermionic degrees of freedom, which would parametrize the two Majorana-Weyl fermions of 10d type IIB supergravity, are provided by the corresponding anticommuting generators of $PSU(2, 2|4)$.

The next section reviews the basic facts [19] about sigma-model actions on general coset spaces, which will provide a natural way to include the couplings to the background RR fields for the $AdS_5 \times S^5$ case in question. This is also a general strategy for constructing sigma-models in other integrable AdS/CFT systems.

Let us mention that the supercoset general construction naturally applies to the string theory on $AdS_4 \times \mathbb{CP}^3$ [9, 10], $AdS_3 \times S^3 \times S^3 \times S^1$ and $AdS_3 \times S^3 \times T^4$ [20] and also to the η-model [21, 22] and λ-model [23, 24], which are integrable deformations of the Metsaev-Tseytlin supercoset action named after the corresponding deformation parameters. However, they will not be covered in what follows.

2.1.1 String Sigma-Model for Coset Spaces and κ-Symmetry

The construction of the Lagrangian makes use of a general description valid for any homogeneous manifold expressed as coset space G/H, where G is the isometry group of the Killing vectors on such manifold and H is the stabilizer subgroup. The resulting sigma-model is based on the action of the supercoset $\tilde{G}/H$, where $\tilde{G}$ is the supergroup embedding G as its even part. We denote the (super)algebras associated to G, H and $\tilde{G}$ by $\mathfrak{g}$, $\mathfrak{h}$ and $\tilde{\mathfrak{g}}$ respectively. In most of the relevant examples

[1] Indeed, given a bosonic background, one has to determine the exact expressions for the supervielbeins, which is a difficult problem to solve in non-trivial cases.

of AdS backgrounds, we restrict to a Lie superalgebra $\tilde{\mathfrak{g}}$ admitting an order-four *automorphism*, which is a linear map $\Omega : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ with

$$\Omega([a, b]) = [\Omega(a), \Omega(b)] , \qquad a, b \in \tilde{\mathfrak{g}} , \qquad\qquad \Omega^4 = \mathrm{id} \tag{2.1}$$

that decomposes $\tilde{\mathfrak{g}}$ into a direct sum of four graded subspaces

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}^{(0)} \oplus \tilde{\mathfrak{g}}^{(1)} \oplus \tilde{\mathfrak{g}}^{(2)} \oplus \tilde{\mathfrak{g}}^{(3)} , \tag{2.2}$$

where each of them is an eigenspace of Ω

$$\Omega(\tilde{\mathfrak{g}}^{(k)}) = i^k \tilde{\mathfrak{g}}^{(k)} . \tag{2.3}$$

The grading is compatible with the supercommutator $[\tilde{\mathfrak{g}}^{(k)}, \tilde{\mathfrak{g}}^{(m)}] \subset \tilde{\mathfrak{g}}^{(k+m \bmod 4)}$. Note that $\tilde{\mathfrak{g}}^{(0)}$ is a subalgebra and the set of stationary points of Ω. In addition to the $\mathbb{Z}_2$-grading implicit in the definition of superalgebra $\mathfrak{g}$—for which $\tilde{\mathfrak{g}}^{(0)}$, $\tilde{\mathfrak{g}}^{(2)}$ are even and $\tilde{\mathfrak{g}}^{(1)}$, $\tilde{\mathfrak{g}}^{(3)}$ are odd subspaces—the automorphism endows $\mathfrak{g}$ with the structure of a $\mathbb{Z}_4$-graded algebra [25].

The description of the superstring action is given in terms of a *coset representative* $g(\tau, \sigma) \in \tilde{G}$ defined on 2d worldsheet, spanned by the τ and σ coordinates. We such g to build the $\tilde{\mathfrak{g}}$-valued one-form current

$$A \equiv -g^{-1}dg = -(g^{-1}\partial_\tau g)d\tau - (g^{-1}\partial_\sigma g)d\sigma \tag{2.4}$$

and split it as $A = A^{(0)} + A^{(1)} + A^{(2)} + A^{(3)}$ according to the $\mathbb{Z}_4$-decomposition. By construction the current is *flat*, namely it has vanishing two-form curvature $F \equiv dA - A \wedge A = 0$. The current exhibits other important properties:

- it is invariant under the left group action $g \to hg$ of a constant $h \in \tilde{G}$,
- under the local right action $g \to gh$ with $h(\tau, \sigma) \in H$, it undergoes the "gauge" transformation $A \to h^{-1}Ah - h^{-1}dh$, which in components splits into

$$A^{(0)} \to h^{-1}A^{(0)}h - h^{-1}dh , \qquad A^{(k)} \to h^{-1}A^{(k)}h , \quad k = 1, 2, 3 . \tag{2.5}$$

Having fixed the coset representatives g, the supercoset sigma-model action with target superspace $\tilde{G}/H$ is given by

$$S = -\frac{T}{2}\int d\tau d\sigma\, \mathcal{L} , \qquad \mathcal{L} = \sqrt{-g}g^{\alpha\beta}\mathrm{str}\left(A^{(2)}_\alpha A^{(2)}_\beta\right) + \kappa\, \epsilon^{\alpha\beta}\mathrm{str}\left(A^{(1)}_\alpha A^{(3)}_\beta\right) , \tag{2.6}$$

where the dimensionless quantity T is the string tension, $\alpha, \beta = 0, 1$ are the worldsheet indices, $g_{\alpha\beta}$ is the worldsheet metric with $g \equiv \det g_{\alpha\beta}$, the antisymmetric symbol $\epsilon^{\alpha\beta}$ is defined by $\epsilon^{01} = 1$ and we dropped the wedge operator between the one-forms. The Lagrangian density is built by means of the supertrace operator (str) acting on a suitable matrix representation of the $\mathbb{Z}^4$-components of the current $A^{(k)}$ in the algebra $\tilde{\mathfrak{g}}$.

- The first term in $\mathcal{L}$ is the usual kinetic term of a sigma-model. In addition to the proper quadratic kinetic term, it also contains interactions, hence (2.6) has to be regarded as a *non-linear* sigma-model.
- The second addend is a Wess-Zumino (WZ) type term. Although not directly visible in this form, it can be written indeed as the integral of a closed three-form over a 3d space M_3 with the 2d worldsheet $\Sigma = \partial M_3$ as boundary:

$$\frac{1}{2}\int_{\Sigma} \mathrm{str}\Big(A^{(1)}A^{(3)}\Big) = \int_{M_3} \mathrm{str}\Big(A^{(2)}A^{(3)}A^{(3)} - A^{(2)}A^{(1)}A^{(1)}\Big)\,. \tag{2.7}$$

The value of the real coefficient κ can be fixed by the requirement that the action possesses the (κ-symmetry which we discuss below.

Physical motivations for the proposal (2.6) come from the special case when fermionic supermatrix elements of A are set to zero: one can show that S reduces to the standard Polaykov action for bosonic strings in the background G/H. It is also important to observe that the action, although it depends on $g \in \tilde{G}$, it only explicitly depends on the equivalence class of coset elements in $\tilde{G}/H$. This is a consequence of the supertrace in (2.6) being insensible to the similarity transformations (2.5) of the components $A^{(k)}$ with $k = 1, 2, 3$. Different parametrizations of the coset element lead to equivalent Lagrangians that differ in the explicit dependence on the coset degrees of freedom. This observation will motivate a preferred coset parametrization, accompanied by an appropriate gauge-fixing of the bosonic and fermionic symmetries, to drastically simplify the sigma model.

The group of global symmetry of $\mathcal{L}$, acting on the coset space by multiplication from the left, is $\tilde{G}$. In fact, given a coset representative $g \in \tilde{G}$, the action of a constant element $g' \in \tilde{G}$ is the map

$$g(\tau,\sigma) \;\to\; g^{'}g(\tau,\sigma) \equiv g^{''}(\tau,\sigma)h(\tau,\sigma)\,, \tag{2.8}$$

where the image of g is rewritten as the *right* action of a suitable local element $h \in H$ on the new coset representative $g^{''}$. Then the global $\tilde{G}$-invariance of the Lagrangian $\mathcal{L}$ is a consequence of the local H-symmetry of the action that gauges away $h(\tau, \sigma)$.

As already anticipated, the symmetry group of the action can be enhanced to include κ-symmetry, a local fermionic symmetry first discovered for massive and massless supersymmetric particles [4, 5] and then found in the GS superstring in flat space [2]. It plays a crucial role for the consistency of the supercoset model, ensuring that there is the correct number of physical fermionic degrees of freedom. It is realized as a certain right local action of $g^{'}(\tau,\sigma) \equiv \exp\varepsilon(\tau,\sigma) \in \tilde{G}$, with parameter $\varepsilon \in \tilde{\mathfrak{g}}$, on the coset element

$$g(\tau,\sigma) \;\to\; g(\tau,\sigma)g^{'}(\tau,\sigma) \equiv g^{''}(\tau,\sigma)h(\tau,\sigma), \tag{2.9}$$

where we needed a compensating element $h(\tau, \sigma) \in H$ as done in (2.8) and g'' is a new coset element. The fundamental difference with the case of global symmetry analyzed above is the fact that invariance of the action under (2.9) is guaranteed only for some parametrizations of the infinitesimal fermionic parameter $\varepsilon \equiv \varepsilon^{(1)} + \varepsilon^{(3)} \in \tilde{\mathfrak{g}}$. Given the variations of the four components of the current, one can prove that the infinitesimal variation of the Lagrangian amounts to

$$\delta_\varepsilon \mathcal{L} = \delta_\varepsilon(\sqrt{-g}g^{\alpha\beta})\text{str}\left(A^{(2)}_\alpha A^{(2)}_\beta\right) - 4\text{str}\left(P^{\alpha\beta}_+\left[A^{(1)}_\beta, A^{(2)}_\alpha\right]\varepsilon^{(1)} + P^{\alpha\beta}_-\left[A^{(3)}_\beta, A^{(2)}_\alpha\right]\varepsilon^{(3)}\right), \tag{2.10}$$

where the two projector operators are

$$P^{\alpha\beta}_\pm = \frac{\sqrt{-g}g^{\alpha\beta} \pm \kappa\, \epsilon^{\alpha\beta}}{2}\,. \tag{2.11}$$

Finding those ε that make the variation vanish needs an ansatz appropriate to the particular supercoset under investigation. Here we just want to point out that a necessary requirement for the invariance under κ-symmetry of the supercoset sigma-model for $AdS_5 \times S^5$ is that the projectors (2.11) are orthogonal to each other $g_{\beta\gamma}P^{\alpha\beta}_+ P^{\gamma\delta}_- = 0$, which is true only if the real parameter in the WZ term is either $\kappa = 1$ or $\kappa = -1$.

Another crucial question for the consistency of the model is to understand the number of independent parameters of ε, which equals the numbers of fermionic degrees of freedom that can be gauged away by a κ-symmetry transformation. One can show that it is always possible to reduce the 32 real degrees of freedom in the coset element by a factor of one half. Therefore the gauge-fixed coset model involves 16 real physical fermions, which indeed coincides with the number of fermionic degrees of freedom in the κ-symmetry gauge-fixed GS action in ten dimensions.

2.1.2 Classical Integrability of the Supercoset Model

Classical integrability is a bonus symmetry[2] in all supercoset sigma-models with $\mathbb{Z}_4$-grading. This property[3] is equivalent to the statement that the Euler-Lagrange equations of motion following from (2.6) can be cast into the zero-curvature condition

$$dL + L \wedge L = 0 \tag{2.12}$$

for the one-parameter family of *Lax connection* (or *Lax pair*) one-forms $L(\tau, \sigma, z)$, functions of the fields of the theory and spanned by the *spectral parameter* $z \in \mathbb{C}$.

[2]This does not obviously mean that integrability is not spoiled by quantum effects [26, 27].

[3]Most of the discussion here is tailored to Sect. 2.1.1. A primer on classical and quantum field theories with emphasis on modern AdS/CFT applications can be found in [28].

It is worth appreciating that (2.12) is a strong condition on the classical dynamics of the model because it has to be satisfied for any value of z. There is, however, a certain level of arbitrariness in constructing L, as reflected by the observation that that gauge transformations

$$L \;\to\; h^{-1}Lh - h^{-1}\,dh \tag{2.13}$$

leaves the vanishing of the "field strength" built out of L unaffected. Both the local parameter h and the Lax connection are generically square matrices taking values in the some non-abelian algebra.

The existence of a Lax connection is sufficient condition to write the integrals of motion of the action. One can construct the *monodromy matrix*

$$T(z) \equiv \mathcal{P}\exp\left(\int_0^{2\pi} d\sigma L_\sigma(\tau,\sigma,z)\right) \tag{2.14}$$

as the path-ordered exponential of L parallel-transported along a closed path encircling the spacelike direction of the worldsheet cylinder, where for definiteness we assumed that all the quantities are periodic in $\sigma \in [0, 2\pi)$. The flatness condition (2.12) guarantees that (2.14) does not depend on the timeslice at fixed τ of such loop. In other words, the monodromy matrix encapsulates the time-independent information of the model. Practically, the Taylor expansion of T in the *continuous* spectral parameter delivers the *infinite* number of local conserved charges.

The Lax pair formulation of integrability at our disposal is very convenient, but it requires some effort to be set up for the supercoset sigma models with $\mathbb{Z}_4$-grading, as pedagogically shown in [19]. One can start with an ansatz for the Lax connection in terms of the components $A^{(k)}$ of the one-form current (2.4) and constrain it by imposing the flatness condition (2.12). At the end the string equations of motion lead to the zero-curvature condition for

$$L_\alpha = A^{(0)}_\alpha + \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)A^{(2)}_\alpha - \frac{1}{2\kappa}\left(z^2 - \frac{1}{z^2}\right)\gamma_{\alpha\beta}\epsilon^{\beta\gamma}A^{(2)}_\gamma + zA^{(1)}_\alpha + \frac{1}{z}A^{(3)}_\alpha . \tag{2.15}$$

An intriguing byproduct of integrability is that the zero-curvature condition for the Lax connection (2.15) implies $\kappa = \pm 1$ in (2.6). This is exactly the same condition that is found by imposing the κ-symmetry invariance of the sigma-model.

2.1.3 The $AdS_5 \times S^5$ String Action in the AdS Light-Cone Gauge

The formulation of a supercoset non-linear sigma-model on the curved $AdS_5 \times S^5$ background space is a special case of (2.6). We recall that the bosonic part of the supercoset where the string moves

$$AdS_5 \times S^5 = \frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}\,. \tag{2.16}$$

is the quotient between the isometry group $G = SO(2,4) \times SO(6)$ and isotropy group $H = SO(1,4) \times SO(5)$ of $AdS_5 \times S^5$. The group G is enlarged to $\tilde{G} = PSU(2,2|4)$ to endow the model with fermionic degrees of freedom.

The action constructed in [7] is the unique generalization of the flat-space GS action [3] that meets the following conditions:

- the bosonic reduction of the action is the usual Polyakov action in $AdS_5 \times S^5$,
- it is invariant under global $PSU(2,2|4)$ transformations and (local) κ-symmetry,
- it reduces to the type IIB GS action in flat space (after an appropriate rescaling of fermionic variables) in the limit of infinite AdS_5 and S^5 radius,
- it has the general form of the GS action corresponding to the $AdS_5 \times S^5$ supergravity solution.

The form of the action (2.6) is convenient to make the symmetries of the model apparent, but it is not amenable to direct sigma-model computations. The explicit expansion in terms of the bosonic and fermionic degrees of freedom is generally highly non-linear and dependent on the particular embedding of the coset element g into $PSU(2,2|4)$ written in terms of a set of generators of the superalgebra $\mathfrak{psu}(2,2|4)$.

Secondly, substantial simplifications occur when κ-symmetry is gauge-fixed. This is achieved in the flat-space GS action by selecting a light-cone gauge [2, 29] for which the action becomes at most quadratic in fermions. At variance with flat space, two possible ways of imposing the light-cone gauge in $AdS_5 \times S^5$ are available, depending on the choice of a light-cone direction in the background geometry: we can pick either a null geodesic wrapping a great circle of S^5 or one lying entirely in AdS_5. The former is equivalent to expand near the pp-wave geometry [30–38] and it is related to the ferromagnetic vacuum of the spin-chain picture for local operators in $\mathcal{N} = 4$ SYM. However, it is also known to lead to a complicated non-polynomial form of the action that is not suitable for direct calculations in sigma-model perturbation theory. On the other hand, the latter option was considered in [8] and is known to produce a gauge-fixed form of the action with at most quartic powers of the physical fermions. Since our eventual aim in the following chapters will be to quantize the theory perturbatively around some particular vacua, in this section we will proceed with the form of the AdS light-cone gauge that selects a massless geodesic entirely in AdS_5.

The AdS light-cone gauge is conveniently described in the Poincaré patch of AdS_5 (see 3.5 below), where the radial AdS coordinate is $z = e^{\phi}$ and we define the light-cone directions $x^{\pm}$ running in the AdS boundary and their transversal complex coordinates $x, \bar{x}$:

$$x^\pm \equiv \frac{x^3 \pm x^0}{\sqrt{2}}\,, \qquad x = \frac{x^1 + i x^2}{\sqrt{2}}\,, \tag{2.17}$$
$$\bar{x} = \frac{x^1 - i x^2}{\sqrt{2}}\,, \qquad x^a = (x^+, x^-, x, \bar{x})\,, \quad a = 1, 2, 3, 4\,,$$

The appropriate *light-cone basis* of $\mathfrak{psu}(2, 2|4)$[4] corresponds to a set of generators that respect the splitting of its even part into $\mathfrak{so}(2, 4) \oplus \mathfrak{so}(6)$. If we identity $\mathfrak{so}(2, 4)$ with the superconformal algebra in four dimensions, then this algebra comprises the momenta P^a, the angular momenta J^{ab}, the conformal boosts K^μ and the dilatation generator D, with the indices $a, b = +, -, x, \bar{x}$ being in the light-cone frame notation of (2.17). In addition to this, we need to include the $\mathfrak{so}(6) \sim \mathfrak{su}(4)$ rotations $J^i{}_j$ ($i, j = 1, \ldots 4$). The odd part of the superalgebra is spanned by 32 generators spinors $\{Q^{\pm i}, Q^\pm_i, S^{\pm i}, S^\pm_i\}$ labeled by upper (lower) index $i = 1, \ldots 4$ transforming in the (anti-)fundamental of the $SU(4)$ R-symmetry. The super-Poincaré Q's and superconformal S's account for the 32 Killing spinors preserved by the (maximally supersymmetric) supergravity background of $AdS_5 \times S^5$.

A natural choice for the coset representative is given by[5]

$$g = e^{x^a P_a + \theta\cdot Q}\, e^{\eta\cdot S}\, e^{y^i{}_j J^j{}_i}\, e^{\phi D}\,, \tag{2.18}$$

in terms of linear combinations $y^i{}_j = \frac{i}{2} y^{A'} (\gamma^{A'})^i{}_j$ of the S^5 coordinates $y^{A'}$ involving the $SO(5)$ Dirac matrices $\gamma^{A'}$ ($A' = 1, \ldots 5$), in addition to the combinations $\theta \cdot Q \equiv \theta^{-i} Q^+_i + \theta^-_i Q^{+i} + \theta^+_i Q^{-i} + \theta^{+i} Q^-_i$ and $\eta \cdot S \equiv \eta^{-i} S^+_i + \eta^-_i S^{+i} + \eta^+_i S^{-i} + \eta^{+i} S^-_i$. The 16 θ- and 16 η-variables are the Grassmann-odd partners of the 10 bosonic coordinates $x^\pm, x, \bar{x}, \phi, y_{A^{A'}}$ and all the supercoordinates are in correspondence with a generator of the superalgebra. Raising and lowering $SU(4)$ indices in the supercharges correspond to take the operation of complex conjugation.

We impose the *κ-symmetry light-cone gauge* by setting to zero the half of the fermions that have positive charges under the generator J^{+-}, namely

$$\theta^+_i = \theta^{+i} = \eta^+_i = \eta^{+i} = 0\,. \tag{2.19}$$

The non-vanishing fermions $\theta^-_i\,, \theta^{-i}\,, \eta^-_i\,, \eta^{-i}$ (we drop the minus label from now on) are the 16 physical degrees for freedom and their number matches the fermionic content of a gauge-fixed GS action in ten dimensions. The induced coordinate parametrization of the one-form current (2.18) can be read off by projecting the $\mathfrak{psu}(2, 2|4)$-invariant current A on the light-cone basis[6]:

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[4]Commutation relation of the $\mathfrak{psu}(2, 2|4)$ superalgebra and further details of derivation of the action are given in the seminal paper [8].

[5]This form of the expression defining g goes under the name of *Killing parametrization* of the superspace spanned by the supercoodinates $(x, \phi, y, \theta, \eta)$. The action can be equivalently put in the *Wess-Zumino parametrization* discussed in (2.33) below.

[6]We changed the sign of the current compared to (2.4).

$$A = g^{-1}dg \equiv A_{_P}^{a}P_a + A_{_K}^{a}K_a + A_{_D}D + \frac{1}{2}A^{ab}J_{ab} + A^i{}_j J^j{}_i + A_{_Q}^{-i}Q_i^+ + A_{_Q\,i}^{-}Q^{+i}$$
$$+ A_{_Q}^{+i}Q_i^- + A_{_Q\,i}^{+}Q^{-i} + A_{_S}^{-i}S_i^+ + A_{_S\,i}^{-}S^{+i} + A_{_S}^{+i}S_i^- + A_{_S\,i}^{+}S^{-i}\,, \tag{2.20}$$

where the non-vanishing *Cartan one-forms* are

$$\begin{aligned}
&A_{_P}^{+} = e^{\phi}dx^+\,, \qquad A_{_P}^{-} = e^{\phi}(dx^- - \frac{i}{2}\tilde{\theta}^i\tilde{d\theta}_i - \frac{i}{2}\tilde{\theta}_i\tilde{d\theta}^i)\,,\\
&A_{_P}^{x} = e^{\phi}dx\,, \qquad A_{_P}^{\bar{x}} = e^{\phi}d\bar{x}\,, \qquad A_{_D} = d\phi\,,\\
&A_{_K}^{-} = e^{-\phi}\big[\frac{1}{4}(\tilde{\eta}^2)^2dx^+ + \frac{i}{2}\tilde{\eta}^i\tilde{d\eta}_i + \frac{i}{2}\tilde{\eta}_i\tilde{d\eta}^i\big]\,,\\
&A^i{}_j = (dUU^{-1})^i{}_j + i(\tilde{\eta}^i\tilde{\eta}_j - \frac{1}{4}\tilde{\eta}^2\delta^i_j)dx^+\,,\\
&A_{_Q}^{-i} = e^{\phi/2}(\tilde{d\theta}^i + i\tilde{\eta}^i dx)\,, \qquad A_{_Qi}^{-} = e^{\phi/2}(\tilde{d\theta}_i - i\tilde{\eta}_i d\bar{x})\,,\\
&A_{_Q}^{+i} = -ie^{\phi/2}\tilde{\eta}^i dx^+\,, \qquad A_{_Qi}^{+} = ie^{\phi/2}\tilde{\eta}_i dx^+\,,\\
&A_{_S}^{-i} = e^{-\phi/2}(\tilde{d\eta}^i + \frac{i}{2}\tilde{\eta}^2\tilde{\eta}^i dx^+)\,, \qquad A_{_Si}^{-} = e^{-\phi/2}(\tilde{d\eta}_i - \frac{i}{2}\tilde{\eta}^2\tilde{\eta}_i dx^+)\,.
\end{aligned} \tag{2.21}$$

We introduced the shorthand $\tilde{\eta}^2 \equiv \tilde{\eta}^i\tilde{\eta}_i$, the fermions

$$\tilde{\theta}^i \equiv U^i{}_j\theta^j\,, \qquad \tilde{\theta}_i \equiv \theta_j(U^{-1})^j{}_i\,, \tag{2.22}$$

$$\tilde{d\theta}^i \equiv U^i{}_j d\theta^j\,, \qquad \tilde{d\theta}_i \equiv d\theta_j(U^{-1})^j{}_i\,, \tag{2.23}$$

and similar ones for η, all obtained through the local unitary matrix

$$U \equiv \cos\frac{|y|}{2} + i\gamma^{A'}n^{A'}\sin\frac{|y|}{2}\,, \qquad |y| \equiv \sqrt{y^{A'}y^{A'}}\,, \quad n^{A'} \equiv \frac{y^{A'}}{|y|}\,. \tag{2.24}$$

After a field-dependent rescaling $\eta^i \to \sqrt{2}e^{\phi}\eta^i$, $\eta_i \to \sqrt{2}e^{\phi}\eta_i$ to have fermions of homogeneous conformal dimension and the sign flip $x^a \to -x^a$ to change the sign of the kinetic term, one eventually reaches a form of the Lagrangian (2.6) that reads

$$S = T\int d\tau d\sigma\,\mathcal{L}\,, \tag{2.25}$$

$$\begin{aligned}
\mathcal{L} = {}&\sqrt{-g}g^{\mu\nu}\Big[e^{2\phi}(\partial_\mu x^+\partial_\nu x^- + \partial_\mu x\partial_\nu\bar{x}) + \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}G_{\mathcal{AB}}(y)D_\mu y^{\mathcal{A}}D_\nu y^{\mathcal{B}}\Big]\\
&+ \frac{i}{2}\sqrt{-g}g^{\mu\nu}e^{2\phi}\partial_\mu x^+\Big[\theta^i\partial_\nu\theta_i + \theta_i\partial_\nu\theta^i + \eta^i\partial_\nu\eta_i + \eta_i\partial_\nu\eta^i + ie^{2\phi}\partial_\nu x^+(\eta^2)^2\Big]\\
&- \Big\{\epsilon^{\mu\nu}e^{2\phi}\partial_\mu x^+\eta^i\Big[C'_{ij}\cos|y| + i(C'\gamma^{A'})_{ij}n^{A'}\sin|y|\Big](\partial_\nu\theta^j - i\sqrt{2}e^{\phi}\eta^j\partial_\nu x) + \text{h.c.}\Big\}\,.
\end{aligned} \tag{2.26}$$

A few explanations of the objects in the Lagrangian follow in order. The S^5 metric has a factorized form in terms of the vielbien

$$G_{\mathcal{AB}} = e_{\mathcal{A}}^{\mathrm{A}'} e_{\mathcal{B}}^{\mathrm{A}'} , \qquad e_{\mathcal{A}}^{\mathrm{A}'} = \frac{\sin|y|}{|y|} (\delta_{\mathcal{A}}^{\mathrm{A}'} - n_{\mathcal{A}} n^{\mathrm{A}'}) + n_{\mathcal{A}} n^{\mathrm{A}'} , \tag{2.27}$$

and we made a distinction between curved $\mathcal{A}, \mathcal{B}, \ldots$, and flat indices $\mathrm{A}', \mathrm{B}', \ldots$, both with range from 1 to 5, while $\mu, \nu = 0, 1$ denote the worldsheet indices. C' is the constant charge conjugation matrix of the $SO(5)$ Dirac matrix algebra generated by $\gamma^{\mathrm{A}'}$. The differential $D_\mu y^{\mathcal{A}}$ is defined by

$$D_\mu y^{\mathcal{A}} = \partial_\mu y^{\mathcal{A}} - 2i\eta_i (V^{\mathcal{A}})^i{}_j \eta^j e^{2\phi} \partial_\mu x^+ , \tag{2.28}$$

where $(V^{\mathcal{A}})^i{}_j$ are the components of the $su(4)$-valued Killing vectors of S^5

$$\begin{aligned}(V^{\mathcal{A}})^i{}_j \partial_{y^{\mathcal{A}}} &= \frac{1}{4} (\gamma^{\mathrm{A}'\mathrm{B}'})^i{}_j [y^{\mathrm{A}'} \partial_{y^{\mathrm{B}'}} - y^{\mathrm{B}'} \partial_{y^{\mathrm{A}'}}] \\ &\quad + \frac{i}{2} (\gamma^{\mathrm{A}'})^i{}_j \Big[|y| \cot|y| (\delta^{\mathrm{A}'\mathcal{A}} - n^{\mathrm{A}'} n^{\mathcal{A}}) + n^{\mathrm{A}'} n^{\mathcal{A}} \Big] \partial_{y^{\mathcal{A}}} ,\end{aligned} \tag{2.29}$$

with $\delta^{\mathrm{A}'\mathcal{A}}$ being the Kronecker delta symbol and $y^{\mathcal{A}} = \delta^{\mathcal{A}}_{\mathrm{A}'} y^{\mathrm{A}'}$, $n^{\mathcal{A}} = \delta^{\mathcal{A}}_{\mathrm{A}'} n^{\mathrm{A}'}$, $n^{\mathcal{A}} = n_{\mathcal{A}}$.

In (2.26) the kinetic term is proportional to the Weyl-invariant combination $\sqrt{-g} g^{\mu\nu}$, while the Wess-Zumino term contains the antisymmetric symbol ϵ with $\epsilon^{01} = 1$. The formula shows also that in the light-cone κ-symmetry gauge-fixed action, together with the choice of supercoodinates implicit in the supercoset parametrization (2.18), no fermionic interactions with powers higher than four appear, as the action is quadratic in one half of them (θ) and quartic in the other one (η).

The gauge-fixed action can be expressed in a related form, in which the matrix (2.24) can be either incorporated in the definition of a covariant derivative for fermions (Wess-Zumino parametrization) or eliminated through a change of coordinates in S^5 that manifestly realizes the $SO(6)$ symmetry of its coordinates. We proceed to present only the latter strategy, as it will lead to the from of the action needed in Chap. 7.

The Lagrangian can be put into a manifestly $SU(4)$-invariant form by combining the $y^{\mathrm{A}'}$ and the radial coordinate ϕ into an $SO(6)$ vector z^M ($M = 1, \ldots 6$)

$$z^{\mathrm{A}'} = e^{-\phi} \sin|y|\, n^{\mathrm{A}'} , \qquad z^6 = e^{-\phi} \cos|y| , \quad z \equiv \sqrt{z^M z^M} = e^{-\phi} , \tag{2.30}$$

in terms of which the metric appears in the "4+6 parametrization"

$$ds^2_{AdS_5 \times S^5} = \frac{dx^a dx_a + dz^M dz_M}{z^2} . \tag{2.31}$$

If we start again with (2.26) and use the identifications $(\gamma^{A'})^i{}_j = i(\rho^{A'})^{il}\rho^6_{lj}$ and $C'_{ij} = \rho^6_{ij}$ with the ρ-matrices in (F.2), we arrive to the final form of the AdS light-cone gauge-fixed action

$$S = T\int d\tau d\sigma\, \mathcal{L} \tag{2.32}$$

$$\begin{aligned}\mathcal{L} = &\sqrt{-g}g^{\mu\nu}z^{-2}\Big[\partial_\mu x^+\partial_\nu x^- + \partial_\mu x\partial_\nu\bar{x} + \frac{1}{2}(\partial_\mu z^M + i\eta_i(\rho^{MN})^i{}_j z^N\eta^j z^{-2}\partial_\mu x^+) \\ &\times(\partial_\nu z^M + i\eta_i(\rho^{MP})^i{}_j z^P\eta^j z^{-2}\partial_\nu x^+)\Big] + \frac{i}{2}\sqrt{-g}g^{\mu\nu}z^{-2}\partial_\mu x^+\Big[\theta^i\partial_\nu\theta_i + \theta_i\partial_\nu\theta^i + \eta^i\partial_\nu\eta_i \\ &+ \eta_i\partial_\nu\eta^i + iz^{-2}\partial_\nu x^+(\eta^2)^2\Big] - \Big[\epsilon^{\mu\nu}z^{-3}\partial_\mu x^+\eta^i\rho^M_{ij}z^M\Big(\partial_\nu\theta^j - i\sqrt{2}z^{-1}\eta^j\partial_\nu x\Big) + \text{h.c.}\Big]\,.\end{aligned} \tag{2.33}$$

2.2 The $AdS_4 \times \mathbb{CP}^3$ String Action in the AdS Light-Cone Gauge

The $AdS_4 \times \mathbb{CP}^3$ space is another prominent example of AdS background in the context of integrable systems in the gauge/gravity duality. The background, supported by with RR four-form flux through AdS_4 and a RR two-form flux through a $\mathbb{CP}_1 \subset \mathbb{CP}_3$, arises as a ten-dimensional solution of the IIA supergravity equations. The crucial difference with $AdS_5 \times S^5$ is the absence of maximal supersymmetry, since $AdS_4 \times \mathbb{CP}^3$ preserves only 24 out of 32 supersymmetries [39]. The Green-Schwarz approach to find the superstring action suffers from an obstruction: although the formal expression of the GS action is known for any type IIA supergravity background, it is was explicitly written up to quartic terms only [40], so the complete form remains unknown for the $AdS_4 \times \mathbb{CP}^3$ space.

The supercoset approach outlined in Sect. 2.1.1 provides a pragmatic strategy that was pursued in [9, 10]. In fact, $AdS_4 \times \mathbb{CP}^3$ is an homogeneous space

$$AdS_4 \times \mathbb{CP}^3 = \frac{SO(2,3)}{SO(1,3)} \times \frac{SO(6)}{U(3)} \tag{2.34}$$

written as the quotient of $G = USp(2,3)\times SO(6)$, locally isomorphic to $SO(2,3)\times SO(6)$ and $H = SO(1,3)\times U(3)$. Together with the observation that G is the bosonic subgroup of the orthosymplectic group $\tilde{G} = OSp(4|6)$, the coset structure of the space (2.34) hinted at the formulation of a sigma-model on the supercoset $OSp(4|6)/(SO(1,3)\times U(3))$. The construction of the sigma-model parallels the case of the $AdS_5 \times S^5$ superstring in many aspects. It is again possible to define a $\mathbb{Z}_4$-grading of $\mathfrak{osp}(4|6)$ to decompose the zero-curvature current one-form (2.4) and plug its components into (2.6) after an appropriate choice of the coset representative g. The supercoset straightforwardly inherits the classical integrability—as a mean to write the superstring equations of motion in a flat Lax connection (2.12)—found in [41] for the case of $AdS_5 \times S^5$. The standard kinetic term is supplemented with a Wess-Zumino term so that it was possible to establish κ-symmetry transformations similar to the usual one in $AdS_5 \times S^5$ [7].

On the other hand, there is a problem that did not exist in the other case since here the background is not maximally supersymmetric. The superspace consists of 24 fermionic directions in correspondence with an equal number of supercharges preserved by the background space. The other 8 fermionic coordinates, required to match the total number of 32 degrees of freedom of the GS string, were argued to be gauged away by half of the 16 parameters of κ-symmetry. In [10, 42] the supercoset sigma-model was interpreted as equivalent to a *partially κ-symmetry gauge-fixed* GS action in relation to these missing 8 coordinates associated to the odd $\mathfrak{osp}(4|6)$-generators broken by the background.

For this interpretation to be correct, for a *generic* configuration where the string motion occurs in both AdS_4 and $\mathbb{CP}^3$ the supercoset enjoys a (residual) κ-symmetry of rank 8, namely capable of eliminating precisely 8 of the 24 fermionic degrees of freedom in the supercoset to yield a *totally κ-symmetry gauge-fixed* action with only 16 physical fermionic degrees of freedom.

This consideration is not true for a string embedded purely in the AdS_4 sector[7] of the background, in which case the rank of the (residual) κ-symmetry is enhanced to 12 and the supercoset action with κ-symmetry totally gauge-fixed would have only 12 fermionic degrees of freedom [10, 42]. The supercoset formulation is not equivalent to the GS action for these "singular" configurations: the implicit κ-symmetry gauge choice puts the 8 broken fermions to zero in the supercoset action, and this turns out to be incompatible in some string configurations.

The null cusp classical string studied in Chap. 6 is an example of these singular backgrounds, being embedded only in the AdS_4 part, and cannot be properly described[8] by the correct number of physical fermions within the supercoset formalism. In other words, the trouble of the semiclassically quantization of this string would be a sudden change in the number of fermionic degrees of freedom (from 12 to 8) as soon as the classical solution is perturbed in $\mathbb{CP}^3$. Since the work in of Chap. 6 will be the only instance of strings in the $AdS_4 \times \mathbb{CP}^3$ geometry, we will rather concentrate[9] on an alternative non-coset form of the $AdS_4 \times \mathbb{CP}^3$ action—developed by Uvarov in [11, 12]—that is capable of capturing the dynamics of all bosonic string configurations.

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[7] The same situation occurs when a string forms a worldsheet instanton in $\mathbb{CP}^3$ [43].

[8] Despite this obstruction, one explicit calculation suggests that the supercoset action can be still used for one-loop calculations for classical solutions that are point-like in $\mathbb{CP}^3$ [44]. This is possible because a "regularization" parameter (non-vanishing angular momentum J) is kept throughout the calculation and the observable (one-loop string energy) admits a smooth limit when the classical string becomes a singular configuration ($J \to 0$).

[9] Let us mention that an alternative κ-symmetry gauge-fixing was considered in [45], based on the the complete $AdS_4 \times \mathbb{CP}^3$ superspace with 32 fermionic directions [42] that allows to cover regions of the space that are not reachable by the supercoset sigma model of [9, 10]. A different ("superconformal") realization of the κ-symmetry gauge-fixing was presented in [46].

The starting point is the supersymmetric membrane action [47] based on the super-coset $OSp(4|8)/(SO(1,3) \times SO(7))$ in the maximally supersymmetric background $AdS_4 \times S^7$. The string action is eventually obtained performing a double dimensional reduction to $AdS_4 \times \mathbb{CP}^3$ and choosing a κ-symmetry light-cone gauge for which both light-like directions lie in AdS_4. The result is an action that is a close counterpart of the sigma-model action for type IIB superstrings in the $AdS_5 \times S^5$ background [8, 48].

In the construction of [11, 12], the space $AdS_4 \times S^7$ is seen as the bosonic coset

$$AdS_4 \times S^7 = \frac{SO(2,3)}{SO(1,3)} \times \frac{SO(8)}{SO(7)} \tag{2.35}$$

which admits the supersymmetric extension $OSp(4|8)/\,(SO(1,3) \times SO(7))$. The latter includes 32 fermionic coordinates (called θ and η) associated to the each of the (respectively super-Poincaré and superconformal) Grassmann-odd generators in the algebra $\mathfrak{osp}(4|8)$. An analogue of the AdS κ-symmetry light-cone gauge sets to zero half of the fermionic coordinates, where the "half" corresponds to those 16 fermionic generators carrying negative charge under the $SO(1,1)$ generator M^{+-} from the Lorentz group of the 3d boundary of AdS_4.

The output is an action including physical fermions up to the fourth power[10] that is able to capture the string dynamics in any submanifold of $AdS_4 \times \mathbb{CP}^3$. As for classical integrability, the standard Lax pair construction of [10, 41] does not clearly carry over to this *non-coset* model. The zero-curvature Lax pair was built for any string configuration in the full $AdS_4 \times \mathbb{CP}^3$ superspace up to quadratic order in the fermionic degrees of freedom [49].[11]

Here we will be mostly interested in presenting the final gauge-fixed string action. The starting point is the metric of $AdS_4 \times \mathbb{CP}^3$

$$ds^2_{AdS_4 \times \mathbb{CP}^3} = R^2_{\mathbb{CP}^3} \left(\frac{1}{4}\, ds^2_{AdS_4} + ds^2_{\mathbb{CP}^3} \right) , \tag{2.36}$$

where we factored out the $\mathbb{CP}^3$ radius $R_{\mathbb{CP}^3}$, which can be then set to 1 for simplicity. The factor of $(1/2)^2$ accounts for the relative size of the radius of $\mathbb{CP}^3$ being twice the one of AdS_4. We parametrize AdS_4 in Poincaré patch

$$ds^2_{AdS_4} = \frac{dw^2 + dx^+ dx^- + dx^1 dx^1}{w^2} , \qquad x^{\pm} \equiv x^2 \pm x^0 , \tag{2.37}$$

[10] This is a feature shared with the κ-symmetry light-cone gauge-fixed action in $AdS_5 \times S^5$ action (2.26).

[11] In [50] is was shown that the string is classically integrable up to quadratic order in fermions *before* fixing κ-symmetry, for the full $AdS_4 \times \mathbb{CP}^3$ and other AdS backgrounds relevant in the AdS/CFT correspondence.

where $w \equiv e^{2\varphi}$ is the radial coordinate in AdS_4 and we pick two light-cone directions $x^{\pm}$ in the three-dimensional boundary of AdS_4. For the moment we do not specify the $\mathbb{CP}^3$ coordinates z^M ($M = 1, \ldots 6$):

$$ds^2_{\mathbb{CP}^3} = g_{MN}\, dz^M dz^N \,. \tag{2.38}$$

In addition to the embedding coordinates of $AdS_4 \times \mathbb{CP}^3$, the model has 16 physical complex fermions: the 3+3 η_a and θ_a (and their 3+3 conjugates $\bar{\eta}^a$ and $\bar{\theta}^a$) transform in the fundamental (anti-fundamental) representation of $SU(3)$ ($a = 1, 2, 3$) and stem from those 24 supersymmetries of type IIA supergravity unbroken by the $AdS_4 \times \mathbb{CP}^3$ background, while the broken 8 supersymmetries bring the remaining 1+1 fermions η_4, θ_4 and their 1+1 conjugates η^4, θ^4.

The κ-symmetry light-cone gauge-fixed Lagrangian [11, 12] reads

$$\begin{aligned}
S &= \frac{T}{2} \int d\tau\, d\sigma\, \mathcal{L}\,, \\
\mathcal{L} &= g^{ij} \Big[\frac{e^{-4\varphi}}{4} \left(\partial_i x^+ \partial_j x^- + \partial_i x^1 \partial_j x^1 \right) + \partial_i \varphi \partial_j \varphi + g_{MN} \partial_i z^M \partial_j z^N \\
&\quad + e^{-4\varphi} \left(\partial_i x^+ \varpi_j + \partial_i x^+ \partial_j z^M h_M + e^{-4\varphi} B \partial_i x^+ \partial_j x^+ \right) \Big] \\
&\quad - 2\, \epsilon^{ij} e^{-4\varphi} \left(\omega_i \partial_j x^+ + e^{-2\varphi} C \partial_i x^1 \partial_j x^+ + \partial_i x^+ \partial_j z^M \ell_M \right) .
\end{aligned} \tag{2.39}$$

Some brief explanations of the objects present in the Lagrangian are in order. The string tension T was discussed around (1.9) and g_{ij} ($i, j = 0, 1$) is the usual auxiliary metric field in the Polyakov part of the action. The action exhibits highly non-linear bosonic interactions with fermionic fields in the coefficients

$$\varpi_i = i \left(\partial_i \theta_a \bar{\theta}^a - \theta_a \partial_i \bar{\theta}^a + \partial_i \theta_4 \bar{\theta}^4 - \theta_4 \partial_i \bar{\theta}^4 + \partial_i \eta_a \bar{\eta}^a - \eta_a \partial_i \bar{\eta}^a + \partial_i \eta_4 \bar{\eta}^4 - \eta_4 \partial_i \bar{\eta}^4 \right), \tag{2.40}$$

$$\omega_i = \hat{\eta}_a \hat{\partial}_i \bar{\theta}^a + \hat{\partial}_i \theta_a \hat{\bar{\eta}}^a + \frac{1}{2} \left(\partial_i \theta_4 \bar{\eta}^4 - \partial_i \eta_4 \bar{\theta}^4 + \eta_4 \partial_i \bar{\theta}^4 - \theta_4 \partial_i \bar{\eta}^4 \right) , \tag{2.41}$$

$$B = 8 \left[(\hat{\eta}_a \hat{\bar{\eta}}^a)^2 + \epsilon_{abc} \hat{\bar{\eta}}^a \hat{\bar{\eta}}^b \hat{\bar{\eta}}^c \bar{\eta}^4 + \epsilon^{abc} \hat{\eta}_a \hat{\eta}_b \hat{\eta}_c \eta_4 + 2 \eta_4 \bar{\eta}^4 \left(\hat{\eta}_a \hat{\bar{\eta}}^a - \theta_4 \bar{\theta}^4 \right) \right] , \tag{2.42}$$

$$C = 2\, \hat{\eta}_a \hat{\bar{\eta}}^a + \theta_4 \bar{\theta}^4 + \eta_4 \bar{\eta}^4 \,, \tag{2.43}$$

$$h_M = 2 \left[\Omega^a_M \epsilon_{abc} \hat{\bar{\eta}}^b \hat{\bar{\eta}}^c - \Omega_{aM} \epsilon^{abc} \hat{\eta}_b \hat{\eta}_c + 2 \left(\Omega_{aM} \hat{\bar{\eta}}^a \bar{\eta}^4 - \Omega^a_M \hat{\eta}_a \eta_4 \right) + 2 \left(\theta_4 \bar{\theta}^4 + \eta_4 \bar{\eta}^4 \right) \tilde{\Omega}_a{}^a{}_M \right] , \tag{2.44}$$

$$\ell_M = 2\, i \left[\Omega_{aM} \hat{\bar{\eta}}^a \bar{\theta}^4 + \Omega^a_M \hat{\eta}_a \theta_4 + \left(\theta_4 \bar{\eta}^4 - \eta_4 \bar{\theta}^4 \right) \tilde{\Omega}_a{}^a{}_M \right] . \tag{2.45}$$

At variance with [11, 12], we operated a field-dependent rescaling of the fermionic fields

$$\theta_a \to \sqrt{2}\, \theta_a \qquad \theta_4 \to \sqrt{2}\, e^{-\varphi} \theta_4 \qquad \eta_a \to \sqrt{2}\, e^{-2\varphi} \eta_a \qquad \eta_4 \to \sqrt{2}\, e^{-\varphi} \eta_4 \tag{2.46}$$

and similarly for their complex conjugates, inspired by the analogue one below (2.24) for $AdS_5 \times S^5$. The η_a and θ_a appear in fully contracted combinations in (2.40)–(2.45). The manifest symmetry of the action is therefore only the $SU(3)$ subgroup of the $SU(4)$ global symmetry of $\mathbb{CP}^3 = SU(4)/U(3)$ that rotates the unbroken fermions into themselves.[12]

The Ω^a_M and Ω_{aM} are the complex vielbein of $\mathbb{CP}^3$ which satisfy $ds^2_{\mathbb{CP}^3} \equiv \Omega^a_M \Omega_{aN}\, dz^M\, dz^N$, namely the components of the Cartan one-forms of $SU(4)/U(3)$, $\Omega^a = \Omega^a_M\, dz^M$ and $\Omega_a = \Omega_{aM}\, dz^M$. In the Uvarov's supercoset construction [11], $\tilde{\Omega}_a{}^a$ is related to a one-form corresponding to the reduction direction coordinate in S^7 and its explicit expression can be found below in terms of the $\mathbb{CP}^3$ coordinates. The Ω^a_M and $\tilde{\Omega}_a{}^a$ appear in [11] in a "dressed" supercoset representative for $OSp(4|6)/(SO(1,3) \times U(3)))$ where the dressing incorporates the information on the broken supersymmetries and the $U(1)$ fiber direction.

Hatted variables in the Lagrangian are related to unhatted ones through a local rotation depending on the $\mathbb{CP}^3$ coordinates

$$\hat{\eta}_a \equiv T_a{}^b\, \eta_b + T_{ab}\, \bar{\eta}^b\,, \qquad \hat{\bar{\eta}}^a \equiv T^a_{\ b}\, \bar{\eta}^b + T^{ab}\, \eta_b \tag{2.47}$$

and similarly for the θ fermions, in the same spirit of the unitary in U matrices (2.24) above.

We can now parametrize $\mathbb{CP}^3$ with with complex variables z^a (and their conjugates $\bar{z}_a$) transforming in the fundamental (anti-fundamental) of the symmetry group $SU(3)$ [51]. Then (2.38) can be expanded as

$$ds^2_{\mathbb{CP}^3} = g_{ab}\, dz^a\, dz^b + g^{ab}\, d\bar{z}_a\, d\bar{z}_b + 2\, g_a{}^b\, dz^a\, d\bar{z}_b\,, \tag{2.48}$$

with coefficients[13]

$$\begin{aligned} g_{ab} &= \frac{1}{4|z|^4}\left(|z|^2 - \sin^2|z| + \sin^4|z|\right)\bar{z}_a\,\bar{z}_b\,,\\ g^{ab} &= \frac{1}{4|z|^4}\left(|z|^2 - \sin^2|z| + \sin^4|z|\right) z^a\, z^b\,,\\ g_a{}^b &= \frac{\sin^2|z|}{2|z|^2}\,\delta^b_a + \frac{1}{4|z|^4}\left(|z|^2 - \sin^2|z| - \sin^4|z|\right)\bar{z}_a\, z^b\,, \end{aligned} \tag{2.49}$$

where $|z|^2 \equiv z^a\, \bar{z}_a$ for short. We read off the one-forms

$$\Omega^a = \Omega^a_{\ ,b}\, dz^b + \Omega^{a,b}\, d\bar{z}_b\,, \qquad \Omega_a = \Omega_{a,b}\, dz^b + \Omega_a^{\ ,b}\, d\bar{z}_b\,, \qquad \tilde{\Omega}^a_a = \tilde{\Omega}^a_{a\,,b}\, dz^b + \tilde{\Omega}^{a,b}_a\, d\bar{z}_b\,, \tag{2.50}$$

[12] The broken symmetry will remain visible, for instance, at the level of semiclassical fluctuations around the light-like cusp in Sect. 6.2.

[13] They are related to the conventional Fubini-Study metric of $\mathbb{CP}^3$, see [12].

from the vielbein of this metric

$$\Omega_a = d\bar{z}_a \tfrac{\sin|z|}{|z|} + \bar{z}_a \tfrac{\sin|z|(1-\cos|z|)}{2|z|^3}(dz^c\bar{z}_c - z^c d\bar{z}_c) + \bar{z}_a \left(\tfrac{1}{|z|} - \tfrac{\sin|z|}{|z|^2}\right) d|z|\,, \tag{2.51}$$

$$\Omega^a = dz^a \tfrac{\sin|z|}{|z|} + z^a \tfrac{\sin|z|(1-\cos|z|)}{2|z|^3}(z^c d\bar{z}_c - dz^c\bar{z}_c) + z^a \left(\tfrac{1}{|z|} - \tfrac{\sin|z|}{|z|^2}\right) d|z| \tag{2.52}$$

and

$$\tilde{\Omega}_a{}^a = i\,\frac{\sin^2|z|}{|z|^2}\left(dz^a\,\bar{z}_a - z^a\,d\bar{z}_a\right)\,. \tag{2.53}$$

The matrices (2.47) can be usefully incapsulated into a 6×6 unitary matrix $T_{\hat{a}}{}^{\hat{b}}$ [51]

$$T_{\hat{a}}{}^{\hat{b}} \equiv \begin{pmatrix} T_a{}^b & T_{ab} \\ T^{ab} & T^a{}_b \end{pmatrix} = \begin{pmatrix} \delta_a^b \cos|z| + \bar{z}_a\, z^b\, \frac{1-\cos|z|}{|z|^2} & i\,\epsilon_{acb}\, z^c\, \frac{\sin|z|}{|z|} \\ -i\,\epsilon^{acb}\,\bar{z}_c\,\frac{\sin|z|}{|z|} & \delta_b^a \cos|z| + z^a\,\bar{z}_b\,\frac{1-\cos|z|}{|z|^2} \end{pmatrix}. \tag{2.54}$$

In Sect. 6.1 we will be interested in Wick-rotating the gauge-fixed Lagrangian (2.39) to Euclidean AdS_4 and perform a diagrammatical computation at two loops. One of the main motivations behind this analysis will be also to put the action in this gauge to a stringent test at the quantum level, where the $\mathbb{CP}^3$ geometry indirectly manifests in more complicated structures (2.40)–(2.45) compared to the compact form of the $AdS_5 \times S^5$ supercoset action (2.33). Let us also remark that the action (2.39) can be rewritten in a more compact form that resembles the Wess-Zumino type parametrization of [8, 48] by the introduction of a covariant derivative for the terms quadratic in fermions.[14]

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[14]This was first illustrated in appendix A of [52] and reported with further details in [53].

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