An Open Problem in the Theory of Regularization by Noise for Nonlinear PDEs

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Abstract Stochastic 2D Euler equations with transport noise are considered; more precisely, a variant with regularization of Biot-Savart law is investigated. The parameter of regularization is chosen so that the equation is not well posed without noise. An attempt to prove uniqueness due to noise is shown but a full solution remains open and the difficulties and partial results are discussed.

Keywords 2D Euler equations · Transport noise · Uniqueness · Girsanov transform · Commutators

1 Introduction

It is well known that the addition of noise to a deterministic ODE has a regularizing effect in terms of well posedness: among several results, let us recall the celebrated work of Veretennikov (1981) where it is proved that an SDE with only bounded measurable drift and additive noise has the properties of pathwise uniqueness and strong existence, false without noise. Additive non degenerate noise is the easiest noise which allows one to reach this result.

In recent years there has been a considerable effort to improve and extend these results, both to even more singular SDEs or more refined properties like the existence of a stochastic flow, and to SPDEs, see a review in Flandoli (2011). Additive noise remains the best choice in infinite dimensions when the drift has a generic kind of irregularity, like being bounded measurable (see for instance Da Prato et al. (2013)) and it has been the main choice to attempt proving well posedness of stochastic 3D Navier–Stokes equations (see for instance Albeverio and Ferrario (2008); Da Prato and Debussche (2003); Flandoli and Romito (2008)). But its effect on specific PDEs of fluid dynamics remain relatively unclear.

If we start from a deterministic PDE without parabolic regularization, like Euler equations or the simpler linear inviscid transport equations, additive noise does not

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seem to introduce interesting new phenomena, perhaps because it breaks useful conservation laws. From the viewpoint of the question of regularization by noise, linear inviscid transport equations with irregular drift have been proved to be regularized by a Stratonovich multiplicative noise of transport type: uniqueness of weak solutions and no blow-up of regular solutions have been proved, in the noisy case, under assumptions on the drift which, in the deterministic case, allow for non-uniqueness and blow-up examples; see Flandoli (2011), Maurelli (2016). For nonlinear inviscid equations, like 2D Euler equations and 1D Vlasov–Poisson equation, in the special case of distributional solutions concentrated in point masses, again we have observed a regularization by noise due to a Stratonovich multiplicative noise of transport type, see Flandoli et al. (2011), Delarue et al. (2014). Therefore this seems to be the most promising noise for regularization purposes and it is the case investigated in the present work. Let us recall that this kind of noise has been used in the theory of turbulence and of advection of passive scalars (see for instance Falkovich et al. (2001)), and occupies a special position also in the geometric studies of fluid mechanics, see Cruzeiro et al. (2007), Holm (2015), Cruzeiro (2015).

The main aim of this note is to present an open problem in this framework. The reason why this particular problem is stated (among so many other open problems concerning fluid dynamic equations) is that there are fragments of solution, which indicate that maybe there is a chance to solve it. We shall present these partial progresses.

The problem stated here is also motivated by the positive result proved by Barbato–Bessaih–Ferrario (2014) for the so called Leray α-model; see also similar results for dyadic models, Barbato et al. (2010), Bianchi (2013).

2 Deterministic 2D Euler Equations

In the sequel, given a closed set $D \subset \mathbb{R}^d$, given $p \geq 1$ and $\alpha \in (0, 1)$, we denote by $L^p(D)$ the usual spaces of Lebesgue p-integrable functions on D, by $C_b^\alpha(D)$ the set of bounded α-Hölder continuous functions $f : D \to \mathbb{R}$, and by $C_b^{1,\alpha}(D)$ the set of differentiable functions $f : D \to \mathbb{R}$, bounded with bounded derivatives, such that the derivatives are α-Hölder continuous.

To understand the relevance of the stochastic open problem formulated below, let us outline a few classical results and open problem for the 2D Euler equations of fluid mechanics. General very useful references on this topic are Majda and Bertozzi (2002), Marchioro and Pulvirenti (1994), Lions (1996). Consider the Euler equations in dimension 2 (u is the velocity and p the pressure):

$$\begin{aligned} \frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{aligned}$$

with, say, periodic boundary conditions on a torus to simplify the exposition. Let us formulate it for the scalar vorticity field

$$\xi = \operatorname{curl} u := \partial_2 u_1 - \partial_1 u_2.$$

The vorticity, being in dimension 2, fulfills the transport equation

$$\frac{\partial \xi}{\partial t} + u \cdot \nabla \xi = 0$$
$$u = \operatorname{curl}^{-1} \xi$$

where here and below we understand that $\operatorname{curl}^{-1} \xi$ is a divergence free field.

A shortlist of well known results and open problem is:

- $\xi(0) \in L^p$ for some $p \in [2, \infty)$ implies existence of solutions $\xi \in L^\infty(0, T; L^p)$; uniqueness is open
- $\xi(0) \in L^\infty$ implies existence and uniqueness of a solution $\xi \in L^\infty(0, T; L^\infty)$.

Let us also mention that in dimension 3 only local results of well posedness for regular initial conditions are known; the theory seems to be too difficult for the exposition of a reasonable open problem having some hope to be solved.

The results recalled above are based on the a priori estimates for the vorticity. If $\xi(0) \in L^p$ for some $p \in [2, \infty)$, we easily have the a priori estimate

$$\sup_{t \in [0,T]} \|\xi(t)\|_{L^p} \leq \|\xi(0)\|_{L^p}$$

namely $\sup_{t \in [0,T]} \|u(t)\|_{W^{1,p}} \leq C$ which gives us compactness for u in suitable topologies, hence strong convergence of u in L^p which is needed to pass to the limit and prove existence of solutions.

The explanation of the result of uniqueness for $\xi(0) \in L^\infty$ is much more difficult but it is easy to understand if we assume a little more: $\xi(0) \in C_b^\varepsilon$. In this case one can build an iteration argument in the class $u \in C\left([0, T]; C_b^{1,\varepsilon}\right)$ based on the fact that the characteristics $X' = u(t, X)$ have good properties; see for instance Majda and Bertozzi (2002). The proof of uniqueness for $\xi(0) \in L^\infty$ is more tricky; beside the celebrated proof given by Yudovich, see Marchioro and Pulvirenti (1994) for a proof based on iteration and characteristics.

3 Stochastic 2D Euler Equations in Vorticity Form

Let us directly start from the equation in vorticity form. We perturb the equation by means of a multiplicative transport term in Stratonovich form:

$$d\xi + u \cdot \nabla \xi dt = \nabla \xi \circ dW \tag{1}$$
$$u = \operatorname{curl}^{-1} \xi.$$

The noise W will, in general, depend on space; the structure is described below. The Stratonovich form is the natural one as a limit of smooth-in-time noise and its precise meaning will be understood below.

Let us consider again the problem with periodic boundary conditions on a torus to simplify the exposition. The theory is at present entirely analogous to the deterministic one, namely:

- if $\xi(0) \in L^p$ for some $p \in [2, \infty)$ then there exists at least one (weak) solution u with trajectories of class $L^\infty(0, T; L^p)$; uniqueness is open
- if $\xi(0) \in L^\infty$ then pathwise uniqueness holds in the class of solutions with trajectories of class $L^\infty(0, T; L^\infty)$.

These results require assumptions on the noise, but, roughly speaking, they are very general. The uniqueness result is proved in Brzezniak et al. (2016) following the proof of Marchioro and Pulvirenti (1994); see references there for other results on stochastic Euler equations.

4 Open Problem and Partial Results

Having in mind the previous results and limitations, a natural open problem could be: *do there exist a noise* W and an exponent $p \in [2, \infty)$ such that *uniqueness holds* for Eq. (1) with initial conditions $\xi(0) \in L^p$?

The problem is still too difficult. In the sequel we modify the equation in such a way that unfortunately the equation is no more a fundamental equation of fluid dynamics - it is similar in spirit to the Leray α model. Technically speaking, we replace the range of $p \in [2, \infty)$ with a range of regularity exponents $\gamma \in [0, 1]$. The integrability index will be always $p = 2$.

Let us introduce a modified version of 2D Euler equations:

$$d\xi + v \cdot \nabla \xi dt = \nabla \xi \circ dW \tag{2}$$
$$v = (1 - \Delta)^{-\gamma/2} \operatorname{curl}^{-1} \xi.$$

Here $\gamma \geq 0$ and the Bessel operator $(1 - \Delta)^{-\gamma/2}$ is a well defined isomorphism (for instance in the case of a torus) between $W^{1,2}$ and $W^{1+\gamma,2}$, preserving divergence free fields. Let us concentrate on the uniqueness issue, for not so regular initial conditions. To identify a correct open problem, assume $\xi(0) \in L^2$. One can prove a bound on trajectories of the solution ξ in $L^\infty\left(0, T; L^2\right)$, hence on trajectories of $\operatorname{curl}^{-1} \xi$ in $L^\infty\left(0, T; W^{1,2}\right)$ and therefore, finally, on trajectories of v in $L^\infty\left(0, T; W^{1+\gamma,2}\right)$. If $\gamma > 1$, then $W^{1+\gamma,2} \subset C_b^{1,\epsilon}$ for some $\epsilon > 0$ (we are in space-dimension 2) and, at least in the deterministic case, characteristic are well defined and reasonably regular to construct a proof of uniqueness. This case is not interesting for our purposes, since uniqueness is not an open problem in the deterministic case. So we work under the restriction

$$\gamma \leq 1$$

and pose the problem: *do there exist a noise W* and an exponent $\gamma \leq 1$ such that *uniqueness holds* for Eq. (2) with initial conditions $\xi(0) \in L^2$?

In the next subsections we show that one can go very close to the solution, but the full picture is missing - we find contradictory conditions on the noise, to solve the different pieces of the story. We start, however, by a counterexample.

4.1 Too Simple Noise Cannot Help

In this section we recall a well known counter-example to a naïve hope for regularization by noise. For simplicity we go back to (1). The hope may come from the fact that the same simple noise regularizes linear inviscid equations, see Flandoli et al. (2010). But for nonlinear problems it has no effect. Consider the case when the noise W in Eq. (1) is just a 2-dimensional Brownian motion

$$W_t = \left(W_t^1, W_t^2\right)$$

independent of space. Set

$$\begin{aligned} \widetilde{u}(t,x) &:= u(t, x - W_t) \\ \widetilde{\xi}(t,x) &:= \xi(t, x - W_t). \end{aligned}$$

A formal computation by Stratonovich calculus gives us

$$\begin{aligned} \frac{\partial \widetilde{\xi}}{\partial t} + \widetilde{u} \cdot \nabla \widetilde{\xi} = 0 \\ \widetilde{u} = \mathrm{curl}^{-1}\, \widetilde{\xi} \end{aligned}$$

because

$$\frac{\partial \widetilde{\xi}}{\partial t} = \frac{\partial \xi}{\partial t} - \nabla \xi \circ \frac{dW}{dt} = -u \cdot \nabla \xi = -\widetilde{u} \cdot \nabla \widetilde{\xi}.$$

The transformation is invertible. Although we have described it only at formal level, it is clear that we cannot expect any improvement by this noise: any kind of pathology like non-uniqueness or singularities shift form one formulation to the other and makes the stochastic Euler equations equivalent to the classical deterministic ones.

The problem is that the noise is just space-independent. Due to the results of Flandoli et al. (2011) and Delarue et al. (2014) we believe that only a noise with very rich space structure can improve the theory.

4.2 The Noise

Let $\mathcal{T} = [0, 2\pi]^2$ be the 2D-torus. The space $H_{\mathbb{C}}$ of $L^2\left(\mathcal{T}; \mathbb{C}^2\right)$ vector fields, closure in $L^2\left(\mathcal{T}; \mathbb{C}^2\right)$ of smooth divergence free, zero average fields, is a Hilbert space with the scalar product $\langle f, g\rangle = \operatorname{Re} \int_{\mathcal{T}} f(x) \overline{g(x)} dx$ and the family $\left\{\frac{k^{\perp}}{|k|} e^{ik\cdot x}, k \in \mathbb{Z}^2 \setminus \{0\}\right\}$ is an orthonormal system (up to a constant). Every element v of the real subspace $H_{\mathbb{C}} \cap L^2\left(\mathcal{T}; \mathbb{R}^2\right)$ can be developed in series, $v(x) = \sum_{k\in\mathbb{Z}^2\setminus\{0\}} v_k \frac{k^{\perp}}{|k|} e^{ik\cdot x}$, with $v_{-k} = \overline{v_k}$ in order to have v real valued. Let Λ be any subset of $\mathbb{Z}^2 \setminus \{0\}$ such that $\{\Lambda, -\Lambda\}$ is a partition of $\mathbb{Z}^2 \setminus \{0\}$. Consider the vector fields $\left\{e_k(x), k \in \mathbb{Z}^2 \setminus \{0\}\right\}$ defined as

$$e_k(x) = \frac{k^{\perp}}{|k|} \frac{e^{ik\cdot x} + e^{-ik\cdot x}}{2} = \frac{k^{\perp}}{|k|} \cos k \cdot x, \qquad k \in \Lambda$$
$$e_k(x) = \frac{k^{\perp}}{|k|} \frac{e^{ik\cdot x} - e^{-ik\cdot x}}{2i} = \frac{k^{\perp}}{|k|} \sin k \cdot x, \qquad k \in \Lambda^c.$$

They are real valued, orthonormal (up to a constant), zero average, divergence free; moreover they are a complete system for $L^2\left(\mathcal{T}; \mathbb{R}^2\right)$:

$$v(x) = \sum_{k\in\mathbb{Z}^2\setminus\{0\}} v_k \frac{k^{\perp}}{|k|} e^{ik\cdot x} = 2 \sum_{k\in\Lambda} \operatorname{Re}(v_k) \frac{k^{\perp}}{|k|} \cos k \cdot x - 2 \sum_{k\in\Lambda_c} \operatorname{Im}\left(v_{-k}\right) \frac{k^{\perp}}{|k|} \sin k \cdot x.$$

Hence $\left\{e_k(x), k \in \mathbb{Z}^2 \setminus \{0\}\right\}$ is a complete orthonormal system of H, the Hilbert space obtained as closure in $L^2\left(\mathcal{T}; \mathbb{R}^2\right)$ of smooth divergence free, zero average vector fields.

We take the $\mathbb{R}^2$-valued random field

$$W(t, x) = \sum_{k\in\mathbb{Z}^2\setminus\{0\}} \sigma_k e_k(x) W_t^k$$

where

$$\sum_{k\in\mathbb{Z}^2\setminus\{0\}} \sigma_k^2 < \infty, \tag{3}$$

and $\left\{W_t^k\right\}_{k\in\Lambda}$ is a family of independent Brownian motions. Since

$$\sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \sigma_k^2 \left| e_k (x) \right|^2 < \infty \text{ for every } x \in \mathcal{T}$$

the series defining the random field W converges in mean square; we may introduce the matrix-valued function $Q(x, y)$ defined componentwise as

$$Q^{\alpha\beta} (x, y) = \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \sigma_k^2 e_k^{\alpha} (x) \, e_k^{\beta} (y)$$

and check it has the form $Q(x - y)$ (this fact is equivalent to ask that the random field $W(t, x)$ has law invariant by space-translation). Multiplying the σ_k's by a constant, we may have

$$Q(0) = Id.$$

In the sequel, to clarify some aspects of the exposition, we assume

$$\sigma_k = |k|^{-\alpha}$$

for some

$$\alpha > 1$$

(needed to have condition (3)).

Remark 1 The restriction $\alpha > 1$, or more precisely the condition (3), seems to be essential everywhere to give a meaning to our objects. However, since we are looking for a regularizing noise, we cannot exclude the possibility that better regularizing properties would be true for a more singular noise, namely when condition (3) is not true. We do not have any precise argument in support of this idea, only the fact that for SPDEs with additive noise, the case of cylindrical, or space-time, noise is the one where the better regularization properties occur, see for instance Da Prato et al. (2013). Here however, in the case of Stratonovich multiplicative noise of transport type, opposite to the additive noise case of Da Prato et al. (2013), there is even a problem of interpretation of the stochastic term; it has a classical meaning only under condition (3). The recent frequent use of renormalization ideas to define rigorously apparently meaningless quantities in the realm of SPDEs arise the question whether there is a suitable - possibly renormalization - procedure which allows to define our equations also when condition (3) is replaced by some weaker one.

4.3 Itô-Stratonovich correction and definition of solution

Consider the formal Eq. (1). Let us clarify the meaning of the Stratonovich term by reformulating it in Itô form. Formally

$$\nabla \xi \circ \frac{\partial W}{\partial t} = \nabla \xi \frac{\partial W}{\partial t} + \frac{1}{2} \Delta \xi. \tag{4}$$

Indeed, for $j = 1, 2$ we have

$$\sum_k \sigma_k e_k \cdot \nabla \xi \circ dW^k = \sum_k \sigma_k e_k \cdot \nabla \xi dW^k + \frac{1}{2} \sum_k \sum_j \sigma_k e_k^j (x) \, d \left[W^k, \partial_j \xi \right]_t$$

$$\frac{\partial \left(\partial_j \xi \right)}{\partial t} = \sum_{k'} \sigma_{k'} \partial_j \left(e_{k'} \cdot \nabla \xi \right) \circ \frac{dW^{k'}}{dt} + BV\text{-terms } = 0$$

$$d \left[W^k, \partial_j \xi \right]_t = \sigma_k \partial_j \left(e_k \cdot \nabla \xi \right) dt$$

$$\begin{aligned} \sum_k \sum_j \sigma_k e_k^j d \left[W^k, \partial_j \xi \right]_t &= \sum_k \sum_j \sigma_k e_k^j \sigma_k \partial_j \left(e_k \cdot \nabla \xi \right) dt \\ &= \sum_k \sum_{i,j} \sigma_k^2 e_k^j \left(\partial_j e_k^i \partial_i \xi + e_k^i \partial_i \partial_j \xi \right) dt \end{aligned}$$

We have

$$\sum_k \sigma_k^2 e_k^i (x) \, e_k^j (x) = Q^{ij} (0) = \delta_{ij}$$

hence

$$\sum_k \sum_{i,j} \sigma_k^2 e_k^j e_k^i \partial_i \partial_j \xi = \Delta \xi.$$

As to the other term, it is zero, because, for each $i = 1, 2$,

$$\sum_k \sum_j \sigma_k^2 e_k^j (x) \, \partial_j e_k^i (x) = \sum_j \partial_j \left(\sum_k \sigma_k^2 e_k^i (x) \, e_k^j (x) \right) = \sum_j \partial_j Q^{ij} (0) = 0.$$

This computation, yielding (4), is not new; see for instance Coghi and Flandoli (2016) for more details.

Thus a more plain formulation of Eq. (1) is

$$\frac{\partial \xi}{\partial t} + v \cdot \nabla \xi = \nabla \xi \frac{\partial W}{\partial t} + \frac{1}{2} \Delta \xi$$
$$v = (1 - \Delta)^{-\gamma/2} \operatorname{curl}^{-1} \xi$$

Let $\mathcal{S}_2(\xi_0)$ be the class of adapted processes ξ such that

$$\sup_{t \in [0,T]} \|\xi(t)\|_{L^2} \leq \|\xi_0\|_{L^2}$$

with probability one. Motivated by the previous computations, we give the following definition of solution.

Definition 2 We call weak solution of class $\mathcal{S}_2(\xi_0)$ of Eq. (1) a filtered probability space (Ω, F, F_t, P), a noise $W(t, x)$ on this probability space satisfying the assumptions of Sect. 4.2, an adapted process $(\xi(t))_{t \in [0,T]}$ of class $\mathcal{S}_2(\xi_0)$ such that

$$\langle \xi_t, \phi \rangle - \langle \xi_0, \phi \rangle - \int_0^t \langle \xi_s, v \cdot \nabla \phi \rangle \, ds = - \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \sigma_k \int_0^t \langle \xi_s, e_k \cdot \nabla \phi \rangle \, dW_s^k + \frac{1}{2} \int_0^t \langle \xi_s, \Delta \phi \rangle \, ds$$

for all $\phi \in C^\infty$, where $v = (1 - \Delta)^{-\gamma/2} \operatorname{curl}^{-1} \xi$.

4.4 Reduction to a Linear Equation by Girsanov

The idea used here is due, in the opinion of the author, to Paul Malliavin, although a precise reference cannot be given; the author became aware of it from Malliavin at the time when the paper Cruzeiro et al. (2007) was written.

Girsanov theorem gives us the following result which looks very close to the solution of the open problem.

Lemma 3 *Assume that the pair (α, γ) satisfies*

$$1 < \alpha \leq 1 + \gamma.$$

Then, in the class $\mathcal{S}_2(\xi_0)$, equation is equivalent in law to the linear SPDE

$$d\xi = \nabla \xi \circ d\widetilde{W} = \frac{1}{2} \Delta \xi dt + \nabla \xi d\widetilde{W} \tag{5}$$

where $\widetilde{W}$ is a new random field with the properties listed in Sect. 4.2. Moreover, a weak solution exists for both equations.

We do not give all the details of the proof but only the idea. Assume v is a solution. Let $\widetilde{W}^k$ be the processes defined as

$$\widetilde{W}_t^k := W_t^k + \int_0^t \left(\frac{1}{\sigma_k} \int v(s,x) e_k(x) \, dx \right) ds$$

so that we have

$$v(t,x) + \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \sigma_k e_k(x) \frac{dW^k}{dt} = \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \sigma_k e_k(x) \frac{d\widetilde{W}^k}{dt}.$$

Novikov condition for Girsanov would be satisfied if the random variable

$$\int_0^T \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{\sigma_k^2} \left| \int v(s,x) e_k(x) \, dx \right|^2 ds \tag{6}$$

is exponentially integrable, multiplied by a suitable constant. From the assumption $\sigma_k = |k|^{-\alpha}$ we have

$$\sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{\sigma_k^2} \left| \int v(s,x) e_k(x) \, dx \right|^2 \sim \sum_k |k|^{2\alpha} |\widehat{v}(s,k)|^2 \sim \|v(s,\cdot)\|_{W^{\alpha,2}}^2$$

where $\sim$ means that the expressions control each other by a constant and $\widehat{v}(s,k)$ denoted Fourier transform of $v(s,x)$ in x. For $\xi \in \mathcal{S}_2(\xi_0)$ we have

$$\sup_{t \in [0,T]} \|v(t)\|_{W^{1+\gamma,2}} \leq C \|\xi_0\|_{L^2}.$$

Hence we have

$$\sup_{t \in [0,T]} \|v(t)\|_{W^{\alpha,2}} \leq C \|\xi_0\|_{L^p}$$

under the assumption $\alpha \leq 1 + \gamma$ and therefore the random variable in (6) is not only exponentially integrable, it is even bounded above with probability one. Hence we may apply Girsanov. The proof of the equivalence claim of the lemma is based on this argument.

Concerning the existence claim, thanks to the property $\operatorname{div} e_k(x) = 0$ and the a priori estimate $\sup_{t \in [0,T]} \|\xi(t)\|_{L^2} \leq \|\xi_0\|_{L^2}$ which holds true for the linear equation, we can prove existence for the linear equation in $\mathcal{S}_2(\xi_0)$ (weak convergence is sufficient to pass to the limit) and thus also for the nonlinear one. Notice that, for the linear equation, also strong existence is true.

4.5 Discussion

Until now we have rigorously formulated the stochastic Euler equation and we have "transformed" it into a linear transport equation, when the pair (α, γ) - which characterize the noise and the regularization - satisfies

$$1 < \alpha \leq 1 + \gamma.$$

Moreover, we impose

$$\gamma \leq 1$$

otherwise the deterministic problem is already well posed. This implies that γ, the regularization parameter of the modified Euler equation (2), must satisfy

$$0 < \gamma \leq 1$$

(in particular it cannot be equal to zero; the true Euler equation is ruled out) and α, the degree of regularity of the noise, must satisfy

$$1 < \alpha \leq 2.$$

It remains to prove a uniqueness result for the linear transport equation, for some $1 < \alpha \leq 2$ (the parameter γ does not enter the linear equation). If we discover $\alpha \in (1, 2]$ such that uniqueness holds for the linear SPDE, then we may choose $\gamma \leq 1$ such that $\alpha \leq 1 + \gamma$, and the pair (α, γ) so determined satisfies all conditions.

Let us give two uniqueness results for the linear SPDE which however, as we shall see, are not sufficient to complete the proof.

4.6 Uniqueness by Commutators

Assume we can prove that, given two solutions $\xi_t^{(1)}, \xi_t^{(2)}$ of the linear equation (5) with initial conditions $\xi_0^{(1)}, \xi_0^{(2)}$, the difference $\xi^{(1)} - \xi^{(2)}$ satisfies a bound like

$$\sup_{t \in [0,T]} \left\| \xi_t^{(1)} - \xi_t^{(2)} \right\|_{L^2} \leq \left\| \xi_0^{(1)} - \xi_0^{(2)} \right\|_{L^2}. \tag{7}$$

Then pathwise uniqueness would hold. Unfortunately, we may prove this fact under a restriction on α which spoils the final result (see the last sentences of the previous section):

Lemma 4 *If*

$$\alpha > 2$$

then (7) is true.

In order to prove (7) we must be able to compute $d \int \xi_t^2 dx$ when ξ_t satisfies in the weak sense Eq. (5) and $\sup_{t\in[0,T]} \|\xi_t\|_{L^p} \leq C$ with probability one for some constant $C > 0$, (then we apply this to $\xi_t = \xi_t^{(1)} - \xi_t^{(2)}$). The problem is that we only know an identity for $\langle \xi_t, \phi \rangle$ over test functions ϕ:

$$\langle \xi_t, \phi \rangle - \langle \xi_0, \phi \rangle - \sum_k \sigma_k \int_0^t \langle \xi_s, e_k \cdot \nabla \phi \rangle \, dW_s^k = \frac{1}{2} \int_0^t \langle \xi_s, \Delta \phi \rangle \, ds.$$

One possibility is to compute

$$d \langle \xi_t, e_k \rangle^2$$

and then take the sum in k, but the rigorous control of this computation is tricky. Another possibility is to mollify and make the computations. The difficulty is similar to the one for deterministic transport equations of the form $\frac{\partial \xi}{\partial t} + b \cdot \nabla \xi = 0$, where it is not sufficient to assume $\operatorname{div} b = 0$ without any regularity, to have uniqueness (recall the theory of DiPerna and Lions (1989); Ambrosio (2004)).

Let $\xi \in \mathcal{S}_p(\xi_0)$ be a weak solution of

$$d\xi + \sum_k \sigma_k e_k \cdot \nabla \xi dW^k = \frac{1}{2} \Delta \xi dt.$$

We have, for $\xi_\epsilon = \theta_\epsilon * \xi$,

$$d\xi_\epsilon + \sum_k \sigma_k e_k \cdot \nabla \xi_\epsilon dW^k = \frac{1}{2} \Delta \xi_\epsilon dt + \sum_k \sigma_k R_\epsilon^k dW^k$$

where

$$R_\epsilon^k = e_k \cdot \nabla \xi_\epsilon - \theta_\epsilon * (e_k \cdot \nabla \xi).$$

For each single k, we have $R_\epsilon^k \to 0$ in quite a strong way. The question is about the series. More precisely, assume we perform the computation

$$\begin{aligned}
d\xi_\epsilon^2 &= 2\xi_\epsilon d\xi_\epsilon + d\left[\xi_\epsilon\right] \\
&= -\sum_k \sigma_k e_k \cdot \nabla \xi_\epsilon^2 dW^k \\
&+ 2\xi_\epsilon \nu \Delta \xi_\epsilon dt \\
&+ 2\sum_k \sigma_k \xi_\epsilon R_\epsilon^k dW^k \\
&+ d\left[\xi_\epsilon\right].
\end{aligned}$$

Integrating, the first term disappears and the second is negative. It remains to understand

$$\sum_k \sigma_k \int_0^t \left(\int \xi_\epsilon R_\epsilon^k dx \right) dW_s^k$$

and $\int_0^t \left[\xi_\epsilon\right]_s ds$. Let us discuss the first one. Unfortunately we have

$$\int \xi_\epsilon R_\epsilon^k dx \sim |k|$$

as one can realize from the computations

$$\begin{aligned}
R_\epsilon^k(x) &= e_k(x) \cdot \nabla \int \theta_\epsilon(x-y)\,\xi(y)\,dy - \int \theta_\epsilon(x-y)\,e_k(y) \cdot \nabla \xi(y)\,dy \\
&= \int (\nabla \theta_\epsilon)(x-y)\,(e_k(x) - e_k(y))\,\xi(y)\,dy - \int \theta_\epsilon(x-y)\,\mathrm{div}\, e_k(y)\,\xi(y)\,dy \\
&= \int |x-y|\,(\nabla \theta_\epsilon)(x-y)\,\frac{e_k(x) - e_k(y)}{|x-y|}\xi(y)\,dy
\end{aligned}$$

and the fact that $|x-y|\,(\nabla\theta_\epsilon)(x-y)$ is of order one and $\frac{e_k(x)-e_k(y)}{|x-y|}$ is of order $|k|$. Therefore we need

$$\sum_k \sigma_k^2 |k|^2 < \infty$$

namely $\alpha > 2$.

4.7 Wiener Uniqueness

The problem of uniqueness for the linear equation

$$\frac{\partial \xi}{\partial t} + \sum_k \sigma_k e_k \cdot \nabla \xi \circ \frac{dW^k}{dt} = 0$$

is not new: see Le Jan–Raimond (2002). Using Wiener-chaos decomposition, they prove uniqueness in the class of solutions adapted to the Brownian motions. This proof has been adapted by Maurelli (2011) to stochastic transport equations with non regular drift and a variant, for such a case, has been developed by Fedrizzi–Neves–Olivera (2017). We follow here, for the case of irregular diffusion coefficients, the idea of proof of Fedrizzi et al. (2017); see also Flandoli and Olivera (2017).

The limitation of these approaches is that it gives us only "Wiener uniqueness".

Lemma 5 *If*

$$\alpha > 1$$

and $\xi_t^{(1)}, \xi_t^{(2)}$ *are two solutions of Eq. (5) corresponding to the same Brownian motions* $\left(W^k\right)$ *and adapted to them, then* $\xi_t^{(1)} = \xi_t^{(2)}$.

Proof For every $n \in \mathbb{N}$ and $h \in L^2\left(0, T; \mathbb{R}^n\right)$ consider the stochastic exponential

$$e_f(t) = \exp\left(\int_0^t h_s \cdot dW_s^{(n)} - \frac{1}{2}\int_0^t |h_s|^2\, ds\right)$$

where $W^{(n)} = \left(W^1, ..., W^n\right)$. Recall that

$$de_f(t) = e_f(t)\, h_t \cdot dW_t^{(n)}.$$

If ξ is a weak solution of the equation

$$d\xi + \sum_k \sigma_k e_k \cdot \nabla\xi dW^k = \frac{1}{2}\Delta\xi dt$$

with $\xi_0 = 0$ then, over test functions that we omit for simplicity,

$$\begin{aligned} d\left(e_f(t)\,\xi_t\right) &= \left(e_f(t)\,\xi_t\right) h_t \cdot dW_t^{(n)} + e_f(t)\left(\frac{1}{2}\Delta\xi dt - \sum_k \sigma_k e_k \cdot \nabla\xi dW^k\right) \\ &\quad + e_f(t)\, h_t \cdot \left(\sigma_k e_k \cdot \nabla\xi\right)_{k=1,\ldots,n} dt \end{aligned}$$

hence $E\left[e_f(t)\,\xi_t\right]$ is a weak solution of the deterministic parabolic equation

$$\frac{\partial}{\partial t}E\left[e_f(t)\,\xi_t\right] = \frac{1}{2}\Delta E\left[e_f(t)\,\xi_t\right] + B_k \cdot \nabla E\left[e_f(t)\,\xi_t\right]$$

for a suitable new regular drift B_k. The advantage of this approach is that this equation is truly parabolic. With proper arguments one can show that $E\left[e_f(t)\,\xi_t\right] = 0$, being $E\left[e_f(0)\,\xi_0\right] = 0$. ∎

Unfortunately, the result of this lemma is not applicable to Euler equation (1). Indeed, assume $\xi_t^{(1)}, \xi_t^{(2)}$ are two solutions of Eq. (1). They are also solutions of Eq. (5) with respect to new stochastic processes $W^{(1)}(t,x)$, $W^{(2)}(t,x)$. A part from the fact that they are different - while the lemma requires they are equal, but maybe this detail can be overcome by the concept of solution in law - the main limitation is that $\xi_t^{(i)}$ is not adapted to $W^{(i)}(t,x)$. The combination of Girsanov and Wiener uniqueness does not work.

4.8 The SDE with that Noise

Let us consider the characteristics associated to Eq. (1) or (5):

$$dX_t = \sum_k \sigma_k e_k(X_t) \circ dW_t^k.$$

The aim of this section is simply to interpret the conditions $\alpha > 2$ and $1 < \alpha < 2$, found above, in terms of solvability of this equation. This may be relevant for other potential approaches to the SPDEs, not discussed further here.

As in Coghi and Flandoli (2016), one can show that under our assumptions they are equivalent to

$$dX_t = \sum_k \sigma_k e_k(X_t)\, dW_t^k. \tag{8}$$

If $\alpha > 2$, this equation is well posed in the classical sense. An easy prototype of computation to see this is the following one. Assume X_t^x, X_t^y are solution associated to the initial conditions x, y. Then

$$X_t^x - X_t^y = x - y + \sum_k \sigma_k \int_0^t \left(e_k\left(X_s^x\right) - e_k\left(X_s^y\right)\right) dW_s^k$$

$$\begin{aligned} E\left[\left|X_t^x - X_t^y\right|^2\right] &\le 2|x-y|^2 + \sum_k \sigma_k^2 \int_0^t E\left[\left|e_k\left(X_s^x\right) - e_k\left(X_s^y\right)\right|^2\right] ds \\ &\le 2|x-y|^2 + \sum_k \sigma_k^2 |k|^2 \int_0^t E\left[\left|X_s^x - X_s^y\right|^2\right] ds \end{aligned}$$

so we may apply Gronwall lemma if $\sum_k \sigma_k^2 |k|^2 < \infty$, namely when $\alpha > 2$. Pathwise uniqueness and existence of stochastic flows can be proved by this or similar computations.

What happens for $\alpha \le 2$ is less clear. Due to certain similarities of Eq. (5) with the problem studied by Le Jan–Raimond (2002), it is perhaps possible to establish some properties for $\alpha \leq 2$ along the lines of that work. This is however a quite delicate issue, not treated here.

4.9 Conclusions

The previous computations show that a suitable regularization of the 2-dimensional Euler equations is a good example where one could investigate the efficacy of different ideas. It looks at the boundary of what can be done.

Let us summarize some of the ideas discovered above.

The case $\alpha > 2$ could be called "regular noise", giving well posedness of characteristics. Girsanov approach requires $\alpha \leq 1 + \gamma$; if $\alpha > 2$ then we must have $\gamma > 1$, but we argued that for $\gamma > 1$ the deterministic regularized Euler equations have already a unique solution. Maybe other approaches, different from Girsanov, for instance based on characteristics, have a chance to work without the condition $\alpha \leq 1 + \gamma$ and thus for some $\gamma \leq 1$. Recall that strong uniqueness for the linear SPDE holds in this case (independently of γ).

The case $1 < \alpha \leq 2$ is very intriguing because it allows us to choose a $\gamma \leq 1$ such that $\alpha \leq 1 + \gamma$ and thus Girsanov approach works. Wiener uniqueness for the linear SPDE also work when $1 < \alpha \leq 2$, but it is not sufficient, because Girsanov requires to be paired to a uniqueness statement for solutions not adapted to the noise. At the Lagrangian level, notice that it is a very delicate case, $1 < \alpha \leq 2$, because diffusion without hitting holds.

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