Preface

Stochastic geometric mechanics is a new area of research in mathematics and physics, which aims at extending geometric mechanics of classical (deterministic) dynamical systems to the case of systems for which random phenomena must be taken into account.

The study of classical dynamical systems with the help of geometrical methods has a long tradition, it suffices perhaps to mention the variational principles (Euler and Lagrange, D'Alembert and Maupertuis…) which were required for their mathematical study, a combination of analytic and geometric methods. Since the work of Lie, ideas and methods coming from the theory of (Lie) groups and differential geometry have played a crucial role in the development of the general theory of dynamical systems. Geometric mechanics incorporates these developments in a unified way, systematically exploiting the intrinsic symmetries of dynamical systems. It successfully describes a large class of natural phenomena, including mechanical systems of rigid bodies and multiparticle systems, as well as continuum systems such as fluids or the continua that are objects of study in acoustic and electromagnetism.

However, several phenomena, both in the sciences of nature and society call for an extension of this description in order to cope with phenomena where random influences, e.g., due to uncertainties in the coefficients entering the equations or in external forces, play an important role. This is particularly the case when the systems are large and of high complexity. The efforts to extend geometric mechanics to cope with such random influences have led to the new research area which we precisely call stochastic geometric mechanics. The creation of this new research area was greatly stimulated by a research semester run in the first half of 2015 at the Centre Interfacultaire Bernoulli (CIB) of the Ecole Polytechnique Fédérale de Lausanne, with the title, "Geometric mechanics-variational and stochastic methods." The research semester gathered mathematicians and scientists from several different areas of mathematics (from analysis, probability, numerical analysis and statistics, to geometry, representation theory, dynamical systems theory) and also areas of physics, control theory, robotics, and life sciences, with the aim of building up in a concentrated joint effort the new research area, both from the theoretical and the applied points of view.

The present book collects contributions of lectures and mini-courses given during that semester program. The lectures were given by leading specialists of different areas of mathematics and its applications, in hopes of building bridges among the communities pertaining to these areas and then working jointly on developing the envisaged new area of research.

Let us briefly describe the contributions present in this book.

The contribution by A. Arnaudon, A.L. De Castro and D. Holm, entitled "Noise and dissipation in rigid body motion" discusses the dynamics of rotating rigid bodies under random noise perturbations. Following a recent very innovative approach initiated by D. Holm, it gives further geometrical motivation for the choice of the multiplicative noise entering the rotational dynamics, based on a stochastic geometric variational principle. A dissipation term which preserves the magnitude of angular momentum is included to guarantee dynamical equilibrium. Stationary solutions in the class of Gibbs probability distributions for the associated Fokker–Planck equation are derived and expressed in terms of the energy and damping parameters of the system. The existence of a random attractor is discussed, and some of its properties are obtained.

The contribution by F. Flandoli, entitled "An open problem in the theory of regularization by noise for nonlinear PDEs," also discusses the choice of noise for perturbations of nonlinear partial differential equation (PDEs), from the point of view of appropriateness relative to achieving particular properties. It discusses both additive noise as well as multiplicative noise of transport type. Open problems for certain fluid dynamics equations, both deterministic and stochastic, are clearly formulated and discussed. In particular, an important open problem for the vorticity dynamics of a two-dimensional inviscid (Euler) incompressible fluid flow, with a novel form of multiplicative transport noise also discussed by D. Holm is pointed out and discussed in detail, by comparing it to the related problems for which results have already been achieved (e.g., the stochastic Leray alpha-model and the stochastic inviscid dyadic model).

The contribution by G. Da Prato, entitled "Surfaces integrals in Hilbert spaces for general measures and applications," presents and discusses infinite dimensional extensions of the important concept of surface measure. The framework is that of a separable Hilbert space H and a probability measure defined on the Borel subsets of H. The surface integral of a given real-valued function ϕ on H, relative to a given real-valued Borel function g, is heuristically the integral of ϕ with respect to ν on the level set $g = r$, for any real r. The author proves the existence of the surface integral under general assumptions on ν and g, and for all ϕ in the space of continuously differentiable functions with bounded derivatives on H. The assumption on ν is essentially, a closure condition in the space of square integrable functions on H with respect to the probability measure of the gradient operator. The assumption on the Borel function g is of Malliavin's type. The second assumption is generalized to a local one in a later section of the paper. The particular case where ν is a symmetric product of Gibbsian measures (depending on two parameters) is investigated and explicit results are derived.

In addition, a conservative transition semigroup is naturally associated with ν, which in turn is associated with a stochastic diffusion process in each direction on the Hilbert space H. A sufficient condition is given for the entire process to constitute a mild solution of a stochastic differential equation on H, with invariant measure given by ν. Explicit examples of surface measures associated with such measures and a quadratic function g on H serve as a detailed illustration of the results achievable by this approach. In the last section of the paper, the results in the general setting are extended to the case where the Malliavin type condition is replaced by a local one which is illustrated by explicit examples. The paper closes by stating an open problem related to surface measures for a stochastic differential equation on the real line, thereby underlining the complexities of the problem of general surface measures.

The contribution of F. Gay-Balmaz and V. Putkaradze, entitled “On noisy extensions of non holonomic constraints,” deals with nonholomic constraints for mechanical systems which have random components. Conservation of energy and other integrals of motion are given particular attention. In particular, the cases of rolling motion and motion on the special Euclidean group *SE(3)* are discussed, as well as stochastic deformations of the Suslov problem. Relations with other types of stochastic perturbations are also discussed as open problems.

Rigid motion on a three-dimensional Euclidean space is treated from another viewpoint in the contribution by G.S. Chirikjian on “Degenerate diffusions and harmonic analysis on *SE(3)* a tutorial.” In this contribution, the class of Fokker–Planck equations associated with degenerate diffusions on the Euclidean group of three-dimensional rotations and translations *SE(3)* is discussed, by exploiting the differential geometry of the Lie group *SE(3)* and the associated harmonic analysis. This model is related to several interesting applications in statistical mechanics, the modeling of DNA proteins, the study of nonholonomic steering, kinematic state estimation, and robotics.

The theory of stochastic processes on infinite dimensional Lie algebras and Lie groups is intimately connected with the representation theory of such algebras and groups, on one hand, and with quantum gauge field theory on the other hand. Two contributions in this volume discuss such problems.

The paper by B. Janssens and K.H. Neeb entitled “Covariant central extensions of gauge Lie algebras” discusses the classification of continuous central extensions of a compactly supported gauge Lie algebra to a locally convex Lie group, which are covariant under a 1-parameter group of transformations of the base manifold. This is an important step in the classification of projective positive energy representations of gauge groups for the smooth action by homomorphisms induced by a smooth 1- parameter group of bundle automorphisms.

The study of gauge groups associated with gauge quantum fields of the Yang-Mills and Chern–Simons types enters the contribution by Th. Lévy and A. Sengupta, entitled “Four chapters on low-dimensional gauge theories.” This is a rich survey paper ranging from classical electromagnetism to classical gauge theories, stressing both their physical and geometrical aspects, especially from the viewpoint of their quantization (in two and three space-time dimensions). It treats in detail the specific example of a two-dimensional Yang–Mills theory with a compact unitary group as a structure group, as well as the Chern–Simons three-dimensional model, with its relations with low-dimensional topology. Extensive bibliographical references are presented and commented upon, so that readers can indeed use this article for further work in this exciting area of research involving geometry, stochastic analysis, and quantum mechanics. In particular, from the references, one can also see how this study can at least heuristically be related to the study of Gibbs-type equilibrium measures for stochastic diffusion dynamics, particularly stochastic quantization of quantum fields. This topic, in turn, relates to the type of evolution described by SPDEs in other contributions of this volume.

The contribution by Y. Brenier, entitled “Some variational and stochastic methods for the Euler equations for incompressible fluid dynamics and related models,” starts with an historical survey about Euler’s equation for inviscid fluids as a prototype of classical field theory and for the application of a corresponding geometrical variational method (principle of least action). It then focuses on three different but strongly connected topics for Euler homogeneous incompressible fluids moving in some three-dimensional convex bounded domain, as well as some approximate models of this motion. The author reports complete existence and uniqueness results for the pressure gradient driving the fluid between two given configurations, without any restrictions on the data. The relation with stochastic analysis consists in relating generalized solutions to probability measures on paths. Extensions of this concept to Navier–Stokes equations are also mentioned. In the second part of his paper, Y. Brenier presents a modification of the classical least action principle in order to take into account some dissipative effects, implemented in an approximate model, called the Vlasov–Monge–Ampère model. In the third part of the paper, recent results are presented showing how the dissipative least action principle can be derived from a stochastic model describing the evolution of a Brownian point cloud. Many open problems are mentioned throughout this contribution.

The contribution of G.A. Chechkin, entitled “Introduction to homogenization theory” exposes the important technique of homogenization, which is of the utmost importance in the study of second order elliptic operators, PDE-boundary value problems and associated processes. For example, homogenization is important in connection with differential operators associated to homogeneous media and the numerical solution of the corresponding elliptic and parabolic boundary value problems. In particular, the case of nonperiodic problems with rapidly alternating boundary conditions is discussed (e. g., vibrations of a membrane with a partially clamped boundary). Applications include problems of elasticity theory and material science.

The contribution by L. Bittner, H. Gottschalk, M. Gröger, N. Moch, M. Saadi and S. Schmitz is entitled “Modeling, minimizing and managing the risk of fatigue for mechanical components.” It discusses probabilistic models of the fatigue phenomenon by which components of a mechanical system that are exposed to a cyclic mechanical loading happen to crack at a certain random time. After reviewing a classical probabilistic local Weibull model for this phenomenon, they concentrate on new models mainly developed by themselves and applied particularly to cracking in gas turbine engineering. In particular, they describe a new and yet unpublished model based on Gompert's law of exponential hazard. Their contribution also discusses how probabilistic models can be used in shape design with the aim of optimizing the components' reliability. In particular, it reviews recent mathematical work on the existence of optimal shapes. Criteria for optimal reliability are presented and discussed, as well as applications to optimal service scheduling. A further aspect discussed addresses micro-models and their integration in macroscopic life time descriptions. The mathematics involved in this paper stems mainly from statistics and the theory of stochastic point processes, but also from the study of PDEs on variable geometries, optimization over infinite dimensional shape manifolds, as well as from numerical analysis and operations research. Important engineering applications are also discussed throughout the paper.

All contributions in this volume have been refereed and we thank all participants in the refereeing procedure for their generous help.

Of course, this volume represents only a fraction of the activities which occurred during the semester program. These activities mainly consisted in the fellowship achieved in relating different communities and initiating a long process of mutual enrichment, in helping one another through continuing discussions to develop the common language needed for handling the problems posed by the phenomena on the agenda of stochastic geometric mechanics. The communities involved ranged from analysis, in its various branches, to probability and statistics, and mathematical physics, with important contributions from areas varying from algebra, geometry and topology to various applied areas, such as control theory and robotics.

Besides the mini-courses in this book, weekly lectures given by the participants of the semester took place. Moreover, Bernoulli lectures were given by Yann Brenier, Martin Hairer, Jean-Michel Bismut, and Peter Constantin.

The semester had over hundred participants, in addition to mathematicians active in the Lausanne area.

Towards the end of the semester, during June 8–11, a workshop entitled "Classic and Stochastic Geometric Mechanics" took place. The speakers at the workshop were: Alexis Arnaudon, Marc Arnaudon, Daniel Beltita, Martin Bruveris, Michael Chekroun, Xin Chen, Dan Crisan, Giuseppe Da Prato, Alexei Daletskii, Shizan Fang, Benedetta Ferrario, Alexandre Grandchamp, Max-Olivier Hongler, Rémi Lassalle, Yves Le Jan, Christian Léonard, Paul Lescot, Xu-Mei Li, Carlo Martinelli, Sonia Mazzuchi, David Meier, Juan-Pablo Ortega, Nicolas Privault, Olga Rozanova, Cesare Tronci, Tomasz Tyranowski, Stefania Ugolini, Laurent Younes, and Jean-Claude Zambrini.

The workshop also hosted the 2015 Marsden Memorial Lecture given by Yan Brenier (see https://www.pims.math.ca/scientific-event/150610-pmmlyb).

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The semester workshop was followed by a special 2-day event on June 12 and 13 : “Conference on Geometric Analysis in honor of the 65th birthday of Tudor Ratiu” organized by A. Alekseev, T. Hausel and J.-P. Ortega, especially dedicated to him.

We are also grateful to the present Director of CIB, Prof. Nicholas Monod for the great hospitality and help at various stages of the semester. We are also indebted to Prof. Max Hongler (EPFL) for advise and generous support throughout the semester. We also acknowledge support in various ways by Profs. Robert Dalang, Manuel Ojanguren, Alfio Quarteroni and John Maddocks, all at EPFL. Last but not least, we would like to warmly thank the Secretarial Staff of CIB, in particular Mrs. Christiane De Paola, Mrs. Isabelle Derivaz-Rabi, Mrs. Rana Gherzeddine and Mrs. Valérie Krier, as well as members of the technical staff in particular Mr. Marc Perraudin and Mr. Julien Junod, for creating and maintaining a very pleasant working atmosphere.

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