

Chapter 16

The Fourier Transform in Schwartz Space

Consider the Euclidean space \mathbb{R}^n , $n \geq 1$, with $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and with $|x| = \sqrt{x_1^2 + \dots + x_n^2}$ and scalar product $(x, y) = \sum_{j=1}^n x_j y_j$. The open ball of radius $\delta > 0$ centered at $x \in \mathbb{R}^n$ is denoted by

$$U_\delta(x) := \{y \in \mathbb{R}^n : |x - y| < \delta\}.$$

Recall the Cauchy–Bunyakovsky–Schwarz inequality

$$|(x, y)| \leq |x||y|.$$

Following Laurent Schwartz, we call an n -tuple $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j \in \mathbb{N} \cup \{0\} \equiv \mathbb{N}_0$ an n -dimensional multi-index. Define

$$|\alpha| = \alpha_1 + \dots + \alpha_n, \quad \alpha! = \alpha_1! \dots \alpha_n!$$

and

$$x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}, \quad 0^0 = 1, \quad 0! = 1.$$

Moreover, multi-indices α and β can be ordered according to

$$\alpha \leq \beta$$

if $\alpha_j \leq \beta_j$ for all $j = 1, 2, \dots, n$. Let us also introduce a shorthand notation

$$\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}, \quad \partial_j = \frac{\partial}{\partial x_j}.$$

Definition 16.1. The *Schwartz space* $S(\mathbb{R}^n)$ of rapidly decaying functions is defined as

$$S(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : |f|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)| < \infty \text{ for any } \alpha, \beta \in \mathbb{N}_0^n\}.$$

The following properties of $S = S(\mathbb{R}^n)$ are readily verified.

- (1) S is a linear space.
- (2) $\partial^\alpha : S \rightarrow S$ for every $\alpha \geq 0$.
- (3) $x^\beta \cdot : S \rightarrow S$ for every $\beta \geq 0$.
- (4) If $f \in S(\mathbb{R}^n)$, then $|f(x)| \leq c_m(1 + |x|)^{-m}$ for every $m \in \mathbb{N}$. The converse is not true (see part (3) of Example 16.2).
- (5) It follows from part (4) that $S(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for every $1 \leq p \leq \infty$.

Example 16.2.

- (1) $f(x) = e^{-a|x|^2} \in S$ for every $a > 0$.
- (2) $f(x) = e^{-a(1+|x|^2)^a} \in S$ for every $a > 0$.
- (3) $f(x) = e^{-|x|} \notin S$.
- (4) $C_0^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n)$, where

$$C_0^\infty(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \text{supp } f \text{ compact in } \mathbb{R}^n\}$$

$$\text{and } \text{supp } f = \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}.$$

The space $S(\mathbb{R}^n)$ is generated by a countable family of seminorms because $|f|_{\alpha,\beta}$ is only a seminorm for $\alpha \geq 0$ and $\beta > 0$, i.e., the condition

$$|f|_{\alpha,\beta} = 0 \quad \text{if and only if} \quad f = 0$$

fails to hold for, e.g., a constant function f . The space (S, ρ) is not normable but it is a metric space if the metric ρ is defined by

$$\rho(f, g) = \sum_{\alpha, \beta \geq 0} 2^{-|\alpha| - |\beta|} \cdot \frac{|f - g|_{\alpha,\beta}}{1 + |f - g|_{\alpha,\beta}}.$$

Exercise 16.1. Prove that ρ is a metric, that is,

- (1) $\rho(f, g) \geq 0$ and $\rho(f, g) = 0$ if and only if $f = g$.
- (2) $\rho(f, g) = \rho(g, f)$.
- (3) $\rho(g, h) \leq \rho(g, f) + \rho(f, h)$.

Prove also that $|\rho(f, h) - \rho(g, h)| \leq \rho(f, g)$.

Theorem 16.3 (Completeness). *The space (S, ρ) is a complete metric space, i.e., every Cauchy sequence converges.*

Proof. Let $\{f_k\}_{k=1}^\infty$, $f_k \in S$, be a Cauchy sequence, that is, for every $\varepsilon > 0$ there exists $n_0(\varepsilon) \in \mathbb{N}$ such that

$$\rho(f_k, f_m) < \varepsilon, \quad k, m \geq n_0(\varepsilon).$$

It follows that

$$\sup_{x \in K} |\partial^\beta (f_k - f_m)| < \varepsilon$$

for every $\beta \geq 0$ and for every compact set K in \mathbb{R}^n . This means that $\{f_k\}_{k=1}^\infty$ is a Cauchy sequence in the Banach space $C^{|\beta|}(K)$. Hence there exists a function $f \in C^{|\beta|}(K)$ such that

$$\lim_{k \rightarrow \infty} f_k \stackrel{C^{|\beta|}(K)}{=} f.$$

Thus we may conclude that our function f is in $C^\infty(\mathbb{R}^n)$. It remains only to prove that $f \in S$. It is clear that

$$\begin{aligned} \sup_{x \in K} |x^\alpha \partial^\beta f| &\leq \sup_{x \in K} |x^\alpha \partial^\beta (f_k - f)| + \sup_{x \in K} |x^\alpha \partial^\beta f_k| \\ &\leq C_\alpha(K) \sup_{x \in K} |\partial^\beta (f_k - f)| + \sup_{x \in K} |x^\alpha \partial^\beta f_k|. \end{aligned}$$

Taking $k \rightarrow \infty$, we obtain

$$\sup_{x \in K} |x^\alpha \partial^\beta f| \leq \limsup_{k \rightarrow \infty} |f_k|_{\alpha, \beta} < \infty.$$

The last inequality is valid, since $\{f_k\}_{k=1}^\infty$ is a Cauchy sequence, so that $|f_k|_{\alpha, \beta}$ is bounded. The last inequality doesn't depend on K , and we may conclude that $|f|_{\alpha, \beta} < \infty$, or $f \in S$. \square

Definition 16.4. We say that $f_k \xrightarrow{S} f$ as $k \rightarrow \infty$ if

$$|f_k - f|_{\alpha, \beta} \rightarrow 0, \quad k \rightarrow \infty$$

for all $\alpha, \beta \geq 0$.

Exercise 16.2. Prove that $\overline{C_0^\infty(\mathbb{R}^n)} = S$, that is, for every $f \in S$ there exists $\{f_k\}_{k=1}^\infty$, $f_k \in C_0^\infty(\mathbb{R}^n)$, such that $f_k \xrightarrow{S} f$, $k \rightarrow \infty$.

Now we are in position to define the Fourier transform in $S(\mathbb{R}^n)$.

Definition 16.5. The *Fourier transform* $\mathcal{F}f(\xi)$ or $\widehat{f}(\xi)$ of the function $f(x) \in S$ is defined by

$$\mathcal{F}f(\xi) \equiv \widehat{f}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i(x, \xi)} f(x) dx, \quad \xi \in \mathbb{R}^n.$$

Remark 16.6. This integral is well defined, since

$$\left| \widehat{f}(\xi) \right| \leq c_m (2\pi)^{-n/2} \int_{\mathbb{R}^n} (1 + |x|)^{-m} dx < \infty,$$

for $m > n$.

Next we prove the following properties of the Fourier transform:

(1) \mathcal{F} is a linear continuous map from S into S .

(2) $\xi^\alpha \partial_\xi^\beta \widehat{f}(\xi) = (-i)^{|\alpha|+|\beta|} \widehat{\partial_x^\alpha (x^\beta f(x))}$.

Indeed, we have

$$\partial_\xi^\beta \widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} (-ix)^\beta e^{-i(x,\xi)} f(x) dx$$

and hence

$$\left\| \partial_\xi^\beta \widehat{f}(\xi) \right\|_{L^\infty(\mathbb{R}^n)} \leq c_m (2\pi)^{-n/2} \int_{\mathbb{R}^n} \frac{|x|^{|\beta|}}{(1 + |x|)^m} dx < \infty$$

if we choose $m > n + |\beta|$. At the same time we have obtained the formula

$$\partial_\xi^\beta \widehat{f}(\xi) = \widehat{(-ix)^\beta f(x)}. \quad (16.1)$$

Further, integration by parts gives us

$$\xi^\alpha \widehat{f}(\xi) = (-i)^{|\alpha|} (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i(x,\xi)} \partial_x^\alpha f(x) dx,$$

from which we have the estimate

$$\left\| \xi^\alpha \widehat{f} \right\|_{L^\infty(\mathbb{R}^n)} \leq c \int_{\mathbb{R}^n} |\partial_x^\alpha f(x)| dx < \infty,$$

since $\partial_x^\alpha f(x) \in S$ for every $\alpha \geq 0$ if $f(x) \in S$. And also we have the formula

$$\xi^\alpha \widehat{f} = \widehat{(-i)^{|\alpha|} \partial_x^\alpha f}. \quad (16.2)$$

If we combine these last two estimates, we may conclude that $\mathcal{F} : S \rightarrow S$ and \mathcal{F} is a continuous map (in the sense of the metric space (S, ρ)), since \mathcal{F} maps every bounded set from S again to a bounded set from S .

The formulas (16.1) and (16.2) show us that it is more convenient to use the following notation:

$$D_j = -i \partial_j = -i \frac{\partial}{\partial x_j}, \quad D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}.$$

For this new derivative the formulas (16.1) and (16.2) can be rewritten as

$$D_{\xi}^{\alpha} \widehat{f} = (-1)^{|\alpha|} \widehat{x^{\alpha} f}, \quad \xi^{\alpha} \widehat{f} = \widehat{D^{\alpha} f}.$$

Example 16.7. It is true that

$$\mathcal{F}(e^{-\frac{1}{2}|x|^2})(\xi) = e^{-\frac{1}{2}|\xi|^2}.$$

Proof. The definition gives us directly

$$\begin{aligned} \mathcal{F}(e^{-\frac{1}{2}|x|^2})(\xi) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i(x,\xi) - \frac{1}{2}|x|^2} dx \\ &= (2\pi)^{-n/2} e^{-\frac{1}{2}|\xi|^2} \int_{\mathbb{R}^n} e^{-\frac{1}{2}(|x|^2 + 2i(x,\xi) - |\xi|^2)} dx \\ &= (2\pi)^{-n/2} e^{-\frac{1}{2}|\xi|^2} \prod_{j=1}^n \int_{-\infty}^{\infty} e^{-\frac{1}{2}(t+i\xi_j)^2} dt. \end{aligned}$$

In order to calculate the last integral, we consider the function $f(z) = e^{-\frac{z^2}{2}}$ of the complex variable z and the domain D_R depicted in Figure 16.1.

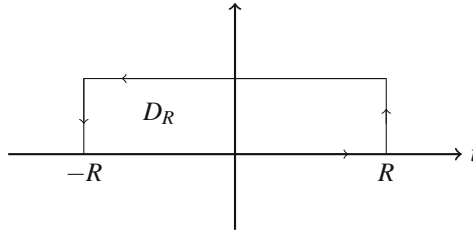


Fig. 16.1 Domain D_R .

We consider the positive direction of going around the boundary ∂D_R . It is clear that $f(z)$ is a holomorphic function in this domain, and by Cauchy's theorem we have

$$\oint_{\partial D_R} e^{-\frac{z^2}{2}} dz = 0.$$

But

$$\begin{aligned} \oint_{\partial D_R} e^{-\frac{z^2}{2}} dz &= \int_{-R}^R e^{-\frac{t^2}{2}} dt + i \int_0^{\xi_j} e^{-\frac{1}{2}(R+i\tau)^2} d\tau \\ &\quad + \int_R^{-R} e^{-\frac{1}{2}(t+i\xi_j)^2} dt + i \int_{\xi_j}^0 e^{-\frac{1}{2}(-R+i\tau)^2} d\tau. \end{aligned}$$

If $R \rightarrow \infty$, then

$$\int_0^{\xi_j} e^{-\frac{1}{2}(\pm R + i\tau)^2} d\tau \rightarrow 0.$$

Hence

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(t+i\xi_j)^2} dt = \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt, \quad j = 1, \dots, n.$$

Using Fubini's theorem and polar coordinates, we can evaluate the last integral as

$$\begin{aligned} \left(\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt \right)^2 &= \int_{\mathbb{R}^2} e^{-\frac{1}{2}(t^2+s^2)} dt ds = \int_0^{2\pi} d\theta \int_0^{\infty} e^{-\frac{r^2}{2}} r dr \\ &= 2\pi \int_0^{\infty} e^{-m} dm = 2\pi. \end{aligned}$$

Thus

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(t+i\xi_j)^2} dt = \sqrt{2\pi}$$

and

$$F(e^{-\frac{|x|^2}{2}})(\xi) = (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2}|\xi|^2} \prod_{j=1}^n \sqrt{2\pi} = e^{-\frac{1}{2}|\xi|^2}.$$

This completes the proof. □

Exercise 16.3. Let $P(D)$ be a differential operator,

$$P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha,$$

with constant coefficients. Prove that $\widehat{P(D)u} = P(\xi)\widehat{u}$.

Definition 16.8. We adopt the following notation for *translation* and *dilation* of a function:

$$(\tau_h f)(x) := f(x - h), \quad (\sigma_\lambda f)(x) := f(\lambda x), \quad \lambda \neq 0.$$

Exercise 16.4. Let $f \in S(\mathbb{R}^n)$, $h \in \mathbb{R}^n$, and $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Prove that

- (1) $\widehat{\sigma_\lambda f}(\xi) = |\lambda|^{-n} \widehat{\sigma_{\frac{1}{\lambda}} f}(\xi)$ and $\widehat{\sigma_\lambda f}(\xi) = |\lambda|^{-n} \widehat{\sigma_{\frac{1}{\lambda}} f}(\xi)$;
- (2) $\widehat{\tau_h f}(\xi) = e^{-i(h, \xi)} \widehat{f}(\xi)$ and $\widehat{\tau_h f}(\xi) = e^{-i(h, \cdot)} \widehat{f}(\xi)$.

Exercise 16.5. Let A be a real-valued $n \times n$ matrix such that A^{-1} exists. Define $f_A(x) := f(A^{-1}x)$. Prove that

$$\widehat{f_A}(\xi) = (\widehat{f})_A(\xi)$$

if and only if A is an orthogonal matrix (a rotation), that is, $A^T = A^{-1}$.

Let us now consider f and g from $S(\mathbb{R}^n)$. Then

$$\begin{aligned} (\mathcal{F}f, g)_{L^2} &= \int_{\mathbb{R}^n} \widehat{f}(\xi) \overline{g(\xi)} d\xi = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \overline{g(\xi)} \left(\int_{\mathbb{R}^n} e^{-i(x, \xi)} f(x) dx \right) d\xi \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) \left(\int_{\mathbb{R}^n} e^{i(x, \xi)} \overline{g(\xi)} d\xi \right) dx = (f, \mathcal{F}^*g)_{L^2}, \end{aligned}$$

where $\mathcal{F}^*g(x) := \mathcal{F}g(-x)$.

Remark 16.9. Here \mathcal{F}^* is the adjoint operator (in the sense of L^2), which maps S into S since $\mathcal{F} : S \rightarrow S$. The inverse Fourier transform \mathcal{F}^{-1} is defined as $\mathcal{F}^{-1} := \mathcal{F}^*$.

In order to justify this definition we will prove the following theorem.

Theorem 16.10 (Fourier inversion formula). *Let f be a function from $S(\mathbb{R}^n)$. Then*

$$\mathcal{F}^* \mathcal{F} f = f.$$

To this end we will prove first the following (somewhat technical) lemma.

Lemma 16.11. *Let $f_0(x)$ be a function from $L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} f_0(x) dx = 1$ and let $f(x)$ be a function from $L^\infty(\mathbb{R}^n)$ that is continuous at $\{0\}$. Then*

$$\lim_{\varepsilon \rightarrow 0+} \int_{\mathbb{R}^n} \varepsilon^{-n} f_0\left(\frac{x}{\varepsilon}\right) f(x) dx = f(0).$$

Proof. Since

$$\int_{\mathbb{R}^n} \varepsilon^{-n} f_0\left(\frac{x}{\varepsilon}\right) f(x) dx - f(0) = \int_{\mathbb{R}^n} \varepsilon^{-n} f_0\left(\frac{x}{\varepsilon}\right) (f(x) - f(0)) dx,$$

we may assume without loss of generality that $f(0) = 0$. Since f is continuous at $\{0\}$, there exists $\delta > 0$ for every $\eta > 0$ such that

$$|f(x)| < \frac{\eta}{\|f_0\|_{L^1}}$$

whenever $|x| < \delta$. Note that

$$\left| \int_{\mathbb{R}^n} f_0(x) dx \right| \leq \|f_0\|_{L^1}.$$

We may therefore conclude that

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} \varepsilon^{-n} f_0\left(\frac{x}{\varepsilon}\right) f(x) dx \right| &\leq \frac{\eta}{\|f_0\|_{L^1}} \cdot \varepsilon^{-n} \int_{|x| < \delta} \left| f_0\left(\frac{x}{\varepsilon}\right) \right| dx \\
&\quad + \|f\|_{L^\infty} \varepsilon^{-n} \int_{|x| > \delta} \left| f_0\left(\frac{x}{\varepsilon}\right) \right| dx \\
&\leq \frac{\eta}{\|f_0\|_{L^1}} \cdot \|f_0\|_{L^1} + \|f\|_{L^\infty} \int_{|y| > \frac{\delta}{\varepsilon}} |f_0(y)| dy \\
&= \eta + \|f\|_{L^\infty} I_\varepsilon.
\end{aligned}$$

But $I_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0+$. This proves the lemma.

Proof (Proof of theorem 16.10). Let us consider $v(x) = e^{-\frac{|x|^2}{2}}$. We know from Example 16.7 that $\int_{\mathbb{R}^n} v(x) dx = (2\pi)^{n/2}$ and $Fv = v$. If we apply Lemma 16.11 with $f_0 = (2\pi)^{-n/2} v(x)$ and $f \in S(\mathbb{R}^n)$, then by Exercise 16.4,

$$\begin{aligned}
(2\pi)^{\frac{n}{2}} f(0) &= \lim_{\varepsilon \rightarrow 0+} \int_{\mathbb{R}^n} \varepsilon^{-n} v\left(\frac{x}{\varepsilon}\right) f(x) dx = \lim_{\varepsilon \rightarrow 0+} \langle f, \varepsilon^{-n} \sigma_{\frac{1}{\varepsilon}} v \rangle_{L^2} \\
&= \lim_{\varepsilon \rightarrow 0+} \langle f, \varepsilon^{-n} \sigma_{\frac{1}{\varepsilon}} \mathcal{F} v \rangle_{L^2} = \lim_{\varepsilon \rightarrow 0+} \langle f, \mathcal{F}(\sigma_{\varepsilon} v) \rangle_{L^2} \\
&= \lim_{\varepsilon \rightarrow 0+} \langle \mathcal{F} f, \sigma_{\varepsilon} v \rangle_{L^2} = \langle \mathcal{F} f, 1 \rangle = \int_{\mathbb{R}^n} \mathcal{F} f(\xi) e^{i(0, \xi)} d\xi,
\end{aligned}$$

where we have used Lebesgue's dominated convergence theorem in the last step. This proves that

$$f(0) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathcal{F} f(\xi) e^{i(0, \xi)} d\xi = (\mathcal{F}^* \mathcal{F} f)(0).$$

The proof is now finished by

$$\begin{aligned}
f(x) &= (\tau_{-x} f)(0) = (\mathcal{F}^* \mathcal{F}(\tau_{-x} f))(0) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathcal{F}(\tau_{-x} f)(\xi) e^{i(0, \xi)} d\xi \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(x, \xi)} \mathcal{F} f(\xi) d\xi = \mathcal{F}^* \mathcal{F} f(x),
\end{aligned}$$

where we have used Exercise 16.4. □

Corollary 16.12. *The Fourier transform is an isometry (in the sense of L^2).*

Proof. The fact that the Fourier transform preserves the norm of $f \in S$ follows from

$$\|\mathcal{F} f\|_{L^2}^2 = (\mathcal{F} f, \mathcal{F} f)_{L^2} = (f, \mathcal{F}^* \mathcal{F} f)_{L^2} = (f, f)_{L^2} = \|f\|_{L^2}^2.$$

This is called *Parseval's equality*. □

Note that

$$(\mathcal{F} f, g)_{L^2} = (f, \mathcal{F}^* g)_{L^2}$$

means that

$$\int_{\mathbb{R}^n} \widehat{f}(\xi) \overline{g(\xi)} d\xi = \int_{\mathbb{R}^n} f(x) \overline{\mathcal{F}^* g(x)} dx = \int_{\mathbb{R}^n} f(x) \mathcal{F}(\overline{g})(x) dx.$$

This implies that

$$\int_{\mathbb{R}^n} \widehat{f}(\xi) g(\xi) d\xi = \int_{\mathbb{R}^n} f(x) \widehat{g}(x) dx,$$

or

$$\langle \mathcal{F}f, g \rangle_{L^2(\mathbb{R}^n)} = \langle f, \mathcal{F}g \rangle_{L^2(\mathbb{R}^n)}.$$

Fourier Series, Fourier Transform and Their Applications
to Mathematical Physics

Serov, V.

2017, XI, 534 p. 4 illus., Hardcover

ISBN: 978-3-319-65261-0