

## Chapter 2

# General Geometric Setup

Goal of this section is to discuss the geometric background on which our investigation is based. As we have already indicated in the introduction, we will not give a full-fledged analysis of all structures, but we will focus on the data essential to us. Yet we will make sure to point an interested reader to the relevant literature. We will also assume that the reader is familiar with the basic tools proper of differential geometry. Within this respect we will follow mainly the notations and conventions of [Wa84].

### 2.1 Globally Hyperbolic Spacetimes

In this subsection we will follow mainly [BGP07, Wa84].

**Definition 2.1** A **spacetime**  $M$  is a connected Hausdorff second-countable orientable 4-dimensional smooth manifold endowed with a smooth Lorentzian metric  $g$  of signature  $(-, +, +, +)$ .

The requirement on the dimensionality of  $M$  is only based on our desire to make a close contact with the examples that we will be discussing in detail. Most of the general results and of the constructive aspects of the theory are valid for Lorentzian manifolds with dimension greater or equal to 2.

The Lorentzian character of the metric  $g$  plays an important distinguishing role for the pair  $(M, g)$ , since we can define two additional concepts: *time orientability* and *causal structures*. We briefly recall that, for any point  $p \in M$  using the scalar product  $g_p$  in  $T_p M$  induced by the metric  $g$ , we can divide the elements of  $T_p M \setminus \{0\}$  into three distinct categories. A vector  $v \in T_p M \setminus \{0\}$  is called **timelike** if  $g_p(v, v) < 0$ , **spacelike** if  $g_p(v, v) > 0$ , **lightlike** (also said **null**) if  $g_p(v, v) = 0$ . **Causal vectors** are those either timelike or lightlike. Co-vectors are classified similarly.

A smooth 3-dimensional embedded submanifold  $S \subseteq M$ , often referred to as a **3-dimensional hypersurface**, is **spacelike**, **timelike**, **lightlike** (also said **null**) if, respectively, its normal co-vector is timelike, spacelike, lightlike.

In addition, for a fixed  $p \in M$  we can define a two-folded **light cone**  $V_p \subseteq T_p M \setminus \{0\}$  made of all causal vectors and we have the freedom to call **future-directed** the non-zero vectors lying in one of the two-folds. If such choice can be made smoothly varying  $p \in M$ , we say that  $(M, g)$  is **time orientable**. It turns out that if a spacetime is time orientable, being connected per assumption, there are only two inequivalent such choices each called **time orientation**.

*Remark 2.1* Henceforth all our spacetimes will be supposed to be **time oriented**, i.e., a choice of time orientation has been made.

The subdivision of each  $T_p M \setminus \{0\}$  into the said three subsets is at the heart of the definition of causal structures for  $(M, g)$ . Consider a *continuous* and *piecewise-smooth* curve  $\gamma : I \rightarrow M$  where, indifferently,  $I = (a, b)$  or  $I = [a, b)$  or  $I = (a, b]$  or  $I = [a, b]$  and we henceforth suppose  $\mathbb{R} \cup \{-\infty\} \ni a < b \in \mathbb{R} \cup \{+\infty\}$ . Assuming that its tangent vector  $\dot{\gamma}$  vanishes nowhere, we shall call  $\gamma$  **timelike** (*resp.* **spacelike**, **lightlike**) **curve** if at each point of the image  $\dot{\gamma}$  is timelike (*resp.* spacelike, lightlike). If the tangent vector to  $\gamma$  is nowhere spacelike and is everywhere future (*resp.* past) directed, we say that it is a **future** (*resp.* **past**) **-directed causal curve**. **Future** (*resp.* **past**) **-directed timelike** curves are defined analogously. Each **causal curve** is either future- or past-directed.

Any such curve, say  $\gamma : I \rightarrow M$  with either  $I = (a, b]$  or  $I = [a, b)$  is said to be **past inextendible** if there is no causal curve  $\gamma' : I' \rightarrow M$  with  $I' \supset I$  and  $\inf I' < a$  such that  $\gamma'|_I = \gamma$ . **Future inextendibility** is analogously defined. Zorn's lemma assures that every causal curve can be completed into an inextendible causal curve.

We have all ingredients to introduce the defining building blocks of the causal structure of  $(M, g)$ . We call

- $J^\pm(p)$  the **causal future (+) / past (-)** of  $p \in M$ , that is the union between  $\{p\}$  and all points  $q \in M$  such that there exists a future- (+) / past- (-) directed, causal curve  $\gamma : [a, b] \rightarrow M$  with  $\gamma(a) = p$  and  $\gamma(b) = q$ ,
- $I^\pm(p)$  the **chronological future (+) / past (-)** of  $p \in M$  that is the collection of all points  $q \in M$  such that there exists a future- (+) / past- (-) directed, timelike curve  $\gamma : [a, b] \rightarrow M$  with  $\gamma(a) = p$  and  $\gamma(b) = q$ .

For subsets  $U \subseteq M$ , we similarly define  $J^\pm(U) \doteq \bigcup_{p \in U} J^\pm(p)$  and  $I^\pm(U) \doteq \bigcup_{p \in U} I^\pm(p)$ .

$N, N' \subseteq M$  are said to be **causally disjoint** if  $(J^+(N) \cup J^-(N)) \cap N' = \emptyset$  which is equivalent to requiring  $(J^+(N') \cup J^-(N')) \cap N = \emptyset$ .

*Remark 2.2* [Wa84] [BEE96] [ON83]

- (1)  $I^\pm(N)$  are always *open* sets, even if  $N \subseteq M$  is not. The topology of  $J^\pm(N)$  is more complicated, however sufficiently close to  $p \in M$ ,  $\partial J^\pm(p) \setminus \{p\}$  is a null

3-dimensional hypersurface. General results for every  $N \subseteq M$  and points in  $M$  are the following:

- (a)  $I^\pm(N) = \text{Int}(J^\pm(N))$  and  $N \subseteq J^\pm(N) \subset \overline{I^\pm(N)}$ , where  $\text{Int}$  indicates the collection of interior points,
- (b) if  $p \in I^\pm(q)$  and  $q \in J^\pm(r)$ , then  $p \in I^\pm(r)$ ,
- (c) if  $p \in J^\pm(q)$  and  $q \in I^\pm(r)$ , then  $p \in I^\pm(r)$ ,
- (d) if  $p \in J^\pm(q)$  and  $q \in J^\pm(r)$ , then  $p \in J^\pm(r)$ .

(2) The notion of causal or chronological past/future strongly depends on the choice of the underlying background. When a disambiguation will be necessary we will employ the more precise notation  $J^\pm(U; N)$  where  $U \subseteq N$  and  $N$  is an open subset of the whole spacetime  $M$  equipped with the restriction of the metric and the relevant causal curves defining  $J^\pm(U; N)$  are those completely included in  $M$ . We shall use a similar notation regarding chronological past/future.

(3) A subset  $O \subset M$  of a spacetime  $(M, g)$  is said to be **causally convex** if every causal curve joining  $p, q \in O$  has image completely enclosed in  $O$ .

(4) A spacetime  $(M, g)$  is said to be **strongly causal** if, for every  $p \in M$  and every open subset  $U \ni p$ , there is an open causally convex subset  $V$  with  $p \in V \subseteq U$ . In strong causal spacetimes, the family of open sets  $I^+(p) \cap I^-(q)$ ,  $p, q \in M$  is a topological basis of the (pre-existent) topology of  $M$ .

Time orientation and causal structures open several possibilities for constructing spacetimes where closed timelike or causal curves exist, hence formalizing at a geometric level the fascinating idea of time travel. From a physical point of view, we are instead interested in identifying a suitable class of spacetimes which, on the one hand, avoids any such pathology, while, on the other hand, it allows for existence and uniqueness theorems for solutions of hyperbolic partial differential equations, such as the scalar D'Alembert wave equation in terms of an initial value problem. The answer to these queries goes under the name of *globally hyperbolic spacetimes* (see, e.g., [Wa84, Ch. 8]).

We begin by introducing an **achronal** subset of a spacetime  $M$ , namely a subset  $\Sigma$  such that  $I^+(\Sigma) \cap \Sigma = \emptyset$ . Subsequently we associate to  $\Sigma$  its **future domain of dependence** as the collection  $D^+(\Sigma)$  of points  $p \in M$  such that every past-inextendible causal curve passing through  $p$  intersects  $\Sigma$  somewhere. One defines analogously the **past domain of dependence**  $D^-(\Sigma)$  of  $\Sigma$  and its **domain of dependence**  $D(\Sigma) \doteq D^+(\Sigma) \cup D^-(\Sigma)$ .

**Definition 2.2** A **Cauchy hypersurface** of a spacetime  $M$  is defined as a closed achronal subset  $\Sigma \subseteq M$  such that  $D(\Sigma) = M$ . A spacetime  $M$  is called **globally hyperbolic** if it possesses a Cauchy hypersurface.

Globally hyperbolic spacetimes represent the canonical class of backgrounds on which quantum field theories are constructed and  $\Sigma$  is the natural candidate to play the role of the hypersurface on which initial data can be assigned, provided  $\Sigma$  is sufficiently smooth. Yet, according to Definition 2.2, only the existence of a single (hopefully smooth) Cauchy hypersurface is guaranteed. This is slightly disturbing

since there is no *a priori* reason why an initial value hypersurface for a certain partial differential equation should be distinguished. In addition Definition 2.2 does not provide any concrete criterion to establish whether a given spacetime  $M$  with an assigned metric  $g$  is globally hyperbolic or not. An important step forward in this direction is represented by the work of Bernal and Sanchez, [BS05, BS06], who, by means of deformation arguments, gave an alternative, very informative additional characterization of globally hyperbolic spacetimes also proving the existence of *smooth spacelike* Cauchy surfaces. We shall report it essentially following the formulation given in Sect. 1.3 of [BGP07]:

**Proposition 2.1** *Let  $(M, g)$  be any time-oriented spacetime. The following two statements are equivalent:*

1.  $(M, g)$  is globally hyperbolic;
2.  $(M, g)$  is isometric to  $\mathbb{R} \times \Sigma$  with metric  $-\beta dt \otimes dt + h_t$ , where  $t : M \rightarrow \mathbb{R}$  plays the role of time coordinate, while  $\beta, h \in C^\infty(\mathbb{R} \times \Sigma)$ ,  $\beta$  is strictly positive. Each  $h_t$  is a smooth Riemannian metric on the smooth manifold  $\Sigma$ , and each  $\Sigma_t \doteq \{t\} \times \Sigma$  identifies a (smooth embedded co-dimension 1) submanifold of  $M$  which is a spacelike Cauchy surface.

**Remark 2.3** The following properties are enjoyed by any globally hyperbolic spacetime  $(M, g)$  [Wa84]:

- (1) It is strongly causal, hence item (4) in Remark 2.2 holds true.
- (2) For every  $p, q \in M$ ,  $J^+(p) \cap J^-(q)$  is either empty or compact.
- (3) For every  $p, q \in M$ ,  $J^\pm(p) \cap J^\mp(\Sigma)$  and thus also  $J^\pm(p) \cap \Sigma$  are compact if  $\Sigma$  is a Cauchy surface in the future (+), resp., past (-) of  $p$ .
- (4) If  $S \subseteq M$  is compact (in particular  $S = \{p\}$ ), then  $J^\pm(S) = \overline{I^\pm(S)}$ ,  $J^\pm(S) \setminus I^\pm(S) = \partial J^\pm(S) = \partial I^\pm(S)$ .

There are several known examples in the literature of globally hyperbolic spacetimes and most of the physically interesting scenarios are based on this notion. In this work we will consider some of these cases in detail and, hence, we will not discuss further this concept. A reader interested in a few concrete examples can consult either [Wa84] or the list given in [BDH13]. It is important to stress that globally hyperbolic spacetimes do not exhaust the collection of spacetimes used in physical applications, even within the algebraic approach of QFT, *e.g.*, the half Minkowski spacetime which appears in the investigation of the Casimir-Polder effect [DNP16] or anti-de Sitter spacetime, the maximally symmetric solution of Einstein's equations with negative cosmological constant [DF16, DR03, DR02, Ri07].

## 2.2 Spacetimes with Distinguished Light-Like 3-Surfaces

The next step consists of introducing the class of spacetimes which we will be considering throughout this text. As mentioned in the introduction we will focus on several specific scenarios where light-like 3-surfaces play a central role.

### 2.2.1 Asymptotically Flat Spacetimes

The first scenario is at the heart of this section. Most notably we will present the spacetimes which are said to be *asymptotically flat at null infinity*. Heuristically speaking, they are those manifolds whose asymptotic behaviour far away along null directions mimics that of Minkowski spacetime. A rigorous mathematical characterization of these backgrounds and a thorough analysis of the associated geometric properties has been subject of investigations starting from the early sixties. A reader who is interested in these aspects can consult [Ge77], [AX78], [Fr86, Fr87, Fr88], and [Wa84, Ch. 8-11]. Our presentation will follow mainly these references and we will review the structures and the related properties which play a key role in the next chapters.

**Definition 2.3** A (4-dimensional) spacetime  $(M, g)$  (dubbed, *physical spacetime* or *bulk*), is **asymptotically flat at future null infinity** if the following objects exist.

- (a) A (4-dimensional) spacetime  $(\tilde{M}, \tilde{g})$  (dubbed, *unphysical spacetime*).
- (b) A smooth embedding  $\psi : M \rightarrow \psi[M] \subset \tilde{M}$ ,  $\psi[M]$  being an open subset of  $\tilde{M}$ .
- (c) A smooth function  $\Omega \in C^\infty(\psi[M])$  fulfilling  $\Omega > 0$  and  $\tilde{g}|_{\psi[M]} = \Omega^2 \psi^* g$ .

The following further five conditions must hold true.

1.  $\psi[M] \subseteq \tilde{M}$  is a manifold with boundary  $\mathfrak{I}^+ \doteq \partial\psi[M]$  given by an embedded three-dimensional submanifold of  $\tilde{M}$  which satisfies  $\mathfrak{I}^+ \cap J^-(\psi[M], \tilde{M}) = \emptyset$ .
2.  $(\tilde{M}, \tilde{g})$  is strongly causal in an open neighborhood of  $\mathfrak{I}^+$ .
3.  $\Omega$  extends (not uniquely in general) to a smooth function on the whole  $\tilde{M}$ , still denoted by  $\Omega$ , such that  $\Omega = 0$  exactly on  $\mathfrak{I}^+$ , whereas  $d\Omega \neq 0$  at each point of  $\mathfrak{I}^+$ .
4. Defining  $n^a := \tilde{g}^{ab} \partial_b \Omega$ , there is a smooth function  $\omega$  defined on  $\tilde{M}$  with  $\omega > 0$  on  $M \cup \mathfrak{I}^+$  such that  $\tilde{\nabla}_a(\omega^4 n^a) = 0$  on  $\mathfrak{I}^+$  and the integral curves of  $\omega^{-1}n$  are complete on  $\mathfrak{I}^+$ . This way  $\mathfrak{I}^+$  turns out to be diffeomorphic to  $\mathbb{S}^2 \times \mathbb{R}$ , the second factor being the range of the parameter of those integral curves.
5. Vacuum Einstein equations are assumed to be fulfilled for  $(M, g)$  in a neighbourhood of the boundary of  $\psi[M]$  (or, more weakly, “approaching” it as discussed at p.278 of [Wa84]).

We call **future null infinity** of  $(M, g)$  the set  $\mathfrak{I}^+$ .

Since the conformal embedding  $\psi$  is a diffeomorphism from  $M$  onto its image, we will henceforth omit to indicate it explicitly writing  $M \subseteq \tilde{M}$ . The index notation has been used in the last part of the definition to make the statements less obscure. We will resort often to this policy, although all the structures that we define and the results that we discuss could be presented all with an index-free notation, never being dependent on the choice of a local chart.

**Remark 2.4** (a) Minkowski spacetime fulfils the given definition [Wa84] and in fact, the geometric structure of  $\mathfrak{I}^+$  described in the definition above is just the one

arising from the analysis of the geometry of null infinity of Minkowski spacetime. In this sense a spacetime is asymptotically flat at null infinity if it “looks like Minkowski spacetime at null infinity”.

(b) More complicated completions of our basic definition exist and concern other types of infinity which may be added to  $(M, g)$ . One could wish to add *future time infinity* of  $M$ . This notion was first introduced by Friedrich in [Fr86, Fr87, Fr88] to describe spacetimes resembling Minkowski space in the far timelike future, and used in a bulk-to-boundary context in [Mo06]. Dealing with this class of backgrounds (which contains Minkowski spacetime) we will refer to them as **asymptotically flat (at future null infinity) with future time infinity**. In this setting in addition to Definition 2.3,

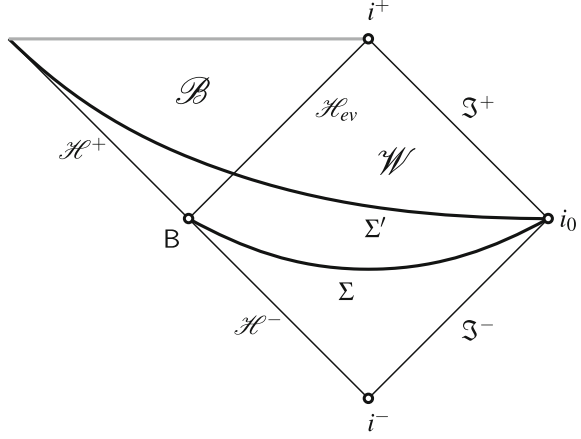
6.  $\tilde{M}$  includes a preferred point  $i^+$ , **future time infinity**, and the embedding of the bulk into the unphysical spacetime is assumed to satisfy  $J^-(i^+; \tilde{M}) \setminus \partial J^-(i^+; \tilde{M}) = \psi[M]$  where now  $J^-(i^+; \tilde{M})$  is supposed to be closed. Future null infinity here satisfies  $\mathfrak{I}^+ = \partial J^-(i^+; \tilde{M}) \setminus \{i^+\}$  so that  $\partial\psi[M] = \mathfrak{I}^+ \cup i^+$ . The extended function  $\Omega$  is again required to satisfy  $\Omega = 0$  exactly on  $\partial\psi[M]$ ,  $d\Omega \neq 0$  on  $\mathfrak{I}^+$ , but  $d\Omega(i^+) = 0$  with  $\tilde{\nabla}_\mu \tilde{\nabla}_\nu \Omega(i^+) = -2\tilde{g}_{\mu\nu}(i^+)$ .

(c) Analogously, one could work replacing  $\mathfrak{I}^+$  and  $i^+$  with the *past null infinity*  $\mathfrak{I}^-$  and **past time infinity**,  $i^-$ , respectively. All constructions, properties and results being unchanged. Alternatively, one can define a spacetime which is asymptotically flat at null and *space* infinity, by adding a (further) point  $i_0$  to  $\tilde{M}$ , **spacelike infinity** of  $M$ , such that around  $i_0$  the spacetime “looks like Minkowski spacetime at spacelike infinity”. A detailed discussion of this sort of asymptotically flat spacetimes appears in [Wa84]. We only remark that the geometric structure close to  $i_0$  is much more delicate than the one around  $\mathfrak{I}^\pm$  or  $i^\pm$  and non-trivial regularity issues arise for  $\tilde{g}$  at  $i_0$ . Assuming the existence of null and spacelike flat infinities, the embedding  $\psi$  is supposed to satisfy  $J^+(i_0; \tilde{M}) \cup J^-(i_0; \tilde{M}) = \tilde{M} \setminus \psi[M]$  and the null infinities are now defined as  $\mathfrak{I}^\pm \doteq \partial J^\pm(i_0; \tilde{M}) \setminus \{i_0\}$ .

### 2.2.2 A Compound Example: Schwarzschild Spacetime

Definition 2.3 with the completion of item (b) in Remark 2.4 seems very hard to check in concrete examples of spacetimes. Yet, this is not really the case and, besides the obvious case of Minkowski spacetime, there are several other known backgrounds admitting null infinities and other types of infinities. The most famous one is the Schwarzschild solution of Einstein equations which also includes other types of light-like 3-surfaces: *Killing horizons*. Since we will be discussing the construction of the Unruh state on this spacetime, it is worth highlighting it more in detail and, to this end, we will follow mainly the notations and conventions of [Wa84, Sect. 6.4]. Our starting point is Kruskal spacetime  $\mathcal{K}$  and we are interested in the subspacetime  $\mathcal{M}$  used to picture a black hole of mass  $M > 0$ . Referring to Fig. 2.1,  $\mathcal{M}$  is made of

**Fig. 2.1** The overall picture represents  $\mathcal{M}$ . The regions  $\mathcal{W}$  and  $\mathcal{B}$  correspond respectively to the regions I and III in Fig. 2.2. The thick horizontal line denotes the metric singularity at  $r = 0$ ,  $\Sigma$  is a smooth spacelike Cauchy surface for  $\mathcal{M}$  while  $\Sigma'$  is a smooth spacelike Cauchy surface for  $\mathcal{W}$



the union of three pairwise disjoint parts, the **Schwarzschild wedge**  $\mathcal{W}$ , the **black hole** region  $\mathcal{B}$ , and their common boundary  $\mathcal{H}_{ev}$  also known as the **(future Killing) event horizon**. Defining the **Schwarzschild radius**  $r_S \doteq 2M$ , the metric outside  $\mathcal{H}_{ev}$  is best defined in terms of the **Schwarzschild coordinates**  $t, r, \theta, \phi$ , where  $t \in \mathbb{R}$ ,  $r \in (r_S, +\infty)$ ,  $(\theta, \phi)$  are standard spherical coordinates over  $\mathbb{S}^2$  in  $\mathcal{W}$ , whereas  $t \in \mathbb{R}$ ,  $r \in (0, r_S)$ ,  $(\theta, \phi)$  cover  $\mathbb{S}^2$  as before in  $\mathcal{B}$ ,

$$g = -\left(1 - \frac{2M}{r}\right) dt \otimes dt + \left(1 - \frac{2M}{r}\right)^{-1} dr \otimes dr + r^2 d\mathbb{S}^2(\theta, \phi), \quad (2.1)$$

where  $d\mathbb{S}^2 \doteq d\theta \otimes d\theta + \sin^2 \theta d\phi \otimes d\phi$  is the standard metric on the unit 2-sphere. We observe that as  $r$  tends to 0 we approach an *intrinsic* (curvature) singularity located at the horizontal upper boundary of  $\mathcal{B}$  (Fig. 2.1), whereas  $r = r_S$  defines an *apparent* singularity on the event horizon  $\mathcal{H}_{ev}$  which actually is just due to the bad choice of coordinates. Another convenient chart over  $\mathcal{W} \cup \mathcal{B}$  is that provided by **Eddington-Finkelstein coordinates** [KW91, Wa84]:  $u, v, \theta, \phi$ , with  $(\theta, \phi)$  standard spherical coordinates over  $\mathbb{S}^2$  and

$$u \doteq t - r^* \text{ in } \mathcal{W}, \quad u \doteq -t - r^* \text{ in } \mathcal{B}, \quad (2.2)$$

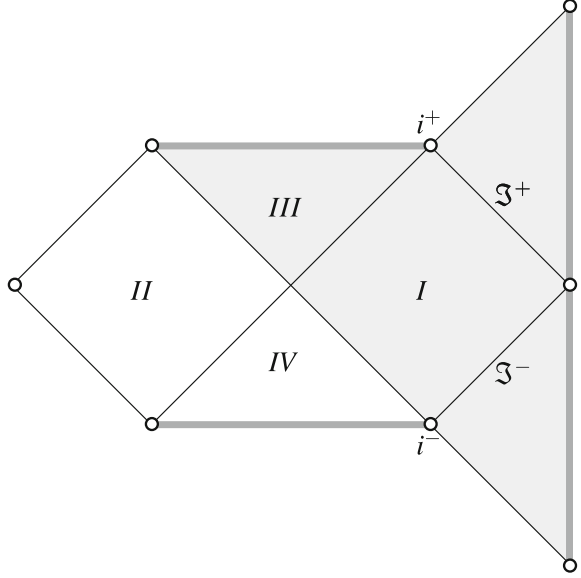
$$v \doteq t + r^* \text{ in } \mathcal{W}, \quad v \doteq t - r^* \text{ in } \mathcal{B},$$

$$r^* \doteq r + 2M \ln \left| \frac{r}{2M} - 1 \right| \in \mathbb{R}. \quad (2.3)$$

A third, related, chart yields the **global null coordinates**  $U, V, \theta, \phi$  which have the advantage of being defined on the whole  $\mathcal{H}$  [Wa84] though here we restrict them to  $\mathcal{M}$  only,

$$U = -e^{-u/(4M)}, \quad V = e^{v/(4M)} \text{ in } \mathcal{W}, \quad U = e^{u/(4M)}, \quad V = e^{v/(4M)} \text{ in } \mathcal{B}. \quad (2.4)$$

**Fig. 2.2** The Kruskal extension of Schwarzschild spacetime, where the physical region is shaded in grey. Following [AH78] one can introduce **spatial infinity**  $i_0$ , while  $i^\pm$  are not part of the conformal diagram and we shall refer to them as (formal) *points at infinity*, often also known as **future** and **past** infinity respectively



In this frame,

$$\mathcal{W} \equiv \{(U, V, \theta, \phi) \in \mathbb{R}^2 \times \mathbb{S}^2 \mid U < 0, V > 0\},$$

$$\mathcal{B} \equiv \{(U, V, \theta, \phi) \in \mathbb{R}^2 \times \mathbb{S}^2 \mid UV < 1, U, V > 0\},$$

$$\mathcal{M} \doteq \mathcal{W} \cup \mathcal{B} \cup \mathcal{H}_{ev} \equiv \{(U, V, \theta, \phi) \in \mathbb{R}^2 \times \mathbb{S}^2 \mid UV < 1, V > 0\}.$$

Each of the three regions is a globally hyperbolic subspacetime of  $\mathcal{K}$ , and they represent the building blocks of our analysis together with  $\mathcal{H}_{ev}$  and with the complete **past (Killing) horizon**  $\mathcal{H}$  of  $\mathcal{M}$  which is part of the boundary of  $\mathcal{M}$  in the Kruskal manifold. These horizons are defined by:

$$\mathcal{H}_{ev} \equiv \{(U, V, \theta, \phi) \in \mathbb{R}^2 \times \mathbb{S}^2 \mid U = 0, V > 0\},$$

$$\mathcal{H} \equiv \{(U, V, \theta, \phi) \in \mathbb{R}^2 \times \mathbb{S}^2 \mid V = 0, U \in \mathbb{R}\}.$$

Following Fig. 2.1 and for future convenience, we decompose  $\mathcal{H}$  into the disjoint union  $\mathcal{H} = \mathcal{H}^- \cup \mathcal{B} \cup \mathcal{H}^+$  where  $\mathcal{H}^\pm$  are defined as the loci  $U > 0$  and  $U < 0$ , while  $\mathcal{B}$  is the **bifurcation surface** at  $U = 0$ . This is a spacelike 2-sphere with radius  $2M$  where  $\mathcal{H}$  meets the closure of  $\mathcal{H}_{ev}$ .

The metric of the whole Kruskal manifold and, per restriction, also of  $\mathcal{M}$ , is

$$g = -\frac{32M^3}{r} e^{-\frac{r}{2M}} dU \otimes dV + r^2 d\mathbb{S}^2(\theta, \phi), \quad (2.5)$$

where the only intrinsic (curvature) singularity of Kruskal spacetime at  $r \rightarrow 0$ , translates to  $UV \rightarrow 1$  while the apparent singularity on  $\mathcal{H}_{ev}$  has disappeared.

We outline succinctly the Killing vector structure since it will play a key role in the next sections. As one can infer directly from (2.1) or from (2.5), besides the Killing fields associated to the spherical symmetry of the background, there is a further smooth Killing field  $X$  which coincides with  $\partial_t$ .  $X$  is timelike in  $\mathcal{W}$  and orthogonal to  $\partial_r, \partial_\theta, \partial_\phi$ , which is thus a *static spacetime*, while  $X$  is spacelike in  $\mathcal{B}$ .  $X$  becomes lightlike and tangent to  $\mathcal{H}$  as well as to both  $\mathcal{H}_{ev}$  and to its completion in the Kruskal manifold, while it vanishes on  $\mathbf{B}$ , giving rise to the structure of a *bifurcate Killing horizon* [KW91]. In terms of the coordinates  $u$  and  $v$ , it turns out that

$$X = \mp \partial_u \text{ on } \mathcal{H}^\pm, \quad X = \partial_v \text{ on } \mathcal{H}_{ev}.$$

To conclude our short survey of Schwarzschild spacetime, we need to make contact with the analysis of the previous section, Definition 2.3 in particular. One follows [SW83] rescaling the metric (2.5) by a factor  $r^{-2}$  after which one can notice that  $(\mathcal{M}, r^{-2}g)$  admits a smooth larger extension  $(\widetilde{\mathcal{M}}, \widetilde{g})$  (see Fig. 2.2) constructed in accordance with Definition 2.3. Here, the geometric singularity in  $(\mathcal{M}, g)$  is pushed at infinity in the sense that the non-null geodesics take an infinite amount of affine parameter to reach a point at  $r = 0$ . The extension of  $(\widetilde{\mathcal{M}}, \widetilde{g})$  obtained in this way does not cover the timelike and spacelike infinities  $i^\pm$  and  $i_0$  in Figs. 2.1 and 2.2 (refer to item (b) in Remark 2.4), though it includes the boundaries given by **future** and **past null infinity** respectively  $\mathfrak{I}^\pm$ . These null 3-submanifolds of  $\widetilde{\mathcal{M}}$  are formally localised at  $r = +\infty$ . A finer extension which includes  $i_0$  is described in great detail in the appendix of [AH78] where it is more generally shown that all Kerr solutions of vacuum Einstein equations are asymptotically flat at null and spacelike infinity. The restriction of the rescaled extended metric  $\widetilde{g}$  to the Killing horizon  $\mathcal{H}$  as well as to the null infinities  $\mathfrak{I}^\pm$  can be written explicitly. More precisely, in the first case,

$$\widetilde{g}|_{\mathcal{H}} = 4M^2 \left( -d\Omega \otimes dU - dU \otimes d\Omega + dS^2(\theta, \phi) \right),$$

where  $\Omega = 2V$  vanishes at  $\mathcal{H}$  since  $V$  is defined as in (2.4). Barring the constant pre-factor  $4M^2$  the metric is said to be in a **geodetically complete Bondi form**, completeness being referred to the complete domain of the affine parameter  $U \in (-\infty, +\infty)$  of the null geodesics forming  $\mathcal{H}$ . The same structure occurs on  $\mathfrak{I}^+$  and on  $\mathfrak{I}^-$ , respectively, formally defined by the limit-value of the Eddington-Finkelstein coordinates  $v = +\infty$  and  $u = -\infty$ . The metric  $\widetilde{g}$  has still a **geodetically complete Bondi form**,

$$\widetilde{g}|_{\mathfrak{I}^+} = -d\Omega \otimes du - du \otimes d\Omega + dS^2(\theta, \phi),$$

where  $\Omega \doteq -2/v$  defines  $\mathfrak{I}^+$  for  $\Omega = 0$ . Similarly

$$\widetilde{g}|_{\mathfrak{I}^-} = -d\Omega \otimes dv - dv \otimes d\Omega + dS^2(\theta, \phi),$$

where  $\Omega \doteq -2/u$  defines  $\mathfrak{I}^-$  for  $\Omega = 0$ . To conclude we observe that the  $g$ -Killing vector field  $X$  coinciding in  $\mathcal{W}$  and in  $\mathcal{B}$  with  $\partial_t$  is an affine Killing vector for  $\tilde{g}$  (as evidently  $\tilde{\nabla}_a X_b + \tilde{\nabla}_b X_a = \mathcal{L}_X(\tilde{g})_{ab} = X(\ln \Omega^2)\tilde{g}_{ab}$  in the bulk) and it extends to an affine  $\tilde{g}$ -Killing vector, still denoted by  $X$ , defined on  $\mathcal{M}$  with

$$X = \partial_u \text{ on } \mathfrak{I}^+, \quad X = \partial_v \text{ on } \mathfrak{I}^-.$$

### 2.2.3 Asymptotic Symmetries: The Bondi-Metzner-Sachs Group

Our next goal is to consider an arbitrary asymptotically flat spacetime at future null infinity  $(M, g)$  as in Definition 2.3 further discussing the geometric properties of future null infinity  $\mathfrak{I}^+$ . For our purposes it suffices that only  $\mathfrak{I}^+$  exists, but, whenever also past null infinity can be defined, a similar analysis for  $\mathfrak{I}^-$  holds true.

As starting point, we remark that, in view of the definition of the unphysical spacetime  $(\tilde{M}, \tilde{g})$ , the metric structure of  $\mathfrak{I}^+$  enjoys a so-called *gauge freedom*. It stems from the observation that we can always rescale  $\Omega \rightarrow \omega\Omega$  in a neighborhood of  $\mathfrak{I}^+$ , where  $\omega$  is a smooth and nowhere vanishing scalar function. Such freedom does not affect the topology of  $\mathfrak{I}^+$ , namely that of  $\mathbb{R} \times \mathbb{S}^2$ , as well as the differentiable structure. Following Definition 2.3, once  $\Omega$  is fixed,  $\mathfrak{I}^+$  turns out to be the union of future-oriented integral lines of the field  $n^a \doteq \tilde{g}^{ab}\tilde{\nabla}_b\Omega$ . This property is, in fact, invariant under gauge transformation, but the field  $n$  depends on the gauge. All relevant information can be encoded in a triple of data  $(\mathfrak{I}^+, \tilde{h}, n)$ , where  $\tilde{h}$  is the degenerate metric induced by  $\tilde{g}$  on  $\mathfrak{I}^+$ . Under a gauge transformations  $\Omega \rightarrow \omega\Omega$ , these data transform as

$$\mathfrak{I}^+ \rightarrow \mathfrak{I}^+, \quad \tilde{h} \rightarrow \omega^2\tilde{h}, \quad n \rightarrow \omega^{-1}n. \quad (2.6)$$

For a given asymptotically flat spacetime  $(M, g)$ , there is no general physical principle which may distinguish one of the above triple of data from another obtained via a gauge transformation. Such property goes together with that of **universality**: It turns out that the geometric structures at future null infinity of a pair of asymptotically flat spacetimes are always isomorphic up to gauge transformations. To wit, if  $C_1$  and  $C_2$  are two equivalence classes under gauge transformation of triples, associated respectively to two asymptotically flat spacetimes  $(M_1, g_1)$  and  $(M_2, g_2)$ , there exists a diffeomorphism  $\gamma : \mathfrak{I}_1^+ \rightarrow \mathfrak{I}_2^+$  such that, for suitable  $(\mathfrak{I}_1^+, \tilde{h}_1, n_1) \in C_1$  and  $(\mathfrak{I}_2^+, \tilde{h}_2, n_2) \in C_2$ ,

$$\gamma(\mathfrak{I}_1^+) = \mathfrak{I}_2^+, \quad \gamma^*\tilde{h}_1 = \tilde{h}_2, \quad \gamma^*n_1 = n_2.$$

The proof of this statement relies on the existence for every asymptotically flat spacetime  $(M, g)$  of a choice of a gauge such that the rescaled unphysical metric

(still indicated by  $\tilde{g}$  instead of  $\omega^2\tilde{g}$ ) reads

$$\tilde{g}|_{\mathfrak{I}^+} = d\Omega \otimes du - du \otimes d\Omega + d\mathbb{S}^2(x_1, x_2) . \quad (2.7)$$

Indeed, Definition 2.3 requires that for a given asymptotically flat spacetime  $(M, g)$  with an initial  $\Omega$ , it is always possible to fix the gauge  $\omega$  such that both  $\tilde{\nabla}_a(\omega^4 n^a) = 0$  and the integral curves of  $\omega^{-1}n$  are complete, *i.e.*, their parameter ranges over the whole  $\mathbb{R}$ . The first condition implies that these curves are lighthlike complete geodesics for the rescaled unphysical metric  $\omega^2\tilde{g}$ . With this choice of  $\omega$ , still denoting by  $\Omega$  (instead of  $\omega\Omega$ ) the gauge-transformed conformal factor, with  $n$  (instead of  $\omega^{-1}n$ ) the gauge-transformed tangent vector and with  $\tilde{g}$  (instead of  $\omega^2\tilde{g}$ ) the gauge-transformed unphysical metric, in a neighbourhood of  $\mathfrak{I}^+$ , there exists a coordinate system  $(u, \Omega, x_1, x_2)$  so that  $d\mathbb{S}^2(x_1, x_2)$  is the standard metric on a unit 2-sphere,  $u \in \mathbb{R}$  is an affine parameter along the *complete* null geodesics forming  $\mathfrak{I}^+$  with tangent vector  $n = \partial/\partial u$ . As in the special example of Schwarzschild spacetime, discussed in the previous section, these are also known as **Bondi coordinates**.  $\mathfrak{I}^+$  is made of the points  $(u, 0, x_1, x_2)$  with  $u \in \mathbb{R}$ ,  $(x_1, x_2) \in \mathbb{S}^2$ . Every asymptotically flat spacetime, using (2.6), admits the triple of data  $(\mathfrak{I}^+, \tilde{h}_B, n_B) \doteq (\mathbb{R} \times \mathbb{S}^2, d\mathbb{S}^2, \partial/\partial u)$ . For more details about the above structures see [Ge77, Wa84].

We focus now on the main topic of this section, namely the so-called **Bondi-Metzner-Sachs (BMS) group**,  $G_{BMS}$  [Pe63, Ge77, AS81]. First introduced at the beginning of the sixties in [BBM62]

**Definition 2.4**  $G_{BMS}$  is the group of diffeomorphisms  $\gamma : \mathfrak{I}^+ \rightarrow \mathfrak{I}^+$  such that  $(\gamma(\mathfrak{I}^+), \gamma^*\tilde{h}, \gamma^*n)$  is identical to  $(\mathfrak{I}^+, \tilde{h}, n)$  up to a gauge transformation (2.6).

In other words we are considering only those diffeomorphisms which generalize the notion of isometry to an asymptotic structure where the metric structures are equivalent up to gauge transformations. (The full group of diffeomorphisms  $\gamma : \mathfrak{I}^+ \rightarrow \mathfrak{I}^+$  without restrictions is the so-called *Newman-Unti group* [NU62].) The following characterization of the one-parameter subgroups of the BMS is rather useful [Wa84]:

**Proposition 2.2** *The smooth one-parameter group of diffeomorphisms which is generated by a smooth vector field  $\xi'$  on  $\mathfrak{I}^+$  is a subgroup of  $G_{BMS}$  if and only if  $\xi'$  can be smoothly extended to a, possibly non unique, vector field  $\xi$  defined in  $M$  in some neighborhood of  $\mathfrak{I}^+$  in such a way that  $\Omega^2\mathcal{L}_\xi g$  has a smooth extension at  $\mathfrak{I}^+$  and vanishes thereon.*

Remembering that  $\tilde{g}$  is a smooth metric also on  $\mathfrak{I}^+$  where  $\Omega$  vanishes ( $\tilde{g} = \Omega^2 g$  is valid *inside*  $M$ ), the condition that  $\Omega^2\mathcal{L}_\xi g$  is smooth and vanishing at  $\mathfrak{I}^+$  can be seen as an interpretation of the heuristic idea of a vector field which becomes an exact symmetry only asymptotically. For this reason, taking the universality property into account, we can say that *the BMS group describes the asymptotic Killing symmetries all asymptotically flat vacuum spacetimes simultaneously*. Proposition 2.2 characterizes also a subalgebra of the smooth vector fields on  $\mathfrak{I}^+$ , which can be

seen as the *Lie algebra* of the BMS group since these vectors generate the smooth one-parameter subgroups. Since  $G_{BMS}$  is not a finite-dimensional Lie group, it is by no means obvious that an exponential map from the Lie algebra to the whole group exists. Although we do not enter into the technical details of the proof, we stress that such exponential map exists in our case and it is a consequence of  $\mathfrak{N}^+$  being generated by a whole family of complete integral curves built out of the vector field  $n$ .

In order to give an explicit representation of  $G_{BMS}$ , we fix the conformal rescaling, so to work in an already defined *Bondi coordinate system*. Recall that this is constructed out of the affine parameter  $u$  of the null integral curves forming  $\mathfrak{N}^+$  together with two additional coordinates on the unit 2-sphere. Starting from the standard ones  $(\theta, \phi)$ , we introduce via a stereographic projection the complex coordinates  $(\zeta, \bar{\zeta}) \in \mathbb{C}^2$  so that  $\zeta = e^{i\phi} \cot(\theta/2)$ . In this frame the set  $G_{BMS}$  is nothing but  $SO(3, 1)_\uparrow \times C^\infty(\mathbb{S}^2)$ , where  $SO(3, 1)_\uparrow$  is the proper orthochronous Lorentz group in four dimensions. A generic element  $(\Lambda, f) \in SO(3, 1)_\uparrow \times C^\infty(\mathbb{S}^2)$  acts on  $p = (u, \zeta, \bar{\zeta}) \in \mathfrak{N}^+$  as [Sa62]

$$u \rightarrow u' \doteq K_\Lambda(\zeta, \bar{\zeta})(u + f(\zeta, \bar{\zeta})), \quad (2.8)$$

$$\zeta \rightarrow \zeta' \doteq \Lambda \zeta \doteq \frac{a_\Lambda \zeta + b_\Lambda}{c_\Lambda \zeta + d_\Lambda}, \quad \bar{\zeta} \rightarrow \bar{\zeta}' \doteq \Lambda \bar{\zeta} \doteq \frac{\bar{a}_\Lambda \bar{\zeta} + \bar{b}_\Lambda}{\bar{c}_\Lambda \bar{\zeta} + \bar{d}_\Lambda}, \quad (2.9)$$

where

$$K_\Lambda(\zeta, \bar{\zeta}) \doteq \frac{(1 + \zeta \bar{\zeta})}{(a_\Lambda \zeta + b_\Lambda)(\bar{a}_\Lambda \bar{\zeta} + \bar{b}_\Lambda) + (c_\Lambda \zeta + d_\Lambda)(\bar{c}_\Lambda \bar{\zeta} + \bar{d}_\Lambda)}, \quad (2.10)$$

while

$$\Pi^{-1}(\Lambda) = \begin{bmatrix} a_\Lambda & b_\Lambda \\ c_\Lambda & d_\Lambda \end{bmatrix},$$

where  $a_\Lambda, b_\Lambda, c_\Lambda, d_\Lambda \in \mathbb{C}$  and  $a_\Lambda d_\Lambda - b_\Lambda c_\Lambda = 1$ .  $\Pi$  is the surjective covering homomorphism from  $SL(2, \mathbb{C})$  to  $SO(3, 1)_\uparrow$  and thus the above matrix is unambiguously fixed up to a global sign, which plays ultimately no role. (2.9) and (2.10) say that  $G_{BMS}$  has the structure of a semidirect product  $SO(3, 1)_\uparrow \ltimes C^\infty(\mathbb{S}^2)$ , the elements of the Abelian subgroup  $C^\infty(\mathbb{S}^2)$  being called **supertranslations**. In particular, if  $\odot$  denotes the product in  $G_{BMS}$ ,  $\circ$  the composition of functions,  $\cdot$  the pointwise product of scalar functions and  $\Lambda$  acts on  $(\zeta, \bar{\zeta})$  as in the right-hand side of (2.9):

$$K_{\Lambda'}(\Lambda(\zeta, \bar{\zeta}))K_\Lambda(\zeta, \bar{\zeta}) = K_{\Lambda'\Lambda}(\zeta, \bar{\zeta}). \quad (2.11)$$

$$(\Lambda', f') \odot (\Lambda, f) = (\Lambda' \Lambda, f + (K_{\Lambda^{-1}} \circ \Lambda) \cdot (f' \circ \Lambda)). \quad (2.12)$$

*Remark 2.5* We underline that in the literature the factor  $K_\Lambda$  does not always have the same definition. In particular, in [MC72, MC73, MC75, GP74, CM74]

$$K_\Lambda(\zeta, \bar{\zeta}) \doteq \frac{(a_\Lambda \zeta + b_\Lambda)(\overline{a_\Lambda \zeta + b_\Lambda}) + (c_\Lambda \zeta + d_\Lambda)(\overline{c_\Lambda \zeta + d_\Lambda})}{(1 + \zeta \bar{\zeta})},$$

but in this paper we stick to the definition (2.10) as in [Sa62] accordingly adapting calculations and results from the above mentioned references.

The following proposition arises from the definition of Bondi frame and from the equations above.

**Proposition 2.3** *Let  $(u, \zeta, \bar{\zeta})$  be a Bondi frame on  $\mathfrak{S}^+$ . The following holds.*

(a) *A global coordinate frame  $(u', \zeta', \bar{\zeta}')$  on  $\mathfrak{S}^+$  is a Bondi frame if and only if*

$$u = u' + g(\zeta', \bar{\zeta}'), \quad (2.13)$$

$$\zeta = \frac{a_R \zeta' + b_R}{c_R \zeta' + d_R}, \quad \bar{\zeta} = \frac{\overline{a_R \zeta' + b_R}}{\overline{c_R \zeta' + d_R}}, \quad (2.14)$$

for  $g \in C^\infty(\mathbb{S}^2)$ , while  $R \in SO(3)$  refers to the canonical inclusion  $SO(3) \hookrightarrow SO(3, 1)_\uparrow$  (i.e. the canonical inclusion  $SU(2) \hookrightarrow SL(2, \mathbb{C})$  for matrices of coefficients  $(a_\Lambda, b_\Lambda, c_\Lambda, d_\Lambda)$  in (2.10).)

(b) *The functions  $K_\Lambda$  are smooth on the Riemann sphere  $\mathbb{S}^2$ . Furthermore  $K_\Lambda(\zeta, \bar{\zeta}) = 1$  for all  $(\zeta, \bar{\zeta})$  if and only if  $\Lambda \in SO(3)$ .*

(c) *Let  $(u', \zeta', \bar{\zeta}')$  be another Bondi frame as in (a). If  $\gamma \in \text{BMS}$  is represented by  $(\Lambda, f)$  in  $(u, \zeta, \bar{\zeta})$ , the same  $\gamma$  is represented by  $(\Lambda', f')$  in  $(u', \zeta', \bar{\zeta}')$  with*

$$(\Lambda', f') = (R, g)^{-1} \odot (\Lambda, f) \odot (R, g). \quad (2.15)$$

A last result, which will play a key role in our analysis, concerns the interplay of the isometries of an asymptotically flat spacetime and the algebra of vector fields on  $\mathfrak{S}^+$  generating the BMS group. We strengthen the results of Proposition 2.2 as follows.

**Proposition 2.4** *For any asymptotically flat spacetime at future null infinity  $(M, g)$ , the following facts hold.*

- (a) *If  $\xi$  is a Killing vector field of  $(M, g)$ , then it extends smoothly to a vector field  $\tilde{\xi}$  on the manifold  $M \cup \mathfrak{S}^+$ . The restriction to  $\mathfrak{S}^+$  of  $\tilde{\xi}$  is uniquely determined by  $\xi$ , and it generates a one-parameter subgroup of  $G_{\text{BMS}}$ .*
- (b) *The linear map  $\xi \rightarrow \tilde{\xi}$  defined in (a) is injective and if, for a fixed  $\xi$  the one-parameter  $G_{\text{BMS}}$ -subgroup generated by  $\tilde{\xi}$  lies in  $C^\infty(\mathbb{S}^2)$  then, more strictly, it must be a subgroup of*

$$T^4 \doteq \left\{ \alpha \in C^\infty(\mathbb{S}^2) \left| \alpha(\zeta, \bar{\zeta}) = \sum_{l=0}^1 \sum_{m=-l}^l \alpha_{lm} Y_{lm}(\zeta, \bar{\zeta}), \quad \alpha_{lm} \in \mathbb{C} \right. \right\}, \quad (2.16)$$

where  $Y_{lm}(\zeta, \bar{\zeta})$  are the standard spherical harmonics.

An explicit proof of item (a) can be found in [Ge77] while that of item (b) in [AX78]. The symbol  $T^4$  has been used on purpose since (2.16) identifies a subgroup of the supertranslations which is isomorphic to the four-dimensional translation group. Two concluding comments are necessary.

1. Although Proposition 2.4 selects a subgroup of the supertranslations isomorphic to the ordinary four-dimensional translation group, we cannot conclude that we can extract a preferred Poincaré subgroup from the BMS group. As a matter of fact, starting from  $\mathcal{P} = SL(2, \mathbb{C}) \ltimes T^4 \subseteq G_{BMS}$  and considering any element of the form  $g = (\mathbb{I}, Y_{lm}(\zeta, \bar{\zeta})) \in G_{BMS}$  with  $l > 1$ , it turns out that  $g^{-1} \odot \mathcal{P} \odot g$  is another subgroup of  $G_{BMS}$  isomorphic to Poincaré group.
2. All our results rely crucially on  $M$  being a four dimensional spacetime. The reasons are manifold, but it is important to notice that, for higher dimensions, the definition itself of an asymptotically flat spacetimes is rather subtle and it offers several difficulties—see [HI05] in particular. In addition, it is possible to impose stronger asymptotic conditions, so to reduce the BMS group to the Poincaré counterpart in higher dimensions—see [HIW06]. Completely different is the situation for asymptotically flat, three dimensional spacetimes for which the BMS group has been studied, proving to be very different from the original four dimensional counterpart. We will not discuss this scenario in details and an interested reader should refer to [ABS97].

## 2.2.4 Expanding Universes with Cosmological Horizon

In this section, we present a second class of backgrounds which are connected to the notion of asymptotic flatness and which will play a key role in our analysis. Our starting point are homogeneous and isotropic four dimensional manifolds  $(M, g_{FRW})$ , also known as **Friedmann-Robertson-Walker** spacetimes. Following [Wa84], their geometry is described by a product manifold  $I \times \Sigma_\kappa$ , where  $I \subseteq \mathbb{R}$  is an open interval and where the metric reads locally

$$g_{FRW} = -dt \otimes dt + a^2(t) \left[ \frac{1}{1 - \kappa r^2} dr \otimes dr + r^2 dS^2(\theta, \phi) \right]. \quad (2.17)$$

Here  $\kappa$  is a constant, which up to a normalization, can take the values  $-1, 0, 1$ . Depending on the choice of these values  $\Sigma_\kappa$ , equipped with the metric in the square bracket, is modelled respectively over one of the three simply-connected manifolds, the hyperbolic space  $\mathbb{H}^3$ ,  $\mathbb{R}^3$ ,  $\mathbb{S}^3$ , or over a non-simply-connected manifold constructed out of one of these via an identification under the action of a discrete group of isometries. In view of this remark and also, taking into account the present, more favoured model in cosmology, we will fix  $\kappa = 0$  and we will assume that  $\Sigma_\kappa$  is isometric to  $\mathbb{R}^3$  with the standard Euclidean flat metric. The only unknown quantity in (2.17) is  $a(t)$ , which is assumed to be a smooth and strictly positive function whose

explicit form has to be determined via the Einstein equations. The coordinate  $t$  is referred as the **proper time** of co-moving observers. By hypothesis,  $\partial_t$  defines the time orientation of  $(M, g_{FRW})$ . We can associate to (2.17) two additional relevant structures: Consider a co-moving observer pictured by an integral line  $\gamma = \gamma(t)$ ,  $t \in I$ , of  $\partial_t$ .

1. If  $J^-(\gamma)$  does not cover the whole spacetime  $M$ , the observer cannot receive information from some points of  $M$ . Using the terminology of [Ri06], we call the three dimensional null hypersurfaces  $\partial J^-(\gamma)$  **cosmological event horizon for  $\gamma$** .
2. If  $J^+(\gamma)$  does not cover the whole spacetime  $M$ , the observer cannot send information to some points of  $M$ . In this case we call the three dimensional null hypersurfaces  $\partial J^+(\gamma)$ , the **cosmological particle horizon for  $\gamma$** .

Another representation of a Friedmann-Robertson-Walker metric (2.17) for  $\kappa = 0$  is

$$g_{FRW} = a^2(\tau) \left[ -d\tau^2 + dr \otimes dr + r^2 d\mathbb{S}^2(\theta, \phi) \right], \quad (2.18)$$

where the **conformal time**  $\tau$  has been defined as

$$\tau(t) = d + \int a^{-1}(t) dt, \quad (2.19)$$

$d \in \mathbb{R}$  being any fixed constant. By construction  $\tau = \tau(t)$  is a diffeomorphism from  $I$  onto a possibly unbounded interval  $(\alpha, \beta) \ni \tau$ .  $\partial_\tau$  is a conformal Killing vector field whose integral lines coincide with those of  $\partial_t$  up to re-parametrisation.

In (2.18),  $a(\tau)$  plays the role of a conformal factor with respect to the Minkowski metric appearing in the square brackets. Since the causal structure is preserved under smooth conformal transformations, we can study  $J^\pm(\gamma)$  referring to the Minkowski metric. A straightforward analysis establishes that  $J^+(\gamma)$  and  $J^-(\gamma)$  do not cover the whole  $M$  respectively whenever  $\alpha > -\infty$  and  $\beta < +\infty$ . In both cases the horizons  $\partial J^-(\gamma)$  and  $\partial J^+(\gamma)$  are null 3-hypersurfaces diffeomorphic to  $\mathbb{R} \times \mathbb{S}^2$ , made of null geodesics of  $g_{FRW}$ . In some models  $\alpha$  and  $\beta$  can be interpreted, when they are finite, as the *big-bang* conformal time or the *big-crunch* conformal time respectively. It is worth noticing that the cosmological horizons introduced above generally depend on the fixed co-moving observer  $\gamma$ . Yet, it is important to bear in mind that the requirement on the finiteness of  $\alpha$  and  $\beta$  are *sufficient* conditions for the existence of the cosmological horizons, but they are by no means necessary.

Concerning cosmological horizons, another more subtle and physically intriguing possibility exists. Following the prototypical example of the cosmological de Sitter spacetime, it may happen that the manifold  $(M, g_{FRW})$  can be realized as an open subset of a larger spacetime  $(\tilde{M}, \tilde{g})$  with physical meaning so that cosmological horizons may exist in  $\tilde{M}$ . Furthermore they coincide with  $\partial M$  which turns out to be a null 3-surface in  $(\tilde{M}, \tilde{g})$  similar to  $\mathfrak{S}^\pm$  for asymptotically flat spacetimes. In these cases the horizons are independent from any choice of co-moving observer  $\gamma$  in  $M$ .

In the following, we shall focus on this type of cosmological horizons and our first goal is their characterization. To this end, let us make more precise the picture outlined above. Starting from (2.18), we rescale  $g_{FRW}$  with the conformal factor  $\Omega \doteq a(\tau)$  and we observe that  $g \doteq \Omega^{-2} g_{FRW}$ , is nothing but (a subset of) Minkowski spacetime which is asymptotically flat at null infinity so that  $\mathfrak{I}^\pm$  can be defined. More precisely, if either  $\alpha = -\infty$  or  $\beta = +\infty$ ,  $(M, g)$  admits a corresponding past or future conformal completion  $(\tilde{M}, \tilde{g})$  in accordance with Definition 2.3 where in particular  $\tilde{g}|_M = \Omega^2 g = g_{FRW}$ . This means that  $(\tilde{M}, \tilde{g})$  extends  $(M, g_{FRW})$  and includes one of the null hypersurfaces  $\mathfrak{I}^\pm$  which is the boundary  $\partial M$  of  $M \subseteq \tilde{M}$ . Since  $\mathfrak{I}^\pm \cap J^\mp(M; \tilde{M}) = \emptyset$ , this hypersurface can be viewed as a cosmological horizon in common with all observers co-moving with the metric  $g_{FRW}$  in  $M$  itself. The following theorem characterizes when the existence of the conformal boundary is guaranteed. We omit the long proof, which can be found in [DMP09a]:

**Theorem 2.1** *Let  $(M, g_{FRW})$  be a simply connected Friedmann-Robertson-Walker spacetime for  $\kappa = 0$ , with*

$$M \simeq (\alpha, \beta) \times \mathbb{R}^3, \quad g_{FRW} = a^2(\tau) [-d\tau \otimes d\tau + dr \otimes dr + r^2 dS^2(\theta, \phi)],$$

where  $\tau \in (\alpha, \beta)$  and where  $r, \theta, \varphi$  are the standard spherical coordinates on  $\mathbb{R}^3$ . Suppose that there exists  $\gamma \in \mathbb{R}$  with

$$a(\tau) = -\frac{1}{H\tau} + O\left(\frac{1}{H^2}\right), \quad \frac{da(\tau)}{d\tau} = \frac{1}{H\tau^2} + O\left(\frac{1}{\tau^3}\right) \quad (2.20)$$

for either  $(\alpha, \beta) \doteq (-\infty, 0)$  and  $H > 0$ , or  $(\alpha, \beta) \doteq (0, +\infty)$  and  $H < 0$ . The above asymptotic values are meant to be taken as  $\tau \rightarrow -\infty$  or  $\tau \rightarrow +\infty$  respectively. The following holds.

- (a) *The spacetime  $(M, g_{FRW})$  extends smoothly to a larger spacetime  $(\tilde{M}, \tilde{g})$ , which is a past conformal completion of the asymptotically flat spacetime at past, or at future, null infinity, respectively,  $(M, a^{-2}g_{FRW})$  with  $\Omega = a$ .*
- (b) *The manifold  $M \cup \mathfrak{I}^\pm$  enjoys the following properties:*

- a. *the vector field  $\partial_\tau$  is a conformal Killing vector for  $\tilde{g}$  in  $M$  with conformal Killing equation*

$$\mathcal{L}_{\partial_\tau} \tilde{g} = -2\partial_\tau(\ln a) \tilde{g}.$$

where the right-hand side vanishes approaching  $\mathfrak{I}^\pm$ .

- b.  *$\partial_\tau$  tends to become tangent to  $\mathfrak{I}^\pm$  approaching it and coincides to  $H^{-1}\widehat{\nabla}^b a$  thereon.*
- c. *The metric on  $\mathfrak{I}^\pm$  takes the geodesically complete Bondi form up to the constant factor  $H^{-2} \neq 0$ :*

$$\widehat{g}|_{\mathfrak{I}^\pm} = H^{-2}(-du \otimes da - da \otimes du + dS^2(\theta, \phi)), \quad (2.21)$$

$u \in \mathbb{R}$  being the parameter of the integral lines of  $n \doteq \nabla a$ .

*Remark 2.6* The statements (a),(b) hold true also if we change  $g_{FRW}$  smoothly inside a region  $M_0 \simeq (\alpha, \beta) \times \Sigma_0$  for a compact  $\Sigma_0 \subseteq \mathbb{R}^3$ . In this case  $\partial_\tau$  is a conformal Killing vector of the metric at least in  $M \setminus M_0$ . In the said hypotheses, one can find an open neighbourhood of  $\mathfrak{N}^\pm$  where the above construction can be adapted.

As an example, consider the metric (2.17) where

$$a(t) = a_0 e^{\alpha t} \quad \text{with } a_0, \alpha > 0 \text{ constant.} \quad (2.22)$$

This represents a conformally static subregion of *de Sitter spacetime* where the cosmological constant is  $\Lambda = 3\alpha^2$ . This can be considered a realistic model of the observed universe assuming, as done nowadays, that dark energy dominates among the various sources to gravitation in the framework of Einstein theory of gravity. In this case one can fix the integration constant in (2.19) so that  $\tau = -e^{-\alpha t}/(a_0\alpha)$  and thus  $\tau \in (-\infty, 0)$ . In this case  $a(\tau) = -c/\tau$ , where  $c = 1/\alpha$ , and thus we can use the obtained result requiring that  $(M, g_{FRW})$  admits a cosmological particle horizon in common with all the observers whose world lines are the integral curves of  $\partial_t$ , and that horizon coincides with  $\mathfrak{N}^-$ .

A more complicated example is a spacetime  $(M, g)$  with metric

$$g(\tau, p) = -H^{-2}(p)d\tau \otimes d\tau + a(t)^2 h(p), \quad \text{for } (t, p) \in \mathbb{R} \times \Sigma \simeq M. \quad (2.23)$$

where  $a(t)$  is as in the previous examples,  $\Sigma$  is diffeomorphic to  $\mathbb{R}^3$  while  $h = dr \otimes dr + r^2 dS^2(\theta, \phi)$  and  $H = -1$ , outside a compact set in  $S$ . As in the previous examples, this spacetime  $(M, g)$  is globally hyperbolic provided the Euclidean metric  $h$  on  $\Sigma$  is complete. This is due to  $g$  being conformally equivalent to the metric whose line element reads

$$g_0(\tau, p) = -H^{-2}(p)d\tau \otimes d\tau + h(p), \quad \text{for } (\tau, p) \in \mathbb{R} \times \Sigma \equiv M. \quad (2.24)$$

Any spacetime  $(M, g)$  with a static metric (2.24) is globally hyperbolic if there exist  $c_1, c_2 \in \mathbb{R}$  with  $c_1 \geq (-H^{-1})(p) \geq c_2 > 0$  for every  $p \in \Sigma$  and if  $(\Sigma, h)$  is complete [Ka78].

### 2.2.5 The Cosmological-Horizon Symmetry Group

In view of the relation established between asymptotically flat spacetimes at null infinity and expanding FRW spacetimes with a suitable rate of expansion  $a$ , we can investigate further the geometric properties of the latter. In particular we want to establish which is the counterpart in this class of spacetimes of the BMS group which we introduced in Sect. 2.2.3. The best course of action is to consider the manifolds introduced in Theorem 2.1 as a specialization of the following more general class:

**Definition 2.5** A globally hyperbolic spacetime  $(M, g)$  equipped with a positive smooth function  $\Omega : M \rightarrow \mathbb{R}^+$  and a future-oriented timelike vector field  $X$  on  $M$ , will be called an **expanding universe with geodesically complete cosmological particle horizon** if

1. **Existence and causal properties of horizon.**  $(M, g)$  can be embedded isometrically as the interior of a submanifold with boundary of a larger spacetime  $(\tilde{M}, \tilde{g})$ , the boundary  $\mathfrak{I}^- \doteq \partial M$  verifying  $\mathfrak{I}^- \cap J^+(M; \tilde{M}) = \emptyset$ .
2.  **$\Omega$ - $\mathfrak{I}^-$  interplay.**  $\Omega$  extends to a smooth function on  $\hat{M}$  such that  $\Omega|_{\mathfrak{I}^-} = 0$  while  $d\Omega \neq 0$  everywhere on  $\mathfrak{I}^+$ .
3.  **$X$ - $\Omega$ - $\tilde{g}$ - $\mathfrak{I}^-$  interplay.**  $X$  is a conformal Killing vector for  $\tilde{g}$  in a neighbourhood of  $\mathfrak{I}^-$  in  $M$ , with

$$\mathcal{L}_X(\tilde{g}) = -2X(\ln \Omega) \tilde{g}, \quad (2.25)$$

where  $X(\ln \Omega) \rightarrow 0$  approaching  $\mathfrak{I}^-$  and where  $X$  does not tend everywhere to the zero vector approaching  $\mathfrak{I}^-$ .

4. **Bondi-form of the metric on  $\mathfrak{I}^-$  and geodesic completeness.**  $\mathfrak{I}^-$  is diffeomorphic to  $\mathbb{R} \times \mathbb{S}^2$  and the metric  $\tilde{g}$  restricted thereon takes the Bondi form up to a possible constant factor  $\gamma^2 > 0$ , that is

$$\tilde{g}|_{\mathfrak{I}^-} = \gamma^2 (-du \otimes d\Omega - d\Omega \otimes du + d\mathbb{S}^2(\theta, \phi)), \quad (2.26)$$

$d\mathbb{S}^2$  being the standard metric on the unit 2-sphere, so that  $\mathfrak{I}^-$  is a null 3-submanifold, and the curves  $\mathbb{R} \ni u \mapsto (u, \theta, \phi)$  are complete null  $\tilde{g}$ -geodesics.

The manifold  $\mathfrak{I}^-$  is called the **cosmological (particle) horizon of  $M$** . The integral parameter of  $X$  is called the **conformal cosmological time**.

A similar definition can be given replacing past null infinity everywhere with  $\mathfrak{I}^+$ . As already mentioned we will not discuss this case any further.

*Remark 2.7* (1) In view of condition 3, the vector  $X$  is a Killing vector of the metric  $g_0 \doteq \Omega^{-2}g$  in a neighbourhood of  $\mathfrak{I}^-$  in  $M$ . In this neighbourhood (which may coincide with the whole  $M$ ), one can think of  $\Omega$  as an expansion scale evolving with rate  $X(\Omega)$  referred to the conformal cosmological time.

(2) By standard properties of causal sets [Wa84], it is possible to prove that  $\mathfrak{I}^- \cap J^+(M; \tilde{M}) = \emptyset$  entails  $M = I^+(M; \tilde{M})$  and  $\mathfrak{I}^- = \partial M = \partial I^+(M; \tilde{M}) = \partial J^+(M; \tilde{M})$ , so that  $\mathfrak{I}^-$  has the proper interpretation as a particle cosmological horizon in common for all the observers in  $(M, g)$  evolving along the integral lines of  $X$ .

(3) It is worth stressing that the spacetimes considered in the given definition are neither homogeneous nor isotropous in general; this is a relevant extension with respect to the family of FRW spacetimes.

(4) Assuming Definition 2.5, the null geodesics in item (4) are the (complete) integral curves,  $u$  is an affine parameter and  $\tilde{\nabla}^a \Omega = -\gamma^{-2} (\partial_u)^a$  on  $\mathfrak{I}^-$ , which is totally geodesic.

(5) In the following, **expanding universe with cosmological horizon** will mean expanding universe with geodesically complete cosmological particle horizon.

An important geometric property of the conformal Killing vector field  $X$  in Definition 2.5 is that it becomes tangent to past null infinity coinciding up a multiplicative factor with  $\partial_u$ . The following proposition establishes this fact and the proof can be found in [DMP09a].

**Proposition 2.5** *If  $(M, g, \Omega, X, \gamma)$  is an expanding universe with cosmological horizon, the following facts hold.*

- (a)  $X$  extends smoothly to a unique smooth vector field  $\tilde{X}$  on  $\mathfrak{I}^-$ , which may vanish on a closed subset of  $\mathfrak{I}^-$  with empty interior at most. The obtained extension of  $X$  to  $M \cup \mathfrak{I}^-$  fulfils the Killing equation on  $\mathfrak{I}^-$  with respect to the metric  $\tilde{g}$ .
- (b)  $\tilde{X} = f\partial_u$ , where  $f$  depends only on the variables on  $\mathbb{S}^2$ , being smooth and nonnegative.

It is important to stress the role of the function  $f$  in the preceding proposition, as  $f = 1$  on FRW spacetimes. Since 2.5 encompasses a class of backgrounds larger than the cosmological ones, we can interpret a non constant function  $f$  as a measure of the failure of homogeneity and isotropy.

We can now look for a subgroup  $SG_{\mathfrak{I}^-}$  of the isometries of  $\mathfrak{I}^-$  with physical relevance. All the results that we will present are proven in [DMP09a]. We start from a preliminary, yet very useful proposition:

**Proposition 2.6** *If  $(M, g, \Omega, X, \gamma)$  is an expanding universe with cosmological horizon and  $Y$  is a Killing vector field of  $(M, g)$ ,  $Y$  can be extended to a smooth vector field  $\hat{Y}$  defined on  $M$ . In addition the following facts hold true:*

1.  $\mathcal{L}_{\hat{Y}}g = 0$  on  $M \cup \mathfrak{I}^-$ ;
2.  $\hat{Y} \doteq \tilde{Y}|_{\mathfrak{I}^-}$  is uniquely determined by  $Y$  and it is tangent to  $\mathfrak{I}^-$  if and only if  $g(Y, X)$  vanishes approaching  $\mathfrak{I}^-$  from  $M$ .

*If we restrict the attention to the linear space of Killing fields  $Y$  on  $(M, g)$  such that  $g(Y, X) \rightarrow 0$  approaching  $\mathfrak{I}^-$ , the following additional facts hold true.*

- If  $\tilde{Y}$  vanishes in some  $A \subseteq \mathfrak{I}^-$  and  $A \neq \emptyset$  is open with respect to the topology of  $\mathfrak{I}^-$ , then  $Y = 0$  everywhere in  $M$ . Hence  $\hat{Y}$  vanishes in the whole  $M \cup \mathfrak{I}^-$ .
- The linear map  $Y \mapsto \tilde{Y}$  is injective.

The most relevant consequence off Proposition 2.6 is that all Killing vectors  $Y$  in  $M$  with  $g(Y, X) \rightarrow 0$  approaching  $\mathfrak{I}^-$  extend to Killing vectors of  $(\mathfrak{I}^-, h)$ ,  $h$  being the degenerate metric on  $\mathfrak{I}^-$  induced by  $\tilde{g}$ . These Killing vectors of  $(M, g)$  are represented on  $\mathfrak{I}^-$  faithfully. In other words all the geometric symmetries of the bulk spacetimes  $(M, g)$  are codified on the boundary  $\mathfrak{I}^-$ .

**Definition 2.6** *If  $(M, g, \Omega, X, \gamma)$  is an expanding universe with cosmological horizon, a Killing vector field of  $(M, g)$ ,  $Y$ , is said to **to preserve**  $\mathfrak{I}^-$  if  $g(Y, X) \rightarrow 0$  approaching  $\mathfrak{I}^-$ . Similarly, the Killing isometries of the (local) one-parameter group generated by  $Y$  are said **to preserve**  $\mathfrak{I}^-$ .*

In the rest of this section we shall consider the one-parameter group of isometries of  $(\mathfrak{I}^-, h)$  generated by such Killing vectors  $\widehat{Y}|_{\mathfrak{I}^-}$ . These account only for part of the isometries of  $(\mathfrak{I}^-, h)$  as one can infer considering the isometry constructed out of the coordinates  $(u, \theta, \phi) \in \mathbb{R} \times \mathbb{S}^2 \equiv \mathfrak{I}^-$ , and of the smooth diffeomorphisms  $f : \mathbb{R} \rightarrow \mathbb{R}$  generating the transformation

$$u \rightarrow f(u), \quad (\theta, \phi) \rightarrow (\theta, \phi). \quad (2.27)$$

These isometries of  $\mathfrak{I}^-$  play a role very similar to those of the elements of the Newman-Unti group in asymptotically flat spacetimes which are not part of the BMS group [NU62]. However only diffeomorphisms of the form  $f(u) = au + b$  with  $a \neq 0$  can be isometries generated by the restriction  $\widehat{Y}|_{\mathfrak{I}^-}$  to  $\mathfrak{I}^-$  of extensions of Killing fields  $Y$  of  $(M, g)$  as in the Proposition 2.6. This is because those isometries are restrictions of isometries of the manifolds with boundary  $(M \cup \mathfrak{I}^-, \widetilde{g}|_{M \cup \mathfrak{I}^-})$ , and thus they *preserve the null  $\widetilde{g}$ -geodesics* in  $\mathfrak{I}^-$ . The requirement that, for all constants  $a, b \in \mathbb{R}$ ,  $a \neq 0$ , there exist constants  $a', b' \in \mathbb{R}$ ,  $a' \neq 0$  such that  $f(au + b) = a'u + b'$  for all  $u$  varying in a fixed nonempty interval  $J$ , is fulfilled only if  $f$  is an affine transformation as said above. If we include also the transformations of angular coordinates, we end up studying the class  $G_{\mathfrak{I}^-}^+$  of diffeomorphisms of  $\mathfrak{I}^-$  of the form

$$u \rightarrow u' \doteq f(u, \theta, \phi), \quad (\theta, \phi) \rightarrow (\theta', \phi') \doteq g(u, \theta, \phi) \quad (2.28)$$

where  $u \in \mathbb{R}$  and  $(\theta, \phi) \in \mathbb{S}^2$  and where these transformations are isometries of the degenerate metric  $h$  induced by  $\widetilde{g}|_{\mathfrak{I}^-}$  (2.26). Following a lengthy analysis, discussed thoroughly in [DMP09a], it is possible to classify all Killing isometries of the degenerate metric  $h$  on  $\mathfrak{I}^-$  which are restrictions of possible Killing  $\widetilde{g}$ -isometries of  $M \cup \mathfrak{I}^-$ . The next definition summarizes the result:

**Definition 2.7** The **horizon symmetry group**  $SG_{\mathfrak{I}^-}$  is the group of all diffeomorphisms of  $\mathbb{R} \times \mathbb{S}^2$ ,

$$F_{(a,b,R)} : \mathbb{R} \times \mathbb{S}^2 \ni (u, \theta, \phi) \mapsto (e^{a(\theta, \phi)} u + b(\theta, \phi), R(\theta, \phi)) \in \mathbb{R} \times \mathbb{S}^2 \quad (2.29)$$

with  $u \in \mathbb{R}$  and  $(\theta, \phi) \in \mathbb{S}^2$ , where  $a, b \in C^\infty(\mathbb{S}^2)$  are arbitrary smooth functions and  $R \in SO(3)$ .

The **Horizon Lie algebra**  $\mathfrak{g}_{\mathfrak{I}^-}$  is the infinite-dimensional Lie algebra of smooth vector fields on  $\mathbb{R} \times \mathbb{S}^2$  generated by the fields

$$S_1, S_2, S_3, \beta \partial_u, \alpha u \partial_u, \quad \text{for all } \alpha, \beta \in C^\infty(\mathbb{S}^2).$$

$S_1, S_2, S_3$  indicate the three smooth vector fields on the unit sphere  $\mathbb{S}^2$  generating rotations about the orthogonal axes, respectively,  $x, y$  and  $z$ .

$SG_{\mathfrak{I}^-}$  depends on the geometry of  $\mathfrak{I}^-$  but not on that of  $(M, g)$ . In this sense it enjoys the same properties of the BMS group, namely it is *universal* for the whole

class of expanding spacetimes with cosmological horizon. As a set  $SG_{\mathfrak{S}^-}$  coincides with  $SO(3) \times C^\infty(\mathbb{S}^2) \times C^\infty(\mathbb{S}^2)$ . In order to unveil the group structure, consider an arbitrary  $F_{a,b,R}$  and indicate it with the triple  $(R, a, b)$ . Per direct inspection of (2.29), we see that the composition between elements in  $SG_{\mathfrak{S}^-}$  can be defined as

$$(R, a, b) \odot (R', a', b') \doteq (RR', a' + a \circ R', e^{a \circ R'} b' + b \circ R')$$

for all  $(R, a, b), (R', a', b') \in SO(3) \times C^\infty(\mathbb{S}^2) \times C^\infty(\mathbb{S}^2)$ .

where  $\circ$  denotes the usual composition of functions.

We state a few additional results aimed at characterizing the properties of  $SG_{\mathfrak{S}^-}$ . In the next proposition we emphasize instead that  $\mathfrak{g}_{\mathfrak{S}^-}$  could be considered as the Lie algebra of  $SG_{\mathfrak{S}^-}$ , although a careful analysis of this point will not be discussed in this work. As usual, the proofs can be found in [DMP09a]:

**Proposition 2.7** *Referring to Definition 2.7, the following facts hold.*

- Each vector field  $Z \in \mathfrak{g}_{\mathfrak{S}^-}$  is complete and it generates a one-parameter group of diffeomorphisms of  $\mathbb{R} \times \mathbb{S}^2$ ,  $\{\exp\{tZ\}\}_{t \in \mathbb{R}}$ , subgroup of  $SG_{\mathfrak{S}^-}$ .
- For every  $F \in SG_{\mathfrak{S}^-}$  there are  $Z_1, Z_2 \in \mathfrak{g}_{\mathfrak{S}^-}$ —with, possibly,  $Z_1 = Z_2$ —such that  $F = \exp\{t_1 Z_1\} \exp\{t_2 Z_2\}$  for some real numbers  $t_1, t_2$ .

**Theorem 2.2** *Let  $(M, g, \Omega, X, \gamma)$  be an expanding universe with cosmological horizon and  $Y$  a Killing vector field of  $(M, g)$  preserving  $\mathfrak{S}^-$ . Then*

1. *The unique smooth extension  $\tilde{Y}$  of  $Y$  to  $\mathfrak{S}^-$  belongs to  $\mathfrak{g}_{\mathfrak{S}^-}$ ,*
2.  *$\exp\{t\tilde{Y}\}_{t \in \mathbb{R}}$  is a subgroup of  $SG_{\mathfrak{S}^-}$ .*

As an example consider the expanding universe  $M$  with cosmological horizon associated with the metric  $g_{FRW}$  (2.18) with  $\kappa = 1$  and  $a$  as in (a) of Theorem 2.1. In this case  $X \doteq \partial_\tau$  and there are a lot of Killing vectors  $Y$  of  $(M, g_{FRW})$  satisfying  $g_{FRW}(Y, X) \rightarrow 0$  approaching  $\mathfrak{S}^-$ . The main ones are those of the surfaces at  $\tau = \text{constant}$  with respect to the induced metric. They form a Lie algebra generated by 6 independent Killing vectors  $Y$  representing, respectively, space translations and space rotations. In this case  $g_{FRW}(Y, X) = 0$  so that the associated Killing vectors  $\tilde{Y}|_{\mathfrak{S}^-}$  belongs to  $\mathfrak{g}_{\mathfrak{S}^-}$ .

We state a last technical result whose proof is in [DMP09a].

**Proposition 2.8** *Let  $(M, g, \Omega, X, \gamma)$  be an expanding universe with cosmological horizon and  $Y$  a smooth vector field of  $(M, g)$  which tends to the smooth field  $\tilde{Y} \in \mathfrak{g}_{\mathfrak{S}^-}$  pointwisely. If there exists an open set  $A \subseteq \tilde{M}$  with  $A \supseteq \mathfrak{S}^-$  and such that  $Y|_{A \cap M}$  is timelike and future directed, then, everywhere on  $\mathfrak{S}^-$ ,*

$$\tilde{Y}(u, \theta, \phi) = f(\theta, \phi) \partial_u, \quad \text{for some } f \in C^\infty(\mathbb{S}^2), \text{ with } f(\theta, \phi) \geq 0 \text{ on } \mathbb{S}^2. \quad (2.30)$$

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Dappiaggi, C.; Moretti, V.; Pinamonti, N.

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