Chapter 1
Main Definitions and Basic Results

In this preliminary chapter, we define pseudotrajectories and various shadowing properties for dynamical systems with discrete and continuous time (Sects. 1.1 and 1.2), study the notion of chain transitivity (Sect. 1.1), describe hyperbolicity, Ω-stability, and structural stability (Sect. 1.3), and prove a lemma on finite Lipschitz shadowing in a neighborhood of a hyperbolic set (Sect. 1.4).

1.1 Pseudotrajectories and Shadowing in Dynamical Systems with Discrete Time: Chain Transitive Sets

Consider a metric space (M, dist). Everywhere below (if otherwise is not stated), we denote by $N(a, x)$ and $N(a, A)$ the open a-neighborhoods of a point $x \in M$ and a set $A \subset M$, respectively. For a set $A \subset M$, $\mathrm{Int}(A)$, $\mathrm{Cl}(A)$, and ∂A denote the interior, closure, and boundary of A, respectively.

Let f be a homeomorphism of the metric space M. As usual, we identify the homeomorphism f with the dynamical system with discrete time generated by f on M.

We denote by

$$O(x,f) = \left\{ f^k(x) : \ k \in \mathbb{Z} \right\}$$

the trajectory (orbit) of a point $x \in M$ in the dynamical system f.

We also consider positive and negative semitrajectories of a point x,

$$O^+(x,f) = \left\{ f^k(x) : \ k \geq 0 \right\} \text{ and } O^-(x,f) = \left\{ f^k(x) : \ k \leq 0 \right\}.$$

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Similar notation is used for trajectories of sets;

$$O(A,f) = \left\{f^k(A) : \ k \in \mathbb{Z}\right\}$$

is the trajectory of a set $A \subset M$ in the dynamical system f, etc.

We denote by $\mathrm{Per}(f)$ the set of periodic points of f.

Remark 1.1.1 We give the main definitions in this section for the most general case of dynamical system with discrete time generated by homeomorphisms; in fact, the main results of this book are related to smooth dynamical systems – either to systems with discrete time generated by diffeomorphisms or to systems with continuous time (flows) generated by smooth vector fields on manifolds.

If M is a smooth closed (i.e., compact and boundaryless) manifold with Riemannian metric dist, we denote by TM the tangent bundle of M and by T_xM the tangent space of M at a point x, respectively. For a vector $v \in T_xM$, $|v|$ is its norm induced by the metric dist.

If f is a diffeomorphism of a smooth manifold M, we denote by

$$Df(x) : \ T_xM \to T_{f(x)}M$$

its derivative at a point $x \in M$.

Let us give the main definition in the case of a homeomorphism of a metric space (M, dist).

Definition 1.1.1 Fix a $d > 0$. A sequence

$$\xi = \{x_k \in M : \ k \in \mathbb{Z}\} \tag{1.1}$$

is called a *d-pseudotrajectory* of the dynamical system f if the following inequalities hold:

$$\mathrm{dist}(x_{k+1}, f(x_k)) < d, \quad k \in \mathbb{Z}. \tag{1.2}$$

Sometimes, d-pseudotrajectories are called *d-orbits*.

The basic property of dynamical systems related to the notion of a pseudotrajectory is called shadowing (or tracing).

Definition 1.1.2 We say that a dynamical system f has the *shadowing property* if for any $\varepsilon > 0$ we can find a $d > 0$ such that for any d-pseudotrajectory ξ of f there exists a point $p \in M$ such that

$$\mathrm{dist}\left(x_k, f^k(p)\right) < \varepsilon, \quad k \in \mathbb{Z}. \tag{1.3}$$

In this case, we say that the pseudotrajectory ξ is ε-*shadowed* by the exact trajectory of the point p, and the trajectory $O(p,f)$ is called the *shadowing trajectory*.

Sometimes, this property is called the *standard shadowing property* or the POTP (*pseudoorbit tracing property*, see [5] and [6]).

In addition to infinite pseudotrajectories, we consider also finite pseudotrajectories, i.e., sets of points

$$\xi = \{x_k \in M : \ l \le k \le m\}$$

such that analogs of inequalities (1.2) hold for $l \le k \le m-1$.

The corresponding shadowing property called *finite shadowing property* means that for any $\varepsilon > 0$ we can find a $d > 0$ such that for any finite d-pseudotrajectory ξ of f as above there exists a point $p \in M$ such that analogs of inequalities (1.3) hold for $l \le k \le m-1$. Here it is important to emphasize that d depends on ε and does not depend on the number $m-l$.

In what follows, it will be convenient for us to introduce special notation for sets of dynamical systems having some shadowing properties. Let us denote by SSP_D the set of systems with discrete time having the standard shadowing property (of course, any time, using a notation of that kind, we will indicate the phase space and the class of smoothness of the considered dynamical systems).

In this book, we also consider several modifications of the standard shadowing property.

The first of these modifications is a property that is weaker than the standard shadowing property. First let us recall the definition of the Hausdorff metric.

Denote by $\mathscr{C}(M)$ the set of all nonempty compact subsets of M. Let $x \in M$ and $K \in \mathscr{C}(M)$; set

$$\mathrm{dist}(x, K) = \min_{y \in K} \mathrm{dist}(x, y).$$

The Hausdorff metric dist_H on $\mathscr{C}(X)$ is defined as follows:

$$\mathrm{dist}_H(A, B) = \max\left(\max_{a \in A} \mathrm{dist}(a, B), \max_{b \in B} \mathrm{dist}(b, A) \right)$$

for $A, B \in \mathscr{C}(X)$.

The next result which we use below is well known (see p. 47 of [32]).

Lemma 1.1.1 *If the space M is compact, then $(\mathscr{C}(M), dist_H)$ is a compact metric space.*

Definition 1.1.3 We say that a dynamical system f has the *orbital shadowing property* if for any $\varepsilon > 0$ we can find a $d > 0$ such that for any d-pseudotrajectory ξ of f there exists a point $p \in M$ such that

$$\mathrm{dist}_H(\mathrm{Cl}(\xi), \mathrm{Cl}(O(p, f))) < \varepsilon. \tag{1.4}$$

We denote by OSP_D the set of systems with discrete time having the orbital shadowing property.

One more shadowing property is defined below.

Definition 1.1.4 We say that f has the *Lipschitz shadowing property* if there exist $\mathscr{L}, d_0 > 0$ such that for any d-pseudotrajectory $\{x_k\}$ with $d \le d_0$ there exists an exact trajectory $\{f^k(p)\}$ satisfying the inequalities

$$\operatorname{dist}\left(x_k, f^k(p)\right) \le \mathscr{L}d, \quad k \in \mathbb{Z}. \tag{1.5}$$

One can define the *finite Lipschitz shadowing property* similarly to the finite shadowing property (we leave details to the reader).

Let us denote by LSP_D the set of systems with discrete time having the Lipschitz shadowing property.

Obviously, the following inclusions hold:

$$\mathrm{LSP}_D \subset \mathrm{SSP}_D \subset \mathrm{OSP}_D \tag{1.6}$$

(of course, here we have in mind that we consider dynamical systems with the same phase spaces).

Simple examples show that all the inclusions in (1.6) are strict.

To show that $\mathrm{SSP}_D \setminus \mathrm{LSP}_D \neq \emptyset$, consider a North Pole – South Pole diffeomorphism f of the circle S^1 that has two fixed points, an asymptotically stable fixed point s and a completely unstable (i.e., asymptotically stable for f^{-1}) fixed point u and such that $f^k(x) \to s, k \to \infty$, for any $x \neq u$, and $f^k(x) \to u, k \to -\infty$, for any $x \neq s$. It is easy to show that such a diffeomorphism f has the standard shadowing property. Theorem 1.4.1 (1) implies that if the fixed points s and u are hyperbolic (in this case, f is structurally stable), then f has the Lipschitz shadowing property. At the same time, it is an easy exercise to show that f does not have the Lipschitz shadowing property if one of the fixed points s or u is not hyperbolic.

It is also an easy exercise to show that irrational rotation of the circle gives us an example of a diffeomorphism belonging to $\mathrm{OSP}_D \setminus \mathrm{SSP}_D$.

It is possible to study shadowing properties dealing with pseudotrajectories that are subjected to some additional restrictions. In this book, we consider the case of periodic pseudotrajectories.

Definition 1.1.5 We say that f has the *periodic shadowing property* if for any $\varepsilon > 0$ we can find a $d > 0$ such that for any periodic d-pseudotrajectory ξ of f there exists a periodic point p of f such that inequalities (1.3) hold.

Remark 1.1.2 Note that it is not assumed in the above definition that the periods of the pseudotrajectory ξ and periodic point p coincide.

Let us denote by PerSP_D the set of systems with discrete time having the periodic shadowing property.

Definition 1.1.6 We say that f has the *Lipschitz periodic shadowing property* if there exist positive constants $\mathscr{L}, d_0$ such that if $\xi = \{x_k\}$ is a periodic

d-pseudotrajectory with $d \leq d_0$, then there exists a periodic point p of f such that inequalities (1.5) hold.

Let us denote by LPerSP_D the set of systems with discrete time having the Lipschitz periodic shadowing property.

As was mentioned, we also consider pseudotrajectories defined not on $\mathbb{Z}$ but on some subsets of $\mathbb{Z}$. Such pseudotrajectories will appear, for example, in the study of the following property.

Definition 1.1.7 We say that f has the *Hölder shadowing property on finite intervals* with constants $\mathscr{L}, C, d_0, \theta, \omega > 0$ if for any d-pseudotrajectory

$$\xi = \{x_k : \ 0 \leq k \leq Cd^{-\omega}\}$$

of f with $d \leq d_0$ there exists a point p such that

$$\operatorname{dist}\left(x_k, f^k(p)\right) \leq \mathscr{L} d^{\theta}, \quad 0 \leq k \leq Cd^{-\omega}. \tag{1.7}$$

We denote by $\mathrm{FHSP}_D(\mathscr{L}, C, d_0, \theta, \omega)$ the set of systems with discrete time having the property formulated in Definition 1.1.7.

An important application of pseudotrajectories defined on subsets of $\mathbb{Z}$ is the theory of chain recurrence and chain transitivity.

The main tools in this theory are ε-chains (finite ε-trajectories joining points of the phase space; following tradition, we preserve this terminology and use ε instead of d in analogs of inequalities (1.2)).

Until the end of this section, we assume, in addition, that M is a compact metric space.

Let C be a subset of M and let $p, q \in C$.

Definition 1.1.8 For $\varepsilon > 0$, a sequence $\{x_0, x_1, \dots, x_m\}$ of points of the subset C is called an *ε-chain* in C of length $m + 1$ from p to q if $x_0 = p$, $x_m = q$, and $\operatorname{dist}(f(x_i), x_{i+1}) < \varepsilon$ for $0 \leq i < m$.

If there is an ε-chain in C from p to q, then we write $p \rightsquigarrow_C^{\varepsilon} q$.

Let us also write

$$p \leftrightsquigarrow_C^{\varepsilon} q \text{ if both } p \rightsquigarrow_C^{\varepsilon} q \text{ and } q \rightsquigarrow_C^{\varepsilon} p,$$

$$p \rightsquigarrow_C q \text{ if } p \rightsquigarrow_C^{\varepsilon} q \text{ for any } \varepsilon > 0,$$

$$p \leftrightsquigarrow_C q \text{ if } p \leftrightsquigarrow_C^{\varepsilon} q \text{ for any } \varepsilon > 0.$$

In the above notation, we omit C if $C = M$.

Definition 1.1.9 A point $x \in M$ is called a *chain recurrent point* if $x \leftrightsquigarrow x$.

Definition 1.1.10 The set

$$\mathscr{R}(f) = \{x \in M : \ x \leftrightsquigarrow x\}$$

of all chain recurrent points of f is called the *chain recurrent set of* f.

Definition 1.1.11 Two points x and y of M are called *chain equivalent* if $x \leftrightsquigarrow y$.

Note that if $x, y \in M$ and $x \leftrightsquigarrow y$, then $x, y \in \mathscr{R}(f)$.

Clearly, the chain equivalence is an equivalence relation on $\mathscr{R}(f)$.

Definition 1.1.12 Each equivalence class of the above equivalence relation is called a *chain recurrence class*.

We note that $\mathscr{R}(f)$ and chain recurrence classes are closed f-invariant sets (see Lemma 1.1.5 below).

Definition 1.1.13 We say that a closed f-invariant set Λ is *chain transitive* if $x \rightsquigarrow_{\Lambda} y$ for any $x, y \in \Lambda$.

A chain recurrence class $\mathscr{R}$ is called a *maximal chain transitive set* if the inclusion $\mathscr{R} \subset C$, where C is a chain transitive set, implies that $\mathscr{R} = C$.

The main statement which we prove in this section is the following proposition.

Proposition 1.1.1 *Any chain recurrence class is a maximal chain transitive set.*

The next convention will be frequently used in this section. For $\varepsilon > 0$, $\delta(\varepsilon)$ denotes a real number such that $0 < \delta(\varepsilon) < \varepsilon$ and the inequality $\operatorname{dist}(x, y) < \delta(\varepsilon)$ implies that $\operatorname{dist}(f(x), f(y)) < \epsilon$.

We prove a sequence of lemmas which we need.

Lemma 1.1.2 *The relation*

$$\mathscr{R}(\rightsquigarrow) = \{(x, y) \in M \times M : x \rightsquigarrow y\}$$

is closed in $M \times M$.

Proof Let a sequence $\{(x_i, y_i) : 1 \le i < \infty\}$ in $\mathscr{R}(\rightsquigarrow)$ converge to $(x, y) \in M \times M$. We show that $(x, y) \in \mathscr{R}(\rightsquigarrow)$. For $\varepsilon > 0$, let $\delta = \delta(\varepsilon/3)$. Fix an index $i \ge 1$ such that $\max(\operatorname{dist}(x_i, x), \operatorname{dist}(y_i, y)) < \delta$. Since $x_i \rightsquigarrow y_i$, there is a δ-chain $\{z_0, \dots, z_m\}$ from x_i to y_i. Assume that $m = 1$. Then

$$\operatorname{dist}(f(x), y) \le \operatorname{dist}(f(x), f(z_0)) + \operatorname{dist}(f(z_0), z_1) + \operatorname{dist}(z_1, y) <$$
$$< \varepsilon/3 + \delta + \delta < \varepsilon.$$

Thus, $x \rightsquigarrow^{\epsilon} y$. Next assume that $m \ge 2$. Then it is easy to see that

$$\{x, z_1, z_2, \dots, z_{m-1}, y\}$$

is an ε-chain from x to y. Hence, $x \rightsquigarrow^{\epsilon} y$ in any case. Since $\varepsilon > 0$ is arbitrary, $x \rightsquigarrow y$, and $(x, y) \in \mathscr{R}(\rightsquigarrow)$. □

The following statement is an obvious corollary of Lemma 1.1.2.

Lemma 1.1.3 *The relation*

$$\mathscr{R}(\leftrightsquigarrow) = \{(x, y) \in M \times M : x \leftrightsquigarrow y\}$$

is closed in $M \times M$.

Lemma 1.1.4

$$(f \times f)(\mathscr{R}(\rightsquigarrow)) \subset \mathscr{R}(\rightsquigarrow)$$

and

$$(f \times f)(\mathscr{R}(\leftrightsquigarrow)) \subset \mathscr{R}(\leftrightsquigarrow).$$

Proof It is enough to prove the first inclusion. Let $(x, y) \in \mathscr{R}(\rightsquigarrow)$; we show that $(f(x), f(y)) \in \mathscr{R}(\rightsquigarrow)$. Fix an $\varepsilon > 0$ and let $\delta = \delta(\varepsilon)$. Since $x \rightsquigarrow y$, there is a δ-chain $\{x_0, \dots, x_m\}$ from x to y. It is easy to see that $\{f(x_0), \dots, f(x_m)\}$ is an ε-chain from $f(x)$ to $f(y)$. Thus, $f(x) \rightsquigarrow f(y)$. Since $\epsilon > 0$ is arbitrary, $(f(x), f(y)) \in \mathscr{R}(\rightsquigarrow)$. $\square$

Lemma 1.1.5 *The set $\mathscr{R}(f)$ and each chain recurrence class are closed f-invariant sets.*

Proof Let A be a chain recurrence class of f. It follows directly from Lemma 1.1.3 that both $\mathscr{R}(f)$ and A are closed. Since $\mathscr{R}(f)$ is a disjoint union of chain recurrence classes, it is enough to show that A is f-invariant.

Let $x \in A$. Then for each $n \geq 1$ there is a $(1/n)$-chain $\{x_0^n, \dots, x_{m_n}^n\}$ from x to itself. Put $y_n = x_{m_n-1}^n$, $n \geq 1$, and let y be one of the limit points of the sequence $\{y_n : n \geq 1\}$. It is easy to see that $x \rightsquigarrow y$. Since $\operatorname{dist}(f(y), x) < 1/n$ for $n \geq 1$, we get the equality $f(y) = x$. Hence, $f(x) \rightsquigarrow f(y) = x$ by Lemma 1.1.4. Since $y \rightsquigarrow f(y) = x \rightsquigarrow f(x)$, we conclude that $x \leftrightsquigarrow y$ and $x \leftrightsquigarrow f(x)$. Thus, both y and $f(x)$ are chain recurrent points and belong to A. Since $x \in A$ is arbitrary, it follows that $f(A) \supset A \supset f(A)$, i.e., $f(A) = A$. $\square$

Let, as above, $\mathscr{C}(M)$ be the set of all nonempty compact subsets of M with the Hausdorff metric dist_H (by Lemma 1.1.1, $(\mathscr{C}(M), \operatorname{dist}_H)$ is a compact metric space).

Consider the map $\mathscr{C}(f) : \mathscr{C}(M) \to \mathscr{C}(M)$ defined by $\mathscr{C}(f)(A) = f(A)$ for $A \in \mathscr{C}(X)$. Clearly, this map is continuous.

Recall that a closed f-invariant subset A is chain transitive if $x \leftrightsquigarrow_A y$ for all $x, y \in A$.

Proof (of Proposition 1.1.1) Let A be a chain recurrence class.

By Lemma 1.1.5, A is closed and $f(A) = A$. We prove the proposition modifying the proof of the result of Robinson [84]. Let $x, y \in A$. For each integer $n \geq 1$, take a $(1/n)$-chain $C_n = \{x_0^n, \dots, x_{m_n}^n\}$ from x through y to x. In particular, $x, y \in C_n$. Since $C_n \in \mathscr{C}(M)$ for any n, there is a subsequence n_k such that $\lim_{k\to\infty} C_{n_k} = C$ for some $C \in \mathscr{C}(M)$. Note that $x, y \in C$. We show that $f(C) = C$. Since $x_0^n = x_{m_n}^n$, we see that $\text{dist}_H(f(C_n), C_n) < 1/n$. Thus,

$$\text{dist}_H(f(C), C) \le \text{dist}_H(f(C), f(C_{n_k})) +$$

$$+\text{dist}_H(f(C_{n_k}), C_{n_k}) + \text{dist}_H(C_{n_k}, C) \le$$

$$\le \text{dist}_H\,(f(C), f(C_{n_k})) + \frac{1}{n_k} + \text{dist}_H(C_{n_k}, C).$$

Letting $k \to \infty$, we conclude that $\text{dist}_H(f(C), C) = 0$, i.e., $f(C) = C$.

Next we show that C is chain transitive. Let $z, w \in C$, and fix any $\varepsilon > 0$. Let $\delta = \delta(\varepsilon/3)$ and take $n = n_k$ such that $1/n < \varepsilon/3$ and $\text{dist}_H(C, C_n) < \delta$. Then

$$C \subset \bigcup_{i=0}^{m_n} N(\delta, x_i^n)$$

(recall that $N(\delta, x) = \{ y \in M : \text{dist}(y, x) < \delta\}$).

Take i, j with $0 \le i, j \le m_n$ such that $\text{dist}(z, x_i^n) < \delta$ and $\text{dist}(w, x_j^n) < \delta$. Since $x_0^n = x_{m_n}^n$, there is a $(1/n)$-chain $\{y_0, y_1, \dots, y_m\} \subset C$ from x_i^n to x_j^n. We now construct an ε-chain $\{z_0, z_1, \dots, z_m\}$ in C from z to w. For $0 < k < m$, take $z_k \in C$ such that $z_k \in N(\delta, y_k)$, and let $z_0 = z$, $z_m = w$. Since $\text{dist}(z_k, y_k) < \delta = \delta(\varepsilon/3)$, it follows that

$$\text{dist}(f(z_k), z_{k+1}) \le \text{dist}(f(z_k), f(y_k)) + \text{dist}(f(y_k), y_{k+1}) +$$

$$+\text{dist}(y_{k+1}, z_{k+1}) < \varepsilon/3 + \frac{1}{n} + \delta < \varepsilon$$

for each $0 \le k < m$. Thus, $\{z_0, \dots, z_m\}$ is an ε-chain in C from z to w. Since $\varepsilon > 0$ is arbitrary, $z \rightsquigarrow_C w$. Since $z, w \in C$ are arbitrary, $z \leftrightsquigarrow_C w$ for any $z, w \in C$, i.e., C is chain transitive. If we take $x = z$, then $x \leftrightsquigarrow_C w$ for all $w \in C$. Thus, $C \subset A$. Since $x \leftrightsquigarrow_C y$, we conclude that $x \leftrightsquigarrow_A y$. Hence, A is chain transitive, as desired. The maximality of A is obvious. □

It is easy to see that the following statement holds (we omit the proof).

Lemma 1.1.6 *For any $x \in M$, the omega-limit set $\omega_f(x)$ of x and alpha-limit set $\alpha_f(x)$ of x are chain transitive.*

Historical Remarks Pseudotrajectories of a special kind (called δ-chains) were considered by G. D. Birkhoff in his study of the last Poincaré geometric theorem [9].

The first basic results related to shadowing were obtained by D. V. Anosov and R. Bowen in [4] and [12] for hyperbolic sets of diffeomorphisms. It is easily seen that both Anosov's and Bowen's proofs, in fact, give Lipschitz shadowing in a neighborhood of a hyperbolic set of a diffeomorphism.

The orbital shadowing property was first considered in the joint paper [65] of the authors of this book and A. A. Rodionova.

The periodic and Lipschitz periodic shadowing were studied by A. V. Osipov, the first author of this book, and S. B. Tikhomirov in [50].

S. B. Tikhomirov studied the Hölder shadowing property on finite intervals in the paper [101].

C. Conley introduced the notion of chain recurrence in [14] and [15]. Most of the results of this section devoted to chain recurrence and chain transitivity, which were reformulated for discrete dynamical systems in Shimomura [93], can be found in [14] and [15] in the case of flows. As far as we know, chain transitive sets of discrete dynamical systems with the standard shadowing property were first considered in [93] from the view point of topological entropy.

1.2 Pseudotrajectories and Shadowing in Dynamical Systems with Continuous Time

Let M be a smooth closed manifold. Consider a C^1 vector field X on M and denote by ϕ the flow of X. We denote by

$$O(x, \phi) = \{\phi(t, x) : \ t \in \mathbb{R}\}$$

the trajectory of a point x in the flow ϕ; $O^+(x, \phi)$ and $O^-(x, \phi)$ are the positive and negative semitrajectories, respectively.

Definition 1.2.1 Fix a number $d > 0$. We say that a mapping $g : \mathbb{R} \to M$ (not necessarily continuous) is a *d-pseudotrajectory* (both for the field X and flow ϕ) if

$$\operatorname{dist}(g(\tau + t), \phi(t, g(\tau))) < d \quad \text{for} \quad \tau \in \mathbb{R},\ t \in [0, 1]. \tag{1.8}$$

Of course, one can also define finite pseudotrajectories defined not on $\mathbb{R}$ but on finite segments $[a, b]$. We leave details to the reader.

It is easy to understand that, defining shadowing properties in the case of flows, it is not reasonable to give a definition parallel to Definition 1.1.2 just replacing inequality (1.3) by an inequality of the form

$$\operatorname{dist}(g(t), \phi(t, p)) < \varepsilon, \quad t \in \mathbb{R}. \tag{1.9}$$

Indeed, consider the following simple example.

Example 1.2.1 Let M be the two-dimensional sphere S^2; consider in a coordinate neighborhood U homeomorphic to $\mathbb{R}^2$ a vector field X having an isolated closed trajectory γ parametrized by

$$\xi(t) = (\sin t, \cos t), \quad t \in \mathbb{R}.$$

Take a small $d > 0$ and let

$$g(t) = \xi(t + kd/2), \quad t \in [2\pi k, 2\pi(k+1)),\ k \in \mathbb{Z}.$$

Since $|X(x)| = 1$ at points of γ, it is easy to understand that g is a d-pseudotrajectory of X.

Assume that there exists a point p such that inequality (1.9) holds with $\varepsilon = \varepsilon(d) \to 0$ as $d \to 0$. Since the trajectory γ is isolated, this is possible (for ε small enough) only if $p \in \gamma$. In this case, there exists a θ such that

$$\phi(t, p) = \xi(t + \theta).$$

Note that

$$\phi(2\pi k, p) = \xi(2\pi k + \theta) = \xi(\theta), \quad k \in \mathbb{Z},$$

while the set of points

$$g(2\pi k) = \xi(2\pi k + kd/2) = \xi(kd/2)$$

is d-dense in γ. Hence, for any d small enough there exists a k such that the distance between $g(2\pi k)$ and $\xi(2\pi k + \theta) = \xi(\theta)$ is larger than $\pi/2$, which contradicts our assumption.

Clearly, a similar construction can be realized in any flow having an isolated closed trajectory, and the set of such flows is large enough.

To avoid problems of that kind, one has to change parametrization of the shadowing trajectories. We introduce the following notion.

Definition 1.2.2 A *reparametrization* is an increasing homeomorphism h of the line $\mathbb{R}$; we denote by Rep the set of all reparametrizations.

For $a > 0$, we denote

$$\mathrm{Rep}(a) = \left\{ h \in \mathrm{Rep} : \left| \frac{h(t) - h(s)}{t - s} - 1 \right| < a, \quad t, s \in \mathbb{R},\ t \neq s \right\}.$$

Definition 1.2.3 We say that a vector field X has the *standard shadowing property* if for any $\varepsilon > 0$ we can find $d > 0$ such that for any d-pseudotrajectory $g(t)$ of X there exists a point $p \in M$ and a reparametrization $h \in \mathrm{Rep}(\varepsilon)$ such that

$$\mathrm{dist}(g(t), \phi(h(t), p)) < \varepsilon \quad \text{for} \quad t \in \mathbb{R}. \tag{1.10}$$

We denote by SSP_F the set of vector fields having the standard shadowing property.

Definition 1.2.4 We say that a vector field X has the *Lipschitz shadowing property* if there exist $d_0 > 0$ and $\mathscr{L} > 0$ such that for any d-pseudotrajectory $g(t)$ of X with

$d \le d_0$ there exists a point $p \in M$ and a reparametrization $h \in \text{Rep}(\mathscr{L}d)$ such that

$$\text{dist}(g(t), \phi(h(t), p)) \le \mathscr{L}d \quad \text{for} \quad t \in \mathbb{R}. \tag{1.11}$$

We denote by LSP_F the set of vector fields having the Lipschitz shadowing property.

Definition 1.2.5 We say that a vector field X has the *oriented shadowing property* if for any $\varepsilon > 0$ we can find $d > 0$ such that for any d-pseudotrajectory $g(t)$ of X there exists a point $p \in M$ and a reparametrization $h \in \text{Rep}$ such that inequalities (1.10) hold (we emphasize that in this case, it is not assumed that the reparametrization h is close to identity).

We denote by OrientSP_F the set of vector fields having the oriented shadowing property.

Definition 1.2.6 We say that a vector field X has the *orbital shadowing property* if for any $\varepsilon > 0$ we can find $d > 0$ such that for any d-pseudotrajectory $g(t)$ of X there exists a point $p \in M$ such that

$$\text{dist}_H(\text{Cl}(\{g(t) : \ t \in \mathbb{R}\}), \text{Cl}(O(p, \phi))) < \varepsilon.$$

We denote by OrbitSP_F the set of vector fields having the orbital shadowing property.

Obviously, the following inclusions hold:

$$\text{OrbitSP}_F \supset \text{OrientSP}_F \supset \text{SSP}_F \supset \text{LSP}_F$$

(of course, here we have in mind that we consider vector fields on the same manifold).

It is easy to show that

$$\text{SSP}_F \setminus \text{LSP}_F \neq \emptyset.$$

It was recently shown by Tikhomirov [100] that

$$\text{OrientSP}_F \setminus \text{SSP}_F \neq \emptyset$$

(this solved the old problem posed by M. Komuro in [29]).

Historical Remarks Let us note that the standard shadowing property of vector fields (and their flows) is equivalent to the strong pseudo orbit tracing property (POTP) in the sense of M. Komuro [29] and [30]; the oriented shadowing property was called the normal POTP by M. Komuro [29] and the POTP for flows by R. F. Thomas in [102].

1.3 Hyperbolicity, Ω-Stability, Structural Stability, Dominated Splittings

Let us shortly recall the definitions of basic notions of the theory of structural stability of dynamical systems which we use in this book.

Let M be a smooth closed manifold and let f be a diffeomorphism of M of class C^1.

Definition 1.3.1 We say that a set $I \subset M$ is a *hyperbolic set* of a diffeomorphism f if the following conditions hold:

(HSD1) the set I is compact and f-invariant;
(HSD2) there exist numbers $C > 0$ and $\lambda \in (0, 1)$ and linear subspaces $S(p)$ and $U(p)$ of the tangent space T_pM defined for any point $p \in I$ such that
(HSD2.1) $S(p) \oplus U(p) = T_pM$;
(HSD2.2) $Df(p)S(p) = S(f(p))$ and $Df(p)U(p) = U(f(p))$;
(HSD2.3) if $v \in S(p)$, then $\left|Df^k(p)v\right| \le C\lambda^k|v|$ for $k \ge 0$;
(HSD2.4) if $v \in U(p)$, then $\left|Df^k(p)v\right| \le C\lambda^{-k}|v|$ for $k \le 0$.

The numbers $C > 0$ and $\lambda \in (0, 1)$ are usually called *hyperbolicity constants* of the set I; the families $S(p)$ and $U(p)$ are called the *hyperbolic structure* on I.

The main objects related to a hyperbolic set I are stable and unstable manifolds of its points.

Definition 1.3.2 The *stable* and *unstable manifolds* of a point $p \in I$ are the sets defined by the equalities

$$W^s(p) = \left\{x \in M : \operatorname{dist}\left(f^k(x), f^k(p)\right) \to 0,\ k \to \infty\right\}$$

and

$$W^u(p) = \left\{x \in M : \operatorname{dist}\left(f^k(x), f^k(p)\right) \to 0,\ k \to -\infty\right\},$$

respectively.

The classical stable manifold theorem (see, for example, [27, 108]) states that if p is a point of a hyperbolic set I as above and $\sigma(p) = \dim S(p)$, then $W^s(p)$ is the image of the Euclidean space $\mathbb{R}^{\sigma(p)}$ under a C^1 immersion α_p^s; this means that the map

$$\alpha_p^s : \mathbb{R}^{\sigma(p)} \to W^s(p)$$

is one-to-one and that

$$\operatorname{rank} D\alpha_p^s(x) = \sigma(p), \quad x \in \mathbb{R}^{\sigma(p)}.$$

In addition, $\alpha_p^s(0) = p$ and

$$T_p W^s(p) = S(p).$$

A similar statement (with $\sigma(p) = \dim U(p)$) is valid for $W^U(p)$.

One more classical definition which we need is the definition of the nonwandering set of a diffeomorphism f.

Definition 1.3.3 A point x is called a *nonwandering* point of f if for any neighborhood U of x and for any number N there exists a number n, $|n| > N$, such that $f^n(U) \cap U \neq \emptyset$. We denote by $\Omega(f)$ the set of nonwandering points of f (sometimes, the set $\Omega(f)$ is called the *nonwandering set* of f).

It is not difficult to show that the set $\Omega(f)$ is nonempty, compact, and f-invariant (see, for example, [71]).

Now we recall the two basic definitions of the theory of structural stability of diffeomorphisms, the definitions of Ω-stability and structural stability.

Let us start with the definition of the C^1 *topology* on the space of diffeomorphisms of a smooth closed manifold M.

First we define a C^0 metric ρ_0 on the space of homeomorphisms of a compact metric space.

Let $(M,\ \mathrm{dist})$ be a compact metric space. If f and g are two homeomorphisms of the space M, we set

$$\rho_0(f, g) = \max_{x \in M} \max \left(\mathrm{dist}(f(x), g(x)), \mathrm{dist}(f^{-1}(x), g^{-1}(x)) \right). \tag{1.12}$$

It is easy to show that ρ_0 is a metric on the space of homeomorphisms of the space M.

We denote by $H(M)$ the space of homeomorphisms of the space M with the metric ρ_0; the topology induced by the metric ρ_0 is called the C^0 *topology*.

It is not difficult to show that the metric space $H(M)$ is complete (see, for example, [71]). At the same time, if we consider the topology on the space of homeomorphisms induced by the standard uniform metric

$$\max_{x \in M}\ \mathrm{dist}(f(x), g(x)), \tag{1.13}$$

then the resulting space is not necessarily complete (see [71]).

Let now M be a smooth closed n-dimensional manifold. To introduce the C^1 topology on the space of diffeomorphisms of M, we assume that M is a submanifold of the Euclidean space $\mathbb{R}^N$ (a different, equivalent, approach to definition of the C^1 topology based on local coordinates is described in [60]).

No generality is lost assuming that M is a submanifold of a Euclidean space since, by the classical Whitney theorem, any smooth closed manifold can be embedded into a Euclidean space of appropriate dimension.

If M is a submanifold of $\mathbb{R}^N$, for any point $x \in M$ we can identify the tangent space T_xM of M at x with a linear subspace of $\mathbb{R}^N$. Consider the metric dist on M

induced by the Euclidean metric of the space $\mathbb{R}^N$. For a vector $v \in T_xM$ we denote by $|v|$ its norm as the norm in the space $\mathbb{R}^N$.

Let f and g be two diffeomorphisms of the manifold M. Define the value $\rho_0(f, g)$ by the same formula (1.12) as for homeomorphisms of a compact metric space.

Take a point x of the manifold M and a vector v from the tangent space T_xM. We consider the vectors $Df(x)v \in T_{f(x)}M$ and $Dg(x)v \in T_{g(x)}M$ as vectors of the same Euclidean space $\mathbb{R}^N$. Hence, the following values are defined: $|Df(x)v - Dg(x)v|$ and

$$\|Df(x) - Dg(x)\| = \max_{v \in T_xM, |v|=1} |Df(x)v - Dg(x)v|.$$

Introduce the number

$$\rho_1(f, g) = \rho_0(f, g) + \max_{x \in M} \max\left(\|Df(x) - Dg(x)\|, \|Df^{-1}(x) - Dg^{-1}(x)\|\right).$$

Clearly, ρ_1 is a metric on the space of diffeomorphisms of the manifold M. We denote by $\mathrm{Diff}^1(M)$ the space of diffeomorphisms of M with the metric ρ_1; the topology induced by the metric ρ_1 is called the C^1 *topology*.

The standard reasoning shows that the topology on $\mathrm{Diff}^1(M)$ does not depend on the embedding of M into a Euclidean space and that $(\mathrm{Diff}^1(M), \rho_1)$ is a complete metric space.

Remark 1.3.1 To explain why it is reasonable to include the term $\|Df^{-1}(x) - Dg^{-1}(x)\|$ in the definition of the C^1 topology on the space of diffeomorphisms, let us consider the following example.

Let $M = S^1$ with coordinate $x \in [0, 1)$, fix a small $t \geq 0$ and define a mapping

$$f_t : S^1 \to S^1$$

by the formula

$$f_t(x) = tx + x^3 + h_t(x),$$

where h_t is of class C^1 in x,

$$h_t(x) = 0, \quad x \leq 1/3,$$

and

$$h_t(x) = 3x(1 - x) - t, \quad x \geq 2/3.$$

Then

$$f_t(x) = tx + x^3, \quad x \leq 1/3,$$

and

$$f_t(x) = 1 + t(x-1) + (x-1)^3, \quad x \geq 2/3.$$

Clearly, one can construct h_t so that

$$f_t'(x) > 0$$

for small $t > 0$ and for all x (thus, any f_t with such t is a diffeomorphism of S^1) and

$$\sup_{0 \leq x < 1} \left(|h_t(x) - h_\tau(x)| + |h_t'(x) - h_\tau'(x)| \right) \to 0, \quad t, \tau \to 0. \tag{1.14}$$

It follows from (1.14) that the family $\{f_t\}$ is a Cauchy sequence as $t \to 0$ with respect to the metric

$$\rho(f, g) = \sup_{0 \leq x < 1} \left(|f(x) - g(x)| + |f'(x) - g'(x)| \right)$$

but, clearly, its limit as $t \to 0$ is not a diffeomorphism of S^1.

Thus, the space of diffeomorphisms of S^1 with the metric ρ is not complete.

In what follows, if A is a subset of $\mathrm{Diff}^1(M)$, then $\mathrm{Int}^1(A)$ denotes the interior of A in $\mathrm{Diff}^1(M)$.

Definition 1.3.4 A diffeomorphism f is called *structurally stable* if there exists a neighborhood W of the diffeomorphism f in $\mathrm{Diff}^1(M)$ such that any diffeomorphism $g \in W$ is topologically conjugate to f (i.e., there exists a homeomorphism $h : M \to M$ such that $h \circ f = g \circ h$).

We denote by $\mathcal{S}_D(M)$ the set of structurally stable diffeomorphisms in $\mathrm{Diff}^1(M)$. We agree to write Diff^1 and $\mathcal{S}_D$ instead of $\mathrm{Diff}^1(M)$ and $\mathcal{S}_D(M)$, respectively, if it is not important for us to indicate the manifold M (as in the remark below).

Remark 1.3.2 Clearly, the set $\mathcal{S}_D$ is open in Diff^1.

Definition 1.3.5 A diffeomorphism f is called *Ω-stable* if there exists a neighborhood W of the diffeomorphism f in $\mathrm{Diff}^1(M)$ such that for any diffeomorphism $g \in W$ there exists a homeomorphism $h : \Omega(f) \to \Omega(g)$ such that

$$h \circ f|_{\Omega(f)} = g \circ h|_{\Omega(f)}.$$

We denote by $\Omega\mathcal{S}_D(M)$ (or simply $\Omega\mathcal{S}_D$) the set of Ω-stable diffeomorphisms. The following statements are also obvious.

Remark 1.3.3

(1) The set $\Omega\mathcal{S}_D$ is open in Diff^1.
(2) $\mathcal{S}_D \subset \Omega\mathcal{S}_D$.

Now we pass to characterization of Ω-stability and structural stability.

S. Smale introduced the following condition.

Axiom A

(AAa) The nonwandering set $\Omega(f)$ is hyperbolic.
(AAb) Periodic points of f are dense in $\Omega(f)$.

This condition played a very important role in the development of the theory of structural stability. First we describe the structure of the nonwandering set of a diffeomorphism that satisfies Axiom A. Smale proved the following statement.

Theorem 1.3.1 (Spectral Decomposition Theorem) *If a diffeomorphism f satisfies Axiom A, then its nonwandering set can be represented in the form*

$$\Omega(f) = \Omega_1 \cup \cdots \cup \Omega_m, \tag{1.15}$$

where the Ω_i are disjoint, compact, invariant sets such that each of these sets contains a dense positive semitrajectory. Representation of the form (1.15) is unique.

The sets Ω_i in representation (1.15) are called *basic*.

We can define analogs of stable and unstable manifolds for basic sets Ω_i:

$$W^s(\Omega_i) = \left\{x \in M : \ \text{dist}\left(f^k(x), \Omega_i\right) \to 0, \quad k \to \infty\right\}$$

and

$$W^u(\Omega_i) = \left\{x \in M : \ \text{dist}\left(f^k(x), \Omega_i\right) \to 0, \quad k \to -\infty\right\}.$$

The following statement holds (one can find a proof, for example, in [60]).

Theorem 1.3.2 *If a diffeomorphism f satisfies Axiom A, then*

$$M = \bigcup_{i=1}^{m} W^s(\Omega_i) = \bigcup_{i=1}^{m} W^u(\Omega_i). \tag{1.16}$$

Thus, any trajectory $f^k(x)$ of a diffeomorphism that satisfies Axiom A tends to a basic set as $|k| \to \infty$.

Now we give definitions which we need to formulate necessary and sufficient conditions of Ω-stability and structural stability of diffeomorphisms.

Let Ω_i and Ω_j be two (not necessarily different) basic sets of a diffeomorphism that satisfies Axiom A. We write $\Omega_i \to \Omega_j$ if there is a point $x \notin \Omega(f)$ such that

$$f^{-k}(x) \to \Omega_i \text{ and } f^k(x) \to \Omega_j, \quad k \to \infty.$$

Definition 1.3.6 We say that a diffeomorphism f has a 1-*cycle* if there exists a basic set Ω_i such that $\Omega_i \to \Omega_i$.

We say that a diffeomorphism f has a *k-cycle*, $k > 1$, if there exist k different basic sets $\Omega_{i_1}, \dots, \Omega_{i_k}$ such that

$$\Omega_{i_1} \to \cdots \to \Omega_{i_k} \to \Omega_{i_1}.$$

We say that a diffeomorphism satisfies the *no cycle condition* if it does not have k-cycles with $k \geq 1$.

Theorem 1.3.3 *A diffeomorphism f is Ω-stable if and only if f satisfies Axiom A and the no cycle condition.*

Definition 1.3.7 Let f be a diffeomorphism satisfying Axiom A. We say that f satisfies the *geometric strong transversality condition* if stable and unstable manifolds of nonwandering points are transverse, i.e., if $p, q \in \Omega(f)$ and $x \in W^u(p) \cap W^s(q)$, then

$$T_x W^u(p) + T_x W^s(q) = T_x M. \tag{1.17}$$

Remark 1.3.4 Usually, the condition introduced in Definition 1.3.7 is called the strong transversality condition; we add the term *geometric* to distinguish this condition and the analytic strong transversality condition introduced below, in Definition 1.3.11.

Theorem 1.3.4 *A diffeomorphism f is structurally stable if and only if f satisfies Axiom A and the geometric strong transversality condition.*

Theorems 1.3.3 and 1.3.4 are classical basic results of the theory of structural stability. Nevertheless, sometimes it is more convenient to use different statements which characterize Ω-stability and structural stability (as we do in this book). Let us formulate some of them.

Recall that $\mathrm{Per}(f)$ denotes the set of periodic points of a diffeomorphism f.

Definition 1.3.8 A periodic point p is called *hyperbolic* if its trajectory $O(p,f)$ is a hyperbolic set. It is easy to see that if p is a periodic point of period m, then p is hyperbolic if and only if the derivative $Df^m(p)$ does not have eigenvalues λ with $|\lambda| = 1$.

Denote by HP_D the set of diffeomorphisms f such that any periodic point of f is hyperbolic.

Theorem 1.3.5 *The sets $Int^1(HP_D)$ and $\Omega\mathscr{S}_D$ coincide.*

Sometimes, the set $\mathrm{Int}^1(\mathrm{HP}_D)$ is denoted by $\mathscr{F}$ and its elements are called *star systems*.

Remark 1.3.5 It follows from Theorem 1.3.5 that to establish the Ω-stability of a diffeomorphism f, it is enough to show that f and its C^1-small perturbations do not have nonhyperbolic periodic points.

Definition 1.3.9 A diffeomorphism $f \in \mathrm{HP}_D$ is called *Kupka–Smale* if stable and unstable manifolds of its periodic points are transverse. We denote by KS_D the set of Kupka–Smale diffeomorphisms.

Definition 1.3.10 A subset A of a topological space X is called *residual* if A contains the intersection of a countable family of open and dense subsets of X. A property P of elements of X is called *generic* if the set

$$\{x \in X : x \text{ satisfies } P\}$$

is residual.

Theorem 1.3.6

(1) The set KS_D is residual in $Diff^1$.
(2) The sets $Int^1(KS_D)$ and $\mathscr{S}_D$ coincide.

Remark 1.3.6 It follows from the second statement of Theorem 1.3.6 that to establish that a diffeomorphism f is structurally stable, it is enough to show that f has a C^1 neighborhood belonging to KS_D.

One more way of proving that a diffeomorphism is structurally stable is based on the result of Theorem 1.3.7 (Mañé's theorem) below. Let us start with a definition.

Fix a point $x \in M$ and consider the following two subspaces of T_xM:

$$B^+(x) = \left\{ v \in T_xM : \lim_{k\to\infty} \inf \left|Df^k(x)v\right| = 0 \right\}$$

and

$$B^-(x) = \left\{ v \in T_xM : \lim_{k\to-\infty} \inf \left|Df^k(x)v\right| = 0 \right\}.$$

Definition 1.3.11 We say that a diffeomorphism f satisfies the *analytic strong transversality condition* if

$$B^+(x) + B^-(x) = T_xM \quad \text{for any} \quad x \in M. \tag{1.18}$$

Theorem 1.3.7 *A diffeomorphism f is structurally stable if and only if f satisfies the analytic strong transversality condition.*

A detailed proof of Theorem 1.3.7 is given in Chap. 2 of this book.

Let us define one more important for us property of invariant sets of diffeomorphisms.

Let Λ be a compact invariant set of a diffeomorphism f.

Definition 1.3.12 We say that f admits a *dominated splitting on Λ* if there exist continuous families of linear subspaces $E(p)$ and $F(p)$ of the tangent spaces T_pM for $p \in \Lambda$ such that

(DS1) $E(p) \oplus F(p) = T_pM$, $p \in \Lambda$;
(DS2) the subspaces $E(p)$ and $F(p)$ are Df-invariant (i.e., analogs of equalities (HSD2.2) from Definition 1.3.1 with $S(p)$ and $U(p)$ replaced by $E(p)$ and $F(p)$ are satisfied);
(DS3) there exist numbers $C > 0$ and $\lambda \in (0, 1)$ such that

$$\left\| Df^k|_{E(p)} \right\| \cdot \left\| Df^{-k}|_{F(f^k(p))} \right\| \leq C\lambda^k, \quad p \in \Lambda,\ k \geq 0. \tag{1.19}$$

One more notion which we use in this book is the notion of a homoclinic point (and homoclinic trajectory).

Let p be a hyperbolic periodic point of a diffeomorphism f.

Definition 1.3.13 A point $q \neq p$ such that

$$q \in W^u(p) \cap W^s(p)$$

is called a *homoclinic* point of the periodic point p.

A homoclinic point q of p is called *transverse* if the stable and unstable manifolds $W^s(p)$ and $W^u(q)$ are transverse at q.

Theorem 1.3.8 *Any neighborhood of a transverse homoclinic point contains an infinite set of different hyperbolic periodic points of f.*

Many notions and statements formulated above for diffeomorphisms have analogs for flows generated by smooth vector fields. Let us give the corresponding definitions and state theorems which we need in what follows (in the case of similar objects, for example, such as the nonwandering set etc., we do not repeat the definitions and leave details to the reader).

Let X be a smooth (of class C^1) vector field on a smooth closed manifold M. Let

$$\phi : \mathbb{R} \times M \to M$$

be the flow generated by X and let, as above,

$$O(x, \phi) = \{\phi(t, x) : t \in \mathbb{R}\}$$

be the trajectory of a point $x \in M$ in the flow ϕ.

We denote by $\Phi(t, p)$ the derivative (in p) of $\phi(t, p)$; thus,

$$\Phi(t, p) : T_pM \to T_{\phi(t,p)}M.$$

Definition 1.3.14 We say that a set $I \subset M$ is a *hyperbolic set* of the vector field X (and its flow ϕ) if I has the following properties:

(HSF1) the set I is compact and ϕ-invariant;
(HSF2) there exist numbers $C > 0$ and $\lambda > 0$ and linear subspaces $S(p)$ and $U(p)$ of the tangent space T_pM defined for any point $p \in I$ such that

(HSF2.1) $S(p) \oplus U(p) \oplus \{X(p)\} = T_pM$, where $\{X(p)\}$ is the subspace spanned by the vector $X(p)$;

(HSF2.2)

$$\Phi(t,p)S(p) = S(\phi(t,p)) \quad \text{and} \quad \Phi(t,p)U(p) = U(\phi(t,p)), \quad t \in \mathbb{R};$$

(HSF2.3) if $v \in S(p)$, then $|\Phi(t,p)v| \leq C\exp(-\lambda t)|v|$ for $t \geq 0$;

(HSF2.4) if $v \in U(p)$, then $|\Phi(t,p)v| \leq C\exp(\lambda t)|v|$ for $t \leq 0$.

Similarly to the case of diffeomorphisms, the main objects related to a hyperbolic set I of a flow ϕ are stable and unstable manifolds of its points (and its trajectories).

Definition 1.3.15 The *stable and unstable manifolds* of a point p are the sets defined by the equalities

$$W^s(p) = \{x \in M \ : \mathrm{dist}(\phi(t,x), \phi(t,p)) \to 0, \ t \to \infty\}$$

and

$$W^u(p) = \{x \in M \ : \mathrm{dist}(\phi(t,x), \phi(t,p)) \to 0, \ t \to -\infty\},$$

respectively.

One uses these objects to define the stable and unstable manifolds of the trajectory of a point p:

$$W^s(O(p,\phi)) = \bigcup_{t\in\mathbb{R}} W^s(\phi(t,p))$$

and

$$W^u(O(p,\phi)) = \bigcup_{t\in\mathbb{R}} W^u(\phi(t,p)).$$

The stable manifold theorem for flows states that if p is a point of a hyperbolic set I as above and $\sigma(p) = \dim S(p)$, then the structure of $W^s(O(p,\phi))$ can be described as follows:

- if p is a rest point (i.e., $\phi(t,p) \equiv p, \ t \in \mathbb{R}$), then $W^s(O(p,\phi)) = W^s(p)$ is the image of the Euclidean space $\mathbb{R}^{\sigma(p)}$ under a C^1 immersion;
- if $O(p,\phi)$ is a closed trajectory that is not a rest point (i.e., $\phi(t,p)$ is periodic in t with a nonzero minimal period), then $W^s(O(p,\phi))$ is the image under a C^1 immersion of a fiber bundle over the circle with fibers $\mathbb{R}^{\sigma(p)}$;
- if $O(p,\phi)$ is a trajectory such that $\phi(t_1,p) \neq \phi(t_2,p)$ for $t_1 \neq t_2$, then $W^s(O(p,\phi))$ is the image of the Euclidean space $\mathbb{R}^{\sigma(p)+1}$ under a C^1 immersion.

Similar statements hold for the unstable manifolds of trajectories of a hyperbolic set.

Now we recall the two basic definitions of the theory of structural stability of vector fields, the definitions of Ω-stability and structural stability.

Let us start with the definition of the C^1 topology on the space of vector fields of class C^1 on a smooth closed manifold M (everywhere below, a vector field is a vector field of class C^1).

Let X and Y be two such vector fields; define the number

$$\rho_1(X, Y) = \max_{x \in M} \left(|X(x) - Y(x)| + \left\| \frac{\partial X}{\partial x}(x) - \frac{\partial Y}{\partial x}(x) \right\| \right).$$

It is easily seen that ρ_1 is a metric on the space of vector fields of class C^1; we denote by $\mathscr{X}^1(M)$ (or simply by $\mathscr{X}^1$) the space of vector fields with this metric (and with the induced topology which we call C^1 topology). As in the case of diffeomorphisms, if A is a subset of $\mathscr{X}^1(M)$, then $\mathrm{Int}^1(A)$ denotes the interior of A in $\mathscr{X}^1(M)$.

Remark 1.3.7 Let X and Y be two vector fields and let ϕ and ψ be their flows, respectively. Consider the diffeomorphisms $f(x) = \phi(1, x)$ and $g(x) = \psi(1, x)$. It is not difficult to show that if $\rho_1(X, Y) \to 0$, then $\rho_1(f, g) \to 0$ (see, for example, Chap. 2 of [71]).

Let us denote by $\mathrm{Per}(X)$ (or $\mathrm{Per}(\phi)$) the set of rest points and closed trajectories of X (and its flow ϕ) and by $\Omega(X)$ ($\Omega(\phi)$) the nonwandering set (the definition of the nonwandering set for a flow is similar to that for a diffeomorphism, and we omit it).

Definition 1.3.16 A vector field X (and its flow ϕ) is called *structurally stable* if there exists a neighborhood W of X in $\mathscr{X}^1(M)$ such that for any vector field $Y \in W$, its flow ψ is *topologically equivalent* to the flow ϕ, i.e., there exists a homeomorphism $h : M \to M$ that maps trajectories of X to trajectories of Y preserving the orientation of trajectories.

Let us denote by $\mathscr{S}_F(M)$ (or $\mathscr{S}_F$) the set of structurally stable vector fields (and flows).

Remark 1.3.8 Let us note that, in contrast to Definition 1.3.4, it is not assumed in Definition 1.3.16 that h is a topological conjugacy of the flows ϕ and ψ of X and Y (the latter means that

$$h(\phi(t, x)) = \psi(t, h(x))$$

for all t and x).

In fact, the homeomorphism h in Definition 1.3.16 must have the following property: There exists a function $\tau : \mathbb{R} \times M \to \mathbb{R}$ such that

(1) for any $x \in M$, the function $\tau(\cdot, x)$ increases and maps $\mathbb{R}$ onto $\mathbb{R}$;
(2) $\tau(0, x) = x$ for any $x \in M$;
(3) $h(\phi(t, x)) = \psi(\tau(t, x), h(x))$ for any $(t, x) \in \mathbb{R} \times M$.

Clearly, the necessity of time reparametrization of shadowing trajectories in the case of shadowing for flows (see Sect. 1.2) is caused by the same reasons as the replacement of topological conjugacy by topological equivalence in Definition 1.3.16.

Definition 1.3.17 A vector field X (and its flow ϕ) is called *Ω-stable* if there exists a neighborhood W of X in $\mathscr{X}^1(M)$ such that for any vector field $Y \in W$, its flow ψ is *Ω-equivalent* to the flow ϕ, i.e., there exists a homeomorphism $h : \Omega(\phi) \to \Omega(\psi)$ that maps trajectories of $\Omega(\phi)$ to trajectories of $\Omega(\psi)$ preserving the orientation of trajectories.

Let us denote by $\Omega\mathscr{S}_F(M)$ (or $\Omega\mathscr{S}_F$) the set of Ω-stable vector fields (and flows).

The following condition (also introduced by Smale) is an analog of Axiom A for the case of vector fields and flows.

Axiom A′

(AA′a) The nonwandering set $\Omega(\phi)$ of the flow ϕ is hyperbolic.
(AA′b) The set $\Omega(\phi)$ is the union of two disjoint compact ϕ-invariant sets Q_1 and Q_2, where Q_1 consists of a finite number of rest points, while Q_2 does not contain rest points, and points of closed trajectories are dense in Q_2.

If a flow ϕ satisfies Axiom A′, then the following analog of Theorem 1.3.1 holds.

Theorem 1.3.9 *The nonwandering set $\Omega(\phi)$ has a unique representation of the form*

$$\Omega(\phi) = \Omega_1 \cup \cdots \cup \Omega_m,$$

where the Ω_i are disjoint, compact, ϕ-invariant sets such that each of these sets contains a dense positive semitrajectory.

As in the case of a diffeomorphism, the sets Ω_i are called *basic*. A basic set of a flow ϕ that satisfies Axiom A′ is either a rest point or a closed invariant set that does not contain rest points and such that points of closed trajectories are dense in it.

Let Ω_i and Ω_j be two different basic sets of a flow ϕ that satisfies Axiom A′. We write $\Omega_i \to \Omega_j$ if there exists a point x such that

$$\phi(t,x) \to \Omega_i,\ t \to -\infty, \quad \text{and} \quad \phi(t,x) \to \Omega_j,\ t \to \infty.$$

The no cycle condition for a flow ϕ literally repeats the corresponding condition for a diffeomorphism.

The following statement is an analog of Theorem 1.3.3.

Theorem 1.3.10 *A flow ϕ is Ω-stable if and only if ϕ satisfies Axiom A′ and the no cycle condition.*

If a flow ϕ satisfies Axiom A′, then hyperbolic trajectories $\phi(t,p)$, $p \in \Omega(\phi)$, have stable and unstable manifolds $W^s(O(p,\phi))$ and $W^u(O(p,\phi))$, respectively.

Definition 1.3.18 We say that such a flow ϕ satisfies the *geometric strong transversality condition* if for any points $p, q \in \Omega(\phi)$, the manifolds $W^s(O(q,\phi))$ and $W^u(O(p,\phi))$ are transverse at any point of their intersection.

The following statement is an analog of Theorem 1.3.4.

Theorem 1.3.11 *A flow ϕ is structurally stable if and only if ϕ satisfies Axiom A′ and the geometric strong transversality condition.*

Definition 1.3.19 A rest point or a closed trajectory of a flow ϕ is called *hyperbolic* if it is a hyperbolic set of ϕ.

Remark 1.3.9 Condition under which a rest point or a closed trajectory is hyperbolic are well-known:

- a rest point p of a flow ϕ generated by a vector field X is hyperbolic if and only if any eigenvalue of the Jacobi matrix $DX(p)$ has nonzero real part;
- a closed trajectory γ of a flow ϕ is hyperbolic if and only if, for any transverse section Σ at any point of γ, the zero point of the section (corresponding to the intersection of γ with Σ) is a hyperbolic fixed point of the Poincaré map generated by Σ (see [71] for details).

Denote by HP_F the set of flows ϕ such that any rest point and closed trajectory of ϕ is hyperbolic.

A complete analog of Theorem 1.3.5 for vector fields (and flows) is not correct (see Historical Remarks at the end of this section). Only the following partial analog is valid.

Theorem 1.3.12 *A nonsingular vector field in $\mathrm{Int}^1(HP_F)$ belongs to $\Omega\mathscr{S}_F$.*

Definition 1.3.20 A flow $\phi \in \mathrm{HP}_F$ is called *Kupka–Smale* if stable and unstable manifolds of its rest points and closed trajectories are transverse. We denote by KS_F the set of Kupka–Smale flows.

The following statement is an analog of Theorem 1.3.6.

Theorem 1.3.13

(1) The set KS_F is residual in $\mathscr{X}^1$.
(2) The sets $\mathrm{Int}^1(KS_F)$ and $\mathscr{S}_F$ coincide.

Let us describe one more approach for establishing the structural stability of a flow.

Let, as above, ϕ be the flow generated by a vector field X.

Definition 1.3.21 A point $x \in M$ is called a *chain recurrent point* of the flow ϕ if for any $d, T > 0$ there exists a d-pseudotrajectory g of ϕ (in the sense of Definition 1.2.1) such that $g(0) = x$ and $g(t) = x$ for some $t \geq T$.

In this case, similarly to Sect. 1.1, we write $x \leftrightsquigarrow x$.

Definition 1.3.22 The set

$$\mathscr{R}(\phi) = \{x \in M : \ x \leftrightsquigarrow x\}$$

of all chain recurrent points of ϕ is called the *chain recurrent set of* ϕ.

It is easy to show (compare with Sect. 1.1) that in our case (where M is a compact manifold), the set $\mathscr{R}(\phi)$ is nonempty, compact, and ϕ-invariant.

In Sect. 2.7, we refer to the following two results.

Theorem 1.3.14 *If X is a vector field of class C^1 such that the chain recurrent set $\mathscr{R}(\phi)$ of its flow ϕ is hyperbolic and stable and unstable manifolds of trajectories in $\mathscr{R}(\phi)$ are transverse, then X is structurally stable.*

Now we formulate a theorem which allows one to show that components of the set $\mathscr{R}(\phi)$ are hyperbolic.

Let Σ be a compact, ϕ-invariant component of $\mathscr{R}(\phi)$ that does not contain rest points of ϕ. Denote $f(x) = \phi(1, x)$.

For a point $x \in \Sigma$, denote by $P(x)$ the orthogonal projection in T_xM with kernel spanned by $X(x)$ and by $V(x)$ the orthogonal complement to $X(x)$ in T_xM. Consider the normal subbundle $\mathscr{V}(\Sigma)$ of the tangent bundle $TM|_\Sigma$ which is the set of pairs $(x, V(x))$, where $x \in \Sigma$.

Define a mapping π on the normal subbundle $\mathscr{V}(\Sigma)$ over Σ by

$$\pi(x, v) = (f(x), B(x)v)\,, \text{ where } B(x) = P(f(x))Df(x)$$

(recall that $f(x) = \phi(1, x)$).

The hyperbolicity of π on $\mathscr{V}(\Sigma)$ is defined similarly to the usual hyperbolicity of a diffeomorphism on a compact invariant set. It means that there exist numbers $C > 0$ and $\lambda \in (0, 1)$ and linear subspaces $S(p), U(p)$ of $V(p)$ for $p \in \Sigma$ such that

- $S(p) \oplus U(p) = V(p)$;
- $B(p)S(p) = S(f(p))$ and $B(p)U(p) = U(f(p))$;
- if $v \in S(p)$, then $\left|B^k(p)v\right| \le C\lambda^k|v|$ for $k \ge 0$;
- if $v \in U(p)$, then $\left|B^k(p)v\right| \le C\lambda^{-k}|v|$ for $k \le 0$.

Theorem 1.3.15 *If π is hyperbolic on $\mathscr{V}(\Sigma)$, then Σ is a hyperbolic set of the flow ϕ.*

If p is a rest point of a flow ϕ (i.e., $O(p, \phi) = \{p\}$), then we denote by $W^s(p)$ and $W^u(p)$ (instead of $W^s(O(p, \phi))$ etc.) its stable and unstable manifolds, respectively.

If γ is a closed trajectory of a flow ϕ (i.e., $O(p, \phi) = \gamma$ for any $p \in \gamma$), then we denote by $W^s(\gamma)$ and $W^u(\gamma)$ its stable and unstable manifolds, respectively.

Let p be a hyperbolic rest point (or let γ be a hyperbolic closed trajectory) of a flow ϕ.

Definition 1.3.23 A point $q \neq p$ such that

$$q \in W^u(p) \cap W^s(p)$$

is called a *homoclinic* point of the rest point p.

A point $q \notin \gamma$ such that

$$q \in W^u(\gamma) \cap W^s(\gamma)$$

is called a *homoclinic* point of the closed trajectory γ.

A homoclinic point q of γ is called *transverse* if the stable and unstable manifolds $W^s(p)$ and $W^u(q)$ are transverse at q.

Remark 1.3.10 Let us note that a homoclinic point q of a hyperbolic rest point p cannot be transverse. Indeed, such a point q cannot be a rest point (otherwise, $q = p$); hence, $X(q) \neq 0$ (where X is the vector field which generates the flow ϕ).

Since

$$\dim W^s(p) + \dim W^u(p) = \dim M$$

and

$$0 \neq X(q) \in T_q W^s(p) \cap T_q W^u(p),$$

the equality

$$T_q W^s(p) + T_q W^s(p) = T_q M$$

is impossible.

An analog of Theorem 1.3.8 for flows can be formulated as follows.

Theorem 1.3.16 *If q is a transverse homoclinic point of a hyperbolic closed trajectory γ of a flow ϕ, then any neighborhood of $O(q, \phi)$ contains an infinite set of different hyperbolic closed trajectories of ϕ.*

Historical Remarks The general definition of a hyperbolic set is usually attributed to D. V. Anosov [3].

The stable manifold theorem has a long history; usually, one refers to the names of J. Hadamard and O. Perron (one can find an interesting discussion concerning the theory of stable and unstable manifolds in D. V. Anosov's monograph [3]; there he mentiones also G. Darboux, H. Poincaré, and A. M. Lyapunov).

The notions of nonwandering points and other classical objects of the global theory of dynamical systems were introduced and studied by G. Birkhoff [10].

The theory of structural stability originates from the A. A. Andronov and L. S. Pontryagin's paper [2] in which they defined a kind of such a property for vector fields in a two-dimensional disk or on the two-dimensional sphere.

A very important role was played by S. Smale's paper [95] in which he introduced the notions of Ω-stability, Axioms A and A$'$, proved the spectral decomposition theorem (Theorem 1.3.1), gave first sufficient conditions of Ω-stability, etc.

Later, S. Smale proved the sufficiency of conditions of Theorem 1.3.3 [98].

The basic results of the theory of Ω-stability and structural stability were formulated as conjectures by J. Palis and S. Smale [52].

The sufficiency statement in Theorem 1.3.4 was first proved by J. Robbin in [78] for diffeomorphisms of class C^2 and later by C. Robinson [81] in the general case.

The necessity of conditions of Theorem 1.3.4 was established by R. Mañé in [45]; later, the necessity of conditions of Theorem 1.3.5 was proved by J. Palis [53].

The set HP was studied by many authors; the set $\mathrm{Int}^1(\mathrm{HP})$ is sometimes denoted by $\mathscr{F}$ (or $\mathscr{F}^1$), and its elements are called star systems (both in the case of diffeomorphisms and in the case of vector fields).

Theorem 1.3.5 was proved by Aoki [7] and S. Hayashi [25].

The complete analog of Theorem 1.3.5 for vector fields (and flows) is not correct. A vector field in $\mathrm{Int}^1(\mathrm{HP}_F)$ may fail to have a hyperbolic nonwandering set, as the famous Lorenz attractor shows [22], or fail to have rest points and closed trajectories dense in the nonwandering set [17], or, even if Axiom A$'$ is satisfied, still fail to satisfy the no cycle condition [37].

R. Mañé proved Theorem 1.3.7 in [39].

Theorem 1.3.12 was proved S. Gan and L. Wen in [21].

Kupka–Smale systems were independently defined and studied by I. Kupka [31] and S. Smale [94]. They proved Theorem 1.3.6 (1) and Theorem 1.3.13 (1).

Theorem 1.3.6 (2) follows from the results of [7] (where it was proved that $\mathrm{Int}^1(\mathrm{KS}_D) \subset \mathscr{S}_D$) and [82], where the inverse inclusion was established.

The inclusion $\mathrm{Int}^1(\mathrm{KS}_F) \subset \mathscr{S}_F$ was proved by H. Toyoshiba [103] and C. Robinson [80]; the inverse inclusion was established by C. Robinson [79] and S. Hayashi [26].

Homoclinic points were first studied by H. Poincaré [75]; Theorem 1.3.8 (as well as Theorem 1.3.13) belongs to S. Smale [96, 97].

The sufficiency of conditions of Theorem 1.3.10 was established by C. Pugh and M. Shub [76]; the sufficiency of conditions of Theorem 1.3.11 was proved by C. Robinson [79].

The necessity of conditions in these theorems follows from results of L. Wen [106] and S. Hayashi [26].

It was shown by J. E. Franke and J. F. Selgrade in [18] that for a flow ϕ, the set $\mathscr{R}(\phi)$ is hyperbolic if and only if ϕ satisfies Axiom A$'$ and the no cycle condition. Theorem 1.3.14 follows from this result combined with Theorem 1.3.11.

R. Sacker and G. Sell studied in detail dichotomies and invariant splittings in linear differential systems [86]; in particular, they proved Theorem 1.3.15 in [85].

1.4 Hyperbolic Shadowing

As we wrote in the Preface, one of the main goals of this book is to study relations between shadowing and basic notions of the theory of structural stability. It was known that structural stability implies Lipschitz shadowing both for diffeomorphisms and vector fields; let us formulate this as a theorem.

Theorem 1.4.1 *The following inclusions hold:*

(1) $\mathscr{S}_D \subset LSP_D$;
(2) $\mathscr{S}_F \subset LSP_F$.

We show in Chap. 2 that the inverse inclusions hold as well, so that structural stability is equivalent to Lipschitz shadowing.

An important part in the proof of Theorem 1.4.1 is the statement that a diffeomorphism or a vector field has the Lipschitz shadowing property in a neighborhood of its hyperbolic set.

In this section, we prove that a diffeomorphism has the finite Lipschitz shadowing property in a neighborhood of a hyperbolic set (in this book, we refer to this statement in Sect. 2.4). This is a classical result having a lot of different proofs. The proof which we give here is of a geometric origin; its modification can be applied in the absence of hyperbolicity as well (see, for example, [58]).

To simplify presentation, we consider a diffeomorphism f of $\mathbb{R}^n$ and its hyperbolic set Λ.

Our proof applies the existence of a so-called *adapted* (or *Lyapunov*) norm in a neighborhood of Λ (with respect to this norm, the constant C in inequalities (HSD2.3) and (HSD2.4) of Sect. 1.3 equals 1); a proof of this result can be found in [71].

Lemma 1.4.1 *Let Λ be a hyperbolic set of a diffeomorphism f. There exist constants $\nu \geq 1$ and $\lambda \in (0, 1)$ such that for any $\varepsilon > 0$ we can find a neighborhood W of the set Λ having the following property. There exists a positive constant δ, a C^∞ norm $|\cdot|_x$ for $x \in W$, and continuous (but not necessarily Df-invariant) extensions S' and U' of the families S and U of the given hyperbolic structure to the neighborhood W such that*

(1) $S'(p) \oplus U'(p) = \mathbb{R}^n, \quad p \in W;$

(2) if $p, q \in W$, $|f(p) - q| \leq \delta$, and $P(q)$ is the projection onto $S'(q)$ parallel to $U'(q)$, then the mapping $P(q)Df(p)$ is a linear isomorphism between $S'(p)$ and $S'(q)$ (respectively, if $Q(q) = Id - P(q)$, then the mapping $Q(q)Df(p)$ is a linear isomorphism between $U'(p)$ and $U'(q)$) and the following inequalities hold:

$$|P(q)Df(p)v|_q \leq \lambda|v|_p \text{ and } |Q(q)Df(p)v|_q \leq \varepsilon|v|_p, \quad v \in S'(p), \tag{1.20}$$

and

$$\lambda|Q(q)Df(p)v|_q \geq |v|_p \text{ and } |P(q)Df(p)v|_q \leq \varepsilon|v|_p, \quad v \in U'(p); \tag{1.21}$$

(3)

$$\frac{1}{\nu}|v|_p \leq |v| \leq \nu|v|_p, \quad p \in W, \ v \in \mathbb{R}^n; \tag{1.22}$$

(4)

$$\|P(p)\|, \|Q(p)\| \leq \nu, \quad p \in W \tag{1.23}$$

(in inequalities (1.23), we have in mind the operator norm related to the norm $|.|_p$).

Remark 1.4.1

(1) Since the adapted norm is Lipschitz equivalent to the standard norm (see inequalities (1.22)), f has (or does not have) the finite Lipschitz shadowing property with respect to these norms simultaneously. For that reason, to simplify presentation, we assume that the standard Euclidean norm is adapted. Similarly, we write $S(p)$ and $Q(p)$ instead of $S'(p)$ and $U'(p)$ for $p \in W$.
(2) In addition, we may assume that the neighborhoods W corresponding to small enough ε are subsets of a fixed closed neighborhood of Λ.

This allows us to assume that the norm $\|Df(p)\|$ is bounded for $p \in W$ and to use uniform estimates of the remainder term of the first-order Taylor formula for f in the proof of property (P$'$4) and in formula (1.35).

Thus, we assume that

$$\|Df(p)\| \leq M_0, \quad p \in W,$$

and set $M = \nu(1 + 12M_0)$.

Take

$$\mathscr{L} = 2\nu/(1 - \lambda) \tag{1.24}$$

and note that

$$\mathscr{L} > \lambda\mathscr{L} + \nu > 1 \text{ and } \mathscr{L}/\lambda > \mathscr{L} + \nu. \tag{1.25}$$

There exists an $\varepsilon > 0$ such that

$$\mathscr{L} > \nu + \varepsilon M(1 + \nu)\mathscr{L}, \tag{1.26}$$

$$\mathscr{L} > \lambda\mathscr{L} + \nu + \varepsilon(1 + 2\nu)\mathscr{L}. \tag{1.27}$$

Note that (1.27) implies the inequality

$$\mathscr{L}/\lambda > \mathscr{L} + \nu + \varepsilon(1 + 2\nu)\mathscr{L}. \tag{1.28}$$

Let W be a neighborhood of Λ corresponding by Lemma 1.4.1 to this ε.

Our main result in this section is as follows.

Theorem 1.4.2 *The diffeomorphism f has the finite Lipschitz shadowing property in W.*

Proof First we define several geometric objects related to the introduced structure.

Fix a point p in W; we represent points q close to p in the form $p + v$ and define our objects by imposing conditions on the projections $P(p)v$ and $Q(p)v$.

Let Δ' and Δ be positive numbers; consider the sets

$$R(\Delta', \Delta, p) = \left\{q = p + v : \ |P(p)v| \le \Delta', |Q(p)v| \le \Delta\right\};$$

we write $R(\Delta, p)$ instead of $R(\Delta, \Delta, p)$. Let

$$V(\Delta, p) = \{q = p + v \in R(\Delta, p) : \ |Q(p)v| = \Delta\}$$

and

$$T(\Delta, p) = \{q = p + v \in R(\Delta, p) : \ Q(p)v = 0\}.$$

Let us note several obvious geometric properties of the introduced objects.

(P1) $V(\Delta, p)$ is not a retract of $R(\Delta, p)$.
(P2) $V(\Delta, p)$ is a retract of $R(\Delta, p) \setminus T(\Delta, p)$.
(P3) If $\Delta' > \Delta$, then there exists a retraction

$$\sigma : R(\Delta', p) \to R(\Delta, p)$$

such that if

$$q = p + v \text{ and } Q(p)v \ne 0,$$

then

$$\sigma(q) = p + v', \text{ where } Q(p)v' \ne 0.$$

Now we prove several properties of the images of the introduced sets under f.

(P4) There exists a $\Delta_1 > 0$ such that if $p, r, f(p) \in W$, $\Delta \le \Delta_1$, and $|r - f(p)| < \Delta$, then

$$f(R(\Delta, p)) \subset R(M_1\Delta, r) \tag{1.29}$$

and

$$f^{-1}(R(\Delta, r)) \subset R(M_1\Delta, p), \tag{1.30}$$

where $M_1 = 4\nu M_0$.

We prove only the part of property (P4) related to inclusion (1.29); the part related to inclusion (1.30) is proved by a similar reasoning (possibly, with different constants M_1 and Δ_1).

First we prove an auxiliary statement:

(P4′) There exists a $\varDelta_1 > 0$ such that if $p, f(p) \in W$ and $\varDelta \le \varDelta_1$, then

$$f(R(\varDelta, p)) \subset R(M_1\varDelta, f(p)), \tag{1.31}$$

where $M_1 = 4\nu M_0$.

Indeed, take a point $q = p + v \in R(\varDelta, p)$; then

$$|v| \le |P(p)v| + |Q(p)v| \le 2\varDelta.$$

Since

$$f(q) = f(p) + Df(p)v + o(p, v),$$

where

$$|o(p, v)|/|v| \to 0, \quad |v| \to 0,$$

uniformly in p and $\|Df(p)\| \le M_0$, there exists a $\varDelta_1 > 0$ such that if $\varDelta \le \varDelta_1$, then

$$|f(q) - f(p)| \le 2M_0|v|, \quad |v| \le 2\varDelta.$$

If $f(q) = f(p) + w$, then

$$|P(f(p))w|, |Q(f(p))w| \le 2\nu M_0|v| \le 4\nu M_0\varDelta,$$

which proves (P4′) with $M_1 = 4\nu M_0$.

Now we prove (1.29). Since the projections P and Q are uniformly continuous, we can reduce, if necessary, $\varDelta_1$ so that

$$\|P(x) - P(y)\|, \|Q(x) - Q(y)\| < 1, \quad x, y \in W, \ |x - y| < \varDelta_1. \tag{1.32}$$

Let $\varDelta \le \varDelta_1$. Take a point $q \in f(R(\varDelta, p))$ and let

$$q = f(p) + v = r + w.$$

Then $|v - w| < \varDelta$ and

$$|P(f(p))v|, |Q(f(p))v| \le M_1\varDelta$$

by (P4′).

Let us estimate

$$|P(r)w| \le |P(r)w - P(r)v| + |P(r)v - P(f(p))v| + |P(f(p))v| \le$$

$$\le \nu\varDelta + 2M_1\varDelta + M_1\varDelta = (\nu + 3M_1)\varDelta = M\varDelta$$

(estimating the second term, we take the inequality $|v| \leq 2M_1\Delta$ and (1.32) into account).

A similar estimate holds for $|Q(r)w|$, which proves (1.29).

Of course, without loss of generality, we may assume that

$$M \geq 1. \tag{1.33}$$

Now we fix a

$$d_0 \in (0, \Delta_1/\mathscr{L})$$

with the following properties:

(1) if $p, f(p), r \in W$ and $|r - f(p)| < d_0$, then inequalities (1.20) and (1.21) are satisfied with the chosen ε;
(2) in the representation

$$f(p+v) = f(p) + Df(p)v + o(p,v), \tag{1.34}$$

the estimate

$$|o(p,v)| \leq \varepsilon|v|, \quad |v| \leq 2M\mathscr{L}d_0, \tag{1.35}$$

holds.

Now we prove one more statement.

(P5) If $d \leq d_0$, $p, f(p), r \in W$, $|r - f(p)| < d$, and $\Delta = \mathscr{L}d$, then

$$f(T(M\Delta, p)) \cap V(\Delta, r) = \emptyset, \tag{1.36}$$

$$f(T(\Delta, p)) \subset \mathrm{Int}(R(\Delta, r)), \tag{1.37}$$

$$f(R(\Delta, p)) \cap \partial R(\Delta, r) \subset V(\Delta, r), \tag{1.38}$$

and

$$f(V(\Delta, p)) \cap R(\Delta, r) = \emptyset. \tag{1.39}$$

First we prove relation (1.36).

If $q = p + v \in T(M\Delta, p)$, then $v = P(p)v \in S(p)$, $|P(p)v| \leq M\Delta = M\mathscr{L}d$, and $Q(p)v = 0$. Hence, it follows from representation (1.34) and estimates (1.26) and (1.35) that

$$|Q(r)(f(q) - r)| \leq |Q(r)(f(p) - r)| + |Q(r)Df(p)P(p)v| + |Q(r)o(p,v)| \leq$$

$$\leq \nu d + \varepsilon M\mathscr{L}d + \nu\varepsilon M\mathscr{L}d = (\nu + \varepsilon M(1+\nu)\mathscr{L})d < \mathscr{L}d = \Delta,$$

which proves relation (1.36).

Let us prove relations (1.37) and (1.38).

First we note that inequality (1.33) implies the inclusion

$$T(\Delta, p) \subset T(M\Delta, p),$$

and it follows from the above inequality that

$$|Q(r)(f(q) - r)| < \Delta, \quad q \in T(\Delta, p). \tag{1.40}$$

Now we consider a point $q = p + v \in R(\Delta, p)$, represent $v = P(p)v + Q(p)v$, and estimate

$$|P(r)(f(q) - r)| \leq |P(r)(f(p) - r)| + |P(r)Df(p)P(p)v| +$$

$$+ |P(r)Df(p)Q(p)v| + |P(r)o(p, v)| \leq$$

$$\leq \nu d + \lambda \mathscr{L} d + \varepsilon \mathscr{L} d + 2\nu\varepsilon \mathscr{L} d = (\nu + \lambda \mathscr{L} + \varepsilon(1 + 2\nu)\mathscr{L})d < \mathscr{L} d = \Delta$$

(here we refer to the estimate $|v| \leq 2\mathscr{L} d$ and to inequality (1.27)).

The above inequality proves relation (1.38). Combining it with inequality (1.40), we get a proof of relation (1.37).

Finally, we prove relation (1.39). If $q = p + v \in V(\Delta, p)$, then $|P(p)v| \leq \Delta = \mathscr{L} d$ and $|Q(p)v| = \Delta = \mathscr{L} d$. Then

$$|Q(r)(f(q) - r)| \geq$$

$$\geq |Q(r)Df(p)(P(p)v + Q(p)v)| - |Q(r)(f(p) - r)| - |Q(r)o(p, v)| \geq$$

$$\geq |Q(r)Df(p)Q(p)v| - |Q(r)Df(p)Q(p)v| - |Q(r)(f(p) - r)| - |Q(r)o(p, v)| \geq$$

$$\geq \mathscr{L} d/\lambda - \varepsilon \mathscr{L} d - \nu d - 2\varepsilon\nu \mathscr{L} d = (\mathscr{L}/\lambda - \nu - \varepsilon(1 + 2\nu))d > \mathscr{L} d = \Delta$$

(here we refer to inequality (1.28)). This proves relation (1.39).

Now we consider points $p_0, \ldots, p_m \in W$ such that

$$f(p_k) \in W, \quad k = 0, \ldots, m - 1, \tag{1.41}$$

and

$$|f(p_k) - p_{k+1}| < d \leq d_0, \quad k = 0, \ldots, m - 1,$$

and prove that there exists a point $r \in R(\Delta, p_0)$ such that

$$f^k(r) \in R(\Delta, p_k), \quad k = 1, \ldots, m, \tag{1.42}$$

where $\Delta = \mathscr{L}d$.

Let us note that condition (1.41) is not a real restriction since we can guarantee it reducing W, if necessary.

For brevity, we denote $R_k = R(\Delta, p_k), V_k = V(\Delta, p_k), T_k = T(\Delta, p_k)$.

Consider the sets

$$A_k = R_k \setminus \bigcap_{l=k+1}^{m} f^{-(l-k)}(\mathrm{Int}(R_l)), \quad k = 0, \dots, m-1.$$

It follows from equality (1.39) that

$$f(V_k) \cap R_{k+1} = \emptyset.$$

Hence, $V_k \subset A_k$.

We claim that there exist retractions

$$\rho_k : A_k \to V_k, \quad k = 0, \dots, m-1.$$

This is enough to prove our statement since the existence of ρ_0 means that

$$\bigcap_{l=0}^{m} f^{-l}(\mathrm{Int}(R_l)) \neq \emptyset$$

(otherwise there exists a retraction of R_0 to V_0, which is impossible by property (P1)), which, in turn, means that there exists a point $r \in R_0$ such that

$$f^k(r) \in R_k, \quad k = 0, \dots, m,$$

or

$$\left| f^k(r) - p_k \right| \leq 2\nu \mathscr{L}d, \quad k = 0, \dots, m.$$

Thus, our claim implies the finite Lipschitz shadowing property of f in W with constants d_0 and $2\nu\mathscr{L}$.

Let us prove our claim. The existence of ρ_{m-1} is obvious since inclusion (1.37) implies that

$$T_{m-1} \subset f^{-1}(\mathrm{Int}(R_m)),$$

and hence,

$$R_{m-1} \setminus f^{-1}(\mathrm{Int}(R_m)) \subset R_{m-1} \setminus T_{m-1},$$

while V_{m-1} is a retract of the latter set by property (P2).

Let us assume that the existence of retractions $\rho_{k+1}, \dots, \rho_{m-1}$ has been proved. Let us prove the existence of ρ_k.

The definition of the sets A_k implies that

$$A_k \cap f^{-1}(R_{k+1}) \subset f^{-1}(A_{k+1}) \tag{1.43}$$

since

$$f(A_k) \cap f^{-(l-k)+1}((\mathrm{Int}(R_l)) = \emptyset \text{ for } l \geq k+2.$$

Define a mapping θ on A_k by setting

$$\theta(q) = f^{-1} \circ \rho_{k+1} \circ f(q), \quad q \in A_k \cap f^{-1}(R_{k+1}),$$

and

$$\theta(q) = q, \quad q \in A_k \setminus f^{-1}(R_{k+1}).$$

Inclusion (1.43) shows that the mapping θ is properly defined.

Let us show that this mapping is continuous. Clearly, it is enough to show that $\rho_{k+1}(r) = r$ for $r \in f(A_k \cap f^{-1}(\partial R_{k+1}))$.

For this purpose, we note that

$$f(A_k \cap f^{-1}(\partial R_{k+1})) = f(A_k) \cap \partial R_{k+1} \subset f(R_k) \cap \partial R_{k+1} \subset V_{k+1}$$

(we refer to inclusion (1.38)) and $\rho_{k+1}(r) = r$ for $r \in V_{k+1}$.

Clearly, θ maps A_k into the set

$$B_k = [R_k \setminus f^{-1}(R_{k+1})] \cup f^{-1}(V_{k+1}). \tag{1.44}$$

Since $d < \Delta_1$ by our choice of d_0, it follows from property (P4) that

$$B_k \subset R(M\Delta, p_k).$$

Let us consider a retraction

$$\sigma : \; R(M\Delta, p_k) \to R_k$$

given by property (P3).

If

$$q = p_k + v \in f^{-1}(V_{k+1}) \setminus R_k,$$

then $q \notin T(M\Delta, p_k)$ by (1.36); thus, $Q(p)v \neq 0$. It follows from property (P3) that in this case,

$$\sigma(q) \in C_k := R_k \setminus T(\Delta, p_k).$$

If

$$q \in R_k \setminus f^{-1}(R_{k+1}),$$

then the above inclusion follows from (1.37).

Condition (P2) implies that there exists a retraction

$$\rho : \ C_k \to V_k.$$

It remains to note that $\theta(q) = q$ for $q \in V_k$ due to relation (1.39). Thus,

$$\rho_k = \rho \circ \sigma \circ \theta : \ A_k \to V_k$$

is the required retraction. □

Historical Remarks There exist several proofs of the inclusion

$$\mathscr{S}_D \subset \mathrm{SSP}_D$$

based on different ideas.

This statement was proved by A. Morimoto [46], K. Sawada [92], and C. Robinson [83] (note that the proof in [83] is not complete).

As far as the authors know, the first statement of Theorem 1.4.1 was first proved in the book [61] of the first author, and the second statement was proved in his paper [62].

Lemma 1.4.1 belongs to D. V. Anosov [3].

As was mentioned in Historical Remarks to Sect. 1.1, both classical proofs of the shadowing property in a neighborhood of a hyperbolic set of a diffeomorphism given by D. V. Anosov in [4] and R. Bowen in [12] show that shadowing is Lipschitz.

Our proof of Theorem 1.4.2 published in the joint paper [58] of the first author and A. A. Petrov mostly follows the ideas of the joint paper [63] of the first author and O. B. Plamenevskaya.

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