

Chapter 2

The Group $U(1)$ and its Representations

The simplest example of a Lie group is the group of rotations of the plane, with elements parametrized by a single number, the angle of rotation θ . It is useful to identify such group elements with unit vectors $e^{i\theta}$ in the complex plane. The group is then denoted $U(1)$, since such complex numbers can be thought of as 1 by 1 unitary matrices. We will see in this chapter how the general picture described in chapter 1 works out in this simple case. State spaces will be unitary representations of the group $U(1)$, and we will see that any such representation decomposes into a sum of one-dimensional representations. These one-dimensional representations will be characterized by an integer q , and such integers are the eigenvalues of a self-adjoint operator we will call Q , which is an observable of the quantum theory.

One motivation for the notation Q is that this is the conventional physics notation for electric charge, and this is one of the places where a $U(1)$ group occurs in physics. Examples of $U(1)$ groups acting on physical systems include:

- Quantum particles can be described by a complex-valued “wavefunction” (see chapter 10), and $U(1)$ acts on such wavefunctions by point-wise phase transformations of the value of the function. This phenomenon can be used to understand how particles interact with electromagnetic fields, and in this case, the physical interpretation of the eigenvalue of the Q operator will be the electric charge of the state. We will discuss this in detail in chapter 45.
- If one chooses a particular direction in three-dimensional space, then the group of rotations about that axis can be identified with the group $U(1)$. The eigenvalues of Q will have a physical interpretation as the quantum version of angular momentum in the chosen direction. The fact that such eigenvalues are not continuous, but integral, shows that quantum angular momentum has quite different behavior than classical angular momentum.

- When we study the harmonic oscillator (chapter 22), we will find that it has a $U(1)$ symmetry (rotations in the position-momentum plane) and that the Hamiltonian operator is a multiple of the operator Q for this case. This implies that the eigenvalues of the Hamiltonian (which give the energy of the system) will be integers times some fixed value. When one describes multiparticle systems in terms of quantum fields, one finds a harmonic oscillator for each momentum mode, and then the Q for that mode counts the number of particles with that momentum.

We will sometimes refer to the operator Q as a “charge” operator, assigning a much more general meaning to the term than that of the specific example of electric charge. $U(1)$ representations are also ubiquitous in mathematics, where often the integral eigenvalues of the Q operator will be called “weights.”

In a very real sense, the reason for the “quantum” in “quantum mechanics” is precisely because of the role of $U(1)$ groups acting on the state space. Such an action implies observables that characterize states by an integer eigenvalue of an operator Q , and it is this “quantization” of observables that motivates the name of the subject.

2.1 Some representation theory

Recall the definition of a group representation:

Definition (Representation). *A (complex) representation (π, V) of a group G on a complex vector space V (with a chosen basis identifying $V \simeq \mathbf{C}^n$) is a homomorphism*

$$\pi : G \rightarrow GL(n, \mathbf{C})$$

This is just a set of n by n matrices, one for each group element, satisfying the multiplication rules of the group elements. n is called the dimension of the representation.

We are mainly interested in the case of G a Lie group, where G is a differentiable manifold of some dimension. In such a case, we will restrict attention to representations given by differentiable maps π . As a space, $GL(n, \mathbf{C})$ is the space \mathbf{C}^{n^2} of all n by n complex matrices, with the locus of non-invertible (zero determinant) elements removed. Choosing local coordinates on G , π will be given by $2n^2$ real functions on G , and the condition that G is a differentiable manifold means that the derivative of π is consistently defined. Our focus will be not on the general case, but on the study of certain specific Lie groups and representations π which are of central interest in quantum mechanics. For these representations, one will be able to readily see that the maps π are differentiable.

To understand the representations of a group G , one proceeds by first identifying the irreducible ones:

Definition (Irreducible representation). *A representation π is called irreducible if it has no subrepresentations, meaning nonzero proper subspaces $W \subset V$ such that $(\pi|_W, W)$ is a representation. A representation that does have such a subrepresentation is called reducible.*

Given two representations, their direct sum is defined as:

Definition (Direct sum representation). *Given representations π_1 and π_2 of dimensions n_1 and n_2 , there is a representation of dimension $n_1 + n_2$ called the direct sum of the two representations, denoted by $\pi_1 \oplus \pi_2$. This representation is given by the homomorphism*

$$(\pi_1 \oplus \pi_2) : g \in G \rightarrow \begin{pmatrix} \pi_1(g) & \mathbf{0} \\ \mathbf{0} & \pi_2(g) \end{pmatrix}$$

In other words, representation matrices for the direct sum are block-diagonal matrices with π_1 and π_2 giving the blocks. For unitary representations

Theorem 2.1. *Any unitary representation π can be written as a direct sum*

$$\pi = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_m$$

where the π_j are irreducible.

Proof. If (π, V) is not irreducible, there exists a $W \subset V$ such that $(\pi|_W, W)$ is a representation, and

$$(\pi, V) = (\pi|_W, W) \oplus (\pi|_{W^\perp}, W^\perp)$$

Here, W^\perp is the orthogonal complement of W in V (with respect to the Hermitian inner product on V). $(\pi|_{W^\perp}, W^\perp)$ is a subrepresentation since, by unitarity, the representation matrices preserve the Hermitian inner product. The same argument can be applied to W and W^\perp , and continue until (π, V) is decomposed into a direct sum of irreducibles. \square

Note that non-unitary representations may not be decomposable in this way. For a simple example, consider the group of upper triangular 2 by 2 matrices, acting on $V = \mathbf{C}^2$. The subspace $W \subset V$ of vectors proportional to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a subrepresentation, but there is no complement to W in V that is also a subrepresentation (the representation is not unitary, so there is no orthogonal complement subrepresentation).

Finding the decomposition of an arbitrary unitary representation into irreducible components can be a very non-trivial problem. Recall that one gets explicit matrices for the $\pi(g)$ of a representation (π, V) only when a basis for V is chosen. To see if the representation is reducible, one cannot just look to see if the $\pi(g)$ are all in block-diagonal form. One needs to find out whether there is some basis for V for which they are all in such form, something very non-obvious from just looking at the matrices themselves.

The following theorem provides a criterion that must be satisfied for a representation to be irreducible:

Theorem (Schur's lemma). *If a complex representation (π, V) is irreducible, then the only linear maps $M : V \rightarrow V$ commuting with all the $\pi(g)$ are $\lambda \mathbf{1}$, multiplication by a scalar $\lambda \in \mathbf{C}$.*

Proof. Assume that M commutes with all the $\pi(g)$. We want to show that (π, V) irreducible implies $M = \lambda \mathbf{1}$. Since we are working over the field \mathbf{C} (this does not work for \mathbf{R}), we can always solve the eigenvalue equation

$$\det(M - \lambda \mathbf{1}) = 0$$

to find the eigenvalues λ of M . The eigenspaces

$$V_\lambda = \{v \in V : Mv = \lambda v\}$$

are *nonzero* vector subspaces of V and can also be described as $\ker(M - \lambda \mathbf{1})$, the kernel of the operator $M - \lambda \mathbf{1}$. Since this operator and all the $\pi(g)$ commute, we have

$$v \in \ker(M - \lambda \mathbf{1}) \implies \pi(g)v \in \ker(M - \lambda \mathbf{1})$$

so $\ker(M - \lambda \mathbf{1}) \subset V$ is a representation of G . If V is irreducible, we must have either $\ker(M - \lambda \mathbf{1}) = V$ or $\ker(M - \lambda \mathbf{1}) = 0$. Since λ is an eigenvalue, $\ker(M - \lambda \mathbf{1}) \neq 0$, so $\ker(M - \lambda \mathbf{1}) = V$, and thus, $M = \lambda \mathbf{1}$ as a linear operator on V . \square

More concretely Schur's lemma says that for an irreducible representation, if a matrix M commutes with all the representation matrices $\pi(g)$, then M must be a scalar multiple of the unit matrix. Note that the proof crucially uses the fact that eigenvalues exist. This will only be true in general if one works with \mathbf{C} and thus with complex representations. For the theory of representations on real vector spaces, Schur's lemma is no longer true.

An important corollary of Schur's lemma is the following characterization of irreducible representations of G when G is commutative.

Theorem 2.2. *If G is commutative, all of its irreducible representations are one dimensional.*

Proof. For G commutative, $g \in G$, any representation will satisfy

$$\pi(g)\pi(h) = \pi(h)\pi(g)$$

for all $h \in G$. If π is irreducible, Schur's lemma implies that, since they commute with all the $\pi(g)$, the matrices $\pi(h)$ are all scalar matrices, i.e., $\pi(h) = \lambda_h \mathbf{1}$ for some $\lambda_h \in \mathbf{C}$. π is then irreducible when it is the one-dimensional representation given by $\pi(h) = \lambda_h$. \square

2.2 The group $U(1)$ and its representations

One might think that the simplest Lie group is the one-dimensional additive group \mathbf{R} , a group that we will study together with its representations beginning in chapter 10. It turns out that one gets a much easier to analyze Lie group by adding a periodicity condition (which removes the problem of what happens as you go to $\pm\infty$), getting the “circle group” of points on a unit circle. Each such point is characterized by an angle, and the group law is addition of angles.

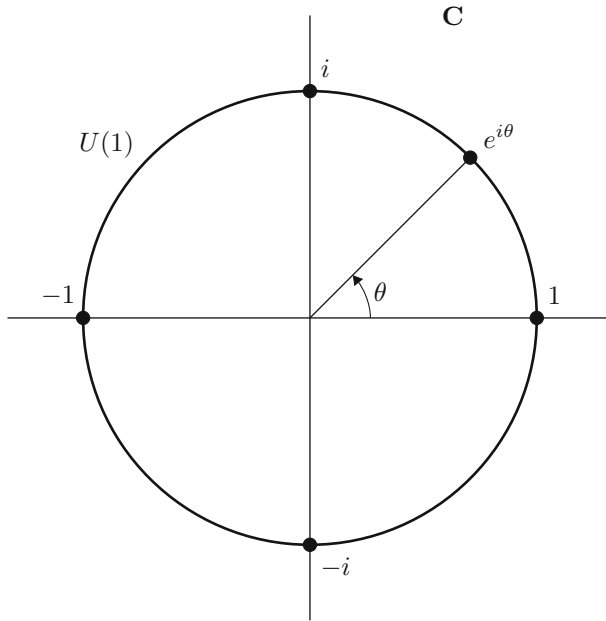
The circle group can be identified with the group of rotations of the plane \mathbf{R}^2 , in which case it is called $SO(2)$, for reasons discussed in chapter 4. It is quite convenient, however, to identify \mathbf{R}^2 with the complex plane \mathbf{C} and work with the following group (which is isomorphic to $SO(2)$):

Definition (The group $U(1)$). *The elements of the group $U(1)$ are points on the unit circle, which can be labeled by a unit complex number $e^{i\theta}$, or an angle $\theta \in \mathbf{R}$ with θ and $\theta + N2\pi$ labeling the same group element for $N \in \mathbf{Z}$. Multiplication of group elements is complex multiplication, which by the properties of the exponential satisfies*

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$

so in terms of angles the group law is addition (mod 2π).

The name “ $U(1)$ ” is used since complex numbers $e^{i\theta}$ are 1 by 1 unitary matrices.

Figure 2.1: $U(1)$ viewed as the unit circle in the complex plane \mathbf{C}

By theorem 2.2, since $U(1)$ is a commutative group, all irreducible representations will be one dimensional. Such an irreducible representation will be given by a differentiable map

$$\pi : U(1) \rightarrow GL(1, \mathbf{C})$$

$GL(1, \mathbf{C})$ is the group of invertible complex numbers, also called \mathbf{C}^* . A differentiable map π that is a representation of $U(1)$ must satisfy homomorphism and periodicity properties which can be used to show:

Theorem 2.3. *All irreducible representations of the group $U(1)$ are unitary and given by*

$$\pi_k : e^{i\theta} \in U(1) \rightarrow \pi_k(\theta) = e^{ik\theta} \in U(1) \subset GL(1, \mathbf{C}) \simeq \mathbf{C}^*$$

for $k \in \mathbf{Z}$.

Proof. We will write the π_k as a function of an angle $\theta \in \mathbf{R}$, so satisfying the periodicity property

$$\pi_k(2\pi) = \pi_k(0) = 1$$

Since it is a representation, π will satisfy the homomorphism property

$$\pi_k(\theta_1 + \theta_2) = \pi_k(\theta_1)\pi_k(\theta_2)$$

We need to show that any differentiable map

$$f : U(1) \rightarrow \mathbf{C}^*$$

satisfying the homomorphism and periodicity properties is of this form. Computing the derivative $f'(\theta) = \frac{df}{d\theta}$, we find

$$\begin{aligned} f'(\theta) &= \lim_{\Delta\theta \rightarrow 0} \frac{f(\theta + \Delta\theta) - f(\theta)}{\Delta\theta} \\ &= f(\theta) \lim_{\Delta\theta \rightarrow 0} \frac{(f(\Delta\theta) - 1)}{\Delta\theta} \quad (\text{using the homomorphism property}) \\ &= f(\theta)f'(0) \end{aligned}$$

Denoting the constant $f'(0)$ by c , the only solutions to this differential equation satisfying $f(0) = 1$ are

$$f(\theta) = e^{c\theta}$$

Requiring periodicity, we find

$$f(2\pi) = e^{c2\pi} = f(0) = 1$$

which implies $c = ik$ for $k \in \mathbf{Z}$, and $f = \pi_k$ for some integer k . \square

The representations we have found are all unitary, with π_k taking values in $U(1) \subset \mathbf{C}^*$. The complex numbers $e^{ik\theta}$ satisfy the condition to be a unitary 1 by 1 matrix, since

$$(e^{ik\theta})^{-1} = e^{-ik\theta} = \overline{e^{ik\theta}}$$

These representations are restrictions to the unit circle $U(1)$ of irreducible representations of the group \mathbf{C}^* , which are given by

$$\pi_k : z \in \mathbf{C}^* \rightarrow \pi_k(z) = z^k \in \mathbf{C}^*$$

Such representations are not unitary, but they have an extremely simple form, so it sometimes is convenient to work with them, later restricting to the unit circle, where the representation is unitary.

2.3 The charge operator

Recall from chapter 1, the claim of a general principle that, when the state space \mathcal{H} is a unitary representation of a Lie group, we get an associated self-adjoint operator on \mathcal{H} . We will now illustrate this for the simple case

of $G = U(1)$, where the self-adjoint operator we construct will be called the charge operator and denoted Q .

If the representation of $U(1)$ on \mathcal{H} is irreducible, by theorem 2.2 it must be one dimensional with $\mathcal{H} = \mathbf{C}$. By theorem 2.3, it must be of the form (π_q, \mathbf{C}) for some $q \in \mathbf{Z}$. In this case, the self-adjoint operator Q is multiplication of elements of \mathcal{H} by the integer q . Note that the integrality condition on q is needed because of the periodicity condition on θ , corresponding to the fact that we are working with the group $U(1)$, not the group \mathbf{R} .

For a general $U(1)$ representation, by theorems 2.1 and 2.3 we have

$$\mathcal{H} = \mathcal{H}_{q_1} \oplus \mathcal{H}_{q_2} \oplus \cdots \oplus \mathcal{H}_{q_n}$$

for some set of integers q_1, q_2, \dots, q_n (n is the dimension of \mathcal{H} , the q_j may not be distinct), where \mathcal{H}_{q_j} is a copy of \mathbf{C} , with $U(1)$ acting by the π_{q_j} representation. One can then define

Definition. *The charge operator Q for the $U(1)$ representation (π, \mathcal{H}) is the self-adjoint linear operator on \mathcal{H} that acts by multiplication by q_j on the irreducible subrepresentation \mathcal{H}_{q_j} . Taking basis elements in \mathcal{H}_{q_j} it acts on \mathcal{H} as the matrix*

$$Q = \begin{pmatrix} q_1 & 0 & \cdots & 0 \\ 0 & q_2 & \cdots & 0 \\ \cdots & & & \cdots \\ 0 & 0 & \cdots & q_n \end{pmatrix}$$

Thinking of \mathcal{H} as a quantum mechanical state space, Q is our first example of a quantum mechanical observable, a self-adjoint operator on \mathcal{H} . States in the subspaces \mathcal{H}_{q_j} will be eigenvectors for Q and will have a well-defined numerical value for this observable, the integer q_j . A general state will be a linear superposition of state vectors from different \mathcal{H}_{q_j} , and there will not be a well-defined numerical value for the observable Q on such a state.

From the action of Q on \mathcal{H} , the representation can be recovered. The action of the group $U(1)$ on \mathcal{H} is given by multiplying by i and exponentiating, to get

$$\pi(e^{i\theta}) = e^{iQ\theta} = \begin{pmatrix} e^{iq_1\theta} & 0 & \cdots & 0 \\ 0 & e^{iq_2\theta} & \cdots & 0 \\ \cdots & & & \cdots \\ 0 & 0 & \cdots & e^{iq_n\theta} \end{pmatrix} \in U(n) \subset GL(n, \mathbf{C})$$

The standard physics terminology is that “ Q is the generator of the $U(1)$ action by unitary transformations on the state space \mathcal{H} .”

The general abstract mathematical point of view (which we will discuss in much more detail in chapter 5) is that a representation π is a map between manifolds, from the Lie group $U(1)$ to the Lie group $GL(n, \mathbf{C})$, that takes the identity of $U(1)$ to the identity of $GL(n, \mathbf{C})$. As such it has a differential π' ,

which is a linear map from the tangent space at the identity of $U(1)$ (which here is $i\mathbf{R}$) to the tangent space at the identity of $GL(n, \mathbf{C})$ (which is the space $M(n, \mathbf{C})$ of n by n complex matrices). The tangent space at the identity of a Lie group is called a “Lie algebra.” In later chapters, we will study many different examples of such Lie algebras and such maps π' , with the linear map π' often determining the representation π .

In the $U(1)$ case, the relation between the differential of π and the operator Q is

$$\pi' : i\theta \in i\mathbf{R} \rightarrow \pi'(i\theta) = iQ\theta$$

The following drawing illustrates the situation:

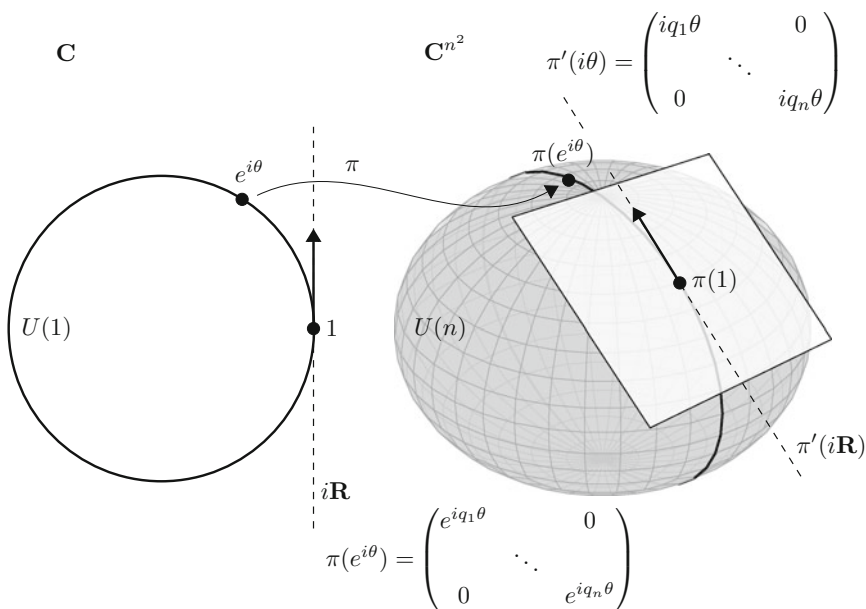


Figure 2.2: Visualizing a representation $\pi : U(1) \rightarrow U(n)$, along with its differential

The spherical figure in the right-hand side of the picture is supposed to indicate the space $U(n) \subset GL(n, \mathbf{C})$ ($GL(n, \mathbf{C})$ is the n by n complex matrices, \mathbf{C}^{n^2} , minus the locus of matrices with zero determinant, which are those that cannot be inverted). It has a distinguished point, the identity. The representation π takes the circle $U(1)$ to a circle inside $U(n)$. Its derivative π' is a linear map taking the tangent space $i\mathbf{R}$ to the circle at the identity to a line in the tangent space to $U(n)$ at the identity.

In the very simple example $G = U(1)$, this abstract picture is over-kill and likely confusing. We will see the same picture though occurring in many other much more complicated examples in later chapters. Just like in this $U(1)$ case, for finite dimensional representations the linear maps π' will be matrices, and the representation matrices π can be found by exponentiating the π' .

2.4 Conservation of charge and $U(1)$ symmetry

The way we have defined observable operators in terms of a group representation on \mathcal{H} , the action of these operators has nothing to do with the dynamics. If we start at time $t = 0$ in a state in \mathcal{H}_{q_j} , with definite numerical value q_j for the observable, there is no reason that time evolution should preserve this. Recall from one of our basic axioms that time evolution of states is given by the Schrödinger equation

$$\frac{d}{dt}|\psi(t)\rangle = -iH|\psi(t)\rangle$$

(we have set $\hbar = 1$). We will later more carefully study the relation of this equation to the symmetry of time translation (the Hamiltonian operator H generates an action of the group \mathbf{R} of time translations, just as the operator Q generates an action of the group $U(1)$). For now though, note that for time-independent Hamiltonian operators H , the solution to this equation is given by exponentiating H , with

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle$$

where

$$U(t) = e^{-itH} = 1 - itH + \frac{(-it)^2}{2!}H^2 + \dots$$

The commutator of two operators O_1, O_2 is defined by

$$[O_1, O_2] := O_1O_2 - O_2O_1$$

and such operators are said to commute if $[O_1, O_2] = 0$. If the Hamiltonian operator H and the charge operator Q commute, then Q will also commute with all powers of H

$$[H^k, Q] = 0$$

and thus with the exponential of H , so

$$[U(t), Q] = 0$$

This condition

$$U(t)Q = QU(t) \quad (2.1)$$

implies that if a state has a well-defined value q_j for the observable Q at time $t = 0$, it will continue to have the same value at any other time t , since

$$Q|\psi(t)\rangle = QU(t)|\psi(0)\rangle = U(t)Q|\psi(0)\rangle = U(t)q_j|\psi(0)\rangle = q_j|\psi(t)\rangle$$

This will be a general phenomenon: if an observable commutes with the Hamiltonian observable, one gets a conservation law. This conservation law says that if one starts in a state with a well-defined numerical value for the observable (an eigenvector for the observable operator), one will remain in such a state, with the value not changing, i.e., “conserved.”

When $[Q, H] = 0$, the group $U(1)$ is said to act as a “symmetry group” of the system, with $\pi(e^{i\theta})$ the “symmetry transformations.” Equation 2.1 implies that

$$U(t)e^{iQ\theta} = e^{iQ\theta}U(t)$$

so the action of the $U(1)$ group on the state space of the system commutes with the time evolution law determined by the choice of Hamiltonian. It is only when a representation determined by Q has this particular property that the action of the representation is properly called an action by symmetry transformations and that one gets conservation laws. In general $[Q, H] \neq 0$, with Q then generating a unitary action on \mathcal{H} that does not commute with time evolution and does not imply a conservation law.

2.5 Summary

To summarize the situation for $G = U(1)$, we have found

- Irreducible representations π are one dimensional and characterized by their derivative π' at the identity. If $G = \mathbf{R}$, π' could be any complex number. If $G = U(1)$, periodicity requires that π' must be iq , $q \in \mathbf{Z}$, so irreducible representations are labeled by an integer.
- An arbitrary representation π of $U(1)$ is of the form

$$\pi(e^{i\theta}) = e^{i\theta Q}$$

where Q is a matrix with eigenvalues a set of integers q_j . For a quantum system, Q is the self-adjoint observable corresponding to the $U(1)$ group action on the system and is said to be a “generator” of the group action.

- If $[Q, H] = 0$, the $U(1)$ group acts on the state space as “symmetries.” In this case, the q_j will be “conserved quantities,” numbers that characterize the quantum states, and do not change as the states evolve in time.

Note that we have so far restricted attention to finite dimensional representations. In section 11.1, we will consider an important infinite dimensional case, a representation on functions on the circle which is essentially the theory of Fourier series. This comes from the action of $U(1)$ on the circle by rotations, giving an induced representation on functions by equation 1.3.

2.6 For further reading

I’ve had trouble finding another source that covers the material here. Most quantum mechanics books consider it somehow too trivial to mention, starting their discussion of group actions and symmetries with more complicated examples.

<http://www.springer.com/978-3-319-64610-7>

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An Introduction

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