

Random Motions with Space-Varying Velocities

Roberto Garra and Enzo Orsingher

Abstract Random motions on the line and on the plane with space-varying velocities are considered and analyzed in this paper. On the line we investigate symmetric and asymmetric telegraph processes with space-dependent velocities and we are able to present the explicit distribution of the position $\mathcal{T}(t)$, $t > 0$, of the moving particle. Also the case of a nonhomogeneous Poisson process (with rate $\lambda = \lambda(t)$) governing the changes of direction is analyzed in three specific cases. For the special case $\lambda(t) = \alpha/t$, we obtain a random motion related to the Euler–Poisson–Darboux (EPD) equation which generalizes the well-known case treated, e.g., in (Foong, S.K., Van Kolck, U.: Poisson random walk for solving wave equations. *Prog. Theor. Phys.* **87**(2), 285–292, 1992, [6], Garra, R., Orsingher, E.: Random flights related to the Euler–Poisson–Darboux equation. *Markov Process. Relat. Fields* **22**, 87–110, 2016, [8], Rosencrans, S.I.: Diffusion transforms. *J. Differ. Equ.* **13**, 457–467, 1973, [16]). A EPD-type fractional equation is also considered and a parabolic solution (which in dimension $d = 1$ has the structure of a probability density) is obtained. Planar random motions with space-varying velocities and infinite directions are finally analyzed in Sect. 5. We are able to present their explicit distributions, and for polynomial-type velocity structures we obtain the hyper- and hypoelliptic form of their support (of which we provide a picture).

Keywords Planar random motions · Damped wave equations
Euler–Poisson–Darboux fractional equation

MSC 2010 60G60 · 35R11

R. Garra · E. Orsingher (✉)
“Sapienza” Università di Roma, Rome, Italy
e-mail: enzo.orsingher@uniroma1.it

R. Garra
e-mail: roberto.garra@sbai.uniroma1.it

© Springer International Publishing AG 2017
V. Panov (ed.), *Modern Problems of Stochastic Analysis and Statistics*,
Springer Proceedings in Mathematics & Statistics 208,
DOI 10.1007/978-3-319-65313-6_2

1 Introduction

The telegraph process represents a simple prototype of finite velocity random motions on the line, whose probability law is governed by a hyperbolic partial differential equation that is the classical telegraph equation, widely used in mathematical physics both in problems of electromagnetism and heat conduction (see for example [2]). In [13], the authors studied a generalization of the classical telegraph process with space-time-varying propagation speed. Within this framework, the probabilistic model is based on the limit of a persistent random walk on a nonuniform lattice. The consequence of the assumption of a space-time-depending velocity $c(x, t)$ is that the probability law of the corresponding finite velocity random motion is governed by the following telegraph equation with variable coefficients

$$\frac{\partial}{\partial t} \left[\frac{1}{c(x, t)} \frac{\partial p}{\partial t} \right] + 2\lambda \frac{1}{c(x, t)} \frac{\partial p}{\partial t} = \frac{\partial}{\partial x} \left[c(x, t) \frac{\partial p}{\partial x} \right]. \quad (1.1)$$

In some cases, it is possible to find the explicit form of the probability law of this generalization of the telegraph process, by solving equation (1.1) subject to suitable initial conditions. In particular, we focus our attention on the case of space-depending velocity, where (1.1) becomes

$$\frac{\partial^2 p}{\partial t^2} + 2\lambda \frac{\partial p}{\partial t} = c(x) \frac{\partial}{\partial x} \left[c(x) \frac{\partial p}{\partial x} \right]. \quad (1.2)$$

The function $c(x) \in C^1(\mathbb{R})$ represents the velocity of a particle running through point x and thus must be $c(x) \geq 0$. The transformation

$$y = \begin{cases} \int_0^x \frac{dw}{c(w)}, & x > 0 \\ -\int_x^0 \frac{dw}{c(w)}, & x < 0 \end{cases} \quad (1.3)$$

implies that

$$\frac{\partial}{\partial x} = \frac{1}{c(x)} \frac{\partial}{\partial y} \quad (1.4)$$

and converts (1.2) into the classical telegraph equation. Provided that

$$\int_{\min\{0, x\}}^{\max\{0, x\}} \frac{dw}{c(w)} < +\infty, \quad (1.5)$$

we take $y = 0$ for $x = 0$. These conditions on $c = c(x)$ must hold in all parts of the paper, suitably adapted to the specific cases.

In principle, the transformation (1.3) is sufficient for converting (1.2) into the telegraph equation with constant velocity in the frame (y, t) but we need also (1.3)

for the necessary changes of the probability distributions. For the case where $c(x) \notin C^1(\mathbb{R})$ in some isolated points, we can replace it with a suitable smoothed version $c_\epsilon(x)$ and then apply the procedure just described and finally take $\epsilon \rightarrow 0$.

A possible example of velocity function is $c(x) = |x|^\alpha$ which for $0 < \alpha < 1$ denotes a moderately increasing velocity and for $\alpha < 0$ has fading off effect on motions. Of course for $\alpha \geq 1$, the particle undergoes an accelerating process and loses the character of a finite velocity motion.

In this paper, we consider the asymmetric telegraph process with space-varying velocity and also the symmetric telegraph process with a nonhomogeneous Poisson process governing the changes of space-dependent velocities.

A section is devoted to a fractional Euler–Poisson–Darboux-type equation and to the discussion of a special class of nonnegative solutions.

While the telegraph process on the line is essentially a persistent random walk with only two possible directions, the picture of finite velocity random motions on the plane and in the space is more complicated and gives rise to the studies of random flights (see for example [3–5, 15]). An interesting result, in this context, was proved by Kolesnik and Orsingher in [12], where the connection between planar random motions with an infinite number of possible directions and the damped wave equation was discussed. In their model, the motion is described by a particle taking directions θ_j , $j = 1, 2, \dots$, uniformly distributed in $[0; 2\pi)$ at Poisson paced times. The orientations θ_j are i.i.d. r.v.'s independent from the homogeneous Poisson process $N(t)$ of rate λ governing the changes of direction. The particle starts off at time $t = 0$ from the origin and moves with constant velocity c . At the epochs of the Poisson process, the particle takes new directions (uniformly distributed in $[0, 2\pi)$), independent from its previous evolution. Under these assumptions, it is possible to prove that the explicit probability law of the current position $(X(t), Y(t))$ of the randomly moving particle is a solution of the damped wave equation

$$\frac{\partial^2 p}{\partial t^2} + 2\lambda \frac{\partial p}{\partial t} = c^2 \left[\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \right]. \quad (1.6)$$

In the last part of this paper, we consider the effect of a space-varying speed of propagation on the model of planar random motions with infinite possible directions, leading to the equation

$$\frac{\partial^2 p}{\partial t^2} + 2\lambda \frac{\partial p}{\partial t} = c_1(x) \frac{\partial}{\partial x} \left(c_1(x) \frac{\partial p}{\partial x} \right) + c_2(y) \frac{\partial}{\partial y} \left(c_2(y) \frac{\partial p}{\partial y} \right). \quad (1.7)$$

We show the consequence of assuming space-varying velocities on the form of the support \mathcal{D} of the distribution of $(X(t), Y(t))$. By means of the transformation (suitably extended as in the one-dimensional case)

$$u = \int_0^x \frac{dw}{c_1(w)}$$

$$v = \int_0^y \frac{dz}{c_2(z)},$$

The Eq. (1.7) is reduced to the form (1.6) and thus we can obtain the explicit distribution $p(x, y, t)$ of $(X(t), Y(t))$. We then examine the form of the support of $p = p(x, y, t)$ and analyze its dependence on the space-varying velocity.

In the special case where $c_1(x) = |x|^\gamma/c_1$, $c_2(y) = |y|^\beta/c_2$, $\gamma, \beta < 1$, we obtain that the boundary of \mathcal{D} is hyperelliptic for $\gamma = \beta < 0$ and hypoelliptic for $1 > \gamma = \beta > 0$ and elliptic for $\gamma = \beta = 0$.

2 Telegraph Process with Drift and Space-Varying Velocity

In this section, we consider a generalization of the telegraph process with drift considered by Beghin et al. (see Ref. [1]) in the case where the velocity is assumed to be space-varying. In particular, here we consider the random motion of a particle moving on the line and switching from the space-varying (positive) velocity $c(x)$ to $-c(x)$ after an exponentially distributed time with rate λ_1 and from $-c(x)$ to $c(x)$ after an exponential time with a different rate λ_2 . For the description of the random position of the particle $X(t)$ at time $t > 0$, we use the following probability densities:

$$\begin{cases} f(x, t)dx = P\{X(t) \in dx, V(t) = c(x)\} \\ b(x, t)dx = P\{X(t) \in dx, V(t) = -c(x)\}, \end{cases} \quad (2.1)$$

satisfying the system of partial differential equations (see [14] for a detailed probabilistic derivation)

$$\begin{cases} \frac{\partial f}{\partial t} = -c(x) \frac{\partial f}{\partial x} - \lambda_1 f + \lambda_2 b \\ \frac{\partial b}{\partial t} = c(x) \frac{\partial b}{\partial x} + \lambda_1 f - \lambda_2 b. \end{cases} \quad (2.2)$$

Defining

$$p(x, t) = f + b, \quad w = f - b, \quad (2.3)$$

we have the following system of equations

$$\begin{cases} \frac{\partial p}{\partial t} = -c(x) \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial t} = -c(x) \frac{\partial p}{\partial x} + \lambda_2(p - w) - \lambda_1(p + w). \end{cases} \quad (2.4)$$

Therefore, the probability law $p(x, t)$ is governed by the following telegraph-type equation with space-varying velocity and drift

$$\frac{\partial^2 p}{\partial t^2} + (\lambda_1 + \lambda_2) \frac{\partial p}{\partial t} = c(x) \frac{\partial}{\partial x} \left(c(x) \frac{\partial p}{\partial x} \right) + c(x) (\lambda_1 - \lambda_2) \frac{\partial p}{\partial x}. \quad (2.5)$$

In order to eliminate the drift term and to find the explicit form of the probability law, we now introduce the following Lorentz-type transformation of variables:

$$\begin{cases} x' = A \int_0^x \frac{dw}{c(w)} + Bt \\ t' = C \int_0^x \frac{dw}{c(w)} + Dt. \end{cases} \quad (2.6)$$

By means of some calculation we obtain that, by taking the following choice of the coefficients appearing in (2.6)

$$A = D = 1, \quad B = C = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}, \quad (2.7)$$

equation (2.5) becomes the classical telegraph equation

$$\frac{\partial^2 p}{\partial t'^2} + (\lambda_1 + \lambda_2) \frac{\partial p}{\partial t'} = \frac{\partial^2 p}{\partial x'^2} \quad (2.8)$$

and we can therefore find the following explicit probability law, starting from that of the classical telegraph process (with $\lambda = \frac{\lambda_1 + \lambda_2}{2}$)

$$\begin{aligned} p(x, t) = & \frac{e^{-\lambda t}}{2} \left\{ \delta \left(t - \left| \int_0^x \frac{dx'}{c(x')} \right| \right) + \delta \left(t + \left| \int_0^x \frac{dx'}{c(x')} \right| \right) \right\} \\ & + \frac{e^{-\lambda t}}{2c(x)} \left[\lambda I_0 \left(\lambda \sqrt{t^2 - \left| \int_0^x \frac{dx'}{c(x')} \right|^2} \right) + \frac{\partial}{\partial t} I_0 \left(\lambda \sqrt{t^2 - \left| \int_0^x \frac{dx'}{c(x')} \right|^2} \right) \right] \times \mathbf{1}_D(x), \end{aligned}$$

where $\mathbf{1}_D$ is the characteristic function of the set

$$D := \left\{ x \in \mathbb{R} : \left| \left(\int_0^x \frac{dx'}{c(x')} \right) \right| < t \right\}$$

and $I_0(\cdot)$ is the modified Bessel function of order zero.

3 Nonhomogeneous Telegraph Processes with Space-Varying Velocities

Let us recall that a telegraph process $\mathcal{T}(t)$, $t > 0$, where changes of direction are paced by a nonhomogeneous Poisson process, denoted by $\mathcal{N}(t)$, with time-dependent rate $\lambda(t)$, $t > 0$, has distribution $p(x, t)$ satisfying the Cauchy problem (see e.g., [10]):

$$\begin{cases} \frac{\partial^2 p}{\partial t^2} + 2\lambda(t) \frac{\partial p}{\partial t} = c^2 \frac{\partial^2 p}{\partial x^2}, \\ p(x, 0) = \delta(x), \quad \frac{\partial p}{\partial t}(x, t)|_{t=0} = 0. \end{cases} \quad (3.1)$$

In order to obtain explicit distributions in some specific cases, we observe that the transformation

$$p(x, t) = e^{-\int_0^t \lambda(s) ds} v(x, t), \quad (3.2)$$

converts (3.1) into

$$\frac{\partial^2 v}{\partial t^2} - [\lambda'(t) + \lambda^2(t)]v = c^2 \frac{\partial^2 v}{\partial x^2}. \quad (3.3)$$

In (3.2), we exclude the initial time instant (which, however, does not play any role in the subsequent differential transformations) in order to avoid pathologies at $t = 0$. Functions of the form $\lambda(t) = \alpha/t$ and $\lambda(t) = \lambda \coth \lambda t$, for $t > 0$, display an initial high-valued intensity of the Poisson events which hinder the particle to reach the endpoints of the support interval. Then, in order to find the explicit probability law of $T(t)$ from (3.1), a mathematical trick is to solve the following Riccati equation emerging from (3.3) (see [8, 9]):

$$\lambda'(t) + \lambda^2(t) = \text{const}. \quad (3.4)$$

In this way, it is possible to find, in particular, the following probability laws with absolutely continuous components given by

$$P\left\{Y(t) \in dx\right\}/dx = \frac{1}{2c \cosh \lambda t} \frac{\partial}{\partial t} I_0\left(\frac{\lambda}{c} \sqrt{c^2 t^2 - x^2}\right), \quad |x| < ct, \quad (3.5)$$

and

$$P\left\{X(t) \in dx\right\}/dx = \frac{\lambda I_0\left(\frac{\lambda}{c} \sqrt{c^2 t^2 - x^2}\right)}{2c \sinh \lambda t}, \quad |x| < ct, \quad (3.6)$$

corresponding to the cases

$$\begin{cases} \lambda(t) = \lambda \tanh \lambda t, \\ \lambda(t) = \lambda \coth \lambda t, \end{cases}$$

respectively. We observe that the process $Y(t)$ has a discrete component of the distribution concentrated at $x = \pm ct$ (see [9]), while $X(t)$ has only an absolutely continuous distribution (see [8]).

Starting from (3.5) and (3.6), we can clearly build other families of explicit probability laws of the form

$$\left\{ \begin{array}{l} P\left\{Y(t) \in dx\right\}/dx = \frac{1}{2c(x) \cosh \lambda t} \frac{\partial}{\partial t} I_0 \left(\lambda \sqrt{t^2 - \left| \int_0^x \frac{dx'}{c(x')} \right|^2} \right), \\ P\left\{X(t) \in dx\right\}/dx = \frac{\lambda}{2c(x) \sinh \lambda t} I_0 \left(\lambda \sqrt{t^2 - \left| \int_0^x \frac{dx'}{c(x')} \right|^2} \right), \\ \text{for } \left\{x : \left| \int_0^x \frac{dx'}{c(x')} \right| < t\right\}, \end{array} \right. \quad (3.7)$$

which depend on the particular choice of $c(x) > 0$ (s.t. condition (1.5) is fulfilled for all x). These probability laws are clearly related to the following partial differential equations:

$$\left\{ \begin{array}{l} \frac{\partial^2 p}{\partial t^2} + 2\lambda \tanh \lambda t \frac{\partial p}{\partial t} = c(x) \frac{\partial}{\partial x} c(x) \frac{\partial p}{\partial x}, \\ \frac{\partial^2 p}{\partial t^2} + 2\lambda \coth \lambda t \frac{\partial p}{\partial t} = c(x) \frac{\partial}{\partial x} c(x) \frac{\partial p}{\partial x}, \end{array} \right. \quad (3.8)$$

respectively.

Another interesting case is $\lambda(t) = \frac{\alpha}{t}$, which converts Eq. (3.1) into the classical Euler–Poisson–Darboux equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{2\alpha}{t} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R}, t > 0. \quad (3.9)$$

The first probabilistic interpretation of the fundamental solution of the EPD equation was given by Rosencrans in [16] and some of its generalizations have been considered in [8]. In the spirit of the previous observations, we have that the solution of the Cauchy problem

$$\left\{ \begin{array}{l} \frac{\partial^2 v}{\partial t^2} + \frac{2\alpha}{t} \frac{\partial v}{\partial t} = c(x) \frac{\partial}{\partial x} c(x) \frac{\partial v}{\partial x}, \\ v(x, 0) = \delta(x), \\ \left. \frac{\partial v}{\partial t} \right|_{t=0} = 0 \end{array} \right. \quad (3.10)$$

can be written as

$$v(x, t) = \frac{1}{B(\alpha, \frac{1}{2}) c(x) t} \left(1 - \frac{\left| \int_0^x \frac{dx'}{c(x')} \right|^2}{t^2} \right)^{\alpha-1}, \quad \text{for } \left\{x : \left| \int_0^x \frac{dw}{c(w)} \right| < t\right\}. \quad (3.11)$$

We finally observe that it is possible to consider other cases of nonhomogeneous telegraph processes with space-dependent velocities according to the following simple steps:

- Consider the equation

$$\frac{\partial^2 p}{\partial t^2} + 2\lambda(t) \frac{\partial p}{\partial t} = c(x) \frac{\partial}{\partial x} c(x) \frac{\partial p}{\partial x}, \quad (3.12)$$

governing a telegraph process on the line, where the changes of direction are given by a nonhomogeneous Poisson process with a deterministic time-dependent rate $\lambda(t)$ and with space-dependent velocity $c(x)$.

- Define the new variables $x' = \int_0^x \frac{du}{c(u)}$ and $t' = \int_0^t \gamma(s) ds$, where $\gamma(t)$ is a $C^1[0, +\infty)$ function that will be defined in the next step;
- In the new variables, we have that $p(x', t')$ satisfies the equation

$$\gamma^2(t') \frac{\partial^2 p}{\partial t'^2} + (\gamma' + 2\lambda\gamma) \frac{\partial p}{\partial t'} = \frac{\partial^2 p}{\partial x'^2}; \quad (3.13)$$

- Take $\gamma(t)$ such that $\frac{\gamma'}{\gamma} = -2\lambda(t)$. Then the problem is finally reduced to the following D'Alembert equation with a time-dependent coefficient

$$\frac{\partial^2 p}{\partial t'^2} = \frac{1}{\gamma^2(t')} \frac{\partial^2 p}{\partial x'^2}. \quad (3.14)$$

- By taking the further change of variable $(x', t') \rightarrow (\gamma(t')x', t')$ and calling $x'' = \gamma(t')x'$ we finally reduce Eq.(3.14) to the classical D'Alembert equation in the variables (x'', t')

$$\frac{\partial^2 u}{\partial t'^2} = \frac{\partial^2 u}{\partial x''^2}. \quad (3.15)$$

Thus an observer in the framework (x'', t') sees the original random motion transformed into a deterministic one governed by the classical D'Alembert equation.

4 Time-Fractional Euler–Poisson–Darboux Equation with Variable Velocity

We here provide some new results about the Euler–Poisson–Darboux equation involving time-fractional derivatives in the sense of Riemann–Liouville (see [11]) and with space-varying velocity. It is well known that the EPD equation governs a telegraph process with time-dependent rate $\lambda(t) = \alpha/t$. As far as we know this is the first investigation about the time-fractional EPD equation.

Theorem 4.1 *The d -dimensional time-fractional EPD-type equation*

$$\left(\frac{\partial^{2\nu}}{\partial t^{2\nu}} + \frac{C_1}{t^\nu} \frac{\partial^\nu}{\partial t^\nu} \right) u = \Delta u, \quad (4.1)$$

with $\nu \in (0, 1) \setminus \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ and

$$C_1 = -\frac{\Gamma(1-4\nu)}{\Gamma(1-5\nu)}, \quad (4.2)$$

admits the following nonnegative solution:

for $C_2 > 0$

$$u(\mathbf{x}_d, t) = \begin{cases} \frac{1}{t^\nu} \left[1 - C_2 \frac{\|\mathbf{x}_d\|^2}{t^{2\nu}} \right], & \|\mathbf{x}_d\| < \frac{t^\nu}{C_2^{1/2}}, \\ 0 & \text{elsewhere,} \end{cases} \quad (4.3)$$

while for $C_2 < 0$

$$u(\mathbf{x}_d, t) = \frac{1}{t^\nu} \left[1 - C_2 \frac{\|\mathbf{x}_d\|^2}{t^{2\nu}} \right], \quad \forall \mathbf{x}_d \in \mathbb{R}^d \quad (4.4)$$

where $\mathbf{x}_d = (x_1, x_2, \dots, x_d)$, $d \in \mathbb{N}$ and

$$C_2 = -\frac{1}{2d} \left[\frac{\Gamma(1-\nu)}{\Gamma(1-3\nu)} - \frac{\Gamma(1-4\nu)}{\Gamma(1-5\nu)} \frac{\Gamma(1-\nu)}{\Gamma(1-2\nu)} \right]$$

Proof By considering that (4.1) has the structure of an EPD equation, we determine a parabolic-type solution. By using the well-known fact that (see [11], p. 71)

$$\frac{\partial^\alpha t^\beta}{\partial t^\alpha} = \frac{\Gamma(\beta+1)t^{\beta-\alpha}}{\Gamma(\beta+1-\alpha)}, \quad \text{for } \alpha > 0 \text{ and } \beta > -1, \quad (4.5)$$

we can calculate the exact form of the coefficient C_2 such that (4.3) is a solution of (4.1). We assume that $\nu \neq \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ in order to avoid the singularities in the coefficients appearing in C_1 and C_2 . \square

Remark 4.2 It is possible to construct a probability law with compact support, starting from the general Theorem 4.1 in the one-dimensional case, assuming that ν is such that C_2 is positive. In this case, we have that the probability law

$$p(x, t) = \frac{N}{t^\nu} \left[1 - C_2 \frac{|x|^2}{t^{2\nu}} \right], \quad |x| < \frac{t^\nu}{C_2^{1/2}}, \quad (4.6)$$

with

$$C_2 = -\frac{1}{2} \left[\frac{\Gamma(1-\nu)}{\Gamma(1-3\nu)} - \frac{\Gamma(1-4\nu)}{\Gamma(1-5\nu)} \frac{\Gamma(1-\nu)}{\Gamma(1-2\nu)} \right]$$

and $N = \frac{3}{4}\sqrt{C_2}$ the normalizing constant satisfies the one-dimensional time-fractional EPD-type Eq. (4.1).

We remark that it is not a trivial matter to find the explicit values of $\nu \in (0, 1)$ such that the coefficient $C_2 > 0$.

Notice that it is extremely hard to ascertain that functions of the form

$$u(x, t) = \frac{N}{t^\beta} \left(1 - \frac{\|\mathbf{x}_d\|^2}{t^\alpha} \right)^\gamma \quad (4.7)$$

are solutions of (4.1) for $\gamma \neq 1$ and suitable β and α .

We can also observe, with the following Proposition, that we are able to find a solution for a time-fractional EPD-type equation of higher order.

Proposition 4.3 *The d -dimensional time-fractional EPD-type equation*

$$\left(\frac{\partial^{2\nu}}{\partial t^{2\nu}} + \frac{C_1}{t^\nu} \frac{\partial^\nu}{\partial t^\nu} \right) u = \sum_{j=1}^d \frac{\partial^{2n} u}{\partial x_j^{2n}}, \quad n \in \mathbb{N}, \quad (4.8)$$

with $\nu \in (0, 1) \setminus \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ and

$$C_1 = -\frac{\Gamma(1-4\nu)}{\Gamma(1-5\nu)}, \quad (4.9)$$

admits the following nonnegative solution:

for $C_2 > 0$

$$u(x_1, \dots, x_d, t) = \begin{cases} \frac{1}{t^\nu} \left[1 - C_2 \frac{\sum_{j=1}^d x_j^{2n}}{t^{2\nu}} \right], & \sum_{j=1}^d x_j^{2n} < \frac{t^{2\nu}}{C_2}, \\ 0 & \text{elsewhere} \end{cases} \quad (4.10)$$

and for $C_2 < 0$

$$u(x_1, \dots, x_d, t) = \frac{1}{t^\nu} \left[1 - C_2 \frac{\sum_{j=1}^d x_j^{2n}}{t^{2\nu}} \right], \quad \forall \mathbf{x}_d \in \mathbb{R}^d \quad (4.11)$$

where $d \in \mathbb{N}$ and

$$C_2 = -\frac{1}{(2n)!d} \left[\frac{\Gamma(1-\nu)}{\Gamma(1-3\nu)} - \frac{\Gamma(1-4\nu)}{\Gamma(1-5\nu)} \frac{\Gamma(1-\nu)}{\Gamma(1-2\nu)} \right].$$

Starting from (4.6), we have the following corollary.

Corollary 4.4 *Taking $\nu \in (0, 1)$ such that $C_2 > 0$, the probability law*

$$p(x, t) = \frac{N}{c(x)t^\nu} \left[1 - C_2 \frac{\left| \int_0^x \frac{dx'}{c(x')} \right|^2}{t^{2\nu}} \right], \quad \text{for } \left\{ x : \left| \int_0^x \frac{dx'}{c(x')} \right| < \frac{t^\nu}{C_2^{1/2}} \right\}, \quad (4.12)$$

with $N = \frac{3}{4}\sqrt{C_2}$ and

$$C_2 = -\frac{1}{2} \left[\frac{\Gamma(1-\nu)}{\Gamma(1-3\nu)} - \frac{\Gamma(1-4\nu)}{\Gamma(1-5\nu)} \frac{\Gamma(1-\nu)}{\Gamma(1-2\nu)} \right],$$

satisfies the time-fractional EPD-type equation with nonconstant coefficients

$$\left(\frac{\partial^{2\nu}}{\partial t^{2\nu}} + \frac{C_1}{t^\nu} \frac{\partial^\nu}{\partial t^\nu} \right) p = c(x) \frac{\partial}{\partial x} c(x) \frac{\partial p}{\partial x}, \quad (4.13)$$

with

$$C_1 = -\frac{\Gamma(1-4\nu)}{\Gamma(1-5\nu)}.$$

5 Planar Random Motions with Space-Varying Velocity

We start our analysis from the damped wave equation with space-depending velocities as follows:

$$\frac{\partial^2 p}{\partial t^2} + 2\lambda \frac{\partial p}{\partial t} = c_1(x) \frac{\partial}{\partial x} c_1(x) \frac{\partial p}{\partial x} + c_2(y) \frac{\partial}{\partial y} c_2(y) \frac{\partial p}{\partial y}. \quad (5.1)$$

By taking the change of variables

$$\begin{cases} z = \int_0^x \frac{dx'}{c_1(x')} \\ w = \int_0^y \frac{dy'}{c_2(y')} \end{cases}, \quad (x, y) \in \mathbb{R}^2, \quad (5.2)$$

we obtain

$$\frac{\partial^2 p}{\partial t^2} + 2\lambda \frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial z^2} + \frac{\partial^2 p}{\partial w^2}. \quad (5.3)$$

The transformation (5.2) must be extended on the whole plane $(x, y) \in \mathbb{R}^2$ by suitably adapting the considerations discussed in the introduction. The absolutely continuous

component of the distribution of the position $(X(t), Y(t))$ of the moving particle performing the planar motion described in the introduction satisfies (5.3) (see [12]). Therefore, returning to the original variables (x, y) , we are able to understand the role played by the variable velocity on the model considered in [12]. The absolutely continuous component of the probability law is given by

$$p(x, y, t) = \frac{\lambda}{2\pi c_1(x)c_2(y)} \frac{\exp \left\{ -\lambda t + \lambda \sqrt{t^2 - \left| \int_0^x \frac{dx'}{c_1(x')} \right|^2 - \left| \int_0^y \frac{dy'}{c_2(y')} \right|^2} \right\}}{\sqrt{t^2 - \left| \int_0^x \frac{dx'}{c_1(x')} \right|^2 - \left| \int_0^y \frac{dy'}{c_2(y')} \right|^2}}, \quad (5.4)$$

provided that both $c_1(x)$ and $c_2(y)$ are positive and such that $\int_{\min\{0,x\}}^{\max\{0,x\}} \frac{dx'}{c_1(x')} < \infty$ for all $x \in \mathbb{R}$ and $\int_{\min\{0,y\}}^{\max\{0,y\}} \frac{dy'}{c_2(y')} < \infty$ for all $y \in \mathbb{R}$, respectively.

Therefore, the support of $p(x, y, t)$ is given by the set

$$\mathcal{D} := \left\{ (x, y) : \left| \int_0^x \frac{dx'}{c_1(x')} \right|^2 + \left| \int_0^y \frac{dy'}{c_2(y')} \right|^2 < t^2 \right\}. \quad (5.5)$$

The set \mathcal{D} is therefore a *deformation* of the circle representing the support of $(X(t), Y(t))$ in the case of constant velocity. From formula (5.4), we can extract the conditional distribution of this class of generalized planar random motions. Since

$$P\{X(t) \in dx, Y(t) \in dy\} = \sum_{n=0}^{\infty} P\{X(t) \in dx, Y(t) \in dy | N(t) = n\} P\{N(t) = n\} dx dy, \quad (5.6)$$

where $P\{N(t) = n\}$ is the homogeneous Poisson distribution of rate λ , we have that the conditional distribution is obviously given by

$$\begin{aligned} & P\{X(t) \in dx, Y(t) \in dy | N(t) = n\} \\ &= \frac{n}{2\pi t^n} \left[t^2 - \left| \int_0^x \frac{dx'}{c_1(x')} \right|^2 - \left| \int_0^y \frac{dy'}{c_2(y')} \right|^2 \right]^{\frac{n}{2}-1} \frac{dx dy}{c_1(x)c_2(y)}. \end{aligned} \quad (5.7)$$

The planar motion with space-varying velocity after n changes of direction can be described as

$$\begin{cases} X(t) = \sum_{j=1}^{n+1} \left(\int_{t_{j-1}}^{t_j} c_1(X(s)) ds \right) \cos \theta_j \\ Y(t) = \sum_{j=1}^{n+1} \left(\int_{t_{j-1}}^{t_j} c_2(Y(s)) ds \right) \sin \theta_j, \end{cases} \quad (5.8)$$

where $0 = t_0 < t_1 < \dots < t_j < \dots < t_n < t_{n+1} = t$ are the epochs of the Poisson process and θ_j are the directions of motion assumed at times t_j . The reader can ascertain that (5.8) coincides with equation (12) of [12] in the case $c_1 = c_2 = \text{const.}$

The intuitive idea underlying (5.8) is that at Poisson times t_j the particle chooses its direction randomly and the displacement performed is determined by the local velocity field. During $(s, s + ds)$, for example, the x -coordinate makes a step of length $c_1(X(s))ds$ depending on the position occupied at time s .

We also observe that we can arrive at (5.7) by expanding (5.4) and then using (5.6).

In order to understand the role of considering different velocities on both axes, we consider a general domain that includes some interesting cases. It corresponds to taking the space-dependent velocities of the form $c_1(x) = \frac{|x|^\gamma}{c_1}$ and $c_2(y) = \frac{|y|^\beta}{c_2}$, with $\gamma, \beta < 1$ and $c_1, c_2 > 0$. The functions $c_1(x)$ and $c_2(x)$ considered here can be regarded as the limit of approximating smooth functions excluding $x = y = 0$. With this choice, we obtain a family of probability laws concentrated inside domains of the form

$$\mathcal{D}_{\gamma,\beta} := \left\{ (x, y) : \left(\frac{c_1 |x|^{1-\gamma}}{1-\gamma} \right)^2 + \left(\frac{c_2 |y|^{1-\beta}}{1-\beta} \right)^2 < t^2 \right\}. \quad (5.9)$$

This means that the boundary of the support of this family of probability laws is given by a superellipse, also known as a Lamé curves including a wide class of geometrical figures like hypoellipses (for $\gamma = \beta < 0$) and hyperellipses (for $\gamma = \beta > 0$). We consider, in particular, two interesting cases.

The first one is the case in which $c_1(x) = c_1$ and $c_2(y) = c_2$ and $c_1 \neq c_2 \neq 0$. In this case, the support of the probability law is clearly given by the ellipse:

$$\mathcal{D}_{0,0} : \left\{ (x, y) : c_1^2 |x|^2 + c_2^2 |y|^2 < t^2 \right\}. \quad (5.10)$$

The second interesting case is given by the choice $c_1(x) = c_1 |x|^{2/3}$ and $c_2(y) = c_2 |y|^{2/3}$, leading to the compact support

$$\mathcal{D}_{2/3,2/3} : \left\{ (x, y) : 9c_1^2 |x|^{2/3} + 9c_2^2 |y|^{2/3} < t^2 \right\}. \quad (5.11)$$

For $c_1 = c_2 = 1/3$, we obtain as boundary of $\mathcal{D}_{2/3,2/3}$ the astroid (see Fig. 1). For $c_1 \neq c_2 \neq 1$, we have instead a squeezed astroid, possibly on both axes.

Another interesting class of d -dimensional random motions at finite velocities is related to the EPD equation

$$\frac{\partial^2 v}{\partial t^2} + \frac{2\alpha + d - 1}{t} \frac{\partial v}{\partial t} = \sum_{j=1}^d c_j(x_j) \frac{\partial}{\partial x_j} c_j(x_j) \frac{\partial v}{\partial x_j}. \quad (5.12)$$

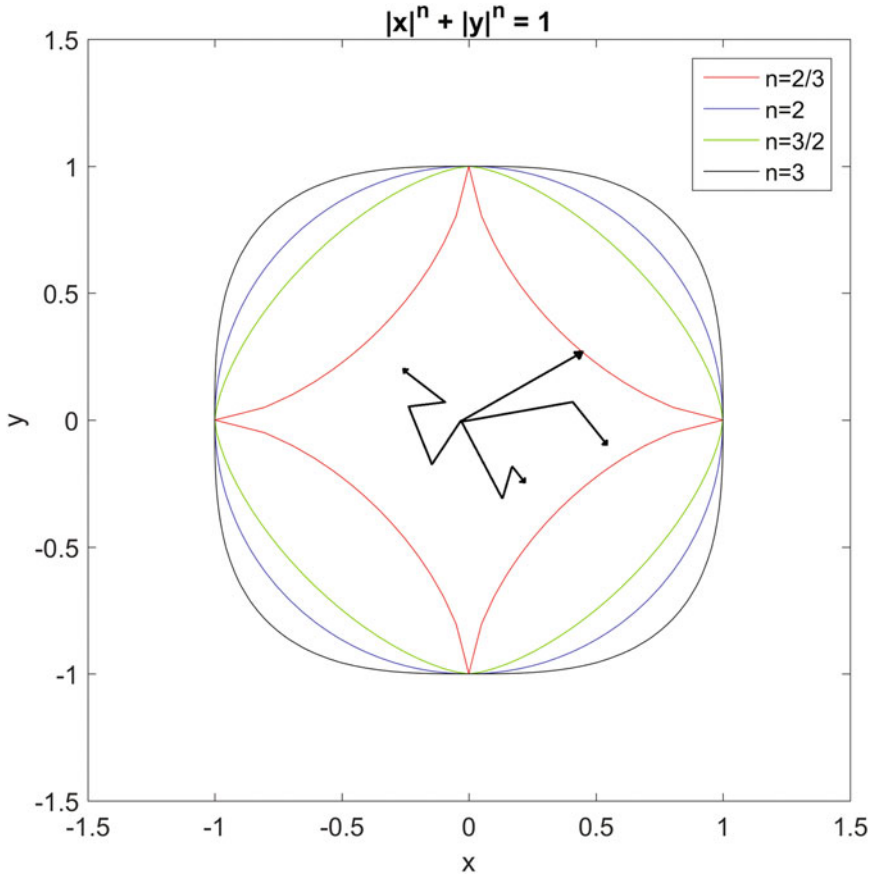


Fig. 1 We represent the boundary of the compact support $\mathcal{D}_{\gamma, \beta}$ of the family of probability laws (5.4), in the hypoelliptic ($n = 2/3$ leading to the astroid and $n = 3/2$), elliptic ($n = 2$), and hyperelliptic case ($n = 3$). Here, we assume that $c_1 = c_2 = 1$ and $n = 2(1 - \gamma) = 2(1 - \beta)$. We show, as an example, four sample paths with zero, two, three, and four changes of direction, in the case when the random motion takes place in the astroid

In this case, the probability law $p(x_1, x_2, \dots, x_d, t)$ of the particle moving in the d -dimensional space has the form

$$p(x_1, x_2, \dots, x_d, t) = \frac{1}{\prod_{j=1}^d c_j(x_j)} \frac{\Gamma(\alpha + \frac{d}{2})}{\pi^{d/2} \Gamma(\alpha) t^{d+2\alpha-2}} \left(t^2 - \sum_{j=1}^d \left| \int_0^{x_j} \frac{du_j}{c_j(u_j)} \right|^2 \right)^{\alpha-1}, \quad (5.13)$$

$$\text{for } \sum_{j=1}^d \left| \int_0^{x_j} \frac{du_j}{c_j(u_j)} \right|^2 < t^2$$

and represents a solution of (5.12). The projection on the axes x_1, \dots, x_d of the probability law (5.13) coincides with the one-dimensional motion dealt with in Sect. 3.

A more general random motion in \mathbb{R}^d with space-varying velocities leads to equation

$$\frac{\partial^2 p}{\partial t^2} + 2\lambda(t) \frac{\partial p}{\partial t} = \sum_{j=1}^d c_j(x_1, \dots, x_j, \dots, x_d) \frac{\partial}{\partial x_j} c_j(x_1, \dots, x_j, \dots, x_d) \frac{\partial}{\partial x_j} p. \quad (5.14)$$

This can be object of future research.

Acknowledgements We are very grateful to both referees for their suggestions and comments.

References

1. Beghin, L., Nieddu, L., Orsingher, E.: Probabilistic analysis of the telegrapher's process with drift by means of relativistic transformations. *Int. J. Stoch. Anal.* **14**(1), 11–25 (2001)
2. Cattaneo, C.R.: Sur une forme de l' équation de la chaleur éliminant le paradoxe d'une propagation instantanée. *Comptes Rendus* **247**(4), 431–433 (1958)
3. De Gregorio, A.: Transport processes with random jump rate. *Stat. Probab. Lett.* **118**, 127–134 (2016)
4. De Gregorio, A., Orsingher, E.: Flying randomly in \mathbb{R}^d with dirichlet displacements. *Stoch. Process. Appl.* **122**(2), 676–713 (2012)
5. D'Ovidio, M., Orsingher, E., Toaldo, B.: Time-changed processes governed by space-time fractional telegraph equations. *Stoch. Anal. Appl.* **32**(6), 1009–1045 (2014)
6. Foong, S.K., Van Kolck, U.: Poisson random walk for solving wave equations. *Prog. Theor. Phys.* **87**(2), 285–292 (1992)
7. Garra, R., Orsingher, E., Polito, F.: Fractional Klein-Gordon equations and related stochastic processes. *J. Stat. Phys.* **155**(4), 777–809 (2014)
8. Garra, R., Orsingher, E.: Random flights related to the Euler-Poisson-Darboux equation. *Markov Process. Relat. Fields* **22**, 87–110 (2016)
9. Iacus, S.M.: Statistical analysis of the inhomogeneous telegrapher's process. *Stat. Probab. Lett.* **55**, 83–88 (2001)
10. Kaplan, S.: Differential equations in which the poisson process plays a role. *Bull. Am. Math. Soc.* **70**(2), 264–268 (1964)
11. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: *Theory and Applications of Fractional Differential Equations*, vol. 204. Elsevier Science Limited, Amsterdam (2006)
12. Kolesnik, A.D., Orsingher, E.: A planar random motion with an infinite number of directions controlled by the damped wave equation. *J. Appl. Probab.* **42**(4), 1168–1182 (2005)
13. Masoliver, J., Weiss, G.H.: Telegraphers equations with variable propagation speeds. *Phys. Rev. E* **49**(5), 3852–3854 (1994)
14. Orsingher, E.: Probability law, flow function, maximum distribution of wave-governed random motions and their connections with Kirchoff's laws. *Stoch. Process. Appl.* **34**(1), 49–66 (1990)
15. Pogorui, A.A., Rodríguez-Dagnino, R.M.: Random motion with uniformly distributed directions and random velocity. *J. Stat. Phys.* **147**(6), 1216–1225 (2012)
16. Rosencrans, S.I.: Diffusion transforms. *J. Differ. Equ.* **13**, 457–467 (1973)

Modern Problems of Stochastic Analysis and Statistics

Selected Contributions In Honor of Valentin Konakov

Panov, V. (Ed.)

2017, XII, 511 p. 15 illus., 8 illus. in color., Hardcover

ISBN: 978-3-319-65312-9