Preface

This book began as course notes prepared for a class taught at Columbia University during the 2012–13 academic year. The intent was to cover the basics of quantum mechanics, up to and including relativistic quantum field theory of free fields, from a point of view emphasizing the role of unitary representations of Lie groups in the foundations of the subject. It has been significantly rewritten and extended since that time, partially based upon experience teaching the same material during 2014–15.

The approach to this material is simultaneously rather advanced, using crucially some fundamental mathematical structures discussed, if at all, only in graduate mathematics courses, while at the same time trying to do this in as elementary terms as possible. The Lie groups needed are (with one crucial exception) ones that can be described simply in terms of matrices. Much of the representation theory will also just use standard manipulations of matrices. The only prerequisite for the course as taught was linear algebra and multivariable calculus (while a full appreciation of the topics covered would benefit from quite a bit more than this). My hope is that this level of presentation will simultaneously be useful to mathematics students trying to learn something about both quantum mechanics and Lie groups and their representations, as well as to physics students who already have seen some quantum mechanics, but would like to know more about the mathematics underlying the subject, especially that relevant to exploiting symmetry principles.

The topics covered emphasize the mathematical structure of the subject and often intentionally avoid overlap with the material of standard physics courses in quantum mechanics and quantum field theory, for which many excellent textbooks are available. This document is best read in conjunction with such a text. In particular, some experience with the details of the physics not covered here is needed to truly appreciate the subject. Some of the main differences with standard physics presentations include the following:

- The role of Lie groups, Lie algebras, and their unitary representations is systematically emphasized, including not just the standard use of these to derive consequences for the theory of a "symmetry" generated by operators commuting with the Hamiltonian.
- Symplectic geometry and the role of the Lie algebra of functions on phase space in the classical theory of Hamiltonian mechanics are emphasized. "Quantization" is then the passage to a unitary representation (unique by the Stone–von Neumann theorem) of a subalgebra of this Lie algebra.
- The role of the metaplectic representation and the subtleties of the projective factor involved are described in detail. This includes phenomena depending on the choice of a complex structure, a topic known to physicists as "Bogoliubov transformations."
- The closely parallel story of the Clifford algebra and spinor representation is extensively investigated. These are related to the Heisenberg Lie algebra and the metaplectic representation by interchanging commutative ("bosonic") and anticommutative ("fermionic") generators, introducing the notion of a "Lie superalgebra" generalizing that of a Lie algebra.
- Many topics usually first encountered in physics texts in the context of relativistic quantum field theory are instead first developed in simpler non-relativistic or finite dimensional contexts. Non-relativistic quantum field theory based on the Schrödinger equation is described in detail before moving on to the relativistic case. The topic of irreducible representations of space–time symmetry groups is first addressed with the case of the Euclidean group, where the implications for the non-relativistic theory are explained. The analogous problem for the relativistic case, that of the irreducible representations of the Poincaré group, is then worked out later on.
- The emphasis is on the Hamiltonian formalism and its representation-theoretical implications, with the Lagrangian formalism (the basis of most quantum field theory textbooks) de-emphasized. In particular, the operators generating symmetry transformations are derived using the moment map for the action of such transformations on phase space, not by invoking Noether's theorem for transformations that leave invariant a Lagrangian.
- Care is taken to keep track of the distinction between vector spaces and their duals. It is the dual of phase space (linear coordinates on phase space) that appears in the Heisenberg Lie algebra, with quantization a representation of this Lie algebra by linear operators.

- The distinction between real and complex vector spaces, along with the role of complexification and choice of a complex structure, is systematically emphasized. A choice of complex structure plays a crucial part in quantization using annihilation and creation operator methods, especially in relativistic quantum field theory, where a different sort of choice than in the non-relativistic case is responsible for the existence of antiparticles.

Some differences with other mathematics treatments of this material are as follows:

- A fully rigorous treatment of the subject is not attempted. At the same time, an effort is made to indicate where significant issues arise should one pursue such a treatment, and to provide references to rigorous discussions of these issues. An attempt is also made to make clear the difference between where a rigorous treatment could be pursued relatively straightforwardly and where there are serious problems of principle making a rigorous treatment hard to achieve.

- The discussion of Lie groups and their representations is focused on specific examples, not the general theory. For compact Lie groups, emphasis is on the groups $U(1), SO(3), SU(2)$ and their finite dimensional representations. Central to the basic structure of quantum mechanics are the Heisenberg group, the symplectic groups $Sp(2n, \mathbf{R})$ and the metaplectic representation, as well as the spinor groups and the spin representation. The geometry of space–time leads to the study of Euclidean groups in two and three dimensions, and the Lorentz (SO (3,1)) and Poincaré groups, together with their representations. These examples of non-compact Lie groups are a fundamental feature of quantum mechanics, but not a conventional topic in the mathematics curriculum.

- A central example studied thoroughly and in some generality is that of the metaplectic representation of the double cover of $Sp(2n, \mathbf{R})$ (in the commutative case), or spin representation of the double cover of $SO(2n, \mathbf{R})$ (anticommutative case). This specific example of a representation provides the foundation of quantum theory, with quantum field theory involving a generalization to the case of n infinite.

- No attempt is made to pursue a general notion of quantization, despite the great mathematical interest of such generalizations. In particular, attention is restricted to the case of linear symplectic manifolds. The linear structure plays a crucial role, with quantization given by a representation of a Heisenberg algebra in the commutative case and a Clifford algebra in the anticommutative case. The very explicit methods used (staying close to the physics formalism) mostly do not apply to more general conceptions of quantization (e.g., geometric quantization) of mathematical interest for their applications in representation theory.

The scope of material covered in later sections of the book is governed by a desire to give some explanation of what the central mathematical objects are that occur in the Standard Model of particle physics, while staying within the bounds of a one-year course. The Standard Model embodies our best current understanding of the fundamental nature of reality, making a better understanding of its mathematical nature a central problem for anyone who believes that mathematics and physics are intimately connected at their deepest levels. The author hopes that the treatment of this subject here will be helpful to anyone interested in pursuing a better understanding of this connection.

0.1 Acknowledgements

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