2

An Activated Elastic Bar: Effective Properties

2.1 Longitudinal Vibrations of Activated Elastic Bar

Consider an immovable elastic bar distributed along the z-axis; the longitudinal wave propagation along the bar is governed by the second order hyperbolic equation

$$(\rho u_t)_t - (k u_z)_z = 0. \tag{2.1}$$

Here, $u = u(z, t)$ denotes a small horizontal displacement depending on z and t, and $\rho = \rho(z, t)$, $k = k(z, t)$ denote, respectively, the (positive) linear density and stiffness (Young modulus) of the bar.

We shall examine the wave propagation along the bar with variable material parameters ρ and k. More specifically, we assume that this dependency is characterized by the following features:

(i) both ρ and k are space *and* time dependent;
(ii) at each point (z, t) the pair (ρ, k) may take either the values (ρ_1, k_1) or the values (ρ_2, k_2) ("materials 1 and 2");
(iii) these values are taken within alternating layers in the (z, t)-plane having the slope $dz/dt = V$ so chosen as to ensure a regular transmission of dynamic disturbances $u(z, t)$ across the interface from one layer to another. In other words, both kinematic and dynamic compatibility conditions must be observed across the interface.

The spatio-temporal variability of both ρ and k may be achieved if we attach (release) some portions of material to (from) the bar wherever and whenever necessary. To illustrate how this operation affects the quantity of motion and the elastic force, consider an infinitesimal material segment dz carrying at time t the mass ρdz, with the absolute velocity u_t of disturbance, and quantity of motion $\rho u_t dz$. Assume that over the time interval dt the mass within the segment receives infinitesimal addition $\rho_t dt dz$; before being

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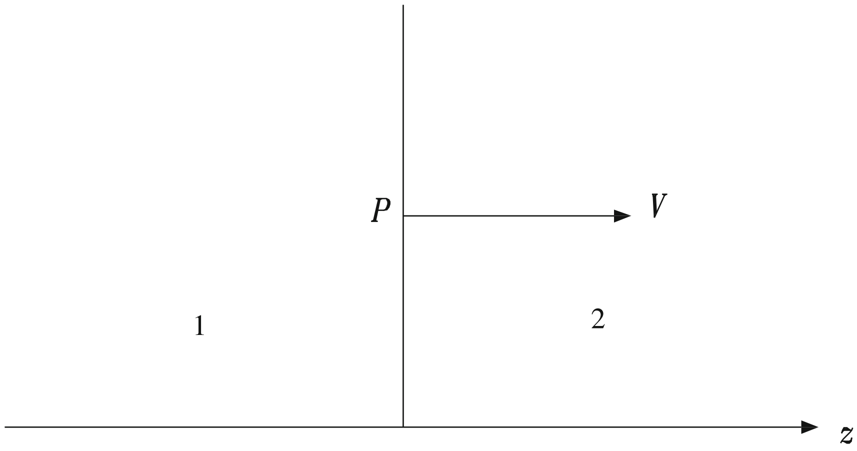

Fig. 2.1. A moving interface

added, this mass is moving with constant absolute velocity V_0 and carrying the quantity of motion $\rho_t V_0 dtdz$. The resultant quantity of motion of two masses before addition is therefore $(\rho u_t + \rho_t V_0 dt)dz$.

After addition, the combined mass $(\rho+\rho_t dt)dz$ carries the disturbance with absolute velocity $u_t + u_{tt}dt$ and quantity of motion $(\rho + \rho_t dt)(u_t + u_{tt}dt)dz$. The difference of these quantities after and before addition equals (with second order terms neglected)

$$(\rho u_{tt} + \rho_t u_t - \rho_t V_0)dzdt.$$

By momentum conservation law, this should be equal to the increment of the impulse of a resultant elastic force

$$(ku_z)_z dzdt,$$

i.e.,

$$\rho u_{tt} = (ku_z)_z - \rho_t(u_t - V_0). \tag{2.2}$$

Remark 2.1.1. Equation (2.2) resembles the fundamental equation [5]

$$m\frac{dv}{dt} = F - \frac{dm}{dt}(v - V_0) \tag{2.3}$$

governing the motion of a particle with variable mass moving at velocity v. The difference between (2.2) and (2.3) is that, in the first of these equations, we deal with the mass densities multiplied by their accelerations, as well as with the densities of force applied to those masses, i.e., with the quantities calculated per unit volume. Unlike that, in (2.3), all symbols are related to the individual material particles. By integrating equation (2.2) over a finite volume, one may arrive at the equation of motion of the center of mass distributed over this volume.

The term—$\rho_t(u_t - V_0)$ represents the density of a reactive force similar to the second term at the rhs of (2.3). For $V_0 = 0$, equation (2.2) reduces to (2.1). If we replace (2.2) by an equivalent system

$$v_t = ku_z, \;\; v_z = \rho(u_t - V_0), \tag{2.4}$$

then it will be easy to introduce the compatibility conditions on the interface P separating two different materials 1 and 2. If V denotes the absolute velocity of the interface (Figure 2.1), then the balance of momenta on the interface separating materials 1 and 2 on the (z, t)-plane is expressed by

$$v_{1t} + Vv_{1z} = v_{2t} + Vv_{2z}; \tag{2.5}$$

where V is the slope of the interface relative to the t-axis, and the subscripts 1, 2 are related to the sides of the interface occupied by materials 1 and 2 (Figure 2.1). One may say that (2.5) is equivalent to the continuity of v.

We must add to (2.4) another compatibility condition

$$u_{1z} + Vu_{1z} = u_{2t} + Vu_{2z}, \tag{2.6}$$

expressing the continuity of a displacement across the interface. Any solution of the system (2.4) satisfying equations (2.5), (2.6) will be called *regular*. The analogy between equations (2.2) and (2.3) introduces a flying rocket as an example of a dynamic material. A system of a rocket plus a reactive gaseous jet is thermodynamically closed; its center of mass remains immovable in space. But the *rocket itself*, treated separately, represents a dynamic material: its inertia (mass) and stiffness depend on a position along the body of a rocket as well as on time; this makes the flying rocket thermodynamically open. Returning to equations (2.4), we characterize them as a hyperbolic system, and caution should be taken to guarantee existence of a regular continuous solution. The problem that arises may be illustrated if we consider, as an example, the case of an *immovable* interface: $V = 0$ (Figure 2.2).

In order to observe the compatibility conditions, we have to make sure that there are precisely two characteristics of the system (2.4) that *depart* from the interface. On an immovable interface, $V = 0$, we have two characteristics, with slopes $\pm a_1$, on the left side occupied by material 1, and two characteristics, with slopes $\pm a_2$, on the right side occupied by material 2; here, $a_i, \;\; i = 1, 2$, denotes the phase velocity $\sqrt{k_i/\rho_i}$ of waves in material i; we assume below, without sacrificing generality, that $a_2 > a_1 > 0$. Clearly, two out of the four characteristics, specifically, those with slopes a_2 and $-a_1$, *depart* from the interface. We conclude that an immovable interface is *admissible*, i.e., it allows for a desired regular solution.

Consider now a *moving* interface ($V \neq 0$), with immovable materials 1 and 2 on opposite sides of it. Instead of Figure 2.2, we now refer to Figure 2.3 as illustration; in this one, the interface is making the angle $\tan^{-1} V$ with the t-axis. Nothing dramatic happens (i.e., the solution remains regular) while $|V|$ is less than the least of the phase velocities: $\mid V \mid < a_1$ (Figure 2.3); if, however, $\mid V \mid$ falls into the interval (a_1, a_2), then the balance of characteristics becomes violated: we have either three (Figure 2.4) or one (Figure 2.5) departing characteristic. In the first case, we have non-uniqueness, in the second case—non-existence of a continuous solution. Both cases will be called *irregular*. The balance will, however, be restored as $\mid V \mid$ becomes greater than both of the phase velocities: $\mid V \mid > a_2$; this case is illustrated in Figure 2.6. We conclude that the condition

$$(V^2 - a_1^2)(V^2 - a_2^2) > 0 \tag{2.7}$$

is necessary for the existence of a regular solution. This condition imposes substantial restriction on spatio-temporal material geometry; particularly, it becomes violated for assemblages that are quite habitual in statics. An example is given by a matrix structure in space-time illustrated in Figure 2.7. In this figure, the matrix and the oval-shaped (shaded) inclusions are occupied by two different materials. Along the oval interfaces, there will always be parts where Ineq. (2.7) is violated. On the other hand, a rectangular microstructure shown in Figure 5.1 is regular because (2.7) is satisfied on both horizontal and vertical interfaces.

In the following sections we will calculate the effective parameters of a regular elastic bar.

Remark 2.1.2. The difference between regular and irregular cases has been extensively discussed in literature, starting with a celebrated paper [4] by I.M. Gel'fand. The mismatch in the number of arriving and departing characteristics has served as an argument for non-existence (non-uniqueness) of weak solutions for linear hyperbolic systems with discontinuous coefficients. On the other hand, the boundedness of energy is an argument often used for the proof of existence. In the context of this book, there is no point to care much about this argument since the energy gain is what is often desirable in applications of dynamic materials. The reader will see illustrations of this in Chapter 5.

The transport of disturbances across the irregular property interface cannot be arranged unless such interface itself demonstrates irregularities like a "toothlike shape" accompanied by formation of shocks, i.e., strong discontinuities in u, v. The relevant analysis has been conducted in [7] for irregular situations illustrated in Figures 2.4 and 2.5. This study has specifically confirmed that infinite energy should then be wasted to support the non-stop wave propagation. For this reason, the standard homogenization technique based on the energy boundedness and, consequently, on G-convergence did not receive extension to laminates with irregular property interfaces. A formal attempt to carry out such extension undertaken in [8] does not seem to be workable for structures other than regular laminates. There have been other attempts to carry out dynamic homogenization for the wave equation with the space-time variable coefficients. For example, in [2,3] there was considered homogenization in the case of time-discontinuous coefficients with inertial parameter bounded in $BV(0,T;L^\infty(\Omega))$. The latter restriction is too strong even in certain regular situations because it forbids some important material geometries, such as a rectangular checkerboard assembled from two distinct ordinary materials in space-time (see Chapter 5). Energy may be unbounded in this example, and no standard homogenization becomes possible. Contrary to the elliptic case, homogenization based on the energy boundedness does not appear to be in hyperbolic problems a single means for relaxation, i.e., for the release of all resources hidden within a spatio-temporal material assembly. There are conceptually different ways toward such release; some of them discussed in Chapter 6.

For many purposes, especially for material optimization in dynamics, we want as much freedom in spatio-temporal material geometry as possible. This freedom, however, can be accompanied by some sacrifice, such as shocks and an extensive energy flow from the external source into the system.

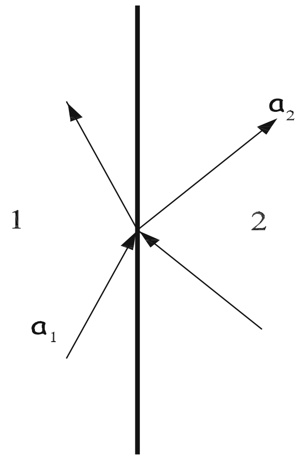

Fig. 2.2. An immovable interface: $V = 0$: two departing characteristics

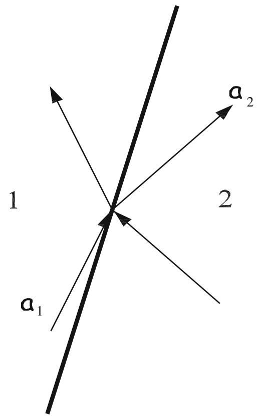

Fig. 2.3. A moving interface: $| V |< a_1$: two departing characteristics

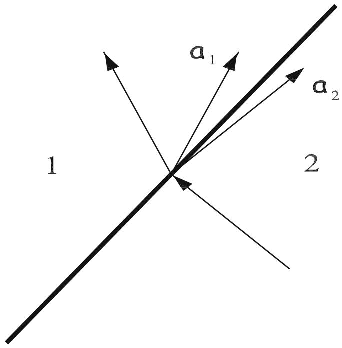

Fig. 2.4. A moving interface: $a_1 < V < a_2$: three departing characteristics

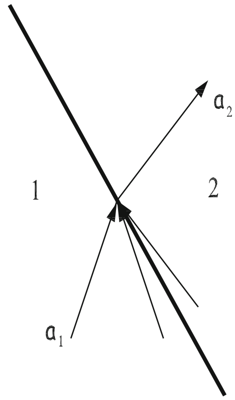

Fig. 2.5. A moving interface: $-a_2 < V < -a_1$: one departing characteristic

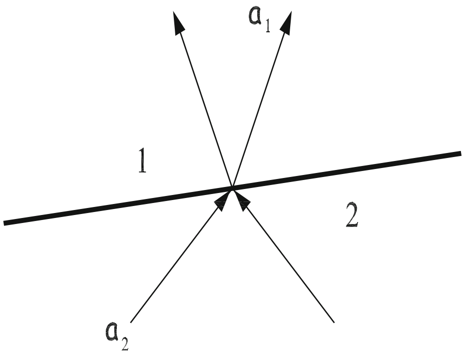

Fig. 2.6. A moving interface: $| V |> a_2$: two departing characteristics

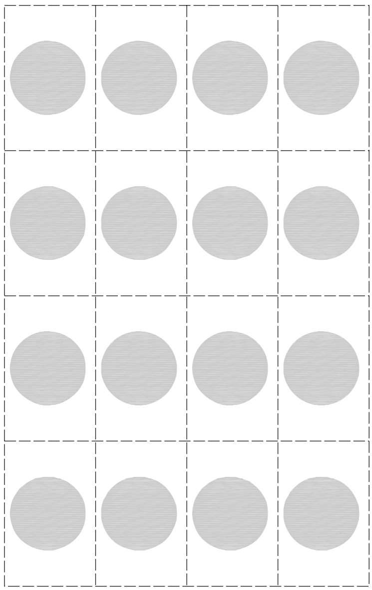

Fig. 2.7. A matrix microstructure in space-time violating Ineq. (2.7)

2.2 The Effective Parameters of Regular Activated Laminate

To determine them, we apply homogenization to the system (2.4) bearing in mind compatibility conditions (2.5) and (2.6). The analysis will be simplified if we assume that $V_0 = 0$ and introduce, instead of (z, t), the Galilean coordinate frame (ζ, τ) specified by

$$\zeta = z - Vt, \quad \tau = t; \tag{2.8}$$

this *co-moving* frame (in which the interface stays immovable) is traveling with velocity V in the positive z-direction.

By using obvious relations

$$\partial/\partial z = \partial/\partial\zeta, \;\; \partial/\partial t = \partial/\partial\tau - V\partial/\partial\zeta, \tag{2.9}$$

we reduce the system (2.4) to the form

$$\rho u_\tau = \rho V u_\zeta + v_\zeta, \;\; v_\tau = k u_\zeta + V v_\zeta;$$

on denoting

$$\Delta = V^2 - a^2, \quad a^2 = k/\rho, \tag{2.10}$$

we rewrite this as

$$u_\zeta = \frac{V}{\Delta}u_\tau - \frac{1}{\rho\Delta}v_\tau, \;\; v_\zeta = -\frac{k}{\Delta}u_\tau + \frac{V}{\Delta}v_\tau. \tag{2.11}$$

Conditions (ii), (iii), Section 2.1, indicate that parameters ρ, k depend on the argument $z - Vt = \zeta$; we shall assume that these parameters are periodic functions, with a unit period, of the *fast variable* $\xi = \zeta/\delta, \; \delta \to 0$. Equation (2.11) will now be averaged over the unit period in ξ. Introduce the symbol $\langle\cdot\rangle = m_1(\cdot)_1 + m_2(\cdot)_2$ for the arithmetic mean of $(\cdot)$, with materials 1 and 2 represented in a unit period with the volume fractions $m_1, \; m_2 \geq 0 \;\; (m_1 + m_2 = 1)$. We apply averaging to equation (2.11) bearing in mind that the derivatives u_τ, v_τ are continuous across the interfaces ζ=const immovable in a new frame; for this reason, they remain unaffected by averaging: $\langle u_\tau\rangle = u_\tau, \;\; \langle v_\tau\rangle = v_\tau$. Preserving symbols $u_\zeta, u_\tau, v_\zeta, v_\tau$ for the averaged quantities $\langle u_\zeta\rangle, \langle u_\tau\rangle, \langle v_\zeta\rangle, \langle v_\tau\rangle$, we arrive at the system

$$u_\zeta = BVu_\tau - Cv_\tau, \;\; v_\zeta = -Du_\tau + BVv_\tau,$$

where we introduced notation

$$B = \left\langle \frac{1}{\Delta} \right\rangle, \;\; C = \left\langle \frac{1}{\rho\Delta} \right\rangle, \;\; D = \left\langle \frac{k}{\Delta} \right\rangle. \tag{2.12}$$

We now go back to z, t, with the reference to (2.8), (2.9); after some calculation, there appears the system

$$v_t = pu_z - qu_t, \quad v_z = qu_z + ru_t. \tag{2.13}$$

The coefficients p, q, r are given by the formulae

$$p = V^2 D - \frac{A^2}{C}, \;\; q = -V\left(D - \frac{AB}{C}\right), \;\; r = \frac{B^2V^2}{C} - D, \tag{2.14}$$

with the symbol A defined as

$$A = BV^2 - 1 = \left\langle \frac{a^2}{V^2 - a^2} \right\rangle. \tag{2.15}$$

A direct calculation shows that

$$\begin{aligned} A &= \frac{a_1^2 a_2^2}{\Delta_1 \Delta_2} \left[V^2 \left(\overline{\frac{1}{a^2}} \right) - 1 \right], \\ B &= \frac{1}{\Delta_1 \Delta_2} (V^2 - \overline{a^2}), \\ C &= \frac{1}{\rho_1 \rho_2 \Delta_1 \Delta_2} (V^2 \bar{\rho} - \bar{k}), \\ D &= \frac{k_1 k_2}{\Delta_1 \Delta_2} \left[V^2 \left(\overline{\frac{1}{k}} \right) - \left(\overline{\frac{1}{\rho}} \right) \right]. \end{aligned}$$

Here, we applied notation

$$\overline{(\cdot)} = m_1 (\cdot)_2 + m_2 (\cdot)_1, \tag{2.16}$$

and an obvious symbol (see (2.10)) $\Delta_i = V^2 - a_i^2, \quad i = 1, 2$.

We shall also use parameters α, β, θ defined by the formulae

$$\begin{aligned} \alpha &= \frac{A}{C} = \frac{\left\langle \frac{a^2}{\Delta} \right\rangle}{\left\langle \frac{1}{\rho \Delta} \right\rangle} = k_1 k_2 \frac{V^2 \left(\overline{\frac{1}{a^2}} \right) - 1}{V^2 \bar{\rho} - \bar{k}}, \\ \beta &= \frac{BV}{C} = V \frac{\left\langle \frac{1}{\Delta} \right\rangle}{\left\langle \frac{1}{\rho \Delta} \right\rangle} = \rho_1 \rho_2 V \; \frac{V^2 - \overline{a^2}}{V^2 \bar{\rho} - \bar{k}}, \\ \theta &= \frac{C}{D} = \frac{\left\langle \frac{1}{\rho \Delta} \right\rangle}{\left\langle \frac{k}{\Delta} \right\rangle} = \frac{1}{k_1 k_2 \rho_1 \rho_2} \; \frac{V^2 \bar{\rho} - \bar{k}}{V^2 \left(\overline{\frac{1}{k}} \right) - \left(\overline{\frac{1}{\rho}} \right)}. \end{aligned} \tag{2.17}$$

The symbols p, q, and r are linked with α, β, θ through the following relations (c.f. (2.14)):

$$p = \frac{V^2 - \theta \alpha^2}{\theta(\beta V - \alpha)}, \quad q = -\frac{V - \theta \alpha \beta}{\theta(\beta V - \alpha)}, \quad r = -\frac{1 - \theta \beta^2}{\theta(\beta V - \alpha)}. \tag{2.18}$$

Parameter $\sqrt{\theta}$ is interpreted as the wave impedance of the laminate: for a pure material with properties ρ, k it becomes equal to $1/\sqrt{k\rho}$. Sometimes we will use the symbol $\gamma = 1/\sqrt{\theta}$ to designate the wave conductance γ. We now return to equation (2.13). This system appeared as a result of homogenization applied to (2.4); the relevant composite is a spatio-temporal laminate in (z, t) of the type illustrated in Figure 1.4. We wish to determine the effective parameters of this laminate.

Consider first the case $V = 0$ when the laminate becomes *spatial*. Then $q = 0$, and p, r become

$$p = \alpha = \frac{1}{\left\langle \frac{1}{\rho a^2} \right\rangle} = \frac{1}{\left\langle \frac{1}{k} \right\rangle} = \langle k^{-1} \rangle^{-1}, \quad r = \frac{1}{\theta\alpha} = \left\langle \frac{k}{a^2} \right\rangle = \langle \rho \rangle. \tag{2.19}$$

Because $q = 0$, we conclude, by comparing (2.13) and (2.4), that parameters p and r specified by (2.19), may be treated, respectively, as the effective stiffness K and density P of a spatial laminate. When $V = 0$, the matrix

$$\begin{vmatrix} p & -q \\ q & r \end{vmatrix} \tag{2.20}$$

of the coefficients in (2.13) becomes diagonal, and its elements are then qualified as effective constants. Another extreme case $V = \infty$ corresponds to what we call a *temporal* laminate. In this case, the terms with q in (2.13) also drop out, and p, r become

$$p = \langle k \rangle, \quad r = \frac{1}{\left\langle \frac{1}{\rho} \right\rangle} = \langle \rho^{-1} \rangle^{-1}, \tag{2.21}$$

with a similar interpretation as the effective stiffness K and density P. The matrix (2.20) also becomes diagonal in this case.

With V being neither zero nor infinity, we diagonalize the matrix (2.20) by introducing a new Galilean frame η, τ, through the formulae

$$\eta = z - wt, \quad \tau = t. \tag{2.22}$$

The frame η, τ is moving along the z-axis with the velocity w specified below. In a new frame, the system (2.13) takes on the form

$$\begin{aligned} v_\tau &= (p + 2qw - rw^2)u_\eta - (q - wr)u_\tau, \\ v_\eta &= (q - wr)u_\eta + ru_\tau. \end{aligned} \tag{2.23}$$

If we now define w as

$$w = q/r, \tag{2.24}$$

then equations (2.23) are reduced to

$$v_\tau = (1/\theta r)u_\eta, \quad v_\eta = ru_\tau. \tag{2.25}$$

Here, we used an easily checked relation (c.f. (2.18))

$$pr + q^2 = 1/\theta. \tag{2.26}$$

The frame (2.22) with w specified by (2.24) with be called a *proper frame.* The matrix (2.20) takes in this frame the diagonal form

$$\begin{vmatrix} \theta^{-1}r^{-1} & 0 \\ 0 & r \end{vmatrix},$$

with the effective stiffness K and density P specified as

$$K = \theta^{-1}r^{-1}, \quad P = r \, . \tag{2.27}$$

Notice that the product of these parameters equals θ^{-1}:

$$KP = \theta^{-1}; \tag{2.28}$$

this is consistent with the remark after equation (2.18).

The symbols p, q, r may be expressed directly through the material parameters ρ_1, k_1, ρ_2, k_2, the velocity V, and the volume fraction m_1. After some calculation in which we use (2.17), we arrive at the formulae:

$$\begin{aligned} \frac{1}{\theta(\beta V - \alpha)} &= \frac{k_1 k_2}{\Delta_1 \Delta_2}\left[V^2\left(\overline{\frac{1}{k}}\right) - \left(\overline{\frac{1}{\rho}}\right)\right], \\ V^2 - \theta\alpha^2 &= \frac{\Delta_1\Delta_2}{F}\bar{\rho}\left(\overline{\frac{1}{k}}\right)\left[V^2 - \frac{1}{\bar{\rho}\left(\overline{\frac{1}{k}}\right)}\right], \\ V - \theta\alpha\beta &= V\frac{\Delta_1\Delta_2}{F}\left[\bar{\rho}\left(\overline{\frac{1}{k}}\right) - \left(\overline{\frac{1}{a^2}}\right)\right], \\ 1 - \theta\beta^2 &= -\frac{\Delta_1\Delta_2}{F a_1^2 a_2^2}\left[V^2 - \bar{k}\left(\overline{\frac{1}{\rho}}\right)\right]. \end{aligned} \tag{2.29}$$

Here, we used notation

$$F = (V^2\bar{\rho} - \bar{k})\left[V^2\left(\overline{\frac{1}{k}}\right) - \left(\overline{\frac{1}{\rho}}\right)\right]. \tag{2.30}$$

Referring to (2.18), we obtain the following expressions for p, q, r, and θ :

$$\begin{aligned} p &= k_1 k_2 \bar{\rho}\left(\overline{\frac{1}{k}}\right)\frac{V^2 - \frac{1}{\bar{\rho}\left(\overline{\frac{1}{k}}\right)}}{V^2\bar{\rho} - \bar{k}} = \langle k\rangle \frac{V^2 - \frac{1}{\bar{\rho}\left(\overline{\frac{1}{k}}\right)}}{V^2 - \frac{\bar{k}}{\bar{\rho}}}, \\ q &= -V k_1 k_2 \frac{\bar{\rho}\left(\overline{\frac{1}{k}}\right) - \left(\overline{\frac{1}{a^2}}\right)}{V^2\bar{\rho} - \bar{k}} = -V\frac{k_1 k_2}{\bar{\rho}}\frac{\bar{\rho}\left(\overline{\frac{1}{k}}\right) - \left(\overline{\frac{1}{a^2}}\right)}{V^2 - \frac{\bar{k}}{\bar{\rho}}}, \\ r &= \rho_1\rho_2\frac{V^2 - \bar{k}\left(\overline{\frac{1}{\rho}}\right)}{V^2\bar{\rho} - \bar{k}} = \frac{\rho_1\rho_2}{\bar{\rho}}\frac{V^2 - \bar{k}\left(\overline{\frac{1}{\rho}}\right)}{V^2 - \frac{\bar{k}}{\bar{\rho}}}, \\ \theta &= \frac{\bar{\rho}}{\rho_1\rho_2\langle k\rangle}\frac{V^2 - \frac{\bar{k}}{\bar{\rho}}}{V^2 - \frac{\left(\overline{\frac{1}{\rho}}\right)}{\left(\overline{\frac{1}{k}}\right)}}. \end{aligned} \tag{2.31}$$

These formulae show that the velocity $w = q/r$ of a proper frame (2.20) is not equal to V unless $V = 0$.

Remark 2.2.1. Note that in this chapter we are working within the Galilean space-time concept represented through equations (2.8), (2.22). We will see in the next chapter how to modify this assumption in a more rigorous relativistic concept.

2.3 The Effective Parameters: Homogenization

To confirm the results of the previous section, we apply a standard homogenization procedure [1] to the system (2.11). It is more convenient, however, to work with an equivalent equation (2.1) in which the coefficients ρ and k are defined as fast periodic functions of the argument $\xi = (z - Vt)/\delta, \quad \delta \to 0$. The period in ξ is taken equal to 1.

We look for solution to (2.1) represented in the form of a power series over δ:

$$u = u_0(z,t,\xi) + \delta u_1(z,t,\xi) + \delta^2 u_2(z,t,\xi) + \dots \tag{2.32}$$

where $u_i, \quad i = 0, 1, 2, \dots$ are assumed 1-periodic in ξ.

The derivatives that participate in (2.1) should be recalculated by the rule of differentiating composite functions:

$$\begin{aligned} \frac{d}{dz} F(z,t,\xi) &= F_z + \delta^{-1} F_\xi, \\ \frac{d}{dt} F(z,t,\xi) &= F_t - V\delta^{-1} F_\xi. \end{aligned} \tag{2.33}$$

The subscripts in these formulae denote partial differentiation over the relevant variables.

By virtue of (2.32), (2.33), we obtain

$$\begin{aligned} \frac{du}{dz} &= u_{0z} + \delta u_{1z} + \delta^2 u_{2z} + \dots + \delta^{-1}(u_{0\xi} + \delta u_{1\xi} + \delta^2 u_{2\xi} + \dots), \\ \frac{du}{dt} &= u_{0t} + \delta u_{1t} + \delta^2 u_{2t} + \dots - V\delta^{-1}(u_{0\xi} + \delta u_{1\xi} + \delta^2 u_{2\xi} + \dots), \\ \frac{d}{dt}\left(\rho(\xi)\frac{du}{dt}\right) &= \left(\rho(\xi)\frac{du}{dt}\right)_t - V\delta^{-1}\left(\rho(\xi)\frac{du}{dt}\right)_\xi \\ &= [\rho(\xi)(u_{0t} + \delta u_{1t} + \delta^2 u_{2t} + \dots) \\ &\quad - V\delta^{-1}\rho(\xi)(u_{0\xi} + \delta u_{1\xi} + \delta^2 u_{2\xi} + \dots)]_t \\ &\quad - V\delta^{-1}[\rho(\xi)(u_{0t} + \delta u_{1t} + \delta^2 u_{2t} + \dots) \\ &\quad - V\delta^{-1}\rho(\xi)(u_{0\xi} + \delta u_{1\xi} + \delta^2 u_{2\xi} + \dots)]_\xi. \end{aligned} \tag{2.34}$$

A similar expansion for

$$\frac{d}{dz}\left(k(\xi)\frac{du}{dz}\right)$$

appears as we formally apply z instead of t, k instead of ρ, and set $V = -1$ in (2.34).

By using these expansions in the lhs of (2.1), we express this one as a power series over δ and require that the coefficients of powers of δ be set equal to zero. This requirement, applied to the coefficient of the lowest power, δ^{-2}, means that

$$V^2(\rho u_{0\xi})_\xi - (ku_{0\xi})_\xi = 0. \tag{2.35}$$

The similar conditions related to the coefficients of δ^{-1} and δ^0 produce the following equations:

$$-V(\rho u_{0\xi})_t - V(\rho u_{0t})_\xi + V^2(\rho u_{1\xi})_\xi - (ku_{0\xi})_z - (ku_{0z})_\xi - (ku_{1\xi})_\xi = 0, \tag{2.36}$$

$$\begin{aligned}(\rho u_{0t})_t - V(\rho u_{1\xi})_t - V(\rho u_{1t})_\xi + V^2(\rho u_{2\xi})_\xi \\ - (ku_{0z})_z - (ku_{1\xi})_z - (ku_{1z})_\xi - (ku_{2\xi})_\xi = 0.\end{aligned} \tag{2.37}$$

We now consider the consequences of (2.35)–(2.37). Integration of (2.35) reveals that

$$(V^2\rho - k)u_{0\xi} = m(z,t).$$

Because u_0 should be 1-periodic in ξ, we get

$$m(z,t)\int_0^1 \frac{d\xi}{V^2\rho - k} = 0,$$

i.e., $m(z,t) = 0$, since the integral equals a non-zero constant C given by (2.12). We conclude that, unless $V^2\rho - k = 0$, u_0 is independent of the fast variable ξ: $u_0 = u_0(z,t)$.

Bearing this in mind and integrating (2.36), we arrive at the relation

$$-V\rho u_{0t} - ku_{0z} + (V^2\rho - k)u_{1\xi} = n(z,t),$$

or, equivalently,

$$u_{1\xi} = \frac{n}{V^2\rho - k} + \frac{V\rho}{V^2\rho - k}u_{0t} + \frac{k}{V^2\rho - k}u_{0z}.$$

We demand, as before, that u_1 be 1-periodic in ξ; this requirement defines n as

$$n = -V\frac{B}{C}u_{0t} - \frac{A}{C}u_{0z},$$

with A, B, C given by (2.15) and (2.12). The expression for $u_{1\xi}$ now takes the form

$$u_{1\xi} = Pu_{0t} + Qu_{0z}, \tag{2.38}$$

with P, Q defined by

$$P = \frac{V}{V^2\rho - k}\left(\rho - \frac{B}{C}\right), \quad Q = \frac{1}{V^2\rho - k}\left(k - \frac{A}{C}\right). \tag{2.39}$$

Note that

$$\langle P \rangle = \int_0^1 P d\xi = \langle Q \rangle = \int_0^1 Q d\xi = 0.$$

We also mention the formulae

$$\begin{aligned} X &= u_{1t} - V u_{2\xi} = -\frac{1}{V^2\rho - k}(kS + VT), \\ Y &= u_{1z} + u_{2\xi} = \frac{1}{V^2\rho - k}(\rho V S + T), \end{aligned} \tag{2.40}$$

where

$$S = \int_0^\xi (N_t - M_z) d\xi, \quad T = \int_0^\xi (-\rho M_t + k N_z) d\xi, \tag{2.41}$$

with M, N defined as

$$\begin{aligned} M &= u_{0t} - V u_{1\xi} = u_{0t}(1 - VP) - u_{0z} VQ, \\ N &= u_{0z} + u_{1\xi} = u_{0t} P + u_{0z}(1 + Q). \end{aligned} \tag{2.42}$$

Equations (2.40)–(2.42) are produced by the same technique as equations (2.38), (2.39).

By integrating (2.37) over the period 1 in ξ and by using the 1-periodicity of u_1 and u_2, we arrive at the relation:

$$\begin{aligned} &(\langle \rho \rangle u_{0t})_t - V \left\langle \frac{V\rho}{V^2\rho - k}\left(\rho - \frac{B}{C}\right) u_{0t} + \frac{\rho}{V^2\rho - k}\left(k - \frac{A}{C}\right) u_{0z} \right\rangle_t \\ &\qquad - (\langle k \rangle u_{0z})_z - \left\langle \frac{Vk}{V^2\rho - k}\left(\rho - \frac{B}{C}\right) u_{0t} + \frac{k}{V^2\rho - k}\left(k - \frac{A}{C}\right) u_{0z} \right\rangle_z = 0. \end{aligned}$$

The symbol $\langle \cdot \rangle$ has been defined in Section 2.2 as $m_1(\cdot)_1 + m_2(\cdot)_2$. Bearing in mind that $u_0 = u_0(z, t)$ and that ρ, k depend on ξ alone, we rewrite the last equation in the form:

$$\left(\frac{B^2V^2}{C} - D\right) u_{0tt} - 2V\left(D - \frac{AB}{C}\right) u_{0zt} - \left(V^2 D - \frac{A^2}{C}\right) u_{0zz} = 0.$$

In view of (2.14), this is reduced to

$$r u_{0tt} + 2q u_{0zt} - p u_{0zz} = 0,$$

which is equivalent to the system (2.13).

Notice that

$$N_t - M_z = u_{0tt}P + u_{0zt}(Q + VP) + u_{0zz}VQ,$$

and, because $\langle P\rangle = \langle Q\rangle = 0$, we have $\langle N_t - M_z\rangle = 0$ identically.

Also,

$$-\rho M_t + kN_z = -\rho u_{0tt}(1 - VP) + u_{0zt}(\rho VQ + kP) + ku_{0zz}(1 + Q).$$

By direct inspection with reference to (2.12)–(2.15) and (2.39), we conclude that

$$p = \langle k(1 + Q)\rangle,\ q = -\frac{1}{2}\langle \rho VQ + kP\rangle,\ r = \langle \rho(1 - VP)\rangle,$$

and, by (2.13),

$$\langle -\rho M_t + kN_z\rangle = -ru_{0tt} - 2qu_{0zt} + pu_{0zz} = 0. \tag{2.43}$$

Now it is easy to see that $S(0) = T(0) = S(1) = T(1) = 0$, and, as a consequence,

$$X(0) = X(1) = Y(0) = Y(1) = 0.$$

2.4 The Effective Parameters: Floquet Theory

Because of a special assumption made about ρ, k as 1-periodic functions of a single argument $\xi = \zeta/\delta$, the system (2.11) may be viewed as a linear system with coefficients that are periodic in ζ with period δ. We shall apply the Floquet theory to this system to obtain its exact solution; the results of Sections 2.2 and 2.3 follow from this solution in a low frequency asymptotic limit. Calculations of this section are detailed in Appendix A.

We first eliminate the τ-variable by applying the Laplace transform:

$$\bar{u}(\zeta, s) = \int_0^\infty e^{-s\tau}u(\zeta,\tau)d\tau.$$

Equation (2.11) then take on the form

$$\begin{aligned}
\bar{u}_\zeta - \frac{s}{V^2 - a^2}\left(V\bar{u} - \frac{1}{\rho}\bar{v}\right) &= 0,\\
\bar{v}_\zeta + \frac{s}{V^2 - a^2}(k\bar{u} - V\bar{v}) &= 0,
\end{aligned} \tag{2.44}$$

where a^2 is defined as k/ρ (see (2.10)).

Assume that $\zeta \geq 0$, and that material 1 occupies the intervals

$$(n - m_1)\delta \leq \zeta \leq n\delta,\ \ n = 0, 1, 2, \ldots, \tag{2.45}$$

while material 2 is concentrated within supplementary intervals

$$n\delta \le \zeta \le (n + m_2)\delta, \quad n = 0, 1, \dots . \tag{2.46}$$

Here m_1 and m_2 denote, as before, the volume fractions of materials 1 and 2 in the period δ.

A general solution to the system (2.44) is given by

$$\begin{aligned} \bar{u} &= A_1 e^{\mu_1 \zeta} P(\mu_1, \zeta) + A_2 e^{\mu_2 \zeta} P(\mu_2, \zeta), \\ \bar{v} &= A_1 e^{\mu_1 \zeta} Q(\mu_1, \zeta) + A_2 e^{\mu_2 \zeta} Q(\mu_2, \zeta), \end{aligned} \tag{2.47}$$

with $P(\mu_1, \zeta), \dots, Q(\mu_2, \zeta)$ being δ-periodic in ζ. In (2.47), A_1 and A_2 denote the coefficients to be determined by the boundary conditions, and μ_1, μ_2 represent the Floquet characteristic exponents given by the formula (see (2.10))

$$\mu_{1,2}\delta = V(\vartheta_1/a_1 + \vartheta_2/a_2) \pm \chi. \tag{2.48}$$

Here, the upper (lower) sign is related to $\mu_1(\mu_2)$, and parameters ϑ_i, χ are defined as

$$\begin{aligned} &\vartheta_i = s\delta\phi_i, \quad \phi_i = m_i a_i/(V^2 - a_i^2), \quad i = 1, 2, \\ &\cosh\chi = \cosh\vartheta_1 \cosh\vartheta_2 + \sigma \sinh\vartheta_1 \sinh\vartheta_2, \\ &\sigma = (\theta_1 + \theta_2)/2\sqrt{\theta_1\theta_2}, \\ &\theta_i = 1/k_i\rho_i, \ i = 1, 2. \end{aligned} \tag{2.49}$$

Clearly, $\sigma \ge 1$. Consider the low frequency case $\mid s\delta/a_i \mid \ll 1$; equation (2.49) then specifies χ approximately as

$$\chi = s\delta\sqrt{\phi_1^2 + \phi_2^2 + 2\sigma\phi_1\phi_2}. \tag{2.50}$$

By (2.7), the quantities ϕ_1, ϕ_2 should be of the same sign; because $\sigma > 0$, the square root in (2.50) is real.

If $s = i\omega$ with ω real, then

$$\cosh\chi = \cos\omega\delta\phi_1 \cos\omega\delta\phi_2 - \sigma \sin\omega\delta\phi_1 \sin\omega\delta\phi_2.$$

If the absolute value of the rhs of this equation exceeds 1, then the roots χ have non-zero real parts, and solution (2.47) contains exponentially increasing terms.

This cannot happen in the low frequency approximation $\omega\delta/a_i \ll 1$, and the corresponding values of χ, as well as μ_1, μ_2, are in this case imaginary.

The functions $P(\mu, \zeta)$ and $Q(\mu, \zeta)$ in (2.47) are given by the formulae

$$P(\mu, \zeta) = \begin{cases} e^{-\left(\mu - \dfrac{s}{V - a_1}\right)(\zeta - n\delta)} + E e^{-\left(\mu - \dfrac{s}{V + a_1}\right)(\zeta - n\delta)}, & \zeta \in \ (2.45) \\ G e^{-\left(\mu - \dfrac{s}{V - a_2}\right)(\zeta - n\delta)} + H e^{-\left(\mu - \dfrac{s}{V + a_2}\right)(\zeta - n\delta)}, & \zeta \in (2.46) \end{cases}$$

$$Q(\mu,\zeta)=\begin{cases}\gamma_1\left[e^{-\left(\mu-\dfrac{s}{V-a_1}\right)(\zeta-n\delta)}+Ee^{\left(\mu-\dfrac{s}{(V+a_1)}\right)(\zeta-n\delta)}\right], & \zeta\in(2.45)\\ \gamma_2\left[-Ge^{-\left(\mu-\dfrac{s}{V-a_2}\right)(\zeta-n\delta)}+He^{-\left(\mu-\dfrac{s}{V+a_2}\right)(\zeta-n\delta)}\right], & \zeta\in(2.46)\end{cases}$$

The constants E, G, and H in these formulae are defined as solutions to the system

$$\begin{aligned}-E+G+H&=1,\\ E+(G-H)(\gamma_2/\gamma_1)&=1\\ -Ee^{\vartheta_1}+Ge^{\vartheta_2\mp\chi}+He^{-\vartheta_2\mp\chi}&=e^{-\vartheta_1},\end{aligned}\tag{2.51}$$

with upper (lower) sign related to $\mu=\mu_1(\mu_2)$. For derivation of (2.51), see Appendix A.

Both $P(\mu,\zeta)$ and $Q(\mu,\zeta)$ are δ-periodic in ζ; these functions actually depend on $\zeta-n\delta$, this argument falling into the ranges:

$$-m_1\le\frac{\zeta-n\delta}{\delta}\le 0 \ \text{ for (2.45)}; \ \ 0\le\frac{\zeta-n\delta}{\delta}\le m_2 \ \text{ for (2.46)}.$$

In both cases, the difference $\zeta-n\delta$ is of order δ. The functions $\bar{u},\bar{v}$ given by (2.47) have the form of modulated waves; when $s=i\omega$ and $\omega\delta/a_i\ll 1$, then $e^{\mu\zeta}$ appears to be the long wave modulation factor, while $P(\mu,\zeta), Q(\mu,\zeta)$ represent the short wave carriers. By averaging $\bar{u}$ and $\bar{v}$ over the period δ, we perform homogenization; this operation eliminates the short wave carriers P, Q, and detects the long wave envelopes $e^{\mu\zeta}$. These envelopes give birth to the original $u(\zeta,\tau)$ taking the form of d'Alembert waves $f(\zeta+\frac{s}{\mu_{1,2}}\tau)$ in the coordinate frame (ζ,τ) linked with the laboratory frame (z,t) through (2.8). In this latter frame, the waves take on the form $f(z-(V-\frac{s}{\mu_{1,2}})t)$, with phase velocities $V-\frac{s}{\mu_{1,2}}$.

In Appendix A, these velocities are calculated for $s=i\omega,\ \ \omega\delta/a_i\ll 1$; they are specified as

$$v_{1,2}=V-\frac{s}{\mu_{1,2}}=-Va_1^2a_2^2\frac{\bar{\rho}\left(\overline{\frac{1}{k}}\right)-\left(\overline{\frac{1}{a^2}}\right)}{V^2-\bar{k}\left(\overline{\frac{1}{\rho}}\right)}\pm a_1a_2\frac{\sqrt{(V^2\bar{\rho}-\bar{k})\left(V^2\left(\overline{\frac{1}{k}}\right)-\left(\overline{\frac{1}{\rho}}\right)\right)}}{V^2-\bar{k}\left(\overline{\frac{1}{\rho}}\right)};\tag{2.52}$$

as before, the upper (lower) sign is related to $\mu_1(\mu_2)$.

On the other hand, if we go back to equation (2.43) and look for its solution $f(z-vt)$, then the phase velocities $v_{1,2}$ appear to be the roots of the equation

$$rv^2 - 2qv - p = 0. \tag{2.53}$$

Referring to (2.31), we conclude that $v_{1,2}$ are identical with the expressions given by (2.52).

2.5 The Effective Parameters: Discussion

Equation (2.27) define the effective parameters K, P of an activated laminate in (z, t). Material properties k, ρ are assumed positive for both of the original substances; as mentioned in Section 2.1, we set $a_2^2 > a_1^2$, i.e., $k_2/\rho_2 > k_1/\rho_1$. Then it is easily checked that, apart from obvious inequalities

$$\begin{aligned} \bar{\rho}\left(\overline{\frac{1}{\rho}}\right) &\geq 1, \\ \bar{k}\left(\overline{\frac{1}{k}}\right) &\geq 1, \end{aligned} \tag{2.54}$$

we have

$$\begin{aligned} a_1^2 &\leq \frac{\bar{k}}{\bar{\rho}} \leq a_2^2, \\ a_1^2 &\leq \frac{\left(\overline{\frac{1}{\rho}}\right)}{\left(\overline{\frac{1}{k}}\right)} \leq a_2^2. \end{aligned} \tag{2.55}$$

Also,

$$\begin{aligned} \frac{1}{\bar{\rho}\left(\overline{\frac{1}{k}}\right)} &\leq a_2^2, \\ a_1^2 &\leq \bar{k}\left(\overline{\frac{1}{\rho}}\right). \end{aligned} \tag{2.56}$$

From this point on, we shall distinguish between two possible situations:

$$(i) \quad k_2 > k_1, \quad \rho_2 < \rho_1; \tag{2.57}$$

in the present discussion, this possibility will be referred to as the *regular* case;

$$(ii) \quad \text{either} \quad k_2 > k_1, \quad \rho_2 > \rho_1, \qquad \text{or} \qquad k_2 < k_1, \quad \rho_2 < \rho_1 \tag{2.58}$$

both of the latter possibilities will be termed *irregular*.

Remark 2.5.1. In this context, the terms *regular* and *irregular* have meaning different from that introduced in Section 2.1.

In a regular case, inequalities (2.56) are complemented by the following:

$$a_1^2 \leq \frac{1}{\bar{\rho}\left(\bar{\frac{1}{k}}\right)}, \qquad \bar{k}\left(\bar{\frac{1}{\rho}}\right) \leq a_2^2. \tag{2.59}$$

In irregular case, however, there exists the range of parameters ρ, k, and m_1, such that

$$\frac{1}{\bar{\rho}\left(\bar{\frac{1}{k}}\right)} \leq a_1^2, \tag{2.60}$$

and the range for which

$$a_2^2 \leq \bar{k}\left(\bar{\frac{1}{\rho}}\right). \tag{2.61}$$

Indeed,

$$\frac{1}{\bar{\rho}\left(\bar{\frac{1}{k}}\right)} - a_1^2 = \frac{k_1 k_2}{\bar{\rho}\langle k\rangle} - \frac{k_1}{\rho_1} = \frac{k_1}{\rho_1 \bar{\rho}\langle k\rangle}\left[k_2\rho_1 - (m_1k_1 + m_2k_2)(m_1\rho_2 + m_2\rho_1)\right]$$

$$= \frac{m_1 k_1}{\rho_1\bar{\rho}\langle k\rangle}\left[\rho_1\Delta k - k_1\Delta\rho - m_2\Delta k\Delta\rho\right]; \tag{2.62}$$

$$a_2^2 - a_1^2 = \frac{k_2}{\rho_2} - \frac{k_1}{\rho_1} = \frac{1}{\rho_1\rho_2}(\rho_1\Delta k - k_1\Delta\rho) \geq 0; \tag{2.63}$$

here we applied notation $\Delta(\cdot) = (\cdot)_2 - (\cdot)_1$. The difference $1/\bar{\rho}\left(\bar{\frac{1}{k}}\right) - a_1^2$ is positive in the regular case when $\Delta k > 0, \Delta\rho < 0$; however, in irregular case, when the signs of Δk and $\Delta\rho$ are the same, this difference may become negative. For example, if $k_2 = 10,\ \ \rho_2 = 9,\ \ k_1 = \rho_1 = 1$, then $\rho_1\Delta k - k_1\Delta\rho - m_2\Delta k\Delta\rho = 9 - 8 - 72m_2$, and this is ≤ 0 if $m_2 \geq 1/72$. At the same time, the difference $k_2/\rho_2 - k_1/\rho_1$ is positive by (2.56), i.e., k/ρ increases as we go from material 1 to material 2. Combined with $\Delta k\Delta\rho > 0$ (irregular case), this means that the increase may be due to that in k and to the less intensive increase (not a decrease) in ρ, or due to the decrease in ρ and the less intensive decrease (not an increase) in k. The possibility for inequality (2.61) to hold is illustrated quite similarly. We calculate the difference

$$\bar{k}\left(\bar{\frac{1}{\rho}}\right) - a_2^2 = \frac{\bar{k}\langle\rho\rangle}{\rho_1\rho_2} - \frac{k_2}{\rho_2} = -\frac{1}{\rho_1\rho_2}\left[k_2\rho_1 - (m_1k_2 + m_2k_1)(m_1\rho_1 + m_2\rho_2)\right]$$

$$= -\frac{m_2}{\rho_1\rho_2}\left[\rho_1\Delta k - k_1\Delta\rho - m_1\Delta k\Delta\rho\right]; \tag{2.64}$$

this difference is negative in a regular case, and may become positive in irregular case. Indeed, for an example cited above ($k_2 = 10,\ \ \rho_2 = 9,\ \ k_1 = \rho_1 = 1$), the difference becomes positive if $m_1 \geq 1/72$, i.e., $m_2 \leq 71/72$. We conclude that, for this example, both inequalities (2.60), (2.61) hold once m_2 falls into the range $(1/72, 71/72)$.

If, as assumed,

$$a_2^2 > a_1^2, \tag{2.65}$$

then, for inequalities (2.60), (2.61) to hold, it is necessary that Δk and $\Delta \rho$ should *both* be non-zero. So the situation in which the original substances differ in *only one* material constant can never maintain (2.60), (2.61).

On making these observations, we may discuss the formulae (2.27) for K and P. Ineq. (2.7) outlines two admissible ranges for V^2 : the *slow* range

$$V^2 < a_1^2, \tag{2.66}$$

and the *fast* range

$$V^2 > a_2^2. \tag{2.67}$$

The last formula (2.17) shows that $\theta \geq 0$ for both ranges once k, ρ are of the same sign for all participating materials. As to the values of r (see (2.31)), they are always positive for the slow range (2.66), but may become negative in the irregular case for the fast range (2.67). Indeed, for this range the denominator $V^2 \bar{\rho} - \bar{k}$ in (2.31) is positive by (2.55), while the numerator $V^2 - \bar{k}\left(\frac{\bar{1}}{\rho}\right)$ may become negative for the fast range: to this end, the value V^2 should fall into the interval $\left(a_2^2, \bar{k}\left(\frac{\bar{1}}{\rho}\right)\right)$; as stated above (see (2.61)), this interval may come into existence in the irregular case.

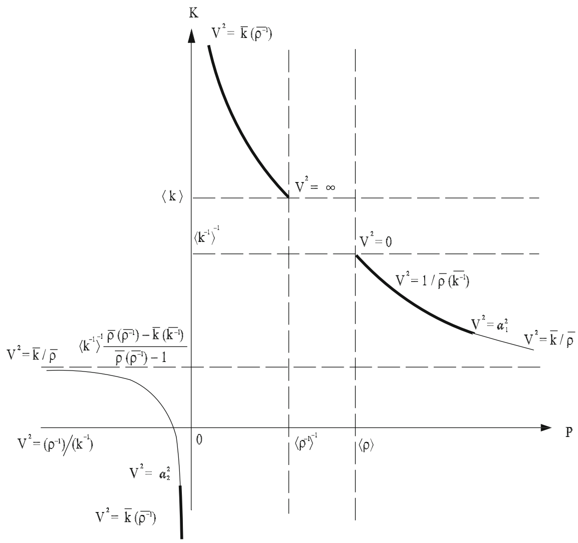

Fig. 2.8. Effective parameters K versus P with variable V (case $\bar{\rho}\left(\frac{\bar{1}}{\rho}\right) - \bar{k}\left(\frac{\bar{1}}{k}\right) \geq 0$)

The plots of K versus P with V variable along the curves are given, respectively, by Figure 2.8 (case $\bar{\rho}\left(\frac{\bar{1}}{\rho}\right) - \bar{k}\left(\frac{\bar{1}}{k}\right) \geq 0$), and Figure 2.9 (case $\bar{\rho}\left(\frac{\bar{1}}{\rho}\right) - \bar{k}\left(\frac{\bar{1}}{k}\right) \leq 0$). Both curves have parametric equations (with parameter V) following from (2.27) and (2.31):

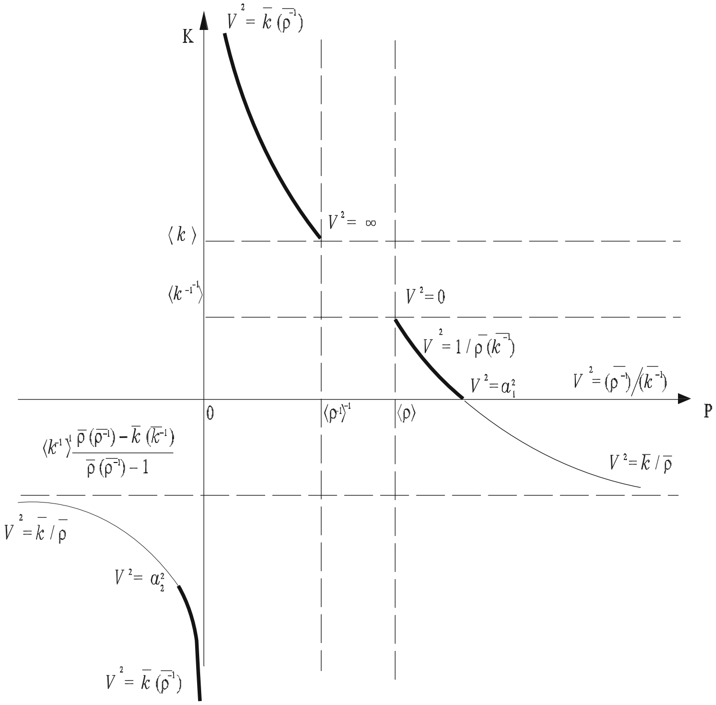

Fig. 2.9. Effective parameters K versus P with variable V (case $\rho\left(\frac{1}{\rho}\right) - k\left(\frac{1}{k}\right) \leq 0$)

$$K = \langle k \rangle \frac{V^2 - \frac{\left(\frac{\bar{1}}{\rho}\right)}{\left(\frac{\bar{1}}{k}\right)}}{V^2 - \bar{k}\left(\frac{\bar{1}}{\rho}\right)}, \quad P = \frac{\rho_1 \rho_2}{\bar{\rho}} \; \frac{V^2 - \bar{k}\left(\frac{\bar{1}}{\rho}\right)}{V^2 - \frac{\bar{k}}{\bar{\rho}}}.$$

Only those parts of the curves are realizable that are consistent with the admissible ranges $V^2 \leq a_1^2$, $V^2 \geq a_2$ of V^2 (see (2.7)); the relevant segments are marked boldface in the figures.

The "averaged" d'Alembert waves, i.e., the low frequency envelopes introduced in Section 2.4, propagate with the phase velocities $v_{1,2}$ specified by (2.52). By (2.53), the product of these velocities equals $-p/r$, or, with reference to (2.31),

$$v_1 v_2 = -\bar{\rho}\left(\frac{\bar{1}}{k}\right) a_1^2 a_2^2 \frac{V^2 - \frac{1}{\bar{\rho}\left(\frac{\bar{1}}{k}\right)}}{V^2 - \bar{k}\left(\frac{\bar{1}}{\rho}\right)}. \tag{2.68}$$

Given the observations made earlier in this section, we conclude that v_1 and v_2 should have opposite signs in a regular case. As to an irregular case, the signs of v_1 and v_2 are the same if V^2 is taken within the interval

$$\left(\frac{1}{\bar{\rho}\left(\overline{\frac{1}{k}}\right)}, a_1^2\right) \quad \text{for the slow range,} \tag{2.69}$$

and within the interval

$$\left(a_2^2, \bar{k}\left(\overline{\frac{1}{\rho}}\right)\right) \quad \text{for the fast range.} \tag{2.70}$$

We have seen in (2.60) and (2.61) that such intervals may exist in irregular case. For each of them, the homogenized waves propagate in the *same* direction relative to a laboratory frame; this direction may be switched to opposite as we go from V to $-V$. We thus arrive at what will be termed *coordinated wave propagation*. The possibility of coordinated wave motion is peculiar to the dynamic materials; this option does not arise if we apply conventional (static) composites. The effects achieved through the use of this phenomenon may be quite unusual as seen from the following example.

Assume that we have a laminate in space-time offering a coordinated wave propagation with both low frequency waves traveling from left to right; we shall term such material a *right* laminate. By switching V to $-V$, the direction of coordinated waves is also switched to opposite, so we obtain a *left* laminate. Now consider the material arrangement produced by placing the left (right) laminate to the left (right) of the point $z = 0$ (see Figure 2.10 representing the relevant families of characteristics).

It is clear that an initial disturbance gives rise to two pairs of d'Alembert waves propagating each in the relevant quadrant of the (z, t)-plane along the characteristics. The interior of the angle AOB in a (z, t)-plane then appears to be a "shadow zone" free from any initially applied disturbance since they are unable to enter this domain due to a special geometry of characteristics. By controlling such geometry, we will selectively screen large domains in space-time from the invasion of long wave dynamic disturbances. With ordinary (static) composites, this *screening* effect is impossible.

Remark 2.5.2. The velocities $v_{1,2}$ specified by (2.52) are the phase velocities of the envelopes $e^{\mu\zeta}$ of the modulated waves that represent the Floquet solutions of equation (2.44); these velocities are defined by (2.52) in a low frequency limit $\omega \to 0$. In this capacity, they represent the group velocities of the low frequency waves propagating through an activated dynamic lamination.

When $V = 0$ (a spatial laminate), the velocities become $\pm v_{sp}$, where, by (2.19),

$$v_{sp}^2 = \langle k^{-1} \rangle^{-1} / \langle \rho \rangle = a_1^2 a_2^2 / \bar{k}\left(\overline{\frac{1}{\rho}}\right). \tag{2.71}$$

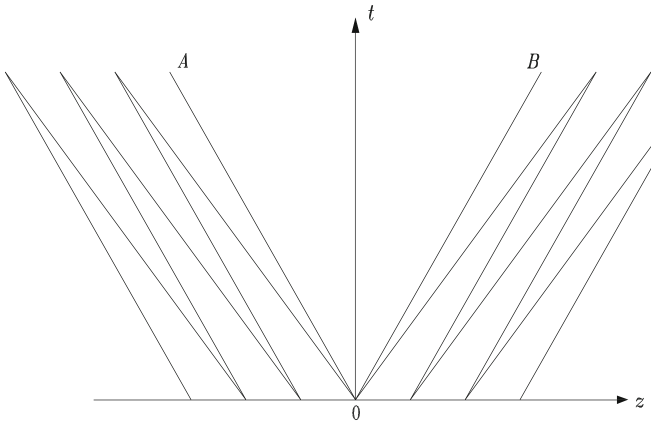

Fig. 2.10. Screening effect produced by a shadow zone

When $V = \infty$ (a temporal laminate), the velocities become $\pm v_{temp}$, where, by (2.21),

$$v_{temp}^2 = \langle k \rangle / \langle \rho^{-1} \rangle^{-1} = a_1^2 a_2^2 \bar{\rho} \left(\overline{\frac{1}{k}} \right). \tag{2.72}$$

Inequalities (2.56) now show that always

$$v_{sp}^2 \le a_2^2, \quad v_{temp}^2 \ge a_1^2.$$

For a regular case, as seen from (2.59),

$$v_{sp}^2 \ge a_1^2, \quad v_{temp}^2 \le a_2^2,$$

whereas for an irregular case, (2.53) and (2.54) show that it is possible that

$$v_{sp}^2 \le a_1^2, \quad v_{temp}^2 \ge a_2^2.$$

Combining these inequalities, we conclude that, in a regular case, both v_{sp} and v_{temp} fall into the interval (a_1, a_2), whereas in irregular case, they may fall outside this interval: v_{sp} may become less than a_1, and v_{temp}—greater than a_2.

Remark 2.5.3. Consider a special case when the wave impedance $\sqrt{\theta} = 1/\sqrt{k\rho}$ takes the same value for both materials; this case belongs with a regular range (2.57). The formula (2.52) for the effective velocities may then be illustrated by the following elementary argument.

When $\theta_1 = \theta_2$, then, at each encounter with the interface separating two adjacent materials in a laminate regular in the sense of Section 2.1., an incident wave propagating through material 1 generates *only one* secondary wave, i.e., a transmitted wave traveling in material 2. The waves propagate through ith material $(i = 1, 2)$ with velocity $\pm a_i - V$ measured in the frame (2.8) where the interfaces stay immovable. An elementary calculation now specifies the average velocity of waves passing through a unit period in ξ in this frame:

$$\frac{1}{\frac{m_1}{\pm a_1 - V} + \frac{m_2}{\pm a_2 - V}} = \frac{(V \mp a_1)(V \mp a_2)}{\pm \bar{a} - V} \ . \tag{2.73}$$

On the other hand, when $\theta_1 = \theta_2$, then a direct inspection indicates that equation (2.52) defines the difference $v_{1,2} - V$ as

$$v_{1,2} - V = \frac{1}{(\bar{a})^2 - V^2}(V \pm \bar{a})(V \mp a_1)(V \mp a_2); \tag{2.74}$$

this difference characterizes the effective velocities of waves measured in the frame (2.8). We see that the values given by (2.73) and (2.74) are identical.

In a laboratory frame (z, t), the effective velocities take the values

$$\frac{(V \mp a_1)(V \mp a_2)}{\pm \bar{a} - V} + V = \frac{a_1 a_2 \mp V\langle a \rangle}{\pm \bar{a} - V}.$$

Particularly, for $V = 0$ we obtain

$$v_{sp} = \pm \langle a^{-1} \rangle^{-1},$$

whereas for $V = \infty$

$$v_{temp} = \pm \langle a \rangle \ .$$

These expressions are identical with those following from the formulae (2.71) and (2.72) for v_{sp} and v_{temp} when we apply them to the case $\theta_1 = \theta_2$.

2.6 Balance of Energy in Longitudinal Wave Propagation Through an Activated Elastic Bar

The differential equation (2.1) governing the wave propagation through an immovable elastic bar represents an Euler equation generated by the action density

$$\Lambda = \frac{1}{2}\,\rho\,\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\,k\,\left(\frac{\partial u}{\partial z}\right)^2 . \tag{2.75}$$

This density defines components of the energy-momentum tensor W according to the formulae

$$\begin{aligned}
W_{tt} &= \frac{\partial u}{\partial t}\,\frac{\partial \Lambda}{\partial\left(\frac{\partial u}{\partial t}\right)} - \Lambda = \frac{1}{2}\rho\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{2}k\left(\frac{\partial u}{\partial z}\right)^2 - \text{the energy density},\\
W_{tz} &= \frac{\partial u}{\partial t}\,\frac{\partial \Lambda}{\partial\left(\frac{\partial u}{\partial z}\right)} = -k\frac{\partial u}{\partial t}\,\frac{\partial u}{\partial z} - \text{the energy flux density},\\
W_{zt} &= \frac{\partial u}{\partial z}\,\frac{\partial \Lambda}{\partial\left(\frac{\partial u}{\partial t}\right)} = \rho\frac{\partial u}{\partial t}\,\frac{\partial u}{\partial z} - \text{the momentum density},\\
W_{zz} &= \frac{\partial u}{\partial z}\,\frac{\partial \Lambda}{\partial\left(\frac{\partial u}{\partial z}\right)} - \Lambda = -\frac{1}{2}\rho\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}k\left(\frac{\partial u}{\partial z}\right)^2 - \text{the momentum flux density}.
\end{aligned} \tag{2.76}$$

These components satisfy the equations

$$\frac{\partial}{\partial t} W_{tt} + \frac{\partial}{\partial z} W_{tz} = -\frac{1}{2}\left[\frac{\partial \rho}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 - \frac{\partial k}{\partial t}\left(\frac{\partial u}{\partial z}\right)^2\right], \qquad (2.77)$$

$$\frac{\partial}{\partial t} W_{zt} + \frac{\partial}{\partial z} W_{zz} = -\frac{1}{2}\left[\frac{\partial \rho}{\partial z}\left(\frac{\partial u}{\partial t}\right)^2 - \frac{\partial k}{\partial z}\left(\frac{\partial u}{\partial z}\right)^2\right], \qquad (2.78)$$

following directly from (2.1), (2.76).

At the lhs of equation (2.77) we have the rate of increase $\frac{DW_{tt}}{Dt}$ of the energy of a unit segment of the bar; this rate is calculated as the sum of the local change $\frac{\partial W_{tt}}{\partial t}$ and the energy $\frac{\partial W_{tz}}{\partial z}$ that is brought into a unit segment through its endpoints per unit time. The net increase $\frac{DW_{tt}}{Dt}$ is equal to the work

$$-\frac{1}{2}\left[\frac{\partial \rho}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 - \frac{\partial k}{\partial t}\left(\frac{\partial u}{\partial z}\right)^2\right], \qquad (2.79)$$

committed, per unit time, by an external agent against the wave $u(z,t)$ to produce the variable property pattern. Equation (2.77) thus expresses the energy balance in the system; the balance of momentum is reflected in equation (2.78).

In this section, we shall see in detail how the energy-momentum balance manifests itself through homogenization. To this end, we apply the analysis of Section 2.3 in order to find an asymptotic form of equations (2.77), (2.78).

For reasons explained in Section 2.3, the derivatives $\partial/\partial t, \partial/\partial z$ entering these equations should be replaced, respectively, by $d/dt, d/dz$, and these latter derivatives calculated by (2.33). We thus reduce (2.77), (2.78) to the following form:

$$\begin{aligned}(W_{tt})_t + (W_{tz})_z - V\delta^{-1}(W_{tt})_\xi + \delta^{-1}(W_{tz})_\xi \\ = \frac{1}{2}V\delta^{-1}\left[\rho_\xi(u_t - V\delta^{-1}u_\xi)^2 - k_\xi(u_z + \delta^{-1}u_\xi)^2\right],\end{aligned} \qquad (2.80)$$

$$\begin{aligned}(W_{zt})_t + (W_{zz})_z - V\delta^{-1}(W_{zt})_\xi + \delta^{-1}(W_{zz})_\xi \\ = -\frac{1}{2}\delta^{-1}[\rho_\xi(u_t - V\delta^{-1}u_\xi)^2 - k_\xi(u_z + \delta^{-1}u_\xi)^2].\end{aligned} \qquad (2.81)$$

Here, as in Section 2.3, we assume that $u = u(z,t,\xi)$, $\rho = \rho(\xi)$, $k = k(\xi)$, $\xi = (z - Vt)/\delta$, with δ being a small parameter; the symbols $(\cdot)_z$, $(\cdot)_t$, $(\cdot)_\xi$ stand for the relevant partial derivatives. We now introduce an asymptotic expansion (2.32) for $u(z,t,\xi)$; as shown in Section 2.3, the function $u_0(z,t,\xi)$ does not depend on ξ, and the derivatives $u_{1\xi}, u_{2\xi}$ are given, respectively, by (2.38) and (2.40).

Bearing this in mind along with (2.34), we reduce the densities W_{tt} and W_{tz} to the form

$$W_{tt} = \frac{1}{2}\rho(u_{0t} - Vu_{1\xi})^2 + \frac{1}{2}k(u_{0z} + u_{1\xi})^2 + \delta[\rho(u_{0t} - Vu_{1\xi})(u_{1t} - Vu_{2\xi}) + k(u_{0z} + u_{1\xi})(u_{1z} + u_{2\xi})] + \ldots, \quad (2.82)$$

$$W_{tz} = -k(u_{0t} - Vu_{1\xi})(u_{0z} + u_{1\xi}) - \delta k[(u_{0t} - Vu_{1\xi})(u_{1z} + u_{2\xi}) + (u_{0z} + u_{1\xi})(u_{1t} - Vu_{2\xi})] + \ldots \ . \quad (2.83)$$

The expression for W_{zt} is produced if we replace k by $-\rho$ in (2.83); the expression for W_{zz} appears to be negative of W_{tt}.

In (2.82) and (2.83), the dots stand for terms of order δ^2 and higher. We drop such terms because we want to calculate both sides of (2.80) up to terms of order δ^0. We also need the expansions

$$(u_t - V\delta^{-1}u_\xi)^2 = [u_{0t} + \delta u_{1t} + \ldots - V\delta^{-1}(\delta u_{1\xi} + \delta^2 u_{2\xi} + \ldots)]^2 = (u_{0t} - Vu_{1\xi})^2 + 2\delta(u_{0t} - Vu_{1\xi})(u_{1t} - Vu_{2\xi}) + \ldots, \quad (2.84)$$

$$(u_z + \delta^{-1}u_\xi)^2 = (u_{0z} + u_{1\xi})^2 + 2\delta(u_{0z} + u_{1\xi})(u_{1z} + u_{2\xi}) + \ldots \ . \quad (2.85)$$

Now as we apply (2.82)–(2.85) toward (2.80), the latter equation includes terms of order $\delta^{-1}, \delta^0, \delta$, etc. The coefficients of such terms taken on both sides of (2.80) should be equal to each other. We are particularly interested in the coefficients of δ^0 because they carry information about the energy flows as we pass to the limit $\delta \to 0$. The balance of δ^{-1} terms yields the equation

$$-V\frac{\partial}{\partial\xi}\left[\frac{1}{2}\rho(u_{0t} - Vu_{1\xi})^2 + \frac{1}{2}k(u_{0z} + u_{1\xi})^2\right] - \frac{\partial}{\partial\xi}k(u_{0t} - Vu_{1\xi})(u_{0z} + u_{1\xi}) = \frac{1}{2}V\rho_\xi(u_{0t} - Vu_{1\xi})^2 - \frac{1}{2}Vk_\xi(u_{0z} + u_{1\xi})^2, \quad (2.86)$$

whereas the balance of δ^0-terms is expressed by

$$\frac{1}{2}\frac{\partial}{\partial t}\left[\rho(u_{0t} - Vu_{1\xi})^2 + k(u_{0z} + u_{1\xi})^2\right] - \frac{\partial}{\partial z}\left[k(u_{0t} - Vu_{1\xi})(u_{0z} + u_{1\xi})\right] - V\frac{\partial}{\partial\xi}\left[\rho(u_{0t} - Vu_{1\xi})(u_{1t} - Vu_{2\xi}) + k(u_{0z} + u_{1\xi})(u_{1z} + u_{2\xi})\right] - \frac{\partial}{\partial\xi}\left[k(u_{0t} - Vu_{1\xi})(u_{1z} + u_{2\xi}) + k(u_{0z} + u_{1\xi})(u_{1t} - Vu_{2\xi})\right] = V\left[\rho_\xi(u_{0t} - Vu_{1\xi})(u_{1t} - Vu_{2\xi}) - k_\xi(u_{0z} + u_{1\xi})(u_{1z} + u_{2\xi})\right].$$

or,

$$\frac{1}{2}\frac{\partial}{\partial t}(\rho M^2 + kN^2) - \frac{\partial}{\partial z}kMN - V\frac{\partial}{\partial \xi}(\rho MX + kNY)$$
$$-\frac{\partial}{\partial \xi}(kMY + kNX) = V(\rho_\xi MX - k_\xi NY). \tag{2.87}$$

Referring to (2.38)–(2.41), we conclude, after some calculation, that both (2.86) and (2.87) are identically satisfied.

To show this for equation (2.87), represent its lhs as $\phi + \psi$, where

$$\phi = \frac{1}{2}\frac{\partial}{\partial t}(\rho M^2 + kN^2) - V\frac{\partial}{\partial \xi}(\rho MX + kNY)$$

and

$$\psi = -\frac{\partial}{\partial z}(kMN) - \frac{\partial}{\partial \xi}(kMY + kNX);$$

with M, N, X, Y introduced by (2.40)–(2.42).

We have

$$\phi = \rho MM_t + kNN_t - V(\rho_\xi MX + K_\xi NY + \rho M_\xi X + kN_\xi Y + \rho MX_\xi + kNY_\xi).$$

Because $u_{0\xi} = 0$, the terms in this expression that do not depend on ρ_ξ, k_ξ, come from its first two members and from $\rho MX_\xi + kNY_\xi$; by (2.40) and (2.41), the sum of such terms equals

$$\rho\ MM_t + kNN_t\frac{\rho V^2}{V^2\rho - k}(\rho MM_t + kNN_t) + \frac{\rho kV}{V^2\rho - k}\left[(MN)_t + V(MN)_z\right]$$
$$-\frac{kV}{V^2\rho - k}(\rho MM_z + kNN_z) = -\frac{k}{V^2\rho - k}\left[\rho MM_t + kNN_t\right.$$
$$\left.+ V(\rho MM_z + kNN_z)\ \right] + \frac{\rho kV}{V^2\rho - k}\left[(MN)_t + V(MN)_z\right]. \tag{2.88}$$

Similarly, the terms in ψ independent of ρ_ξ, k_ξ are combined in the expression

$$-k(MN)_z - \frac{kM}{V^2\rho - k}\left[\rho V(N_t - M_z) + (-\rho M_t + kN_z)\right] +$$
$$+\frac{kN}{V^2\rho - k}\left[k(N_t - M_z) + V(-\rho M_t + kN_z)\right].$$

After simple algebra this expression is reduced to the negative of (2.88), so the sum $\phi + \psi$ does not include terms independent of ρ_ξ, k_ξ.

We now calculate terms with factors ρ_ξ, k_ξ in $\phi + \psi$; by (2.41), their sum equals $V(\rho_\xi MX - k_\xi NY)$. A similar analysis applies to equation (2.81); for this one, the analog of (2.87) is given by

$$\frac{\partial}{\partial t}(\rho MN) - \frac{1}{2}\frac{\partial}{\partial z}(\rho M^2 + kN^2) - V\frac{\partial}{\partial \xi}(\rho MY + \rho NX)$$
$$-\frac{\partial}{\partial \xi}(\rho MX + kNY) = -(\rho_\xi MX - k_\xi NY) \tag{2.89}$$

As before, the terms at the lhs of (2.89) independent of ρ_ξ, k_ξ vanish. The same holds to terms with factor ρ_ξ, as well as with factors k_ξ.

Before we discuss the specifics of such cancellations, it will be appropriate to clarify the physical meaning of various terms participating in (2.87). The first term at the lhs side of (2.87) expresses the local increase of the energy density of *a slow motion*; by (2.82) and (2.42), this density is calculated as

$$\begin{aligned} T_{tt} &= \frac{1}{2}\left[\rho(u_{0t} - Vu_{1\xi})^2 + k(u_{0z} + u_{1\xi})^2\right] \\ &= \frac{1}{2}\left(\rho M^2 + kN^2\right) = \frac{1}{2}\left[\rho(1-VP)^2 + kP^2\right]u_{0t}^2 \\ &- [\rho VQ(1-VP) - kP(1+Q)]u_{0t}u_{0z} \\ &+ \frac{1}{2}\left[\rho V^2 Q^2 + k(1+Q)^2\right]u_{0z}^2. \end{aligned} \tag{2.90}$$

The second term reflects contribution due to the energy flux density of a slow motion; by (2.83) and (2.42), this density is represented as

$$\begin{aligned} T_{tz} = -k(u_{0t} - Vu_{1\xi})(u_{0z} + u_{1\xi}) = -kMN = -k\{P(1-VP)u_{0t}^2 \\ + [(1-VP)(1+Q) - VPQ]u_{0t}u_{0z} - VQ(1+Q)u_{0z}^2\}. \end{aligned} \tag{2.91}$$

The third term at the lhs of (2.87),

$$\begin{aligned} &- V\frac{\partial}{\partial \xi}\left[\rho(u_{0t} - Vu_{1\xi})(u_{1t} - Vu_{2\xi}) + k(u_{0z} + u_{1\xi})(u_{1z} + u_{2\xi})\right] \\ &= -V\frac{\partial}{\partial \xi}\left[\rho MX + kNY\right], \end{aligned} \tag{2.92}$$

represents the local increase of the energy density of a *fast motion*, and the fourth term

$$\begin{aligned} &- \frac{\partial}{\partial \xi}\left[k(u_{0t} - Vu_{1\xi})(u_{1z} + u_{2\xi}) + k(u_{0z} + u_{1\xi})(u_{1t} - Vu_{2\xi})\right] \\ &= -\frac{\partial}{\partial \xi}[k(MY + NX)] \end{aligned} \tag{2.93}$$

reflects contribution due to the energy flux density of such a motion. By the last equation of Section 2.3 we conclude that the averaged values (over period 1) of the terms (2.92) and (2.93) responsible for a fast motion are both equal to zero. As to the rhs of (2.87), it defines the work produced, per unit time, by an external agent against the variable property pattern. This agent is responsible for an external force working against elastic deformations.

When we implement differentiation $\partial/\partial\xi$ in (2.92) and (2.93) and refer to (2.42) and (2.40), there emerge terms with factors ρ_ξ, k_ξ, as well as the terms without such factors. The factored terms are counter-balanced by the rhs of (2.87). The remaining (nonfactored) terms precisely match the expressions in (2.87) generated by (2.88) and (2.91) combined.

All of those reductions occur termwise, with no averaging operation applied whatsoever. The relevant (somewhat cumbersome) calculation is left to the reader.

These observations show that the work of an external force produced over a period is equal to the net increase of the energy of a slow motion.

2.7 Averaged and Effective Energy and Momentum

So far we never referred to the homogenized equation (2.43). This equation appears to be an Euler equation produced by an *effective action density*

$$\bar{\Lambda} = \frac{1}{2}(ru_{0t}^2 + 2qu_{0t}u_{0z} - pu_{0z}^2). \tag{2.94}$$

As in the beginning of Section 2.6, this function generates components of an *effective energy-momentum tensor* $\bar{W}$:

$$\begin{aligned}
\bar{W}_{tt} &= u_{0t}\,\frac{\partial\bar{\Lambda}}{\partial u_{0t}} - \bar{\Lambda} = \frac{1}{2}(ru_{0t}^2 + pu_{0z}^2),\\
\bar{W}_{tz} &= u_{0t}\,\frac{\partial\bar{\Lambda}}{\partial u_{0z}} = qu_{0t}^2 - pu_{0t}u_{0z},\\
\bar{W}_{zt} &= u_{0z}\,\frac{\partial\bar{\Lambda}}{\partial u_{0t}} = ru_{0t}u_{0z} + qu_{0z}^2,\\
\bar{W}_{zz} &= u_{0z}\,\frac{\partial\bar{\Lambda}}{\partial u_{0z}} - \bar{\Lambda} = -\frac{1}{2}(ru_{0t}^2 + pu_{0z}^2).
\end{aligned} \tag{2.95}$$

These components satisfy the system

$$\begin{aligned}
\frac{\partial}{\partial t}\bar{W}_{tt} + \frac{\partial}{\partial z}\,\bar{W}_{tz} = 0,\\
\frac{\partial}{\partial t}\,\bar{W}_{zt} + \frac{\partial}{\partial z}\,\bar{W}_{zz} = 0,
\end{aligned} \tag{2.96}$$

following from (2.43).

Contrary to (2.77), (2.78), the system (2.96) has zero rhs because the coefficients r, q, p in (2.43) are constant while the ρ, k in (2.1) are ξ-dependent. Equation (2.96) therefore express conservation of both energy and momentum for an *effective* motion governed by (2.43). We want to see how the system (2.96) is linked with (2.77), (2.78). To this end, we first rewrite the expression (2.94) for the effective action density in a more convenient form.

Consider the expression

$$\frac{1}{2}[\rho(u_{0t} - Vu_{1\xi})^2 - k(u_{0z} + u_{1\xi})^2] = \frac{1}{2}(\rho M^2 - kN^2). \tag{2.97}$$

Referring to (2.42), we transform it to

$$\frac{1}{2}\left\{u_{0t}^2[\rho(1-VP)^2 - kP^2] - 2u_{0t}u_{0z}[\rho VQ(1-VP) + kP(1+Q)]\right.$$
$$\left. - \ u_{0z}^2[k(1+Q)^2 - \rho V^2Q^2]\right\}.$$

By direct inspection and with reference to (2.39), we get

$$\rho(1-VP)^2 - kP^2 = \frac{1}{V^2\rho - k}\left(\frac{B^2}{C^2}\,V^2 - k\rho\right),$$
$$-[\rho VQ(1-VP) + kP(1+Q)] = \frac{V}{V^2\rho - k}\left(\frac{AB}{C^2} - k\rho\right),$$
$$k(1+Q)^2 - \rho V^2Q^2 = \frac{1}{V^2\rho - k}\left(k\rho V^2 - \frac{A^2}{C^2}\right).$$

Now, by averaging both sides of every equation over the period 1 in ξ and by referring to (2.12), (2.14), and (2.15), we write

$$\langle\rho(1-VP)^2 - kP^2\rangle = \frac{B^2V^2}{C} - D = r,$$
$$-\langle\rho VQ(1-VP) + kP(1+Q)\rangle = V\left(\frac{AB}{C} - D\right) = q,$$
$$\langle k(1+Q)^2 - \rho V^2Q^2\rangle = V^2D - \frac{A^2}{C} = p. \tag{2.98}$$

Combining (2.95), (2.96), and (2.98), we finally obtain

$$\frac{1}{2}\langle\rho M^2 - kN^2\rangle = \frac{1}{2}(ru_{0t}^2 + 2qu_{0t}u_{0z} - pu_{0z}^2),$$

i.e., the effective action density (2.94). We now apply equation (2.95) to calculate the components $\bar{W}_{tt,...}$ of the effective energy momentum tensor.

The component $\bar{W}_{tt}$ may be interpreted as an *effective energy density* represented in the laboratory frame. Given (2.98), this density takes on the form

$$\bar{W}_{tt} = \frac{1}{2}\langle\rho(1-VP)^2 - kP^2\rangle u_{0t}^2 + \frac{1}{2}\langle k(1+Q)^2 - \rho V^2Q^2\rangle u_{0z}^2. \tag{2.99}$$

We now observe that an effective energy density $\bar{W}_{tt}$ is generally *not equal* to the *averaged energy density* $\langle T_{tt}\rangle$ of the slow motion calculated as (c.f. (2.88))

$$\langle T_{tt}\rangle = \frac{1}{2}\langle \rho(1-VP)^2 + kP^2\rangle u_{0t}^2 - \langle \rho VQ(1-VP) - kP(1+Q)\rangle u_{0t}u_{0z}$$
$$+ \frac{1}{2}\langle k(1+Q)^2 + \rho V^2Q^2\rangle u_{0z}^2.$$

The difference between the two densities

$$\bar{W}_{tt} - \langle T_{tt}\rangle = -\langle kP^2\rangle u_{0t}^2 + \langle \rho VQ(1-VP) - kP(1+Q)\rangle u_{0t}u_{0z}$$
$$- \langle \rho V^2Q^2\rangle u_{0z}^2 = \langle \rho VQMu_{0z} - kPNu_{0t}\rangle$$

vanishes when $V = 0$, i.e., for a static laminate.

This difference is non-zero because of a temporal activation. We therefore expect that if we go to a *co-moving* coordinate frame (2.8) in which an interface between layers in an activated composite remains immovable, then the difference between $\bar{W}_{\tau\tau}$ and $\langle T_{\tau\tau}\rangle$ evaluated for this system may vanish.

In the frame (2.8), the components of the energy-momentum tensor are expressed by the formulae:

$$\begin{aligned}
W_{\tau\tau} &= W_{tt} + VW_{zt},\\
W_{\tau\zeta} &= W_{tz} + VW_{zz} - V\left(W_{tt} + VW_{zt}\right),\\
W_{\zeta\tau} &= W_{zt},\\
W_{\zeta\zeta} &= W_{zz} - VW_{zt}.
\end{aligned} \tag{2.100}$$

An asymptotic expression for W_{zt} is produced, as mentioned above, if we replace k by $-\rho$ in (2.83). The term T_{zt} in this expression will be defined as ρMN; it takes the form (c.f. (2.91))

$$\begin{aligned}
T_{zt} = \rho MN = \rho\{&P(1-VP)u_{0t}^2 + [(1-VP)(1+Q) - VPQ]\\
&u_{0t}u_{0z} - VQ(1+Q)u_{0z}^2\};
\end{aligned} \tag{2.101}$$

we may call it the momentum density of a slow motion.

Introduce the quantity similar to $W_{\tau\tau}$:

$$T_{\tau\tau} = T_{tt} + VT_{zt};$$

by direct inspection, with reference to (2.12), (2.15). (2.39), (2.99), and (2.100), we show that

$$\langle T_{\tau\tau}\rangle = \bar{W}_{\tau\tau}.$$

By a similar argument, for the quantity

$$T_{\tau\zeta} = T_{tz} + VT_{zz} - V(T_{tt} + VT_{zt}),$$

we obtain

$$\langle T_{\tau\zeta}\rangle = \bar{W}_{\tau\zeta}.$$

The effective energy density (flux) thus appears to be the same as the averaged energy density (flux) of a slow motion in a co-moving coordinate frame in which the interface remains immovable. We could expect that if we noticed that

$$\frac{\partial}{\partial t}\left(W_{tt}+VW_{zt}\right)+\frac{\partial}{\partial z}\left(W_{tz}+VW_{zz}\right)=0,$$

because of (2.77), (2.78), and due to a supposed dependency of ρ and k on the argument $\xi = z - Vt$ alone. The previous equation is rewritten in a co-moving frame as

$$\frac{\partial W_{\tau\tau}}{\partial \tau}+\frac{\partial W_{\tau\zeta}}{\partial \zeta}=0; \tag{2.102}$$

it shows that the energy is preserved in this frame.

The conservation of energy in a co-moving frame follows from the Noether's theorem applied to the variational principle of stationary action (2.75). For a laminate, the material coefficients ρ and k depend in this frame on ξ alone, and do not depend on the new time variable τ.

Unlike this, the momentum equation (c.f.(2.78)) has non-zero rhs in a co-moving frame, i.e., there is no conservation of momentum. Such conservation holds in a *different* space-time frame; in this one, however, the energy is not preserved.

We will now discuss the momentum equation taking the form (2.81) in a laboratory frame. Following remarks made after Eqn. (2.83), we express the balance of δ^0-terms:

$$\begin{aligned}&\frac{\partial}{\partial t}\rho MN-\frac{1}{2}\frac{\partial}{\partial z}(\rho M^2+kN^2)-V\frac{\partial}{\partial \xi}[\rho(MY+NX)]-\frac{\partial}{\partial \xi}(\rho MX+kNY)\\&=-(\rho_\xi MX-k_\xi NY).\end{aligned} \tag{2.103}$$

As for the energy equation, we use the momentum density T_{zt} of a slow motion defined by (2.101), as well as the momentum flux density T_{zz} of the same motion defined as (see (2.88))

$$T_{zz}=-T_{tt}=-\frac{1}{2}(\rho M^2+kN^2).$$

The third term at the lhs of (2.103) represents contribution due to the momentum density of a fast motion whereas the fourth term reflects a similar contribution produced by the momentum flux density. The averaged (over period 1) values of both terms are equal to zero, just as the averaged values for similar terms in (2.87).

Desiring to arrive at the momentum equation similar to (2.102), we introduce, instead of (2.8), the Galilean frame

$$\eta = z, \quad \theta = t - V^{-1}z. \tag{2.104}$$

The material interface $\theta = const$ becomes temporal in this frame. The transformation formulae for the energy-momentum tensor are similar to (2.100); they read

$$\begin{aligned} W_{\theta\theta} &= W_{tt} - V^{-1} W_{tz}, \\ W_{\theta\eta} &= W_{tz}, \\ W_{\eta\theta} &= W_{zt} + V^{-1} W_{tt} - V^{-1}\left(W_{zz} + V^{-1} W_{tz}\right), \\ W_{\eta\eta} &= W_{zz} + V^{-1} W_{tz}. \end{aligned}$$

The momentum equation in this frame replaces that of (2.89) in the laboratory frame. We get

$$\begin{aligned} \frac{\partial W_{\eta\theta}}{\partial \theta} + \frac{\partial W_{\eta\eta}}{\partial \eta} &= \frac{\partial}{\partial t}\left[W_{zt} + V^{-1} W_{tt} - V^{-1}(W_{zz} + V^{-1} W_{tz})\right] \\ &+ \frac{\partial}{\partial z}(W_{zz} + V^{-1} W_{tz}) + V^{-1}\frac{\partial}{\partial t}(W_{zz} + V^{-1} W_{tz}) = \frac{\partial}{\partial t}(W_{zt} + V^{-1} W_{zz}) \\ &+ \frac{\partial}{\partial z}(W_{zz} + V^{-1} W_{tz}) = 0. \end{aligned} \tag{2.105}$$

This equation means conservation of *momentum* in the frame (2.104), just as the equation (2.102) expresses conservation of *energy* in the frame (2.8).

These observations are related to *exact* values of energy and momentum; in either selected frame, there is no simultaneous conservation of both quantities. Contrary to that, their *effective* values (2.95) are *both preserved*, according to equation (2.96), in any Galilean frame that is *regular*, i.e., consistent with Ineq, (2.7). To show this, multiply both sides of (2.89) by V and add it termwise to (2.87). After averaging the sum over the period 1 in $\xi = (z - Vt)/\epsilon$, we arrive at

$$\left\langle \left[\frac{1}{2}(\rho M^2 + kN^2) + \rho V M N\right]_t - \left[kMN + \frac{1}{2}V(\rho M^2 + kNY\right]_z \right\rangle = 0; \tag{2.106}$$

here, we applied relations $X(0) = X(1) = Y(0) = Y(1) = 0$ mentioned at the end of Section 2.3.

A simple algebra reduces the expression in the corner brackets to the form

$$(\rho M_t - kN_z)(M + VN) + (N_t - M_z)(kN + \rho V M).$$

Referring to equations (2.42) and (2.39), we show that

$$\begin{aligned} M + VN &= u_{0t} + V u_{0z}, \\ kN + \rho V M &= u_{0t} V \frac{B}{C} + u_{0z}\frac{A}{C}, \end{aligned}$$

i.e., these quantities are ξ-independent. For this reason, the averaging in (2.106) becomes possible, and this equation takes the form

$$\langle \rho M_t - kN_z\rangle (u_{0t} + V u_{0z}) + \langle N_t - M_z\rangle \left(u_{0t} V \frac{B}{C} + u_{0z}\frac{A}{C}\right) = 0.$$

As shown at the end of Section 2.3, the factor $\langle N_t - M_z \rangle$ identically vanishes.
We arrive at

$$(ru_{0tt} + 2qu_{ozt} - pu_{0zz})(u_{0t} + Vu_{0z}) = 0,$$

which yields (2.43) because the interface is not a characteristic of this equation. A direct calculation

$$(\bar{W}_{tt})_t + (\bar{W}_{tz})_z = u_{0t}(ru_{0tt} + 2qu_{0zt} - pu_{0zz}),$$

$$(\bar{W}_{zt})_t + (\bar{W}_{zz})_z = u_{0z}(ru_{0tt} + 2qu_{0zt} - pu_{0zz}),$$

shows that equation (2.96) follow from (2.106). In other words, the *effective* energy and momentum are both preserved in a laboratory frame.

2.8 Homogenization of Regular Activated Laminates: Theoretical Motivation

As mentioned in Remark 2.1.2, homogenization is of limited significance for the study of wave equation with variable coefficients. It does not work not only for irregular geometries, but also for some regular situations, such as the checkerboard formation in space-time discussed in Chapter 5. With this background, a successful application of homogenization in Sections 2.2–2.7 to spatio-temporal regular laminates appears to be noteworthy. Below in this section, we explain the reasons for this exception.

The functions u, v in the homogenized system (2.13) are the weak limits of sequences u_n, v_n, that appear in a standard homogenization scheme [1]. This scheme is known to work when the original fields demonstrate the weak compactness, and this property is established as a consequence of Friedrichs inequality. Such inequality follows from the boundedness of energy (or any other positive definite quadratic functional of the derivatives playing the role of energy). Fortunately, such functionals *exist in the case of dynamic laminates.* For the case of *slow* laminates $|V| < a_1$, this has been shown in [9]; for the case of *fast* laminates $|V| > a_2$, the quadratic functional was introduced in [6] and will be demonstrated below.

(i) Case of slow laminates
The wave equation (2.1) with coefficients ρ, k dependent on $(z - Vt)/\epsilon$ is transformed as shown in Section 2.2 to the new variables (2.8)

$$\zeta = z - Vt, \quad \tau = t.$$

The Lagrangian

$$\Lambda = \frac{1}{2}\rho u_t^2 - \frac{1}{2}ku_z^2 = \frac{1}{2}\rho u_\tau^2 - Vu_\tau u_\zeta + \frac{1}{2}(\rho V^2 - k)u_\zeta^2 \qquad (2.107)$$

does not explicitly depend on τ, and the laminate in the frame (2.8) becomes static (immovable). By a standard technique we obtain the *energy* equation

$$\frac{\partial W_{\tau\tau}}{\partial \tau} + \frac{\partial W_{\tau\zeta}}{\partial \zeta} = 0,$$

where $W_{\tau\tau}, W_{\tau\zeta}$ are, respectively, the energy density and the energy flux density in the frame (2.8), calculated as (see(2.95))

$$W_{\tau\tau} = u_\tau \frac{\partial \Lambda}{\partial u_\tau} - \Lambda = \frac{1}{2}\rho u_\tau^2 + \frac{1}{2}(k - \rho V^2)u_\zeta^2, \tag{2.108}$$

$$W_{\tau\zeta} = u_\tau \frac{\partial \Lambda}{\partial u_\zeta} = -\rho V u_\tau^2 - (k - \rho V^2)u_\tau u_\zeta.$$

It is easy to check that $W_{\tau\zeta}$ is continuous across the layers' interfaces $\zeta = \text{const}$; this property together with (2.102) yields

$$\frac{d}{d\tau}\int_{-\infty}^{\infty} W_{\tau\tau} d\zeta = W_{\tau\zeta}\Big|\begin{array}{l}\zeta = -\infty \\ \zeta = \infty\end{array};$$

the rhs vanishes if the field disappears at $z = \pm\infty$. Therefore, the total energy

$$E(\tau) = \int_{-\infty}^{\infty} W_{\tau\tau} d\zeta$$

measured in the frame (2.8) is preserved in time $\tau : E(\tau) = E_0 = \text{const}$. The constant E_0 is positive by (2.108) because $k - \rho V^2 > 0$ for a slow laminate. We conclude that there exists a positive constant μ independent of ϵ such that

$$E_0 > \frac{1}{\mu}\int_{-\infty}^{\infty}(u_\tau^2 + u_\zeta^2)d\zeta.$$

This means that there is a subsequence u_n weakly convergent to u, and homogenization with respect to ζ becomes possible.

(ii) Case of fast laminates

To discuss this case, apply the frame (2.104)

$$\eta = z, \quad \theta = t - \frac{z}{V},$$

with the Lagrangian expressed as

$$\Lambda = \frac{1}{2}\rho u_t^2 - \frac{1}{2}k u_z^2 = \frac{1}{2}(\rho - \frac{k}{V^2})u_\theta^2 + \frac{k}{V}u_\theta u_\eta - \frac{1}{2}k u_\eta^2.$$

This expression does not depend explicitly upon η, and the lamination in coordinates (2.104) becomes purely temporal (dependent upon θ). By a standard technique, we arrive at the *momentum* equation (2.105)

$$\frac{\partial W_{\eta\theta}}{\partial \theta} + \frac{\partial W_{\eta\eta}}{\partial \eta} = 0,$$

with $W_{\eta\theta}, W_{\eta\eta}$ being, respectively, the momentum density and the momentum flux density in coordinates (2.104). We calculate them as

$$W_{\eta\theta} = u_\eta \frac{\partial L}{\partial u_\theta} = \frac{k}{V} u_\eta^2 + (\rho - \frac{k}{V^2}) u_\theta u_\eta,$$

$$W_{\eta\eta} = -\frac{1}{2}(\rho - \frac{k}{V^2}) u_\theta^2 - \frac{1}{2} k u_\eta^2. \tag{2.109}$$

From (2.105) and due to the continuity of $W_{\eta\theta}$ across the interfaces $\theta =$ const, we obtain

$$\frac{d}{d\eta} \int_{-\infty}^{\infty} W_{\eta\eta} d\theta = W_{\eta\theta} \bigg| \begin{matrix} \theta = -\infty \\ \theta = \infty \end{matrix} \;\; ;$$

the rhs vanishes if the field disappears at $z = \pm\infty$ for fixed t.
The net momentum flux

$$M(\eta) = \int_{-\infty}^{\infty} W_{\eta\eta} d\theta$$

in the frame (2.104) is therefore independent of $\eta : M(\eta) = M_0 =$ const. This constant is negative because, by (2.109), $\rho V^2 - k > 0$ for a fast laminate. We now conclude that there is a positive constant ν independent of ϵ such that

$$M_0 < -\frac{1}{\nu} \int_{-\infty}^{\infty} (u_\theta^2 + u_\eta^2) d\theta,$$

or, since $M_0 < 0$,

$$\int_{-\infty}^{\infty} (u_\theta^2 + u_\eta^2) d\theta < -\nu M_0.$$

This guarantees the existence of a subsequence u_n, weakly convergent to u, and homogenization with respect to θ becomes possible. The formulae in Section 2.2 for the effective constants apply to both slow and fast laminates. We conclude that weak compactness property substantial for justification of the homogenization procedure holds with respect to the local fields dependent on the fast variable $(z - Vt)/\epsilon$ in both slow and fast laminates. A laminar microstructure is in this context special because it supports, in appropriate frames, the conservation laws for the original energy (momentum) expressed by equations (2.102) and (2.105), respectively. For a general microstructure, however, the energy-momentum tensor does not satisfy any visible conservation laws, and the homogenization procedure, in its standard version based on weak compactness, cannot, generally speaking, be justified. As mentioned before, this statement will receive confirmation in Chapter 5 through the analysis of a rectangular checkerboard structure in 1D-space and time: a disturbance traveling through such a structure is able to demonstrate the exponential growth of energy.

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