Chapter 2
Continuous Uniform

The continuous uniform distribution applies when the random variable can fall anywhere equally likely between two limits. The distribution is often called when an analyst does not have definitive information on the range and shape of the random variable. For example, management may estimate the time to finish a project is equally likely between 50 and 60 h. A baseball is hit for a homerun and the officials estimate the ball traveled somewhere between 410 and 430 feet. The amount of official snowfall at a location on a wintry day is predicted between 1 and 5 inches. The chapter lists the probability density, cumulative distribution, mean, variance and standard deviation of the random variable. Also described is the α-percent-point of x that identifies the value of x where the cumulative probability is α. When the parameter limit values are not known, and sample data is available, estimates of the parameter values are obtained. Two estimates are described, one by way of the maximum-likelihood method, and the other by the method-of-moments. When sample data is not available, experts are called to obtain the estimates. Often both of the limits are unknown and estimates on both are needed. Sometimes only the low limit is known, and on other occasions, only the upper limit is unknown. The way to estimate the parameter values is described for all three scenarios.

2.1 Fundamentals

The random variable, x, of the continuous uniform distribution, has an admissible range of (a to b), where any value in this range is equally likely to occur. The parameters of the distribution define the range interval and are:

a = min
b = max

N.T. Thomopoulos, *Statistical Distributions*, DOI 10.1007/978-3-319-65112-5_2

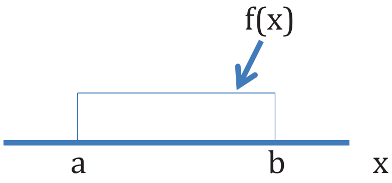

Fig. 2.1 The continuous uniform density

The probability density is below, and a depiction is in Fig. 2.1.

$f(x) = 1/(b - a) \qquad a \le x \le b$

The mean, variance and standard deviation are listed below:

$\mu = (a + b)/2$

$\sigma^2 = (b - a)^2/12$

$\sigma = (b - a)/\sqrt{12}$

The coefficient-of-variation is obtained as below:

$cov = \sigma/\mu$

In the special case when a = 0 and b = 1, cov becomes

$cov = 2/\sqrt{12} = 0.578$

The cumulative distribution function, F(x), becomes:

$F(x) = (x - a)/(b - a) \qquad a \le x \le b$

The α-percent-point on x, denoted as xα, is related to the cumulative probability, α, as below:

$P(x \le x\alpha) = \alpha$

To find the value of xα for the continuous uniform distribution, apply the following:

$x\alpha = a + \alpha(b - a)$

Example 2.1 In a production system, the amount of liquid poured into a container varies and could be anywhere from 7.0 to 7.2 ounces. Hence, for a randomly sampled container, the amount of liquid in the container, noted as x, has an admissible range of: $7.0 \le x \le 7.2$. The probability density becomes:

$f(x) = 1/(7.2 - 7.0) = 5.0 \qquad 7.0 \le x \le 7.2$

The average amount of liquid in a container is $\mu = (7.0 + 7.2)/2 = 7.10$. The variance becomes $\sigma^2 = (7.2–7.0)^2/12 = 0.0033$; and the standard deviation is $\sigma = \sqrt{0.0033} = 0.0577$.

The cumulative probability distribution for any value of x in the admissible range becomes:

$$F(x) = (x - 7.0)/(0.20) \qquad 7.0 \le x \le 7.2$$

Note, for example, the probability of x less or equal to 7.15 is obtained as below:

$$P(x \le 7.15) = F(7.15) = (7.15 - 7.00)/(0.20) = 0.75$$

As an example, the 0.25%-point value of x is computed as follows:

$$x_{.25} = 7.0 + 0.25(7.20 - 7.00) = 7.05$$

2.2 Sample Data

When an analyst wants to apply the continuous uniform distribution in a study and the parameters values, (a, b), are not known, sample data is used to estimate the values of the parameters. The sample data are n randomly drawn observations denoted as: $(x_1, \ldots, x_n)$. To apply the estimates, the following statistics are drawn from the sample data:

$\bar{x}$ = average
s = standard deviation
x(1) = min
x(n) = max

Example 2.2 An experiment yields the following n = 10 observations: (9.1, 3.1, 17.1. 15.8, 12.6. 5.9, 5.1, 14.2, 19.8, 7.3). The analyst assumes the data comes from a continuous uniform distribution and would like to find the mid 50% interval of values. An initial step requires estimates of the parameters, (a, b); and to achieve, the first task is to measure the stats from the data. These are:

$\bar{x} = 11.00$
$s = 5.69$
$x(1) = 3.1$
$x(n) = 19.8$

2.3 Parameter Estimates from Sample Data

Two ways to estimate the parameters are available. One is by the maximum-likelihood, and the other is from the method-of-moments.

The parameter estimates using the maximum-likelihood estimate method become:

$$\hat{a} = \mathrm{x}(1)$$
$$\hat{b} = \mathrm{x}(\mathrm{n})$$

The parameter estimates from the method-of-moments are below:

$$\hat{a} = \bar{x} - \sqrt{12}\mathrm{s}/2$$
$$\hat{b} = \bar{x} + \sqrt{12}\mathrm{s}/2$$

Example 2.3 Continuing with Example 2.2, the estimates for the maximum-likelihood method and by the method-of-moment method are below:

Using the maximum-likelihood method, the parameter estimates become:

$$\hat{a} = \mathrm{x}(1) = 3.1$$
$$\hat{b} = \mathrm{x}(\mathrm{n}) = 19.8$$

The mid-50% interval is computed as below:

$$(\mathrm{x}_{.25} \leq \mathrm{x} \leq \mathrm{x}_{.75})$$

where $\mathrm{x}_{.25 \text{ and }}\mathrm{x}_{.75}$ are the 0.25% and 0.75%-point values of x, respectively. Using the maximum-likelihood estimate method, these become:

$$\mathrm{x}_{.25} = \hat{a} + 0.25\left(\hat{b} - \hat{a}\right) = 3.1 + 0.25(19.8 - 3.1) = 7.27$$
$$\mathrm{x}_{.75} = \hat{a} + 0.75\left(\hat{b} - \hat{a}\right) = 3.1 + 0.75(19.8 - 3.1) = 15.62$$

Hence, the mid-50% interval becomes:

$$(7.27 \leq \mathrm{x} \leq 15.62)$$

and the associated probability estimate on the range is:

$$\mathrm{P}(7.27 \leq \mathrm{x} \leq 15.62) = 0.50$$

Using the data from Example 2.2, the parameter estimates by use of the method-of-moments method are computed below:

$$\hat{a} = \bar{x} - \sqrt{12}s/2 = 11.00 - \sqrt{12} \times 5.69/2 = 1.14$$
$$\hat{b} = \bar{x} + \sqrt{12}s/2 = 11.00 + \sqrt{12} \times 5.69/2 = 20.86$$

Applying the method-of-moment estimates, the mid-50% range is obtained as below:

$$x_{.25} = \hat{a} + 0.25\left(\hat{b} - \hat{a}\right) = 1.14 + 0.25(20.86 - 1.14) = 6.07$$
$$x_{.75} = \hat{a} + 0.75\left(\hat{b} - \hat{a}\right) = 1.14 + 0.75(20.86 - 1.14) = 15.69$$

2.4 Parameter Estimates When No Data

Consider the situation when an analyst wishes to apply the continuous uniform distribution but has no estimates on the parameters (a, b) and has no sample data to draw the estimates. In this situation, the analyst seeks advise from experts who give some estimates on the range of the variable, x.

2.5 When (a, b) Not Known

When both parameter values are not known, the analyst seeks an expert who gives two estimates on the percent-points of x, denoted as: ($x_1 = \alpha_1$-percent-point, and $x_2 = \alpha_2$-percent-point). Note:

$$P(x < x_1) = \alpha_1$$
$$P(x < x_2) = \alpha_2$$

Should $\alpha_1 = 0.0$, the estimate on the min is $\hat{a} = x_1$; and if $\alpha_2 = 1.0$, the estimate on the max is $\hat{b} = x_2$.

Below shows how to estimate the min and max parameter values when $\alpha_1 \geq 0.0$, $\alpha_2 \leq 1.0$ and $\alpha_1 < \alpha_2$. Using x_1, x_2, α_1 and α_2, the estimates of the parameters are obtained as shown below. First observe how x_1 and x_2 are related to the parameters (a, b):

$$x_1 = a + \alpha_1(b - a)$$
$$x_2 = a + \alpha_2(b - a)$$

Also note, the equivalent relations below:

$$x_1 = a(1 - \alpha_1) + \alpha_1 b$$
$$x_2 = a(1 - \alpha_2) + \alpha_2 b$$

Now using some algebra, the estimates on the parameters (a, b) become:

$$\hat{a} = [x_2\alpha_1 - x_1\alpha_2)]/[\alpha_1 - \alpha_2]$$
$$\hat{b} = [x_2 - \hat{a}(1 - \alpha_2)]/\alpha_2$$

Example 2.4 Suppose an analyst wants to use the continuous uniform distribution on some data where experts give estimate values of $x_1 = 10$ and $x_2 = 100$.

If $\alpha_1 = 0.0$ and $\alpha_2 = 1.0$, the above equations yield:

$$\hat{a} = [100 \times 0.0 - 10 \times 1.0]/[0.0 - 1.0] = 10$$
$$\hat{b} = [100 - 10(1 - 1.0)]/1.0 = 100$$

If $\alpha_1 = 0.0$ and $\alpha_1 = 0.8$, the estimates become:

$$\hat{a} = [100 \times 0.0 - 10 \times 0.8]/[0.0 - 0.8] = 10$$
$$\hat{b} = [100 - 10(1 - 0.8)]/0.8 = 122.5$$

If $\alpha_1 = 0.2$ and $\alpha_2 = 1.0$, the estimates become:

$$\hat{a} = [100 \times 0.2 - 10 \times 1.0]/[0.2 - 1.0] = -12.5$$
$$\hat{b} = [100 - (-12.5)(1 - 1.0)]/1.0 = 100$$

If $\alpha_1 = 0.2$ and $\alpha_2 = 0.8$, the estimates become:

$$\hat{a} = [100 \times 0.2 - 10 \times 0.8/[0.2 - 0.8] = -20$$
$$\hat{b} = [100 - (-20)(1 - 0.8)]/0.8 = 130$$

Example 2.5 In a manufacturing system, management needs as estimate on the time to design the configuration on a new product. Inquiring with the engineers, the estimate of the mid-50% interval will be 100–120 h. This implies, the 0.25% point and the 0.75% point are the following:

$$x_{.25} = 100$$
$$x_{.75} = 120$$

Also,

$$\alpha_1 = 0.25$$
$$\alpha_2 = 0.75$$

So now, the estimates of the parameter values are computed as below:

$$\hat{a} = [120 \times 0.25 - 100 \times 0.75]/[0.25 - 0.75] = 90$$
$$\hat{b} = [120 - 90\ (1 - 0.75)]/0.75 = 130$$

Finally, the estimate of the time to complete the design is 90–130 h.

2.6 Summary

The continuous uniform distribution is called when the random variable can equally fall anywhere equally likely between two limits. The distribution is often used when the analyst has little information on the distribution. Estimates on the limits, (a, b), are needed to apply. When sample data is available, the estimates of the parameter values are readily computed, either by the maximum-likelihood method, or by the method-of-moments. When sample data is not available, expert knowledge is taken from which the estimates are obtained.

http://www.springer.com/978-3-319-65111-8

Statistical Distributions
Applications and Parameter Estimates
Thomopoulos, N.T.
2017, XVII, 172 p. 22 illus., 21 illus. in color., Hardcover
ISBN: 978-3-319-65111-8