Preface

It is ingrained in mathematical sciences that any real advance goes hand in hand with the invention of sharper tools and simpler methods which also assist in understanding earlier theories and cast aside earlier more complicated developments.

David Hilbert

In science everything should be made as simple as possible, but not simpler.

Albert Einstein

Metric regularity has emerged during last 2–3 decades as one of the central concepts of *variational analysis*. The roots of this concept go back to a circle of fundamental regularity ideas from classical analysis embodied in such results as the implicit function theorem, the Banach open mapping theorem, and theorems of Lyusternik and Graves, on the one hand, and Sard's theorem and transversality theory, on the other hand.

Smoothness is the key property of the objects to which the classical results are applied. Variational analysis, on the other hand, appeals to objects that may lack this property: functions and maps that are nondifferentiable at points of interest, set-valued mappings, etc. Such phenomena naturally appear in optimization theory and elsewhere.[1]

In traditional nonlinear analysis, regularity of a continuously differentiable mapping (e.g., from a normed space or a manifold to another space or manifold) at a certain point means that its derivative at the point is an operator onto. This property, translated through available analytic or topological means to corresponding local

[1]Grothendieck mentions the "ubiquity of stratified structures in practically all domains of geometry" in his 1984 *Esquisse d'un Programme*, see [140].

properties of the mapping, plays a crucial role in the study of some basic problems of analysis such as the existence and behavior of solutions of a nonlinear equation $F(x) = y$. The most fundamental consequence of regularity of F at some $\bar{x}$ is that the equation has a solution for any y in a neighborhood of $\bar{y} = F(\bar{x})$ and moreover, the distance from the solution to $\bar{x}$ is controlled by $\|y - \bar{y}\|$.

Similar problems appear if, instead of an equation, we consider an inclusion

$$y \in F(x) \tag{1}$$

(with F a set-valued mapping this time) which, in essence, is the main object of study in variational analysis. The challenge here is evident: There is no clear way to approximate the mapping by simple objects, like linear operators in the classical case.

The key step in the answer to this challenge was connected with the understanding of the metric nature of some key phenomena that appear in the classical theory. This eventually led to the choice of the class of metric spaces as the main playground and subsequently to abandoning approximation as the primary tool of analysis in favor of a direct study of the phenomena as such. The "metric theory" offers a rich collection of results that, being fairly general and stated in a purely metric language, are easily adaptable to Banach and finite-dimensional settings (still the most important in applications) and to various classes of mappings with special structure.

Moreover, however surprising this may sound, the techniques coming from the metric theory may appear in certain circumstances more efficient, flexible, and easy to use and at the same time able to produce more precise results than the available Banach techniques (e.g., connected with generalized differentiation), especially in infinite-dimensional Banach spaces. Furthermore, it should be added that the central role played by distance estimates has determined a quantitative character of the theory (contrary to the predominantly qualitative character of the classical theory). Altogether, this opens the gates to a number of new applications, such as, say, metric fixed point theory, differential inclusions, all chapters of optimization theory, and numerical methods.

Our goal is to give a systematic account of the theory of metric regularity. The three principal themes that will be at the focus of our attention are as follows: regularity criteria (containing quantitative estimates for rates of regularity), the effect of perturbations of a mapping on its regularity properties, and the role of metric regularity in analysis and optimization. The structure of this book corresponds to the logical structure of the theory. We start with a thorough study of metric theory that lays a solid foundation for the subsequent study of metric regularity of mappings, first between Banach and then between finite-dimensional spaces. In the last two cases, special attention is paid to mappings with special structures (e.g., mappings with convex graphs, single-valued Lipschitz mappings, polyhedral and semi-algebraic mappings). We also consider a number of applications of the theory to concrete problems of analysis and optimization, including those mentioned in the previous paragraph.

But we begin, in Chap. 1, with a brief survey of the classical theory, providing complete proofs of most of the results. Hopefully, this will help to make the threads connecting the classical and modern theories more visible as far as both the basic ideas and the specific techniques are concerned.

The proper study of the theory of metric regularity starts in Chap. 2. It is concentrated on a direct analysis of the phenomena exhibited by the three equivalent regularity properties: openness at a linear rate, metric regularity proper, and the pseudo-Lipschitz property of the inverse mapping. The main results of the chapter are the regularity criteria and perturbation theorems describing the effect of Lipschitz perturbations of the mapping on the rates of regularity. Both will be systematically used in the sequel. Note that along with the typical local regularity "near a point of the graph" that dominates the research and publications, we thoroughly consider nonlocal metric regularity "on a fixed set," which so far has attracted less attention. Meanwhile, it leads to important applications, especially connected with various existence problems.

The chapter also contains a section in which we introduce and study weaker regularity concepts such as subregularity, calmness, and controllability. These properties may not be stable under small perturbations of the mapping, and hence can hardly be used for practical computations, but nonetheless prove to be extremely useful in some problems of analysis, e.g., in subdifferential calculus and the theory of necessary optimality conditions.

In Chap. 3, we continue to study metric theory, this time its infinitesimal aspects, with the concept of slope at the center. The main results are infinitesimal analogues of the corresponding general results of Chap. 2, actually consequences of the latter. But they are equal in strength only under some restrictions on the class of possible range spaces. The restrictions are not particularly strong. Length spaces, for instance (that is, spaces in which the distance between points is defined by the length of curves joining the points), would work. We also consider in some detail the so-called nonlinear regularity models in which the basic estimates involve certain functions of distances that appear in the definitions of basic regularity properties, rather than distances themselves. The chapter concludes with a study of global regularity which in certain respects is closer to the local theory than to the regularity theory on fixed sets.

Chapter 4 is rather a service chapter providing a bridge between the metric and the Banach space theories. It contains necessary information about tangential set-valued approximations as well as the theory of subdifferentials, mainly relating to the five main types of subdifferentials: Fréchet, Dini–Hadamard, limiting Fréchet, G-subdifferential, and Clarke's generalized gradients. All results are supplied with proofs. The latter makes the chapter, together with § 7.2, in which we consider applications of regularity theory to subdifferential calculus, a reasonably complete, albeit short introduction to the subdifferential theory of variational analysis in arbitrary Banach spaces, not covered, by the way, by the existing literature (except, to a certain extent, in the recent monograph by Penot [265]).

Regularity criteria for set-valued mappings between Banach spaces established in Chap. 5, either dual, using subdifferentials and coderivatives, or primal, using directional derivatives, tangent cones, and contingent derivatives, all follow from the slope-based criteria of Chap. 3 through a series of simple propositions connecting the values of slopes of certain distance functions, naturally connected with the mapping, on the one hand, and norms of suitable elements of subdifferentials, coderivatives, or tangent cones, on the other hand. The propositions also allow us to make a fairly detailed comparison between various Banach space criteria that results in the rather surprising conclusion that certain dual criteria are never worse than their primal counterparts. Another result to be mentioned is the separable reduction theorem, which says that in the Banach case metric regularity of a set-valued mapping near a point is fully determined by its restrictions to separable subspaces of the domain and range spaces. This is a substantial simplification from the theoretical viewpoint, in particular because in separable spaces subdifferential regularity criteria are much more convenient to work with, especially if the space is not reflexive.

In Chap. 6, we turn to the study of regularity properties of some special classes of mappings between Banach spaces. Information about the structure of a mapping may help to use more specialized techniques and obtain more precise results, e.g., better estimates for regularity rates. This is the case we are dealing with in the first three sections devoted, respectively, to error bounds, mappings with convex graphs, and single-valued locally Lipschitz mappings. In the last section, we briefly review implications of regularity for two types of mappings from a Banach space into its dual: monotone operators and subdifferentials of lower semi-continuous functions.

In Chap. 7, we consider a number of applications of regularity theory to analysis and optimization, mainly in infinite-dimensional Banach spaces. We begin with a discussion of possible extensions of the classical transversality concepts to settings of variational analysis. Applications to subdifferential calculus are considered next with fairly short proofs of the strongest available versions of calculus rules for practically all operations of interest in variational analysis. The metric qualification conditions in the statements of the rules are not just the most general. Remarkably (and contrary to popular qualification conditions involving subdifferentials), they are formulated in exactly the same way for all spaces, whether finite or infinite-dimensional, and for all functions, whether convex or not.

We then present a Banach space version of the implicit function theorem for set-valued mappings with special attention to generalized equations. The existence theorem for differential inclusions proved in the fourth section is the first application of the regularity-on-fixed-sets theory. Another application is considered in the seventh section, where we discuss connections between metric regularity and metric fixed point theory. The theorems proved in this section cover a number of well-known and recently established results. But the main innovation is the proofs, void of any iterations and fully based on regularity arguments. It seems that the proofs may substantially change the common perception of the relationship between metric regularity and metric fixed point theories.

The remaining two sections of Chap. 7, the fifth and the sixth, are devoted to necessary conditions in optimization problems. In the fifth section, we discuss two “nonvariational” approaches, both based on regularity theory, and, in particular, demonstrate one of them by giving a nontraditional proof of second-order optimality conditions in smooth optimization problems with equality and inequality constraints. In the sixth section, we give a new proof of Clarke’s necessary conditions for optimal control problems with differential inclusions, so far the strongest for problems of that sort.

The finite-dimensional theory is studied in Chap. 8. Naturally, all regularity criteria here are the best possible and give the exact values of regularity rates. Proofs of the criteria and results relating to stability analysis in finite-dimensional spaces easily follow from what we have already obtained in the previous chapters. We then pass to the study of two classes of finite-dimensional sets that often appear in practice and have many remarkable properties, namely polyhedral sets and their finite unions (called semi-linear sets), on the one hand, and semi-algebraic sets, on the other hand. (The first is, of course, a subclass of the second.) Locally, polyhedral sets have the structure of polyhedral cones, that is, convex hulls of finitely many directions, which tremendously simplifies working with them. The geometry of semi-algebraic sets is more complex. The principal structural property of a semi-algebraic set is that it admits Whitney stratification into a smooth manifold (a sort of stratification in which different strata meet each other in a certain regular way). This makes it possible to obtain a fairly strong version of Sard’s theorem (in which the exceptional set is not just of measure zero but of a smaller dimension) for semi-algebraic set-valued mappings. These structural properties make the regularity theory of semi-linear and semi-algebraic sets and mappings especially rich and interesting.

Finally, in Chap. 9, we apply the theory to a variety of finite-dimensional problems of analysis and optimization. The problems are not essentially connected and cross through a spectrum of disciplines that can be observed in the titles of the sections. In the first section, we offer a new treatment of the theory of variational inequalities over polyhedral sets, fully based on the regularity theory and elementary polyhedral geometry. Some very recent results emphasizing the role of transversality properties for linear convergence of the method of alternating projections for convex and nonconvex sets are presented in the second section. In the third section, we introduce and study a class of curves of “almost steepest descent” for lower semi-continuous functions. We prove the existence of such curves under some natural assumptions on the function and the possibility to obtain them as solutions of the anti-subgradient inclusion involving limiting subdifferentials or generalized gradients. Then, in the fourth section, we return to discussions on the connection between regularity properties of the subdifferential mapping and the characterization of minima of nonconvex functions, in particular tilt stability of the minima under linear perturbations of the function. Finally, in the fifth section, we apply the semi-algebraic Sard's theorem and transversality theorem to prove the typically regular behavior of solutions of nonsmooth optimization problems with semi-algebraic data and of equilibrium prices in (also nonsmooth) models of exchange economies in the spirit of the famous Debreu theorem.

We have substantially benefited from the existing monographs of Klatte and Kummer [200] and, especially, of Dontchev and Rockafellar [96], which have allowed us to mainly concentrate on those basic aspects of the theory that have not been touched upon in these monographs. This is first of all the bulk of the metric theory, including general regularity criteria and all local theory involving slopes, but also quite a bit of the infinite-dimensional Banach space theory, everything connected with semi-algebraic mappings and the majority of applications. On the other hand, there are a number of issues of fundamental importance that have been thoroughly studied in the two quoted monographs and that we address using very different approaches based on the theory developed in this book.

The first to be mentioned is the circle of problems associated with implicit functions. This is, of course, one of the principal themes of any regularity theory and by far the subject of main interest in [96]. We start with a version of the implicit function theorem stated in the most general situation of inclusion (1), where both the domain and the range spaces are metric, and follow the evolution of this result step-by-step as the assumptions on the environments and properties of the mapping change. The proofs at every step are surprisingly simple, and the main idea of the standard proof of the classical implicit function theorem (given in Chap. 1) works already at a very early stage, still in the fully metric setting. Another example is the theory of variational inequalities over polyhedral sets, which we have already mentioned, in many respects very different from the theories available in the existing literature.

Proofs have been the subject of special attention in the process of writing. I have already mentioned that the metric theory offers some new and efficient technical instruments that have been systematically used. To a large extent thanks to them, new, shorter, and simpler proofs have been given to quite a few known results, especially associated with applications. This does not change the fact that in variational analysis we have to deal with rather complicated objects and structural information is often helpful in pursuing simpler and more transparent proofs. This partly explains the close attention paid in this book to classes of objects with special structures. Fortunately, such objects seem to be rather typical in practice.

The book is essentially a research monograph whose aim is to present the state of the art of a fast developing and widely applicable theory. A certain level of advanced knowledge (e.g., in functional analysis and optimization) and mathematical maturity is desirable. But I believe the book will be accessible to a broad audience, including graduate students in mathematical departments and engineering departments with an advanced mathematical education (typical for computer science, electrical engineering, and industrial engineering/operation research departments in many universities).

We conclude with a few technical remarks about the organization of the book, terminology and notation.

How the Book is Organized

Every chapter begins with a short preface explaining the content of the chapter, the main results, techniques, and connections with other parts of the book. Whenever needed, we then add an introduction containing all necessary prerequisite, often with proofs, and the notation that appears in the chapter for the first time. At the end of every chapter (starting with Chap. 2), and in Chaps. 7 and 9 at the end of every section, we add bibliographic comments whose main purpose is not only to indicate the source of one result or another, or the relation of the results presented in the text with those in the literature, but also to give some information about the development of the ideas, the connections with some other related areas of analysis, open questions, etc. There are also many exercises scattered throughout the text.

Terminology and Notation

I have tried to avoid introducing new terminology and notation, unless there was a real necessity (very rarely). Concerning objects, properties, etc., for which there is more than one term often used in the literature, I have usually chosen one for systematic use but mentioned some others as well in definitions (usually in parentheses). The most essential notation is repeatedly reintroduced to free the reader from having to search for its meaning.

It is to be finally mentioned that the number of publications connected with metric regularity is enormous and continues to grow. So the bibliography presented in this book is definitely far from complete. In addition to publications most immediately connected with the results and proofs contained in this book, I have tried to mention works in which, to the best of my knowledge, ideas and results were originated or substantially improved or received new understanding in one way or another, plus of course available monographs, survey articles, and closely related publications from other areas. Needless to say, by doing this, despite all attempts, one cannot avoid being subjective. Therefore, I wish to apologize in advance for (hopefully not many) possible and inevitable omissions, misquotations, and plain mistakes.

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