

# Preface

The modern theory of analysis and differential equations in general certainly includes the Fourier transform, Fourier series, integral operators, spectral theory of differential operators, harmonic analysis and much more. This book combines all these subjects based on a unified approach that uses modern view on all these themes. The book consists of four parts: Fourier series and the discrete Fourier transform, Fourier transform and distributions, Operator theory and integral equations and Introduction to partial differential equations and it outgrew from the half-semester courses of the same name given by the author at University of Oulu, Finland during 2005–2015.

Each part forms a self-contained text (although they are linked by a common approach) and can be read independently. The book is designed to be a modern introduction to qualitative methods used in harmonic analysis and partial differential equations (PDEs). It can be noted that a survey of the state of the art for all parts of this book can be found in a very recent and fundamental work of B. Simon [35].

This book contains about 250 exercises that are an integral part of the text. Each part contains its own collection of exercises with own numeration. They are not only an integral part of the book, but also indispensable for the understanding of all parts whose collection is the content of this book. It can be expected that a careful reader will complete all these exercises.

This book is intended for graduate level students majoring in pure and applied mathematics but even an advanced researcher can find here very useful information which previously could only be detected in scientific articles or monographs.

Each part of the book begins with its own introduction which contains the facts (mostly) from functional analysis used thereafter. Some of them are proved while the others are not.

The first part, Fourier series and the discrete Fourier transform, is devoted to the classical one-dimensional trigonometric Fourier series with some applications to PDEs and signal processing. This part provides a self-contained treatment of all well known results (but not only) at the beginning graduate level. Compared with some known texts (see [12, 18, 29, 35, 38, 44, 45]) this part uses many function spaces such as Sobolev, Besov, Nikol'skii and Hölder spaces. All these spaces are

introduced by special manner via the Fourier coefficients and they are used in the proofs of main results. Same definition of Sobolev spaces can be found in [35]. The advantage of such approach is that we are able to prove quite easily the precise embeddings for these spaces that are the same as in classical function theory (see [1, 3, 26, 42]). In the frame of this part some very delicate properties of the trigonometric Fourier series (Chapter 10) are considered using quite elementary proofs (see also [46]). The unified approach allows us also to consider naturally the discrete Fourier transform and establish its deep connections with the continuous Fourier transform. As a consequence we prove the famous Whittaker-Shannon-Boas theorem about the reconstruction of band-limited signal via the trigonometric Fourier series (see Chapter 13). Many applications of the trigonometric Fourier series to the one-dimensional heat, wave and Laplace equation are presented in Chapter 14. It is accompanied by a large number of very useful exercises and examples with applications in PDEs (see also [10, 17]).

The second part, Fourier transform and distributions, probably takes a central role in this book and it is concerned with distribution theory of L. Schwartz and its applications to the Schrödinger and magnetic Schrödinger operators (see Chapter 32). The estimates for Laplacian and Hamiltonian that generalize well known Agmon's estimates on the continuous spectrum are presented in this part (see Chapter 23). This part can be considered as one of the most important because of numerous applications in the scattering theory and inverse problems. Here we have considered for the first time some classical direct scattering problems for the Schrödinger operator and for the magnetic Schrödinger operator with singular (locally unbounded) coefficients including the mathematical foundations of the classical approximation of M. Born. Also, the properties of Riesz transform and Riesz potentials (see Chapter 21) are investigated very carefully in this part. Before this material could only be found in scientific journals or monographs but not in textbooks. There is a good connection of this part with Operator theory and integral equations. The main technique applied here is the Fourier transform.

The third part, Operator theory and integral equations, is devoted mostly to the self-adjoint but unbounded operators in Hilbert spaces and their applications to integral equations in such spaces. The advantage of this part is that many important results of J. von Neumann's theory of symmetric operators are collected together. J. von Neumann's spectral theorem allows us, for example, to introduce the heat kernel without solving the heat equation. Moreover, we show applications of the spectral theorem of J. von Neumann (for these operators) to the spectral theory of elliptic differential operators. In particular, the existence of Friedrichs extension for these operators with discrete spectrum is provided. Special attention is devoted to the Schrödinger and the magnetic Schrödinger operators. The famous diamagnetic inequality is proved here. We follow in this consideration B. Simon [35] (slightly different approach can be found in [28]). We recommend (in addition to this part) the reader get acquainted with the books [4, 13, 15, 24, 41]. As a consequence of the spectral theory of elliptic differential operators the integral equations with weak singularities are considered in quite simple manner not only in Hilbert spaces but also in some Banach spaces, e.g. in the space of continuous functions on closed

manifolds. The central point of this consideration is the Riesz theory of compact (not necessarily self-adjoint) operators in Hilbert and Banach spaces. In order to keep this part short, some proofs will not be given, nor will all theorems be proved in complete generality. For many details of these integral equations we recommend [22]. We are able to investigate in quite simple manner one-dimensional Volterra integral equations with weak singularities in  $L^\infty(a, b)$  and singular integral equations in the periodic Hölder spaces  $C^\alpha[-a, a]$ . Concerning approximation methods our considerations use the general theory of bounded or compact operators in Hilbert spaces and we follow mostly the monograph of Kress [22].

The fourth part, Introduction to partial differential equations, serves as an introduction to modern methods for classical theory of partial differential equations. Fourier series and Fourier transform play crucial role here too. An important (and quite independent) segment of this part is the self-contained theory of quasi-linear partial differential equations of order one. The main attention in this part is devoted to elliptic boundary value problems in Sobolev and Hölder spaces. In particular, the unique solvability of direct scattering problem for Helmholtz equation is provided. We investigate very carefully the mapping and discontinuity properties of double and single layer potentials with continuous densities. We also refer to similar properties of double and single layer potentials with densities in Sobolev spaces  $H^{1/2}(S)$  and  $H^{-1/2}(S)$ , respectively, but will not prove any of these results, referring for their proofs to monographs [22] and [25]. Here (and elsewhere in the book)  $S$  denotes the boundary of a bounded domain in  $\mathbb{R}^n$  and if the smoothness of  $S$  is not specified explicitly then it is assumed to be such that Sobolev embedding theorem holds. Compared with well known texts on partial differential equations some direct and inverse scattering problems for Helmholtz, Schrödinger and magnetic Schrödinger operators are considered in this part. As it was mentioned earlier this type of material could not be found in textbooks. The presentation in many places of this part has been strongly influenced by the monographs [6, 7, 11] (see also [8, 16, 24, 36, 40]).

In closing we note that this book is not as comprehensive as the fundamental work of B. Simon [35]. But the book can be considered as a good introduction to modern theory of analysis and differential equations and might be useful not only to students and PhD students but also to all researchers who have applications in mathematical physics and engineering sciences. This book could not have appeared without the strong participation, both in content and typesetting, of my colleague Adj. Prof. Markus Harju. Finally, a special thanks to professor David Colton from University of Delaware (USA) who encouraged the writing of this book and who has supported the author very much over the years.

Oulu, Finland  
June 2017

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Fourier Series, Fourier Transform and Their Applications  
to Mathematical Physics

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2017, XI, 534 p. 4 illus., Hardcover

ISBN: 978-3-319-65261-0