1 Prooemium Relations

It is idle to talk always of the alternative of reason and faith. Reason is itself a matter of faith. It is an act of faith to assert that our thoughts have any relation to reality at all.

— G.K. Chesterton (1908)
Orthodoxy
Chapter III. "The Suicide of Thought"

Let me begin with a parody of a few passages from the Prologomenon of *RL*. Expository divergence is, however, imminent …

Sets

1.1 Subset and Superset If A and B are sets and if every element of A is an element of B, then A is a *subset* of B, and B is a *superset* of A, denoted

$$A \subset B \quad (\text{equivalently,} \quad B \supset A). \tag{1}$$

Note that this symbolism of containment means *either* $A = B$ (which means the sets A and B have the same elements; Axiom of Extension, *ML*: 0.2) *or* A is a *proper subset* of B (which means that B contains at least one element that is not in A). Two sets A and B are equal if and only if $A \subset B$ and $B \subset A$ (*ML*: 0.4).

1.2 Inclusion Map For $A \subset B$, the mapping $i : A \to B$ defined by $i(a) = a$ for all $a \in A$ is called the *inclusion map* (of A in B). If the sets involved need to be emphasized, one may use the notation $i_{A \subset B}$ for the inclusion map. The inclusion map of A in A is called the *identity map* on A, denoted 1_A $(= i_{A \subset A})$.

A. H. Louie, *Intangible Life*, Anticipation Science 2,
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1.3 Equipotence Two sets are *equipotent* (to each other) if there exists a bijective mapping, i.e., a one-to-one correspondence, between them.

1.4 Cardinality A set is *finite* if it is either empty or equipotent to the set $\{0, 1, 2, \ldots, n-1\}$ for a natural number n; otherwise it is *infinite*. An infinite set that is equipotent to the set $\mathbb{N}$ of all natural numbers is called *countably infinite*; otherwise the infinite set is *uncountable*. The term *countable* means either finite or countably infinite.

With the formal definition $0 = \varnothing$ and $n = \{0, 1, 2, \ldots, n-1\}$, a finite set is 'equipotent to a whole number'. Each finite set X is equipotent to a *unique* whole number $|X| = n \in \mathbb{N}_0$, the 'number of elements of X'. In short, a finite set is a set consisting of a finite number of elements.

The property that each finite set is equipotent to a unique whole number may be extended to infinite sets. The generalized 'number of elements' of a set is called its *cardinality*, and formally one has the

1.5 Property Every set is equipotent to a unique *cardinal number*.

The usual partial order $\leq$ of whole numbers may be extended to all cardinal numbers. One uses the same notation $|X| = n$ for the cardinality of the set X, where n may be an 'infinite cardinal' in addition to a whole number. Infinite cardinal numbers are usually denoted by the first letter $\aleph$ (*aleph*) of the Hebrew alphabet, ordered by a nonnegative integer subscript. When $|X| = n$, one may simply say 'X has cardinal number n' or 'X has cardinality n'.

1.6 Theorem

i. *Every set has a cardinal number.*
ii. *Two sets A and B are equipotent if and only if they have the same cardinal number, i.e., iff $|A| = |B|$.*
iii. $|A| \leq |B|$ *if and only if A is equipotent to a subset of B (which includes the special case when $A \subset B$).*
iv. $|A| < |B|$ *if and only if A is equipotent to a subset of B but B is not equipotent to a subset of A.*

1.7 Corollaries

i. *Every finite set has a unique number of elements.*
ii. *Two finite sets are equipotent if and only if they have the same number of elements.*
iii. *If a set is finite, then every one of its subsets is finite.*
iv. *If a finite set X has n elements and a subset $A \subset X$ has k elements, then $k \leq n$; further, $k = n$ iff $A = X$.*
v. *If a set is finite, then it is not equipotent to any of its proper subsets.*

Property v, that a finite set is not equipotent to any of its proper subsets, in fact characterizes finite sets. The inverse thus characterizes infinite sets; stated formally:

1.8 Theorem

i. *A set is infinite if and only if it is equipotent to a proper subset of itself.*
ii. *A set is finite if and only if it is not equipotent to any proper subset of itself.*

Sets from Sets

1.9 Complements The *relative complement* of a set A in a set B is the set of elements in B but not in A:

$$B \sim A = \{x \in B : x \notin A\}. \tag{2}$$

Note that this definition does not require that $A \subset B$, and one has $B \sim A = B \sim (A \cap B) \subset B$. If $A \subset B$ and B is finite (whence A also), then $|B \sim A| = |B| - |A|$. When B is the 'universal set' U (of some appropriate universe under study, e.g. the set of all natural systems **N** or the 'largest set' in some field of sets), the set $U \sim A$ is denoted A^c, i.e.

$$A^c = \{x \in U : x \notin A\}, \tag{3}$$

and is called simply the *complement* of the set A. An element of U is either a member of A, or not a member of A, but not both. That is, $A \cup A^c = U$, and $A \cap A^c = \varnothing$.

1.10 De Morgan's Laws Union and intersection interchange under complementation: for sets $\{A_i\}_{i \in I}$ and B,

$$\begin{aligned} B \sim \left(\bigcup_{i \in I} A_i \right) &= \bigcap_{i \in I} (B \sim A_i) \\ B \sim \left(\bigcap_{i \in I} A_i \right) &= \bigcup_{i \in I} (B \sim A_i). \end{aligned} \tag{4}$$

1.11 Inclusion–Exclusion Principle The familiar counting equality for finite sets,

$$(5) \qquad |A \cup B| = |A| + |B| - |A \cap B|,$$

implies, in particular, the inequality $|A \cup B| \leq |A| + |B|$, with $|A \cup B| = |A| + |B|$ iff $|A \cap B| = |\varnothing| = 0$ (i.e., iff sets A and B are *disjoint*). The results generalize for finite sets $A_1, A_2, \ldots, A_n$ to

$$(6) \qquad \begin{aligned} |A_1 \cup A_2 \cup \cdots \cup A_n| = {} & |A_1| + |A_2| + \cdots + |A_n| \\ & - |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots - |A_{n-1} \cap A_n| \\ & + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| \\ & \qquad + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n| \\ & \vdots \\ & + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|, \end{aligned}$$

which may be succinctly written as

$$(7) \qquad \left| \bigcup_{i=1}^{n} A_i \right| = \sum_{k=1}^{n} (-1)^{k-1} \left(\sum_{1 \leq i_1 < \cdots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| \right).$$

Further, $\left| \bigcap_{i=1}^{n} A_i \right| \leq \sum_{i=1}^{n} |A_i|$, with equality iff the sets $A_1, A_2, \ldots, A_n$ are pairwise disjoint.

1.12 Power Set If X is a set, the *power set* $\wp X$ of X is the family of all subsets of X.

The inclusion relation $\subset$ is a *partial order* on the power set $\wp X$; i.e., $\langle \wp X, \subset \rangle$ is a *poset* (*ML*: 1.22). The least element of $\langle \wp X, \subset \rangle$ is $\varnothing$, and the greatest element of $\langle \wp X, \subset \rangle$ is X (*ML*: 1.28). Note that even when $X = \varnothing$, $\varnothing \in \wp X$ (indeed, $\wp X = \{\varnothing\}$) so $\wp X \neq \varnothing$. $\langle \wp X, \cup, \cap \rangle$ is a complete, complemented *lattice* (*ML*: 2.1, 2.12, 3.12). $\langle \wp X, \cup, \cap, {}^c \rangle$ is a Boolean algebra (*ML*: 3.19), called the *power set algebra* of X. A *field of sets* is a subalgebra of a power set algebra. The power set algebra is, indeed, the 'universal' Boolean algebra, in the sense that every Boolean algebra is isomorphic to a field of sets (Stone Representation Theorem, *ML*: 3.20).

1.13 Characteristic Mapping A subset A of X may be identified with its *characteristic mapping*, a mapping χ_A from X to $2 = \{0, 1\}$ defined by

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}. \tag{8}$$

When X is a finite set with n members, there are 2^n different mappings $\chi : X \to 2$, because for each element $x \in X$ there are precisely two choices for the value $\chi(x)$, either 0 or 1. If one defines $A = \chi^{-1}(1) \subset X$, then $\chi = \chi_A$.

1.14 Cardinality of the Power Set Thus if $|X| = n$, then $|\mathrm{P}X| = 2^n$, and the equality may be extended to all cardinal numbers n, finite and infinite. This gives an alternate notation of the power set $\mathrm{P}X$ as 2^X. One may succinctly write

$$|\mathrm{P}X| = \left|2^X\right| = 2^{|X|}. \tag{9}$$

This is consistent even if $X = \varnothing$, when $|X| = 0$ and $|\mathrm{P}X| = 2^0 = 1$. Cantor's Theorem (*RL*: 0.8) states that, for all sets X, $|X| < 2^{|X|}$.

The equivalent notation $\mathrm{P}X = 2^X$ expressing the power set as a 'power' is, of course, the origin of its name.

> **power** (noun): from Old French *poeir*, from Vulgar Latin *potere*, a variant of Classical Latin *posse* "to be able". The Indo-European root is *poti-* "powerful; lord". If you are able to do many things, you are powerful. A powerful person typically has a large number of possessions (a word derived from *posse*) and a large amount of money. In algebra, when even a relatively small number like 2 is multiplied by itself a number of times the result gets large very quickly; metaphorically speaking, the result is powerful. … If the term *power* is used precisely, it refers to the result of multiplying a number by itself a certain number of times. Consider $2^3 = 8$, which says that the 3rd power of 2 is 8. The power is 8. In less precise usage, however, 3 is identified as the power, when it is actually the exponent.
>
> — Steven Schwartzman (1994)
> *The Words of Mathematics: An Etymological Dictionary of Mathematical Terms Used in English*

1.15 Product Given two sets X and Y, one denotes by $X \times Y$ the set of all *ordered pairs* of the form (x, y) where $x \in X$ and $y \in Y$. The set $X \times Y$ is called the *product* (or *Cartesian product*) of the sets X and Y. If either X or Y is empty, then $X \times Y = \varnothing$.

For all sets X and Y, the cardinality of the product set is the product of the cardinalities of the components:

$$|X \times Y| = |X||Y|. \tag{10}$$

1.16 Projections The mappings

$$\pi_1 : X \times Y \to X \quad \text{and} \quad \pi_2 : X \times Y \to Y, \tag{11}$$

defined, for $x \in X$ and $y \in Y$, by

$$\pi_1(x, y) = x \quad \text{and} \quad \pi_2(x, y) = y, \tag{12}$$

are the *canonical projections* (of the product $X \times Y$ onto its components; cf. *ML*: A.22).

For $A \subset X$, the set $\pi_1^{-1}(A)$ of the *inverse image* of A is the subset of $X \times Y$ containing all ordered pairs (x, y) that are sent by π_1 into A:

$$\pi_1^{-1}(A) = \{(x, y) \in X \times Y : \pi_1(x, y) = x \in A\} = A \times Y. \tag{13}$$

Similarly, for $B \subset Y$,

$$\pi_2^{-1}(B) = \{(x, y) \in X \times Y : \pi_2(x, y) = y \in B\} = X \times B. \tag{14}$$

The product set $A \times B \subset X \times Y$ may be identified with the set $\pi_1^{-1}(A) \cap \pi_2^{-1}(B)$ of intersection of inverse images, since

$$\pi_1^{-1}(A) \cap \pi_2^{-1}(B) = (A \times Y) \cap (X \times B) = A \times B. \tag{15}$$

Relations

1.17 Definition A A *relation R* is an ordered triple (X, Y, Γ) where X and Y are sets and Γ is a subset of the Cartesian product $X \times Y$. The sets X and Y are respectively called the *domain* and *codomain* of the relation, and $\Gamma \subset X \times Y$ is called its *graph*.

One may indicate the dependence of X, Y, and Γ on R with the notations $X = \mathrm{dom}(R)$, $Y = \mathrm{cod}(R)$, and $\Gamma(R)$.

According to the formal Definition 1.17, a relation uniquely determines its domain and codomain, so two relations with identical graphs but different domains or different codomains are considered different. [This is, indeed, the category-theoretic requirement that a morphism uniquely entails its domain and codomain; Definitions 0.1, 0.2, and cf. *ML*: A.1; *RL*: 6.7 et seq.] Consider the simple example $\Gamma = \{(2,\mathrm{A}),(1,\mathrm{C}),(2,\mathrm{B})\}$. The relations $R_1 = (\{1,2,3,4,5\}, \{\mathrm{A},\ \mathrm{B},\ \mathrm{C},\ \mathrm{D},\ \mathrm{E},\ \mathrm{F}\},\Gamma)$, $R_2 = (\mathbb{N}, \text{alphanumeric characters}, \Gamma)$, $R_3 = (\mathbb{Z}, \text{Latin alphabet}, \Gamma)$, and $R_4 = (\mathbb{R}, \{\mathrm{A},\ \mathrm{B},\ \mathrm{C}\}, \Gamma)$ are all distinct.

A relation is often identified with its graph (hence the minor equivocation $R = \Gamma(R)$), so one also has the (more common but less rigorous)

1.18 Definition B A *relation* is a set R of ordered pairs; i.e. $R \subset X \times Y$ for some sets X and Y.

Equivalently, a relation R is an element of the power set $\mathfrak{P}(X \times Y)$, i.e., $R \in \mathfrak{P}(X \times Y)$. With domain X and codomain Y, the relation R is *from* X *to* Y. The collection of *all* relations from X to Y is thus the power set $\mathfrak{P}(X \times Y)$, and, in view of (9) and (10) above, the cardinality of this collection is

$$|\mathfrak{P}(X \times Y)| = 2^{|X \times Y|} = 2^{|X||Y|}. \tag{16}$$

If $(x,y) \in R$ (or more precisely $(x,y) \in \Gamma(R)$), then one may say that x *is* R*-related to* y (or simply x *is related to* y when the involved relation R is understood).

There is a chirality inherent in $(x,y) \in R \subset X \times Y$. When $X \neq Y$, the asymmetry between a relation from X to Y and a relation from Y to X are apparent. But even when $R \subset X \times X$ (whence $\mathrm{dom}(R) = \mathrm{cod}(R) = X$ and one says R is a *relation on* X), $(x,y) \in R$ and $(y,x) \in R$ (for $x,y \in X$) are independent statements. (See *ML*: 1.9 et seq. for an exposition of the epistemological consequences of relations on X.) To emphasize the chirality inherent in $(x,y) \in R$, one may also say that x is a *left* R*-relative* (*left relative*) of y, and that y is a *right* R*-relative* (*right relative*) of x.

1.19 External and Internal Entailments Note that even in the formulation 1.18, a relation still has to uniquely determine its domain and codomain, although these two sets cannot be induced from the components of the ordered pairs that are members of $\Gamma(R) = R$. In the example in 1.17, when $\Gamma(R) = \{(2, \mathrm{A}), (1, \mathrm{C}), (2, \mathrm{B})\}$, from $\Gamma(R)$ itself one may only conclude that $X = \mathrm{dom}(R)$ must be a superset of $\{1, 2\}$ and $Y = \mathrm{cod}(R)$ must be a superset of $\{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$; no more information is forthcoming *internally* from $\Gamma(R) = R$.

Thus the domain X and the codomain Y of a relation R have to be *externally* supplied; they are objects extraneous to the graph Γ of R. It is therefore more satisfactory (and more accurate) to apply the term 'relation' to the ordered triple $(X, Y, \Gamma) = R$ rather than $\Gamma = R \subset X \times Y$. But if this sort of thing were systematically done, the mathematical notation would become rather cumbersome. Let me quote Rudin [1986: 1.21] on this issue:

> Most mathematical systems are sets with some class of distinguished subsets or some binary operations or some relations (which are required to have certain properties), and one can list these and then describe the system as an ordered pair, triple, etc., depending on what is needed. For instance, the real line may be described as a quadruple $(\mathbb{R}^1, +, \cdot, <)$, where $+$, $\cdot$, and $<$ satisfy the axioms of a complete archimedean ordered field. But it is a safe bet that very few mathematicians think of the real field as an ordered quadruple.

On this note, I shall henceforth use Definition 1.18 for a relation R, with the tacit assumption that the domain X and codomain Y are known, and use the notation $R \subset X \times Y$ or $R \in \mathrm{P}(X \times Y)$ when these sets need to be emphasized.

1.20 Relation Examples The relation $U = X \times Y \in \mathrm{P}(X \times Y)$ is the *universal relation*, in which every $x \in X$ is related to every $y \in Y$. The relation $\varnothing \in \mathrm{P}(X \times Y)$ is the *empty relation*, in which no $x \in X$ is related to any $y \in Y$. The empty relation $\varnothing \subset X \times Y$ is a relation from X to Y for all sets X and Y, even if either X or Y is empty, in which case $X \times Y = \varnothing$ and $\mathrm{P}(X \times Y) = \{\varnothing\}$. In the partially ordered set $\langle \mathrm{P}(X \times Y), \subset \rangle$, U is the greatest element and $\varnothing$ is the least element (*ML*: 1.28): for all relations $R \in \mathrm{P}(X \times Y)$, $\varnothing \subset R \subset U$.

Let X be a set. *Membership* ("is an element of") is the relation $\in_X \subset X \times \mathrm{P}X$ defined, for $x \in X$ and $A \in \mathrm{P}X$, by

$$(x, A) \in \in_X \quad \text{iff} \quad x \in A. \tag{17}$$

Inclusion ("is a subset of") is the relation $\subset_X \subset \wp X \times \wp X$ defined, for $A, B \in \wp X$, by

$$(A, B) \in \subset_X \quad \text{iff} \quad A \subset B. \tag{18}$$

$\in_X$ contains no members of the form $(x, \varnothing)$ for any $x \in X$. $\subset_X$ contains $(\varnothing, B)$ and (A, X) for all $A, B \in \wp X$.

One may note that if $|X| = n$, then $|\wp X| = 2^n$, whence

$$|X \times \wp X| = n2^n \quad \text{and} \quad |\wp X \times \wp X| = 2^{2n}. \tag{19}$$

For each $i = 0, 1, \ldots, n$, there are $\binom{n}{i}$ subsets A of X with cardinality i. Each of these A s with $|A| = i$ contains i elements $x \in A$, so there are i ordered pairs of the form (x, A). Thus, as the subset $\in_X \subset X \times \wp X$, the cardinality of the membership relation is

$$|\in_X| = \sum_{i=0}^{n} \binom{n}{i} i = n2^{n-1}. \tag{20}$$

One therefore sees that the membership relation contains exactly half of the total number of eligible element–subset ordered pairs:

$$\frac{|\in_X|}{|X \times \wp X|} = \frac{n2^{n-1}}{n2^n} = \frac{1}{2}. \tag{21}$$

Each of the $\binom{n}{i}$ subsets B of X with $|B| = i$ itself contains $|\wp B| = 2^i$ distinct subsets. Thus, as the subset $\subset_X \subset \wp X \times \wp X$, the cardinality of the inclusion relation is

$$|\subset_X| = \sum_{i=0}^{n} \binom{n}{i} 2^i = 3^n. \tag{22}$$

The fraction of subsets that satisfy the inclusion relation $A \subset B$ among all possible subset pairs $A, B \in \wp X$ is thus

$$(23) \qquad \frac{|\subset_X|}{|\wp X \times \wp X|} = \frac{3^n}{2^{2n}} = \left(\frac{3}{4}\right)^n.$$

1.21 Corange and Range The *corange* of a relation $R \subset X \times Y$ is the subset of its domain containing all those $x \in X$ for which there is at least one $y \in Y$ such that $(x, y) \in R$. The *range* of a relation $R \subset X \times Y$ is the subset of its codomain containing all those $y \in Y$ for which there is at least one $x \in X$ such that $(x, y) \in R$.

Stated otherwise,

$$(24) \qquad \begin{array}{l} \operatorname{cor}(R) = \{x \in X : \exists y \in Y \ (x, y) \in R\} \subset \operatorname{dom}(R) = X \\ \operatorname{ran}(R) = \{y \in Y : \exists x \in X \ (x, y) \in R\} \subset \operatorname{cod}(R) = Y \end{array}.$$

For the simple example $\Gamma = \{(2,\ \mathrm{A}), (1,\ \mathrm{C}), (2,\ \mathrm{B})\}$, $\operatorname{cor}(R) = \{1, 2\}$ and $\operatorname{ran}(R) = \{\mathrm{A},\ \ \mathrm{B},\ \ \mathrm{C}\}$. But $\{1,\ 2,\ 3,\ 4,\ 5\} \times \{\mathrm{A, B, C, D, E, F}\}$, $\mathbb{N} \times$ alphanumeric characters, $\mathbb{Z} \times$ Latin alphabet, and $\mathbb{R} \times \{\mathrm{A},\ \ \mathrm{B},\ \ \mathrm{C}\}$ are among an infinitude of valid possibilities for $\operatorname{dom}(R) \times \operatorname{cod}(R)$.

To emphasize the points made previously, let me rephrase the situation as follows. The corange and range are defined in an 'internal' sense; the relation R (material cause) within itself entails the respective sets (final causes). Indeed, $\operatorname{cor}(R)$ and $\operatorname{ran}(R)$ are simply canonically projected images of the set R of ordered pairs onto its first and second components (cf. Definition 1.16 above):

$$(25) \qquad \pi_1(R) = \operatorname{cor}(R) \quad \text{and} \quad \pi_2(R) = \operatorname{ran}(R).$$

In other words, $R \mapsto \operatorname{cor}(R)$ and $R \mapsto \operatorname{ran}(R)$ are well-defined, algorithmic, *material entailments*, information mechanistically caused 'inside' the relation, from an 'intrinsic' perspective. The domain and codomain, on the other hand, are defined only in an 'external' sense; the efficient causes dom and cod must dictate the respective values. These beyond-syntax assignment rules correspond to *functional entailments* $\vdash$ dom and $\vdash$ cod; they are prescriptively caused from 'outside' of the relation, from an 'extrinsic' perspective.

1.22 Analysis and Synthesis With $A = \mathrm{cor}(R) \subset X$ and $B = \mathrm{ran}(R) \subset Y$ in (15), one has

$$\mathrm{cor}(R) \times \mathrm{ran}(R) = \pi_1^{-1}(\pi_1(R)) \cap \pi_2^{-1}(\pi_2(R)). \tag{26}$$

One notes that $R \subset \mathrm{cor}(R) \times \mathrm{ran}(R)$, but in general R is a proper subset of $\mathrm{cor}(R) \times \mathrm{ran}(R)$, so $R \neq \mathrm{cor}(R) \times \mathrm{ran}(R)$. The material entailment $\pi : R \mapsto \{\mathrm{cor}(R), \mathrm{ran}(R)\}$ is *analysis*, while the attempt at reversal $\pi^{-1} : \{\mathrm{cor}(R), \mathrm{ran}(R)\} \mapsto R$ is *synthesis*. The composition $\pi^{-1} \circ \pi$ in (26) illustrates that, in general, from the analytic components (i.e., information on $x \in \mathrm{cor}(R)$ and $y \in \mathrm{ran}(R)$) one may at best synthesize the superset $\mathrm{cor}(R) \times \mathrm{ran}(R) \supset R$; the relational information on $(x, y) \in R$ has been lost so one cannot recover R itself. Thus

$$\text{synthesis} \circ \text{analysis} \ncong \text{identity}. \tag{27}$$

Stated otherwise, once the relation R is broken into its parts, its internal relational connections are lost; it is generically not possible to synthesize R back from its analytic pieces—woe to reductionism! I have explicated this non-invertibility in the context of the amphibology of analysis and synthesis in detail in *ML*: 7.43–7.49; the reader is cordially invited to revise therein. I shall have a lot more to say on invertibility (and the lack thereof that is irreversibility) here in *IL*.

Relational Operations

1.23 Converse Relation The *converse* of a relation $R \subset X \times Y$ is the relation $\breve{R} \subset Y \times X$ such that $(y, x) \in \breve{R}$ if and only if $(x, y) \in R$.

Manifestly, one has

$$\begin{aligned} \mathrm{dom}\left(\breve{R}\right) &= \mathrm{cod}(R) = Y, \quad & \mathrm{cod}\left(\breve{R}\right) &= \mathrm{dom}(R) = X, \\ \mathrm{cor}\left(\breve{R}\right) &= \mathrm{ran}(R), & \mathrm{ran}\left(\breve{R}\right) &= \mathrm{cor}(R). \end{aligned} \tag{28}$$

The graph $\Gamma\left(\breve{R}\right)$ of the converse relation $\breve{R}$ is the *transpose* of the graph $\Gamma(R)$ of R:

$$(29) \qquad \Gamma\left(\breve{R}\right) = \{(y,x) \in Y \times X : (x,y) \in R\} = [\Gamma(R)]^t.$$

The converse operation is an involution; the converse of the converse of a relation is the relation itself:

$$(30) \qquad \breve{\breve{R}} = R.$$

The converse of the membership relation (or simply *converse membership*) is $\ni_X \subset \mathrm{P}X \times X$ defined, for $A \in \mathrm{P}X$ and $x \in X$, by

$$(31) \qquad A \ni_X x, \quad \text{i.e.,}\ (A,x) \in \ \ni_X, \quad \text{iff} \quad x \in X.$$

The converse of the inclusion relation is the relation 'includes' ('is a superset of') $\supset_X \subset \mathrm{P}X \times \mathrm{P}X$ defined, for $A, B \in \mathrm{P}X$, by

$$(32) \qquad (A,B) \in \ \supset_X \quad \text{iff} \quad A \supset B.$$

A converse relation $\breve{R}$ has exactly the same number of ordered pairs as R. So, in particular, when $|X| = n$,

$$(33) \qquad |\ni_X| = \sum_{i=0}^{n} \binom{n}{i} i = n2^{n-1},$$

$$(34) \qquad \frac{|\ni_X|}{|\mathrm{P}X \times X|} = \frac{n2^{n-1}}{n2^n} = \frac{1}{2},$$

$$(35) \qquad |\supset_X| = \sum_{i=0}^{n} \binom{n}{i} 2^i = 3^n,$$

$$(36) \qquad \frac{|\supset_X|}{|\mathrm{P}X \times \mathrm{P}X|} = \frac{3^n}{2^{2n}} = \left(\frac{3}{4}\right)^n.$$

1.24 Relative Product Let $R \subset X \times Y$ and $S \subset Y \times Z$ be relations. Their *relative product* (*RL*: 3.8) $S \circ R \subset X \times Z$ is a relation that is the set of all ordered pairs $(x,z) \in X \times Z$ for which there exists an $y \in Y$ with $(x,y) \in R$ and $(y,z) \in S$:

$$S \circ R = \{(x,z) \in X \times Z : \exists y \in Y \quad (x,y) \in R \wedge (y,z) \in S\}. \tag{37}$$

Consider the triple product $X \times Y \times Z$ equipped with its canonical projections $\pi_{12} : X \times Y \times Z \to X \times Y$, $\pi_{13} : X \times Y \times Z \to X \times Z$, and $\pi_{23} : X \times Y \times Z \to Y \times Z$ (cf. Section 1.16). Then $\pi_{12}^{-1}(R) = R \times Z$ and $\pi_{23}^{-1}(S) = X \times S$, whence

$$\begin{aligned} \pi_{12}^{-1}(R) \cap \pi_{23}^{-1}(S) &= (R \times Z) \cap (X \times S) \\ &= \{(x,y,z) \in X \times Y \times Z : (x,y) \in R \wedge (y,z) \in S\} \\ &\subset X \times Y \times Z. \end{aligned} \tag{38}$$

The projection $\pi_{13} : X \times Y \times Z \to X \times Z$ simply eliminates the intermediary 'y-component', thus one has

$$\begin{aligned} \pi_{13}\left(\pi_{12}^{-1}(R) \cap \pi_{23}^{-1}(S)\right) &= \pi_{13}((R \times Z) \cap (X \times S)) \\ &= \{(x,z) \in X \times Z : \exists y \in Y \quad (x,y) \in R \wedge (y,z) \in S\} \\ &= S \circ R. \end{aligned} \tag{39}$$

One may readily verify that relative product is an associative operation: for relations $Q \subset W \times X$, $R \subset X \times Y$, and $S \subset Y \times Z$,

$$(S \circ R) \circ Q = S \circ (R \circ Q) \subset W \times Z \tag{40}$$

(since $S \circ R \circ Q = \{(w,z) \in W \times Z : \exists x \in X \ \exists y \in Y \quad (w,x) \in Q \wedge (x,y) \in R \wedge (y,z) \in S\}$, in which the 'order of appearance' of the intermediaries $x \in X$ and $y \in Y$ is not important)

1.25 Diagonal The *diagonal* of a set X is the relation

$$\Delta_X = \{(x,x) : x \in X\} \subset X \times X. \tag{41}$$

Since the relation is defined, for $x_1, x_2 \in X$, by

$$(x_1, x_2) \in \Delta_X \quad \text{iff} \quad x_1 = x_2, \tag{42}$$

it is also called the *equality* (or *identity*) *relation* on X, and denoted 1_X.

The reader may have noticed that I have already used the symbol 1_X for the identity *map* on X (Definition 1.2). The equivalence between the mapping $1_X : X \to X$ and the relation $1_X \subset X \times X$ will be explained in the general context in the next chapter. Likewise, for $A \subset X$, the inclusion map $i_{A \subset X} : A \to X$ of A in X corresponds to the 'diagonal inclusion' relation

(43) $$\Delta_{A \subset X} = \{(x,x) : x \in A\} \subset A \times X.$$

Note that $\Delta_{A \subset X}$ and $\Delta_A = \{(x,x) : x \in A\} \subset A \times A$ are different relations. Although they have identical graphs and identical domains, $\mathrm{dom}(\Delta_{A \subset X}) = \mathrm{dom}(\Delta_A) = A$, they have different codomains, $\mathrm{cod}(\Delta_{A \subset X}) = X$ but $\mathrm{cod}(\Delta_A) = A$. In the notation of Definition 1.17, the two relations are $(A, X, \Delta_{A \subset X})$ and (A, A, Δ_A).

1.26 Restrictions A relation may be restricted to a subset of its domain or codomain. The *restriction* of the relation $R \subset X \times Y$ to $A \subset X$ is the relation

(44) $$R|_A = R \cap (A \times Y) = R \cap \pi_1^{-1}(A) = \{(x,y) \in R : x \in A\} \subset A \times Y,$$

with domain A and codomain Y. The *restriction* of the relation $R \subset X \times Y$ to $B \subset Y$ is the relation

(45) $$R|^B = R \cap (X \times B) = R \cap \pi_2^{-1}(B) = \{(x,y) \in R : y \in B\} \subset X \times B,$$

with domain X and codomain B. (The canonical projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are as in Definition 1.16.)

Note that the corange and range of a restriction are restricted accordingly. The components that are directly restricted are

(46) $$\mathrm{cor}\left(R|_A\right) = \mathrm{cor}(R) \cap A \quad \text{and} \quad \mathrm{ran}\left(R|^B\right) = \mathrm{ran}(R) \cap B.$$

The restrictions are indirect for the duals. For example, suppose $a \in A$ is R-related to $b \in Y \sim B$ but to no other elements of B; i.e., $(a,b) \in R$, and for all other $y \in Y$ ($y \neq b$), $(a,y) \notin R$. Since $b \notin B$, $b \notin \mathrm{ran}\left(R|^B\right)$. By removing b, the only member of Y to which a is R-related, a is removed from the corange of the restriction, so also $a \notin \mathrm{cor}\left(R|^{B}\right)$; whence a is not $R|^{B}$-related to b. Thus one sees that, although the restriction is direct on one component, it has an indirect pruning effect on the other component. In general, one has

$$
\begin{aligned}
\mathrm{ran}\left(R|_{A}\right) &= \pi_2\left(R|_{A}\right) = \pi_2\left(R \cap \pi_1^{-1}(A)\right) \\
&= \{y \in Y : \exists x \in A \quad (x,y) \in R\} \subset \mathrm{ran}(R)
\end{aligned} \tag{47}
$$

and

$$
\begin{aligned}
\mathrm{cor}\left(R|^{B}\right) &= \pi_1\left(R|^{B}\right) = \pi_1\left(R \cap \pi_2^{-1}(B)\right) \\
&= \{x \in X : \exists y \in B \ (x,y) \in R\} \subset \mathrm{cor}(R),
\end{aligned} \tag{48}
$$

and the inclusions may be proper. One may also characterize $\mathrm{ran}\left(R|_{A}\right)$ as the set of all right R-relatives of elements of A, and $\mathrm{cor}\left(R|^{B}\right)$ as the set of all left R-relatives of elements of B (Definition 1.18).

Domain and codomain restrictions are related through the converse operation:

$$
R|_{A} = \left(\breve{R}\Big|^{A}\right)^{\smile} \quad \text{and} \quad R|^{B} = \left(\breve{R}\Big|_{B}\right)^{\smile}. \tag{49}
$$

Alternatively, one may say that the converse operation 'commutes dually' with restriction:

$$
\left(R|_{A}\right)^{\smile} = \breve{R}\Big|^{A} \quad \text{and} \quad \left(R|^{B}\right)^{\smile} = \breve{R}\Big|_{B}. \tag{50}
$$

Restrictions may be defined equivalently as relative products with the diagonal inclusion relation:

$$
R|_{A} = R \circ \Delta_{A\subset X} \subset A \times Y \quad \text{and} \quad R|^{B} = \breve{\Delta}_{B\subset Y} \circ R \subset X \times B \tag{51}
$$

(where the 'converse diagonal inclusion relation' $\breve{\Delta}_{B\subset Y} = \{(y,y) : y \in B\} \subset Y \times B$).

1.27 Extensions For a relation $R \subset X \times Y$, an *extension* of R to a larger domain $A \supset X$ is a relation $S \subset A \times Y$ such that $R = S|_{X}$, and an *extension* of R to a larger codomain $B \supset Y$ is a relation $S \subset X \times B$ such that $R = S|^{Y}$. An extension to a larger domain is also called a *continuation* (cf. *ML*: Praefatio).

R and $A \subset X$ intrinsically determine $R|_A$, while R and $B \subset Y$ intrinsically determine $R|^B$; so one says *the* restriction. But R can have more than one extension S to a superset $A \supset X$ (respectively $B \supset Y$). All that is required is for S and R to coincide in $X \times Y$, and S may arbitrarily contain members (in the sense that they are not dictated by R) of $(A \sim X) \times Y$ (respectively $X \times (B \sim Y)$); so one may only say *an* extension rather than *the* extension. In particular, note that an extension of the restriction $R|_A$ or $R|^B$ (back respectively to X or Y) is not necessarily $R \subset X \times Y$ itself (another illustrative example of non-invertiblility; cf. Section 1.22).

Rel

1.28 The Category Rel The category in which the collection of objects is the collection of all sets (in a suitably naive universe of small sets) and where morphisms are relations (as in Definitions 1.17 and 1.18) is denoted **Rel**. Given two sets X and Y, the hom-set $\mathbf{Rel}(X, Y)$ of *all* relations between X and Y is thus the power set $\mathfrak{P}(X \times Y)$.

For relations $R \subset X \times Y$ and $S \subset Y \times Z$, their composite is their *relative product* $S \circ R \subset X \times Z$ (Definition 1.24). The requisite identity morphism in $\mathbf{Rel}(X,X)$ is the diagonal relation Δ_X (= the equality relation 1_X; Definition 1.25). Note that $\varnothing = 1_\varnothing \in \mathbf{Rel}(\varnothing, \varnothing)$.

1.29 Converse Functor The *converse functor* is the contravariant functor $\mathfrak{C} : \mathbf{Rel} \to \mathbf{Rel}$ (or equivalently the covariant functor $\overline{\mathfrak{C}} : \mathbf{Rel} \to \mathbf{Rel}^{op}$; *ML*: A.10) that sends a set X to itself and a relation $R \in \mathbf{Rel}(X, Y)$ to its converse $\breve{R} \in \mathbf{Rel}(Y, X)$; viz.

$$\mathfrak{C} : \begin{cases} X \mapsto X & (X \in \mathfrak{O}\mathbf{Rel}) \\ [R \subset X \times Y] \mapsto \left[\breve{R} \subset Y \times X\right] & (R \in \mathfrak{A}\mathbf{Rel}) \end{cases} . \tag{52}$$

The converse and relative product operations interact thus:

$$(S \circ R)\breve{} = \breve{R} \circ \breve{S}; \tag{53}$$

this is the source of the contravariance, whence

$$\mathsf{C}(S \circ R) = \mathsf{C}(R) \circ \mathsf{C}(S). \tag{54}$$

Since $\breve{1}_X = 1_X$, trivially

$$\mathsf{C}(1_X) = 1_{\mathsf{C}X} = 1_X. \tag{55}$$

Also, the involution (30) implies that

$$\mathsf{C} \circ \mathsf{C} = I_{\mathbf{Rel}} \tag{56}$$

(where $I_{\mathbf{Rel}}$ is the identity functor on **Rel**; cf. Section 0.13 and *ML*: A.12(i)) and C is, naturally, itself an involution (and its own inverse functor). $\mathsf{C} : \mathbf{Rel} \to \mathbf{Rel}$ is thus an isomorphism of categories and is a bijection both on objects and on morphisms (Section 0.13).

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