

Chapter 2

Linear Programming

There are various important features exclusive of linear programming without counterpart in non-linear programming. In this chapter we focus on those, and aim at a first, basic understanding of this special class of optimization problems.

2.1 Some Motivating Examples

As a way to clearly see and understand the scope of linear programming, let us examine three typical situations. We will mimic the passage from the statement of the problem in common language to its precise formulation into mathematical terms. This transformation is as important as the later solution procedure or approximation, to the extent that failure at this step will translate into a waste of time and/or resources. Statements come in a rather typical form of linear programming problems.

Example 2.1 A certain company is willing to invest 75,000 million € in the purchase of new machinery for its business. It can choose among three models A , B , and C , which differ in several features, including price. It is estimated that model A will produce a annual benefit of 315 million €, while the price per unity is 1500 million €; for model B , there will be an estimated benefit of 450 million €, while the price per unity is 1750 million €; likewise for C with 275, and 1375 million €, respectively. Those machines also have requirements and expenses concerning maintenance in terms of facilities and labor; specifically, type A needs 20 million € for labor each, and 34 days per year for maintenance facilities; type B , 27 million € for labor, and 29 days; and type C , 25 million € for labor, and 24 days. There is a maximum allowance of 1000 million € for maintenance purposes, and 800 days for availability of facilities per year. Make a recommendation about how many machines of each type the company should purchase so as to maximize yearly benefits.

This is one of those examples where the information conveyed by the statement can be efficiently recorded in a table where one can clearly see the main ingredients of a linear programming problem.

Type	A	B	C	Resources
Price	1500	1750	1375	75000
Labor	20	27	25	1000
Days	34	29	24	800
Benefit	315	450	275	

Upon reflecting on the content of this table, one realizes that we need to

$$\text{Maximize in } (x_A, x_B, x_C) : 315x_A + 450x_B + 275x_C$$

under the constraints

$$1500x_A + 1750x_B + 1375x_C \leq 75000,$$

$$20x_A + 27x_B + 25x_C \leq 1000,$$

$$34x_A + 29x_B + 24x_C \leq 800.$$

Here variables x indicate number of machines of each type to be purchased. We should also demand, for obvious reasons, that $x_A, x_B, x_C \geq 0$, and even that they ought to take on integer values.

Example 2.2 A scaffolding system. Figure 2.1 represents a system where the sum of the maximum weights of the two black boxes is to be maximized avoiding collapse. Lengths are given in the picture, and cables A, B, C, and D can withstand a maximum tension 200, 100, 200, 100, respectively, in the appropriate units. We should express equilibrium conditions for forces and momenta for each of the two rods. Namely, if

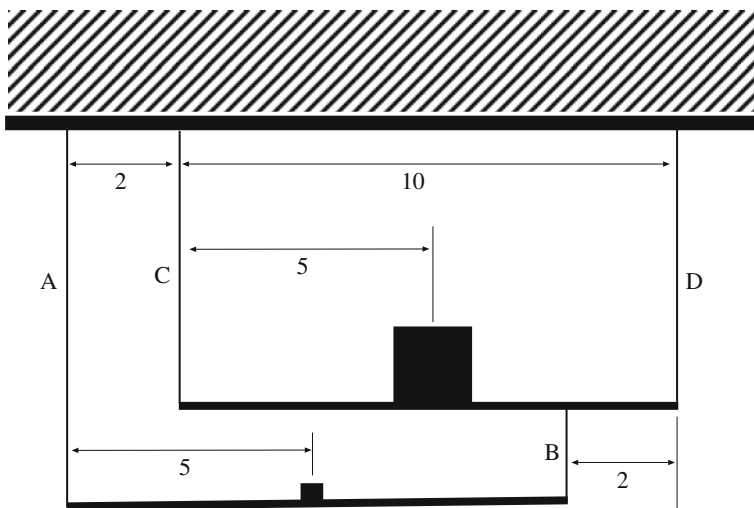


Fig. 2.1 A scaffolding system

we set x_1 and x_2 for the loads of the lower and upper rods, respectively, and T_A, T_B, T_C, T_D the tensions in their respective cables, we must have

$$\begin{aligned}x_1 &= T_A + T_B, & x_2 + T_B &= T_C + T_D, \\5x_1 &= 10T_B, & 5x_2 + 8T_B &= 10T_D.\end{aligned}$$

Therefore the problem may be written in the form

$$\text{Maximize in } (x_1, x_2) : \quad x_1 + x_2$$

subject to

$$\begin{aligned}x_1 &= T_A + T_B, & x_2 + T_B &= T_C + T_D, \\5x_1 &= 10T_B, & 5x_2 + 8T_B &= 10T_D, \\T_A &\leq 200, & T_B &\leq 100, & T_C &\leq 200, & T_D &\leq 100.\end{aligned}$$

This format can be simplified if we use equilibrium conditions to eliminate the tensions $T_X, X = A, B, C, D$. Indeed, it is elementary to find

$$T_A = T_B = \frac{x_1}{2}, \quad T_C = \frac{x_1}{10} + \frac{x_2}{2}, \quad T_D = \frac{2x_1}{5} + \frac{x_2}{2},$$

and imposing the maximum allowable values of these loads, we are led to the constraints

$$x_1 \leq 200, \quad x_1 \leq 400, \quad x_1 + 5x_2 \leq 2000, \quad 4x_1 + 5x_2 \leq 1000. \quad (2.1)$$

The problem now becomes to maximize $x_1 + x_2$ under (2.1). It is, however, interesting to realize that restrictions in (2.1) can be simplified even further because some constraints are more restrictive than others. In the $x_1 - x_2$ plane, it is easy to find that constraints can be reduced as it is indicated in the final form of the problem

$$\text{Maximize in } (x_1, x_2) : \quad x_1 + x_2$$

under

$$0 \leq x_1 \leq 200, \quad 0 \leq x_2, \quad 4x_1 + 5x_2 \leq 1000.$$

Example 2.3 A small electric power supply company owns two generators. The first one yields a net benefit of 3 €/MWh with a maximum exit power of 4 MWh, while the second yields a benefit of 5 €/MWh with a maximal exit power of 6 MWh. Cooling constraints demand that three times the exit power of the first plus twice that of the second cannot exceed, under no circumstances, 18 MWh. The issue is to maximize the global net income coming from the two generators, and the corresponding optimal working conditions. This example is easy to formulate. If we let x_1 and x_2 be the exit

powers of the two generators, respectively, to be determined, the statement clearly translate into the problem

$$\text{Maximize in } (x_1, x_2) : \quad 3x_1 + 5x_2$$

subject to

$$0 \leq x_1 \leq 4, \quad 0 \leq x_2 \leq 6, \quad 3x_1 + 2x_2 \leq 18.$$

Sometimes being able to answer additional questions is of paramount practical importance. In the context of this particular problem, suppose that the company is considering the possibility of expanding the maximal exit power of one of the two generators in order to produce a rise in net income. Which one is the best for a maximum such increment in benefits? What would the expected benefit be if the company would increase in 1 MWh the exit power of such a generator?

There are very classical LP examples that almost every text in Mathematical Programming treats: the diet problem, the transportation problem, the portfolio situation, etc. We will also describe some of them later.

2.2 Structure of Linear Programming Problems

As soon as we ponder a bit on the examples described in the preceding section, we immediately realize that a linear programming problem is one in which we want to

$$\text{Optimize in } \mathbf{x} \in \mathbb{R}^N : \quad \mathbf{c} \cdot \mathbf{x} \quad \text{under} \quad \mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}.$$

The term “optimize” conveys the idea that we could be interested sometimes in maximizing, and some others in minimizing. Constraints could also come in various forms, though we have chosen one in which they often occur. Some of them could also be equalities rather than inequalities. There are then three main ingredients to determine one such linear programming problem:

1. vector $\mathbf{c} \in \mathbb{R}^N$ is the cost vector;
2. vector $\mathbf{b} \in \mathbb{R}^n$ is the constraint vector;
3. matrix $\mathbf{A} \in \mathbb{R}^{n \times N}$ is the constraint matrix.

The important issue is to understand that:

The distinctive feature of every linear programming problem is that every function involved is linear.

Even the sole occurrence of one non-linear function (like an innocent square) among ten million linear functions suffices to discard the problem as a linear programming problem.

With those three ingredients, two main elements are determined. One is the cost function itself, which is the inner product (a linear function) $\mathbf{c} \cdot \mathbf{x}$. There is no much to be said about it. The other main element is the so-called feasible set, which incorporates all of the vectors that comply with the constraints

$$\mathbf{F} = \{\mathbf{x} \in \mathbb{R}^N : \mathbf{A}\mathbf{x} - \mathbf{b} \leq \mathbf{0}, \mathbf{x} \geq \mathbf{0}\}.$$

Let us pause for a moment on this feasible set. Three essentially different cases may occur:

1. \mathbf{F} is the empty set. We have placed so many demands on vectors that we have been left with none. Typically, this situation asks for a revision of the problem so as to relax some of the constraint in order to allow some vectors to be feasible.
2. \mathbf{F} is unlimited through, at least, some direction, and so it cannot be enclosed in a ball no matter how big it is.
3. \mathbf{F} is limited or bounded because it can be put within a certain ball.

When the feasible set \mathbf{F} is bounded, it is easy to visualize its structure, especially in dimensions $N = 2$ and $N = 3$. In fact, because constraints are written in the form of inequalities (or equalities) with linear functions, it is not hard to conclude that \mathbf{F} is the intersection of several semi-spaces, or half-spaces. If we consider a single linear inequality $\mathbf{a} \cdot \mathbf{x} \leq b$, with $\mathbf{a} \in \mathbb{R}^N$, $b \in \mathbb{R}$, we see that it corresponds to a half-space in \mathbb{R}^N whose boundary is the hyper-plane with equality $\mathbf{a} \cdot \mathbf{x} = b$. The other half-space corresponds to the other inequality $\mathbf{a} \cdot \mathbf{x} \geq b$. Hence, the feasible set is like a polyhedron in high dimension, with faces, edges, vertices. This is a heuristic statement that conveys in a intuitive way the structure of the feasible set for a linear programming problem. If the feasible set is not bounded, then that polyhedron is not limited through some direction going all the way to infinity.

Let us now examine the problem through the perspective of the solution. For definiteness, consider the linear programming problem

$$\text{Maximize in } \mathbf{x} \in \mathbb{R}^N : \quad \mathbf{c} \cdot \mathbf{x} \quad \text{under} \quad \mathbf{A}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}. \quad (2.2)$$

Definition 2.1 A vector $\bar{\mathbf{x}}$ is an optimal solution of (2.2) if it is feasible $\mathbf{A}\bar{\mathbf{x}} \leq \mathbf{b}$, $\bar{\mathbf{x}} \geq \mathbf{0}$, and it provides the maximum possible value of the cost among feasible vectors, i.e. $\mathbf{c} \cdot \bar{\mathbf{x}} \geq \mathbf{c} \cdot \mathbf{x}$ for every feasible vector \mathbf{x} .

We may have four different situations:

1. If the problem is infeasible (the admissible set is empty), there is nothing to say, except what we already pointed out earlier. There is no point in talking about solutions here.
2. There is a unique optimal solution. This is, by far, the most desirable situation.
3. There are infinitely many optimal solutions. The point is that as soon as we have two different optimal solution $\bar{\mathbf{x}}_0$, and $\bar{\mathbf{x}}_1$, then we will have infinitely many because every vector in the segment joining those two solutions $t\bar{\mathbf{x}}_1 + (1 - t)\bar{\mathbf{x}}_0$ for arbitrary $t \in [0, 1]$, will again be an optimal solution. This is true because every function involved in a linear programming problem is linear. There is no way for one such problem to have exactly ten, or twenty, or one thousand solutions.
4. There is no optimal solution. This can only happen if the feasible set is unbounded, though some problems with unbounded feasible sets may have optimal solutions. The whole point is that as we move on to infinity through some direction, the cost $\mathbf{c} \cdot \mathbf{x}$ keeps growing or decreasing without limit.

The most important fact for linear programming problems follows.

Proposition 2.1 *Suppose problem (2.2) admits an optimal solution. Then there is always one vertex of the feasible set \mathbf{F} which is also optimal.*

Though we have not defined what we understand by a vertex, it will be clear by the following discussion. Consider problem (2.2), and take a feasible vector \mathbf{x}_0 (provided the feasible set \mathbf{F} is not empty). Its cost is the number $\mathbf{c} \cdot \mathbf{x}_0$. We realize that all feasible vectors in the hyperplane of equation $\mathbf{c} \cdot \mathbf{x} = \mathbf{c} \cdot \mathbf{x}_0$ will also yield the same value of the cost. But we are ambitious, and wonder if we could push a bit upward the cost, and still have feasible vectors furnishing that higher value. For instance, we timidly wonder if we could reach the value $\mathbf{c} \cdot \mathbf{x}_0 + \varepsilon$ for a certain small ε . It will be so if the new hyperplane of equation $\mathbf{c} \cdot \mathbf{x} = \mathbf{c} \cdot \mathbf{x}_0 + \varepsilon$ intersects the feasible set somewhere. This new hyperplane is parallel to the first one. We therefore clearly see that we can keep pushing the first hyperplane to look for higher values of the inner product $\mathbf{c} \cdot \mathbf{x}$ until the resulting intersection becomes empty. But there is definitely a last, maximum value in such a way that if we push a little bit further, then we loose contact with the feasible set. A picture in dimension two may help in understanding the situation. See Fig. 2.2 below, and the discussion in Example 2.4. That maximum value is the maximum of the problem, and most definitely in that last moment, there is always a vertex (perhaps more than one vertex if there are infinitely many solutions) which is in the intersection with the feasible set.

Before discussing a specific example where all of the above material is made fully explicit, two further comments are worth stating.

- A same underlying linear programming problem admits many different forms. In some, only inequalities in both directions are involved; in some others, equalities

seem to be an important part of the problem; sometimes we seek to maximize a cost; some other times, we look for the minimum, etc. It is easy to go from one given formulation to another, equivalent one. It just requires to know how to change inequalities to equalities, and vice versa. For one thing, an equality is the reunion of two inequalities: $\mathbf{a} \cdot \mathbf{x} = b$ is equivalent to $\mathbf{a} \cdot \mathbf{x} \leq b$, $-\mathbf{a} \cdot \mathbf{x} \leq -b$. On the other hand, passing from an inequality to an equality requires to introduce the so-called slack variables: the inequality $\mathbf{a} \cdot \mathbf{x} \leq b$ is equivalent to $\mathbf{a} \cdot \mathbf{x} + x = b$, $x \geq 0$, where the variable x is the slack associated with the inequality. If we are to respect various inequalities, several slack variables (one for each inequality) need to be introduced. The change from a maximum to a minimum, and vice versa, is accomplished by introducing a minus sign. This is standard.

- When the dimension N of the problem is large, or even moderate, there is no way one can examine by hand the problem to find the optimal solution. If the dimension is low (in particular for $N = 2$), knowing that an optimal solution can always be found in a vertex, it is a matter of examining all of them, and decide on the best one. This idea, based on Proposition 2.1, is the basis for the celebrated SIMPLEX method [22, 29] which for many years has been the favorite algorithm for solving linear programming problems. These days interior point algorithms (see [25], for instance) seem to have taken over. In practice, one needs one such algorithm because real-world problems will always depend on many variables. We will not discuss here this procedure, though there are some true elegant ideas behind, but will be contented with a different numerical scheme to approximate optimal solutions not only for linear programming problems but also for non-linear ones.

Example 2.4 We examine the concrete problem

$$\text{Maximize in } (x_1, x_2) \in \mathbb{R}^2 : \quad x_2 - x_1$$

subject to

$$x_1 + x_2 \leq 1, \quad -x_1 + 2x_2 \leq 2, \quad x_1 \geq -1, \quad -x_1 + 2x_2 \geq -1.$$

Even such a simple example, with a bit of reflection, suffices to realize, at least in a heuristic but well-founded way, the structure of a linear programming problem. We refer to Fig. 2.2 for our discussion.

In the first place, the feasible set is limited by the four lines whose equations are obtained by changing the inequality signs in the constraints to equality. It is very easy to represent those lines in \mathbb{R}^2 . Those four lines correspond to the four thick continuous lines in Fig. 2.2. The quadrilateral (irregular polyhedron) limited by the four is the feasible set. It is now much easier to realize what we mean by a polyhedron: a set in \mathbb{R}^N limited by hyperplanes.

We now turn to the cost $x_2 - x_1$, and we want to maximize it within our feasible set: find the maximum possible value of the difference of the two coordinates of feasible points. We start asking: are there feasible points with zero cost? Those points would

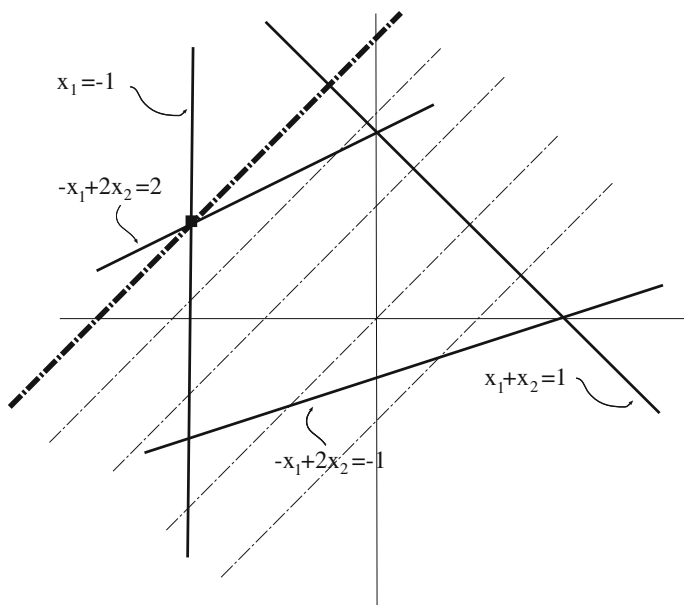


Fig. 2.2 The feasible set for Example 2.4

correspond to having $x_2 - x_1 = 0$, and this equation represents another straight line in \mathbb{R}^2 . It is the line with slope one through the origin. We realize that there are feasible points in this line because that line cuts through our quadrilateral. Those points would have zero cost. Since we try to maximize the cost, we start becoming ambitious, and ask ourselves, first rather timidly, if we could go as high as $x_2 - x_1 = 1$. These points would yield cost 1. But we realize that this line is parallel to the previous one but passing through $(1, 0)$. We again have a non-empty intersection with our quadrilateral. Once we understand the game, we become greedy, and pretend to go all the way up to $x_2 - x_1 = 10$. But this time, we have gone too far: there is no intersection of the line $x_2 - x_1 = 10$ with our feasible set. That means that the maximum sought is smaller than 10. But how smaller? At this stage, we notice that the set of parallel lines $x_2 - x_1 = c$ for different values of c represent points with cost c , and so we ask ourselves what the highest value of c is. As we push upward the iso-cost lines $x_2 - x_1 = c$, there is a final such line which touches for the last time our quadrilateral: that is the point, the vertex, where the maximum is achieved, and the corresponding value of c , the maximum.

As one can easily see, simple problems in two independent variables can always be solved graphically as in this last example.

2.3 Sensitivity and Duality

We have seen that, quite often, linear programming problems come in the form

$$\text{Maximize in } \mathbf{x} : \quad \mathbf{c} \cdot \mathbf{x} \quad \text{subject to} \quad \mathbf{A}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}. \quad (2.3)$$

Vector \mathbf{b} is typically associated with resources constraints, so that it might be interesting to explore how sensitive the value of the previous maximum is with respect to the specific values of the various components of \mathbf{b} . In other words, if $M(\mathbf{b})$ stands for the value of the maximum in the linear programming problem, understood as depending on \mathbf{b} (and also on \mathbf{A} and \mathbf{c} for that matter, though we keep these two ingredients fixed), how does $M(\mathbf{b})$ depend upon \mathbf{b} ? In particular, and from a practical viewpoint, how do changes on \mathbf{b} get reflected on $M(\mathbf{b})$? Is it worth to make an effort in increasing the components of \mathbf{b} because it will be well compensated into a positive increment of the value of M ? Which component is the best to do this? Are there some components of \mathbf{b} upon which the value of M does not depend? All of these rather interesting questions can be summarized mathematically, and responded in practice, once we know the gradient $\nabla M(\mathbf{b})$. This piece of information is what we are asking for.

When computing the gradient of an arbitrary function $f(\mathbf{x})$ of several variables $\mathbf{x} = (x_1, x_2, \dots, x_N)$, we need to examine the incremental quotients

$$\frac{f(\mathbf{x} + h\mathbf{e}_i) - f(\mathbf{x})}{h}$$

for $h \in \mathbb{R}$, small, and every $i \in \{1, 2, \dots, N\}$. Here $\{\mathbf{e}_k\}$ is the canonical basis of \mathbb{R}^N . In our particular situation, when $M(\mathbf{b})$ is defined through a linear programming problem, we immediately find an unexpected difficulty: if we change slightly one component of \mathbf{b} , the feasible set of the linear programming problem changes! It looks as if the ground would move under our feet! Though one could try to understand how that feasible set would change with changes in \mathbf{b} , it looks like a more promising strategy to come up with an equivalent formulation of the problem in such a way that vector \mathbf{b} does not participate in the feasible set, and in this way, it does not change with changes in \mathbf{b} . Is this possible? Fortunately, the answer is yes, and it is one of the most appealing features of linear programming.

One of the best ways to motivate the dual problem and insist that it is like a different side of the same coin, relies on optimality conditions that should be the same for both interpretations of the problem: the primal, and the dual. Here we refer to optimality conditions for programming problems as will be discussed in the next chapter, and so we anticipate those for linear programming problems. Optimality conditions are like special properties that optimal solutions enjoy precisely because of the fact that they are optimal for a particular problem.

Proposition 2.2 *Let $\bar{\mathbf{x}}$ be optimal for (2.3). Then there are vectors $\bar{\mathbf{y}} \in \mathbb{R}^n$, $\bar{\mathbf{z}} \in \mathbb{R}^N$ such that*

$$\begin{aligned} \mathbf{c} - \bar{\mathbf{y}}\mathbf{A} + \bar{\mathbf{z}} &= \mathbf{0}, \\ \bar{\mathbf{y}} \cdot (\mathbf{A}\bar{\mathbf{x}} - \mathbf{b}) &= 0, \quad \bar{\mathbf{z}} \cdot \bar{\mathbf{x}} = 0, \\ \bar{\mathbf{y}} &\geq \mathbf{0}, \quad \bar{\mathbf{z}} \geq \mathbf{0}. \end{aligned}$$

We will come back to this proposition within the next chapter.

The first condition in the conclusion of Proposition 2.2 may be used to eliminate $\bar{\mathbf{z}}$ from the others. If we do so, we are left with

$$\begin{aligned} \bar{\mathbf{y}} \cdot (\mathbf{A}\bar{\mathbf{x}} - \mathbf{b}) &= 0, \quad (\bar{\mathbf{y}}\mathbf{A} - \mathbf{c}) \cdot \bar{\mathbf{x}} = 0, \\ \bar{\mathbf{y}} &\geq \mathbf{0}, \quad \bar{\mathbf{y}}\mathbf{A} - \mathbf{c} \geq \mathbf{0}. \end{aligned}$$

But recall that we also know that

$$\mathbf{A}\bar{\mathbf{x}} - \mathbf{b} \geq \mathbf{0}, \quad \bar{\mathbf{x}} \geq \mathbf{0}.$$

We want to regard \mathbf{y} as the vector variable for a new linear programming problem whose optimal solution be $\bar{\mathbf{y}}$. According to the conditions above, constraints for this new problem will come in the form

$$\mathbf{y}\mathbf{A} - \mathbf{c} \geq \mathbf{0}, \quad \mathbf{y} \geq \mathbf{0}.$$

To deduce the (linear) cost functional for the new problem, suppose that both \mathbf{y} , and \mathbf{x} are feasible for the first, and the new problems, respectively,

$$\begin{aligned} \mathbf{y}\mathbf{A} - \mathbf{c} &\geq \mathbf{0}, \quad \mathbf{y} \geq \mathbf{0}, \\ \mathbf{A}\mathbf{x} - \mathbf{b} &\geq \mathbf{0}, \quad \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

Paying attention to the signs of the various terms involved in the inner products, it is elementary to argue that

$$\mathbf{c} \cdot \mathbf{x} \leq \mathbf{y}\mathbf{A}\mathbf{x} \leq \mathbf{y} \cdot \mathbf{b}.$$

Given that \mathbf{y} , and \mathbf{x} are arbitrary, feasible vectors for their respective problems, we clearly see that the cost for the new (dual) problem should be $\mathbf{y} \cdot \mathbf{b}$, and we ought to look for the minimum instead of the maximum.

Definition 2.2 The two linear programming problems

Maximize in \mathbf{x} : $\mathbf{c} \cdot \mathbf{x}$ subject to $\mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}$,

and

Minimize in \mathbf{y} : $\mathbf{b} \cdot \mathbf{y}$ subject to $\mathbf{yA} \geq \mathbf{c}, \mathbf{y} \geq \mathbf{0}$,

are dual problems of each other.

There is a pretty interesting interpretation of the dual problem from a practical viewpoint which is worth to bear in mind. The dual variables \mathbf{y} can be interpreted as prices of resources (sometimes called shadow prices) so that in the dual problem we would like to minimize the amount of money paid for resources provided those do not lead to cheaper prices for products. Seen those two problems from this perspective, it is enlightening to realize how they are two sides of the same coin.

Probably, the neatest way to express the intimate relationship between these two problems, and, in particular, between the optimal solutions of both is contained in the next statement.

Proposition 2.3 *Two vectors $\bar{\mathbf{x}}$, and $\bar{\mathbf{y}}$ are optimal solutions of the primal and dual problems, respectively, as above, if and only if*

$$\begin{aligned}\bar{\mathbf{y}} \cdot (\mathbf{A}\bar{\mathbf{x}} - \mathbf{b}) &= 0, & (\bar{\mathbf{y}}\mathbf{A} - \mathbf{c}) \cdot \bar{\mathbf{x}} &= 0, \\ \bar{\mathbf{y}}\mathbf{A} - \mathbf{c} &\geq \mathbf{0}, & \bar{\mathbf{y}} &\geq \mathbf{0}, \\ \mathbf{A}\bar{\mathbf{x}} - \mathbf{b} &\leq \mathbf{0}, & \bar{\mathbf{x}} &\geq \mathbf{0}.\end{aligned}$$

There are some relevant consequences from these conditions worth explicitly stating.

1. It is immediate that $\bar{\mathbf{y}} \cdot \mathbf{b} = \mathbf{c} \cdot \bar{\mathbf{x}}$, and so we conclude that the optimal value for both problems (the maximum for the primal, and the minimum for the dual) is the same number.
2. Even though the conditions

$$\bar{\mathbf{y}} \cdot (\mathbf{A}\bar{\mathbf{x}} - \mathbf{b}) = 0, \quad (\bar{\mathbf{y}}\mathbf{A} - \mathbf{c}) \cdot \bar{\mathbf{x}} = 0,$$

may look like just as a couple of identities, they are much more because of the sign conditions coming from feasibility. Let us focus on the first one, as the second is shown in exactly the same way. Note that

$$\bar{\mathbf{y}} \cdot (\mathbf{A}\bar{\mathbf{x}} - \mathbf{b}) = 0, \quad \mathbf{A}\bar{\mathbf{x}} - \mathbf{b} \leq \mathbf{0}, \bar{\mathbf{y}} \geq \mathbf{0}.$$

The sign conditions imply that each term in the inner product $\bar{\mathbf{y}} \cdot (\mathbf{A}\bar{\mathbf{x}} - \mathbf{b})$ is non-positive because it is the product of two numbers of different signs (or zero). How can a sum of non-positive numbers add up to zero, if there is no possibility of cancelation precisely because the sign of all of them is the same? The only possibility is that every single term vanishes $\bar{y}_k((\mathbf{A}\bar{\mathbf{x}})_k - b_k) = 0$ for all k . Similarly, $\bar{x}_j((\bar{\mathbf{y}}\mathbf{A})_j - c_j) = 0$, for all j .

At this stage we are ready to provide an explicit answer to the initial issue which motivated this analysis, and the introduction of the dual problem. Recall that $M(\mathbf{b})$ is the maximum value of the cost $\mathbf{c} \cdot \mathbf{x}$ among feasible vectors \mathbf{x} , i.e., if $\bar{\mathbf{x}}$ is an optimal solution of the primal problem, then $M(\mathbf{b}) = \mathbf{c} \cdot \bar{\mathbf{x}}$. We now know that $M(\mathbf{b}) = \mathbf{b} \cdot \bar{\mathbf{y}}$ if $\bar{\mathbf{y}}$ is an optimal solution for the dual. But, since the feasible set for the dual, $\mathbf{yA} \geq \mathbf{c}$, $\mathbf{y} \geq \mathbf{0}$, does not depend in any way on vector \mathbf{b} , we can conclude that $\nabla M(\mathbf{b}) = \bar{\mathbf{y}}$, the optimal solution of the dual.

Definition 2.3 The optimal solution $\bar{\mathbf{y}}$ of the dual problem is called the vector of sensitivity parameters of the problem. Each of its components furnishes a measure of how the value of the maximum of the primal varies with changes in that particular component for the vector of resources \mathbf{b} .

2.4 A Final Clarifying Example

Given a primal problem like (2.3), dual variables \mathbf{y} are associated, in a one-to-one manner, with constraints in the primal: each y_k corresponds to the k th constraint $(\mathbf{A}\mathbf{x})_k - b_k \leq 0$. Vice versa, each primal variable x_j corresponds to the j th constraint in the dual $(\mathbf{yA})_j - c_j \geq 0$. Optimality is expressed, then, by demanding that the corresponding products variable \times constraint vanish:

$$\bar{y}_k((\mathbf{A}\bar{\mathbf{x}})_k - b_k) = 0, \quad \bar{x}_j((\bar{\mathbf{y}}\mathbf{A})_j - c_j) = 0,$$

for all k , and j , in addition to the sign restrictions.

To see how these important conditions can be used in practice, we are going to work out the following explicit example.

Example 2.5 A farmer uses three types of milk, sheep, goat, and cow, to produce five different dairy products identified as V_i , $i = 1, 2, 3, 4, 5$. The next table sums up all of the relevant information about the proportions of each type of milk for a

unity of each product V_i , the market price of each product, and the total resources, for a given period of time, of each type of milk.

Product	V1	V2	V3	V4	V5	Resources
Sheep	4	10	8	10	5	6000
Goat	2	1	6	30	3	4000
Cow	3	2	5	15	5	5000
Price	12	20	18	40	10	

With all of this information, it is not difficult to write down the optimization problem to find the optimal (maximizing benefits) production of the five dairy products. Namely, if x_i , $i = 1, 2, 3, 4, 5$, indicates the number of units to be produced of each V_i , then we seek to maximize

$$12x_1 + 20x_2 + 18x_3 + 40x_4 + 10x_5$$

restricted to

$$4x_1 + 10x_2 + 8x_3 + 10x_4 + 5x_5 \leq 6000,$$

$$2x_1 + x_2 + 6x_3 + 30x_4 + 3x_5 \leq 4000,$$

$$3x_1 + 2x_2 + 5x_3 + 15x_4 + 5x_5 \leq 5000,$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0.$$

According to the rule to find the dual problem, proceeding with a bit of care, we find that for the dual we should minimize

$$6000y_1 + 4000y_2 + 5000y_3$$

subject to

$$4y_1 + 2y_2 + 3y_3 \geq 12,$$

$$10y_1 + y_2 + 2y_3 \geq 20,$$

$$8y_1 + 6y_2 + 5y_3 \geq 18,$$

$$10y_1 + 30y_2 + 15y_3 \geq 40,$$

$$5y_1 + 3y_2 + 5y_3 \geq 10,$$

$$y_1, y_2, y_3 \geq 0.$$

The whole point of the relationship between these two problems is expressed in the specific and intimate link between constraints and variables in the following way

$$\begin{aligned}
4x_1 + 10x_2 + 8x_3 + 10x_4 + 5x_5 &\leq 6000 \implies y_1 \geq 0, \\
2x_1 + x_2 + 6x_3 + 30x_4 + 3x_5 &\leq 4000 \implies y_2 \geq 0, \\
3x_1 + 2x_2 + 5x_3 + 15x_4 + 5x_5 &\leq 5000 \implies y_3 \geq 0, \\
4y_1 + 2y_2 + 3y_3 &\geq 12 \implies x_1 \geq 0, \\
10y_1 + y_2 + 2y_3 &\geq 20 \implies x_2 \geq 0, \\
8y_1 + 6y_2 + 5y_3 &\geq 18 \implies x_3 \geq 0, \\
10y_1 + 30y_2 + 15y_3 &\geq 40 \implies x_4 \geq 0, \\
5y_1 + 3y_2 + 5y_3 &\geq 10 \implies x_5 \geq 0,
\end{aligned}$$

and at the level of optimality, for optimal solutions of both, we should have

$$\begin{aligned}
y_1(4x_1 + 10x_2 + 8x_3 + 10x_4 + 5x_5 - 6000) &= 0, \\
y_2(2x_1 + x_2 + 6x_3 + 30x_4 + 3x_5 - 4000) &= 0, \\
y_3(3x_1 + 2x_2 + 5x_3 + 15x_4 + 5x_5 - 5000) &= 0, \\
x_1(4y_1 + 2y_2 + 3y_3 - 12) &= 0, \\
x_2(10y_1 + y_2 + 2y_3 - 20) &= 0, \\
x_3(8y_1 + 6y_2 + 5y_3 - 18) &= 0, \\
x_4(10y_1 + 30y_2 + 15y_3 - 40) &= 0, \\
x_5(5y_1 + 3y_2 + 5y_3 - 10) &= 0.
\end{aligned}$$

Assume we are given the following three pieces of information:

1. the maximum possible benefit is 18,400;
2. the optimal solution for the primal is in need of V1;
3. for the optimal solution the total amount of cow milk is not fully spent.

Let us interpret all of this information, and see if we can deduce the optimal solutions for both problems. There is nothing to say about the maximum value, except that is also the optimal value for the minimum in the dual

$$18400 = 6000y_1 + 4000y_2 + 5000y_3 = 12x_1 + 20x_2 + 18x_3 + 40x_4 + 10x_5.$$

The second piece of information is telling us that $x_1 > 0$ because there is no way to find the optimal solution if we insist in being dispensed with V1. But then the restriction corresponding to x_1 in the above list should vanish

$$4y_1 + 2y_2 + 3y_3 = 12.$$

Finally, the restriction for the cow milk is strict, and hence the corresponding dual variable must vanish $y_3 = 0$.

Altogether, we have

$$18400 = 6000y_1 + 4000y_2 + 5000y_3, \quad 4y_1 + 2y_2 + 3y_3 = 12, \quad y_3 = 0.$$

It is easy to find the optimal solution of the dual $(14/5, 2/5, 0)$. If we take back these values of the dual variables to all of the above products, we immediately find that $x_2 = x_3 = x_5 = 0$, and

$$4x_1 + 10x_4 = 6000, \quad 2x_1 + 30x_4 = 4000,$$

i.e. $x_1 = 1400, x_4 = 40$, and the optimal solution for the primal is $(1400, 0, 0, 40, 0)$.

We now know the optimal production to maximize benefits. As we have insisted above the solution of the dual problem $(14/5, 2/5, 0)$ are the sensitivity parameters expressing how sensitive the value 18,400 of the maximum is with respect to the three resources constraints. In particular, it says that it is unsensitive to the cow milk constraint, and so there is no point in trying to increase the amount of cow milk at our disposal (this is also seen in the fact that there is a certain surplus of cow milk as we were informed before). However, we can expect an increment of about $2/5$ monetary units in the benefit for each unity we increase the quantity of goat milk at our disposal, and around $14/5$ units, if we do so for sheep milk.

2.5 Final Remarks

Linear programming looks easy, and indeed understanding its structure is not that hard, if one feels comfortable with the main concepts of Linear Algebra. However, even with linear programming, realistic problems can be very hard to treat mainly because of its size in terms of number of variables, and number of constraints involved. Even for moderate sized problems, it is impossible to find solutions by hand. So, in practice, there is no substitute for a good computational package to find accurate approximation of optimal solutions. We will treat this fundamental issue in the final chapter of this part of the book.

But there is more to linear programming. More often than not, feasible vectors for problems cannot take on non-integer values, so that quantities can only be measured in full units. That is the case of tangible objects (chairs, tables, cars, etc.), or non-tangible ones like full trips, full steps, etc. What we mean is that in these cases, as part of feasibility, variables can only take on integer, or even non-negative integer values. In this case, we talk about integer programming. This apparent innocent condition puts many complex difficulties to the problem. We can always ignore those integer-valued constraints, and cut decimal figures at the end, and this indeed may provide very good solutions, but they may not be the best ones. Some times variables can only take two values: 0, and 1, as a way to model a two-possibility decision, and in this case we talk about binary variables.

The treatment of duality and sensitivity is not complete either. A thorough examination of special situations when one of the two versions, the primal or the dual, turns out to be infeasible or not admit an optimal solution; the possibility of having a gap between the optimal values of the two problems, etc., would require additional effort.

There are also important and difficult issues associated with large scale problems, and how to divide them up into pieces so that the solution procedure is as cheap (in time, in money) as possible. Many other areas of linear programming are reserved to experts.

Despite the simplicity of the structure of LP problems, its applicability is pervasive throughout science and engineering: agriculture, economics, management, manufacturing, telecommunication, marketing, finance, energy,...

2.6 Exercises

2.6.1 Exercises to Support the Main Concepts

1. Find the extreme values of the function

$$f(x_1, x_2, x_3) = x_1 - x_2 + 5x_3$$

in the convex set (polyhedron) generated by the points (vertices)

$$(1, 1, 0), \quad (-1, 0, 1), \quad (2, 1, -1), \quad (-1, -1, -1).$$

Same question for the function

$$g(x_1, x_2, x_3) = x_2 - x_3.$$

2. Consider the LP problem

$$\text{Maximize in } (x_1, x_2) : \quad x_1 + 3x_2$$

under

$$-x_1 + x_2 \leq 2, \quad 4x_1 - x_2 \leq -1, \quad 2x_1 + x_2 \geq 0, \quad x_1, x_2 \geq 0.$$

- (a) Find its optimal solution graphically.
- (b) Solve the three problems consisting in maximizing the same cost in the three different feasible sets

$$\begin{aligned} -x_1 + x_2 &\leq 2 + h, & 4x_1 - x_2 &\leq -1, & 2x_1 + x_2 &\geq 0, \\ -x_1 + x_2 &\leq 2, & 4x_1 - x_2 &\leq -1 + h, & 2x_1 + x_2 &\geq 0, \\ -x_1 + x_2 &\leq 2, & 4x_1 - x_2 &\leq -1, & 2x_1 + x_2 &\geq h, \end{aligned}$$

where h is a real (positive or negative) parameter. Note how the feasible set changes with h in each situation.

- (c) Calculate the derivatives of the optimal value with respect to h for these three cases.
 - (d) Solve the dual problem of the initial one, and relate its optimal solution to the previous item.
3. Some times LP problems may not naturally come in the format we have been using. But with some extra effort, they can be given that form, and then find the dual. Look at the problem

$$\text{Maximize in } (x_1, x_2) : \quad x_1 + 3x_2$$

under

$$-x_1 + x_2 \leq 2, \quad 4x_1 - x_2 \leq -1, \quad 2x_1 + x_2 \geq 0,$$

but with no constraint on the signs of x_1 and x_2 . Make the change of variables

$$X_1 = 2 + x_1 - x_2, \quad X_2 = -1 - 4x_1 + x_2,$$

and write the problem in terms of (X_1, X_2) . Find the dual. Try other possible changes of variables.

4. For the LP problem

$$\text{Maximize in } \mathbf{x} : \quad \mathbf{c} \cdot \mathbf{x} \quad \text{under} \quad \mathbf{Ax} \leq \mathbf{b},$$

write each variable x_i as the difference of two non-negative variables x_{i+}, x_{i-} (its positive and negative parts, respectively) to check that the dual problem is

$$\text{Minimize in } \mathbf{y} : \quad \mathbf{b} \cdot \mathbf{y} \quad \text{under} \quad \mathbf{yA} = \mathbf{c}, \mathbf{y} \geq 0.$$

5. Find the optimal solution of the problem

$$\text{Maximize in } (x_1, x_2, x_3) : \quad x_1 + x_2 - x_3$$

subject to

$$x_1 - x_2 + x_3 \leq 1, \quad x_1 + x_2 + x_3 \leq 1, \quad x_1, x_2, x_3 \geq 0.$$

6. Let's have a look at the problem

$$\text{Maximize in } (x_1, x_2, x_3, x_4) : \quad x_1 + x_2 - x_3 - x_4$$

subject to

$$x_1 - x_2 + x_3 - x_4 \leq 1, \quad x_1 - x_2 - x_3 + x_4 \leq 1, \\ x_1, x_2, x_3, x_4 \geq 0.$$

- (a) Write the dual problem.
- (b) Argue that the dual problem is infeasible.
- (c) Given that the maximum for the primal is the infimum for the dual, conclude that the primal problem cannot have a solution.
- (d) By taking $x_3 = x_4 = 0$, reduce the primal to a problem in dimension two. Check graphically that for this subproblem the maximum is infinite.

7. Find the optimal solution of the problem

$$\text{Minimize in } (x_1, x_2) : \quad 2x_1 - 7x_2$$

under

$$x_1 + hx_2 \leq 0, \quad hx_1 - 2x_2 \geq -1, \quad x_1, x_2 \geq 0,$$

in terms of the parameter $h < 0$.

8. For the LP problem

$$\text{Maximize in } (x_1, x_2, x_3) : \quad x_1 + 5x_2 - 3x_3$$

under

$$-x_1 + x_2 + x_3 \leq 1, \quad x_1 - x_2 + x_3 \leq 1, \quad x_1 + x_2 - x_3 \leq 1, \\ x_1, x_2, x_3 \geq 0,$$

find its dual. Transforming this dual problem in a form like our main model problem in this chapter, find its dual and compare it to the original problem.

9. Find the maximum of the cost function $18x_1 + 4x_2 + 6x_3$ over the set

$$3x_1 + x_2 \leq -3, \quad 2x_1 + x_3 \leq -5, \quad x_1, x_2, x_3 \leq 0,$$

through its dual.

10. Decide for which values of k the LP problem

$$\text{Optimize in } (x_1, x_2) \in \mathbf{F} : \quad x_1 + kx_2,$$

where

$$\mathbf{F} = \{(x_1, x_2) \in \mathbb{R}^2 : 3x_1 + 2x_2 \geq 6, x_1 + 6x_2 \geq 8, x_1, x_2 \geq 0\},$$

has an optimal solution. Discuss when such a solution is unique (optimize means both minimize and maximize).

2.6.2 Practice Exercises

1. Draw the region limited by the conditions

$$x_2 \geq 0, \quad 0 \leq x_1 \leq 3, \quad -x_1 + x_2 \leq 1, \quad x_1 + x_2 \leq 4.$$

For each one of the following objective functions, find the points where the maximum is achieved

$$f_1 = 2x_1 + x_2, \quad f_2 = x_1 + x_2, \quad f_3 = x_1 + 2x_2.$$

2. Solve graphically the following problems

<p>Maximize $2x_1 + 6x_2$ under $-x_1 + x_2 \leq 1,$ $2x_1 + x_2 \leq 2,$ $x_1, x_2 \geq 0,$</p>	<p>Maximize $-3x_1 + 2x_2$ under $x_1 + x_2 \leq 5,$ $0 \leq x_1 \leq 4,$ $1 \leq x_2 \leq 6.$</p>
---	---

3. Find the maximum of the function $240x_1 + 104x_2 + 60x_3 + 19x_4$ over the set determined by the constraints

$$\begin{aligned} 20x_1 + 9x_2 + 6x_3 + x_4 &\leq 20, \\ 10x_1 + 4x_2 + 2x_3 + x_4 &\leq 10, \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

4. For the problem

$$\begin{aligned} &\text{Minimize} \quad 20x_1 - 10x_2 - 6x_3 + 2x_4 \\ &\text{under} \quad 2x_1 + 8x_2 + 6x_3 + x_4 \leq 20 \\ &\quad 10x_1 + 4x_2 + 2x_3 + x_4 \leq 10 \\ &\quad x_1 - 5x_2 + 3x_3 + 8x_4 \geq 1 \\ &\quad x_i \geq 0, \quad i = 1, 2, 3, 4, \end{aligned}$$

someone has provided the following information:

- (a) the minimum value is -22 ;

- (b) in the optimal solution, the second constraint is an strict inequality;
- (c) the third restriction of the corresponding dual problem over its optimal solution is an equality.

Write the dual problem, and find the optimal solutions of both.

5. Solve the problem

$$\max_{(x_1, x_2, x_3) \in \mathbf{F}} 2x_1 + 3x_2 + x_3$$

where \mathbf{F} is the subset of \mathbb{R}^3 defined by

$$x_1 + x_2 + 2x_3 \leq 200, \quad 2x_1 + 2x_2 + 5x_3 \leq 500, \quad x_1 + 2x_2 + 3x_3 \leq 300, \\ x_1, x_2, x_3 \geq 0.$$

6. Find the optimal solution of Example 2.2. Which one of the four participating cables would allow for a greater increase of the overall weight the system can withstand if resistance is increase in a unit?

2.6.3 Case Studies

1. A certain company makes a unit of four different jobs A , B , C , and D , with a specified amount of labor O and resources P according to the table below. Moreover, there is a given number of labor units and resources at its disposal. Market prices are also known.

	Job A	Job B	Job C	Job D	Availability
Labor O	1	2	2	4	200
Resources P	2	1	3	1	150
Price	20	30	30	50	

- (a) Formulate the problem of maximizing benefits. Write its dual.
 - (b) Find the optimal solutions for both, and the maximum benefit possible.
 - (c) Interpret the solution of the dual in terms of the primal. What is more advantageous: to increase labor, or resources?
2. In a carpenter's shop, a clerk may decide to work on tables, chairs or stools. For the first, he receives 3.35 euros per unit; 2, for the each chair; and 0.5 euros for the last. Tables require 1.25 h each; chairs, 0.75 h; and stools, 0.25 h. If each workday amounts to 8 h, and each unfinished element is worthless, decide how many pieces of each kind this clerk has to make in order to maximize salary.
3. A thief, after committing a robbery in a jewelry, hides in a park which is the convex hull of the four points

$$(0, 0), \quad (-1, 1), \quad (1, 3), \quad (2, 1).$$

The police, looking after him, gets into the park and organizes the search according to the function

$$\rho(x_1, x_2) = x_1 - 3x_2 + 10$$

indicating density of vigilance. Recommend the thief the best point through which he can escape, or the best point where he can stay hidden.

4. A farmer wants to customize his fertilizer for his current crop.¹ He can buy plant food mix A and plant food mix B. Each cubic yard of food A contains 20lb of phosphoric acid, 30lb of nitrogen and 5lb of potash. Each cubic yard of food B contains 10lb of phosphoric acid, 30lb of nitrogen and 10lb of potash. He requires a minimum of 460lb of phosphoric acid, 960lb of nitrogen and 220lb of potash. If food A costs 30 per cubic yard and food B costs 35 per cubic yard, how many cubic yards of each food should the farmer blend to meet the minimum chemical requirements at a minimal cost? What is this cost?
5. A company makes three models of desks²: an executive model, an office model and a student model. Each desk spends time in the cabinet shop, the finishing shop and the crating shop as shown in the table:

	Cabinet	Finishing	Crating	Profit
Executive	2	1	1	150
Office	1	2	1	125
Student	1	1	0.5	50
Hours	16	16	10	

How many of each type of model should be made to maximize profits?

6. A small brewery produces ale and beer from corn, hops and barley malt, and sells the product according to the data in the following table³:

	Corn	Hops	Malt	Profit
Available	480	160	1190	
Ale	5	4	35	13
Beer	15	4	20	23

Choose product mix to maximize profits.

7. A calculator company produces a scientific calculator and a graphing calculator.⁴ Long-term projections indicate an expected demand of at least 100 scientific and

¹From <http://www.math.tamu.edu/~Janice.Epstein/141/review/LinProgReview.pdf>.

²From <http://www.math.tamu.edu/~Janice.Epstein/141/review/LinProgReview.pdf>.

³From <https://www.cs.princeton.edu/~rs/AlgsDS07/22LinearProgramming.pdf>.

⁴From <http://www.purplemath.com>.

80 graphing calculators each day. Because of limitations on production capacity, no more than 200 scientific and 170 graphing calculators can be made daily. To satisfy a shipping contract, a total of at least 200 calculators must be shipped each day. If each scientific calculator sold results in a 2 loss, but each graphing calculator produces a 5 profit, how many of each type should be made daily to maximize net profits?

8. You need to buy some filing cabinets.⁵ You know that Cabinet X costs 10 per unit, requires six square feet of floor space, and holds eight cubic feet of files. Cabinet Y costs 20 per unit, requires eight square feet of floor space, and holds twelve cubic feet of files. You have been given 140 for this purchase, though you don't have to spend that much. The office has room for no more than 72 square feet of cabinets. How many of which model should you buy, in order to maximize storage volume?
9. A manufacturer produces violins, guitars, and viola from quality wood, metal and labor according to the table

	Violin	Guitar	Viola
Wood	2	1	1
Labor	2	1	2
Metal	1	1	1

Disposal is 50 units of wood, 60 labor units and 55 units of metal; prices are: 200 for violins, 175 for guitars, and 125 for violas.

- (a) Formulate the problem and find the optimal production plan.
- (b) Write its dual, and find its optimal solution through the optimal solution of the primal problem.
- (c) How much more would the company be willing to pay for an extra unit of wood? For an extra unit of labor? And for one of metal?
- (d) What is the range of prices to which we could sell guitars without compromising the optimal production plan found earlier?

⁵From <http://www.purplemath.com>.



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