

Chapter 2

Spaces of Functions on a Sphere

Noticeable activity in the study of various questions of the theory of Fourier-Laplace series is observed in the recent decades [369]. This is connected mainly with the application of this series for the numerical solution of various problems on a sphere in such applied sciences as meteorology, weather forecast, climate theory, and so forth. In particular, questions such as convergence and summation of Fourier-Laplace series are of great importance when using the spectral method for discretizing partial differential equations on a sphere. The theory of functions on a sphere given in this chapter is then systematically used in the other chapters, in particular, when considering questions of the unique solvability of the vorticity equation and the stability of its solutions.

Section 2.1 surveys briefly the main properties of Legendre polynomials, associated Legendre functions, and spherical harmonics. In Sect. 2.2, geographical coordinates maps for the sphere are defined and the well-known theorem on the partition of unity is given which is an important tool in the theory of integration of functions on smooth compact manifolds [78, 131, 287]. In detail, various maps on the sphere are studied in [32, 78, 131]. The orthogonal projections $Y_n(\psi)$ of a function ψ onto the subspaces $\mathbf{H}_n$ of the homogeneous spherical polynomials of degree n are defined in Sect. 2.3, and then they are used to introduce derivatives D^s and Λ^s of real degree s of a function on the unit sphere. The orthogonal projections $T_N(\psi)$ of a function ψ onto the subspaces $\mathbb{P}^N$ of spherical polynomials of degree $n \leq N$ are also defined in this section.

Spaces of scalar functions having fractional derivatives on the sphere are mainly defined by two ways: either through the modulus of continuity of functions [28, 45, 87, 88, 205, 206, 295–297] or using multiplier operators and geometrical properties of the sphere [104, 105, 151, 152, 160–163]. In Sect. 2.4, we use the second approach, as more simple and constructive, and closely related to the spectral method of discretization of differential equations on a sphere. The method is based

on using the spherical Laplace operator, since it is the unique differential operator invariant with respect to any rotation of the sphere. The fractional derivatives are introduced through real degrees of the Laplace operator, and the smoothness of a function on the sphere is determined by the rate of convergence of its series of spherical harmonics.

A family of Hilbert spaces $\mathbb{H}^s$ of functions having fractional derivatives on the sphere up to real degree s is introduced, besides, $\psi \in \mathbb{H}^s$ if all its fractional derivatives up to order s belong to the Hilbert space $\mathbb{L}^2(S)$. Some structural properties of Hilbert spaces $\mathbb{H}^s$ including various embedding theorems are given. According to the well-known Gibbs phenomenon [219] and Parseval-Steklov identity [213], the convergence of Fourier series is deteriorated near to sharp changes of a continuous function. For the sphere, as a compact manifold, there is a ready instrument of approximation based on the expansion of space $\mathbb{L}^2(S)$ in the orthogonal direct sum of the $(2n + 1)$ -dimensional subspaces $\mathbf{H}_n$ of homogeneous spherical polynomials of degree n which are invariant to the group $SO(3)$ of sphere rotations. The rate of convergence of Fourier-Laplace series of functions from $\mathbb{H}^s$ is estimated in Sect. 2.4 as well. In more detail, the approximation theory of Fourier-Laplace series on the sphere is analyzed in [369]. The space $\mathbb{C}(S)$ of continuous functions on a sphere is considered in Sect. 2.5. The chapter is concluded by Sect. 2.6 where some estimates are given in the norms of Banach spaces $\mathbb{L}^p(S)$ and $\mathbb{L}^p(0, T; \mathbb{X})$.

2.1 Spherical Harmonics

Spherical harmonics are special functions defined on the surface of a sphere. Let $x = (\lambda, \mu)$ be a point of the unit sphere $S = \{x \in \mathbb{R}^3 : |x| = 1\}$ in the three-dimensional Euclidean space $\mathbb{R}^3$ where $\lambda \in [0, 2\pi)$ is the longitude, $\mu = \sin \phi$, the latitude ϕ ranges from $-\pi/2$ at the South Pole to $\pi/2$ at the North Pole, and hence $\mu \in [-1, 1]$. We will indiscriminately use x and spherical coordinates (λ, μ) to denote a point on the sphere S . Let us denote by $\mathbb{C}^\infty(S)$ the set of infinitely differentiable functions on S and by

$$\langle f, g \rangle = \int_S f(x) \overline{g(x)} dS \quad (2.1.1)$$

and

$$\|f\| = \langle f, f \rangle^{1/2} \quad (2.1.2)$$

the inner product and norm of any functions $f(x)$ and $g(x)$ of $\mathbb{C}^\infty(S)$, respectively. Here $dS = d\lambda d\mu$ is an infinitesimal element of the sphere surface, and $\overline{g(x)}$ is the complex conjugate of $g(x)$.

It is well known [257] that the spherical harmonics

$$\begin{aligned} Y_n^m(\lambda, \mu) &= Q_n^m(\mu) e^{im\lambda} \\ &= \left[\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!} \right]^{1/2} P_n^m(\mu) e^{im\lambda}, \quad n \geq 0, \quad |m| \leq n \end{aligned} \quad (2.1.3)$$

form the orthonormal system in $\mathbb{C}^\infty(S)$:

$$\langle Y_n^m, Y_l^k \rangle = \delta_{mk} \delta_{nl} \quad (2.1.4)$$

where

$$\delta_{mk} = \begin{cases} 1, & \text{if } m = k \\ 0, & \text{if } m \neq k \end{cases} \quad (2.1.5)$$

is the Kronecker delta. Thus,

$$Q_n^m(\mu) = c_{nm} P_n^m(\mu) \quad (2.1.6)$$

where

$$P_n^m(\mu) = \frac{(1-\mu^2)^{m/2}}{2^n n!} \frac{d^{n+m}}{d\mu^{n+m}} (\mu^2 - 1)^n \quad (2.1.7)$$

is the associated Legendre function of degree n and zonal number m [215, 287], and

$$c_{nm} = \left[\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!} \right]^{1/2} \quad (2.1.8)$$

It is also known that

$$2\pi \int_{-1}^1 Q_n^m(\mu) Q_l^m(\mu) d\mu = \delta_{nl} \quad (2.1.9)$$

If m is even then $Q_n^m(\mu)$ and $P_n^m(\mu)$ are algebraic polynomials in μ . Each spherical harmonic Y_n^m is the eigenfunction of the eigenvalue problem

$$-\Delta Y_n^m = \chi_n Y_n^m, \quad |m| \leq n \quad (2.1.10)$$

for symmetric and positive definite Laplace operator on S :

$$-\Delta = -\frac{\partial}{\partial \mu} \left[(1-\mu^2) \frac{\partial}{\partial \mu} \right] - \frac{1}{1-\mu^2} \frac{\partial^2}{\partial \lambda^2} \quad (2.1.11)$$

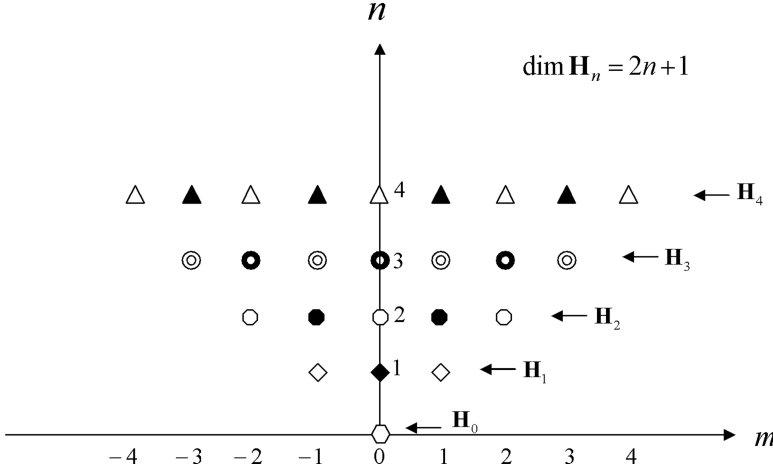


Fig. 2.1 Wave numbers of harmonics $Y_n^m(\lambda, \mu)$

where n and m are integer, and

$$\chi_n = n(n+1) \quad (2.1.12)$$

is the eigenvalue corresponding to $Y_n^m(\lambda, \mu)$.

For each integer $n \geq 0$, χ_n is the eigenvalue of multiplicity $2n+1$, and the span of $2n+1$ spherical harmonics $Y_n^m(\lambda, \mu)$ ($|m| \leq n$) forms a generalized $(2n+1)$ -dimensional eigenspace

$$\mathbf{H}_n = \{\psi : -\Delta\psi = \chi_n\psi\} \quad (2.1.13)$$

corresponding to the eigenvalue χ_n [287, 349]. On the plane of wavenumbers (m, n) shown in Fig. 2.1, the wavenumbers of the spherical harmonics from different subspaces $\mathbf{H}_n$ are on the different parallel lines. Besides, the wavenumbers of the antisymmetric and symmetric spherical harmonics with respect to the equator $\mu = 0$ are marked by black and white symbols, respectively. The spatial structure of spherical harmonics $Y_n^m(\lambda, \mu)$ is shown in Fig. 2.2.

Remark 2.1.1 The subspace $\mathbf{H}_n$ is invariant not only with respect to the Laplace operator but also to any transformations of the $SO(3)$ group of rotations of sphere about its arbitrary axis [131, 287]. The last assertion means that a spherical harmonic $Y_n^m(\lambda, \mu)$ from $\mathbf{H}_n$ can be represented in a new coordinate system (λ', μ') by

$$Y_n^m(\lambda', \mu') = \sum_{k=-n}^n \alpha_k(m, n, N') Y_n^k(\lambda, \mu) \quad (2.1.14)$$

where $\alpha_k(m, n, N')$ is the complex number that depends on the wavenumbers n, m and the pole N' of new coordinate system.

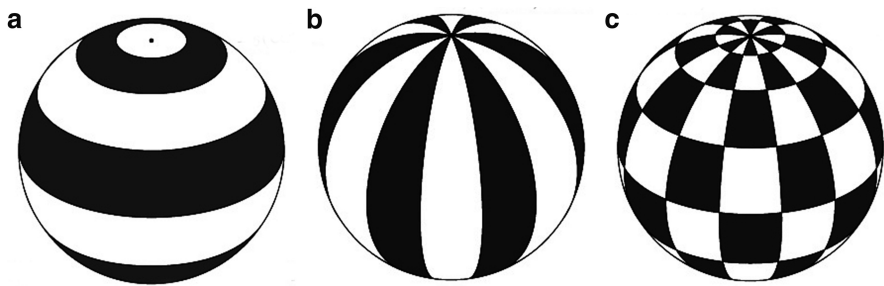


Fig. 2.2 Spatial structure of spherical harmonics $Y_n^m(\lambda, \mu)$: (a) zonal harmonic ($m = 0$), (b) sectorial harmonic ($m = n$) and (c) tesseral harmonic ($|m| < n$)

In order to simplify notation we will widely use a multi-index $\alpha \equiv (m, n) \equiv (m_\alpha, n_\alpha)$ [275]:

$$\bar{\alpha} \equiv (-m, n) \equiv (-m_\alpha, n_\alpha),$$

$$Y_\alpha \equiv Y_n^m, \quad P_\alpha \equiv P_n^m, \quad \chi_\alpha \equiv \chi_n = n(n+1), \quad (2.1.15)$$

$$\sum_{\alpha(k)} \equiv \sum_{n=k}^{\infty} \sum_{m=-n}^n \quad \text{and} \quad \sum_{\alpha(k)} \equiv \sum_{n=k}^N \sum_{m=-n}^n. \quad (2.1.16)$$

We now give the basic properties of the associated Legendre functions [287]:

1. Every function $P_n^m(\mu)$ is the solution of the associated Legendre equation

$$\frac{d}{d\mu} \left[(1 - \mu^2) \frac{d}{d\mu} P_n^m(\mu) \right] + [\chi_n - \frac{m^2}{1 - \mu^2}] P_n^m(\mu) = 0 \quad (2.1.17)$$

where $\chi_n = n(n+1)$. In the particular case when $m = 0$, the solutions of this equation are the Legendre polynomials $P_n(\mu) \equiv P_n^0(\mu)$ (see Figs. 2.3 and 2.4).

2. $P_n^m(\mu) \equiv 0$ if $m > n$.
3. Two functions with complex conjugate wavenumbers α and $\bar{\alpha}$ are related as

$$P_{\bar{\alpha}} = (-1)^{m_\alpha} P_\alpha, \quad Y_{\bar{\alpha}} = (-1)^{m_\alpha} \overline{Y_\alpha} \quad (2.1.18)$$

4. The function $P_n^m(\mu)$ is equal to zero in $(n - m)$ points of interval $(-1, 1)$ (see (2.1.3) and Fig. 2.2).
5. The equality

$$P_n^m(-\mu) = (-1)^{n-m} P_n^m(\mu) \quad (2.1.19)$$

implies that functions $P_n^m(\mu)$ and $Y_n^m(\lambda, \mu)$ are symmetric (or antisymmetric) with respect to the equator $\mu = 0$ if $(n - m)$ is even (odd). At the sphere poles

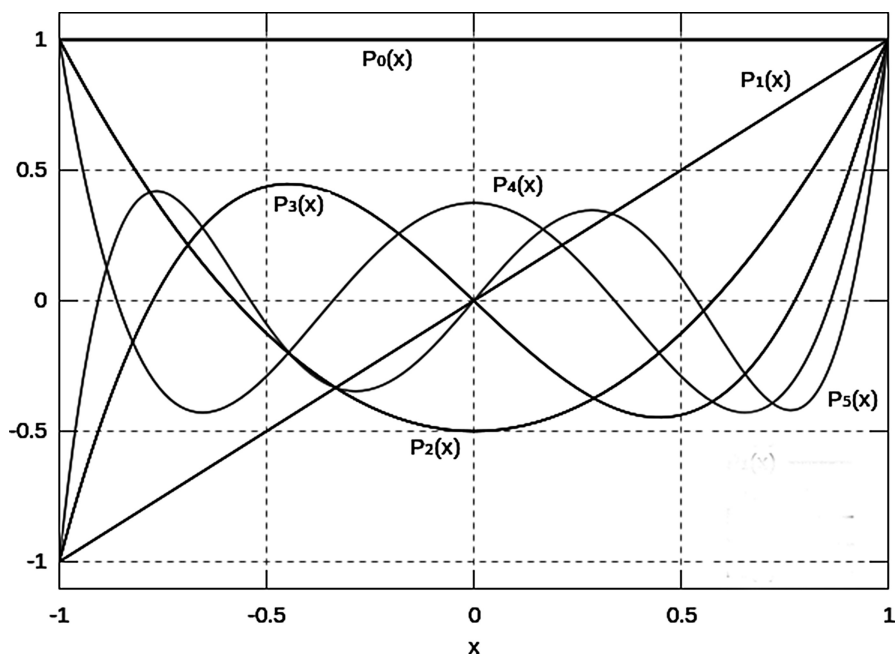


Fig. 2.3 Legendre polynomials $P_n(x)$, $n = 0, 1, \dots, 5$

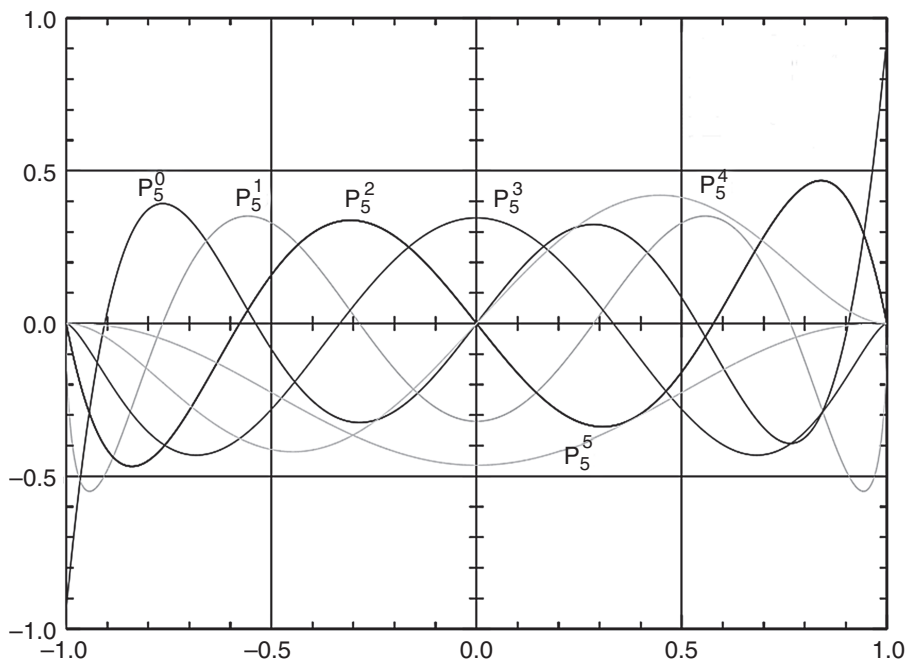


Fig. 2.4 Associated Legendre functions $P_5^m(\mu)$, $m = 0, 1, \dots, 5$

$(\mu = \pm 1)$, $P_n^m(\pm 1) = 0$ if $m \neq 0$, and $P_n(\pm 1) = (\pm 1)^n$ for the Legendre polynomials $P_n(\mu) \equiv P_n^0(\mu)$.

6. The following assertions hold for $P_n^m(\mu)$:

$$\int_{-1}^1 P_n^m(\mu) P_l^m(\mu) d\mu = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \delta_{nl}, \quad (n, l \geq m \geq 0), \quad (2.1.20)$$

$$\int_0^1 \frac{[P_n^m(\mu)]^2}{1-\mu^2} d\mu = \frac{(n+m)!}{2m(n-m)!}, \quad (n \geq 0, \quad 1 \leq m \leq n) \quad (2.1.21)$$

Example 2.1.1 We now prove the orthogonality of $P_n^m(\mu)$ and $P_l^m(\mu)$ if $n \neq l$. Multiplying (2.1.17) by $P_l^m(\mu)$ we obtain one equation. Interchanging the indices n and l in this equation we obtain another equation. Subtracting one equation from the other we get

$$\begin{aligned} & \frac{d}{d\mu} \left[(1-\mu^2) \left\{ P_l^m(\mu) \frac{d}{d\mu} P_n^m(\mu) - P_n^m(\mu) \frac{d}{d\mu} P_l^m(\mu) \right\} \right] \\ &= [\chi_l - \chi_n] P_n^m(\mu) P_l^m(\mu) \end{aligned} \quad (2.1.22)$$

The integral of both parts of (2.1.22) from $\mu = -1$ to $\mu = 1$ leads to

$$\int_{-1}^1 P_n^m(\mu) P_l^m(\mu) d\mu = 0$$

7. The Legendre polynomials $P_n(\mu)$ can be obtained by expanding the function $(1-2\mu z + z^2)^{-1/2}$ in the convergent power series of z for $|z| < 1$ [412]:

$$(1-2\mu z + z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(\mu) z^n \quad (2.1.23)$$

The generating function (2.1.23) is often used in various proofs [352].

Example 2.1.2 Let us show that (2.1.20) holds for $l = n$ and $m = 0$, that is,

$$\int_{-1}^1 [P_n(\mu)]^2 d\mu = \frac{2}{2n+1} \quad (2.1.24)$$

Indeed, if we square both parts of (2.1.23) and then integrate the result from $\mu = -1$ to $\mu = 1$, we obtain

$$\int_{-1}^1 (1-2\mu z + z^2)^{-1} d\mu = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \left\{ \int_{-1}^1 P_n(\mu) P_j(\mu) d\mu \right\} z^{n+j}$$

Integrating the left side of this equality and using the orthogonality of the Legendre polynomials we get

$$\frac{1}{z} \ln\left(\frac{1+z}{1-z}\right) = \sum_{n=0}^{\infty} \left(\frac{2}{2n+1}\right) z^{2n} = \sum_{n=0}^{\infty} \left\{ \int_{-1}^1 [P_n(\mu)]^2 d\mu \right\} z^{2n}$$

The comparison of the corresponding coefficients of the two last series proves (2.1.24).

8. Let $n \geq 0$ be an integer. The inequality $|P_n(\mu)| \leq 1$ is satisfied for every Legendre polynomial. This estimate immediately follows from the integral presentation of the Legendre polynomials (Laplace-Mehler integral):

$$P_n(\mu) = \frac{1}{\pi} \int_0^\pi \left(\mu + i\sqrt{1-\mu^2} \cos \lambda \right)^n d\lambda \quad (2.1.25)$$

where $i = \sqrt{-1}$ [1].

9. Formula

$$\begin{aligned} P_n^m(\mu) &= (-1)^m \frac{(-n)_m (n+1)_m}{m!} \\ &\times \left(\frac{1-\mu}{1+\mu} \right)^{m/2} \sum_{k=0}^{\infty} \left[\frac{(-n)_k (n+1)_k}{(m+1)_k k!} \left(\frac{1-\mu}{2} \right)^k \right] \end{aligned} \quad (2.1.26)$$

holds for every integer m and n such that $|m| \leq n$ [1]. Here $(s)_0 = 1$ and

$$(s)_k = s(s+1) \dots (s+k-1) \quad (2.1.27)$$

is the Pochhammer symbol [1] for any real s and natural k .

10. Let n and m be complex numbers. The following recurrence relations are valid for the associated Legendre functions [1]:

$$\sqrt{1-\mu^2} P_n^{m+1}(\mu) = (n+m) P_{n-1}^m(\mu) - (n-m) \mu P_n^m(\mu) \quad (2.1.28)$$

$$(n-m+1) P_{n+1}^m(\mu) = (2n+1) \mu P_n^m(\mu) - (n+m) P_{n-1}^m(\mu) \quad (2.1.29)$$

$$P_{n+1}^m(\mu) = P_{n-1}^m(\mu) + (2n+1) \sqrt{1-\mu^2} P_n^{m-1}(\mu) \quad (2.1.30)$$

$$\sqrt{1-\mu^2} P_n^{m+1}(\mu) = 2m\mu P_n^m(\mu) - (n+m)(n-m+1) \sqrt{1-\mu^2} P_n^{m-1}(\mu) \quad (2.1.31)$$

$$(1-\mu^2) \frac{d}{d\mu} P_n^m(\mu) = -n\mu P_n^m(\mu) + (n+m) P_{n-1}^m(\mu) \quad (2.1.32)$$

$$(1 - \mu^2) \frac{d}{d\mu} P_n^m(\mu) = m\mu P_n^m(\mu) - (n + m)(n - m + 1) \sqrt{1 - \mu^2} P_n^{m-1}(\mu) \quad (2.1.33)$$

$$\sqrt{1 - \mu^2} P_n^{m+1}(\mu) = m\mu P_n^m(\mu) + (1 - \mu^2) \frac{d}{d\mu} P_n^m(\mu) \quad (2.1.34)$$

$$\frac{d}{d\mu} [P_{n+1}(\mu) - P_{n-1}(\mu)] = (2n + 1) P_n(\mu) \quad (2.1.35)$$

Formula (2.1.29) is the fundamental relationship between associated Legendre functions with equal m values but consecutive n values. In particular, if $m = 0$ then (2.1.29) is a recurrence formula for calculating the Legendre polynomials using the two first polynomials: $P_0(\mu) = 1$ and $P_1(\mu) = \mu$. For example,

$$P_2(\mu) = \frac{1}{2}(3\mu^2 - 1); \quad P_3(\mu) = \frac{1}{2}(5\mu^3 - 3\mu); \quad P_4(\mu) = \frac{1}{8}(35\mu^4 - 30\mu^2 + 3)$$

Formula (2.1.31) is the fundamental relationship linking three associated Legendre functions with the same n values and consecutive m values. It is easily seen that formula (2.1.34) follows from (2.1.28) and (2.1.32), formula (2.1.31) follows from (2.1.33) and (2.1.34), while (2.1.35) is valid due to (2.1.30) and (2.1.34). In particular, for integer m and n when $m \geq 0$, functions P_n^m can be calculated with the recurrence formula (2.1.29) using

$$P_n^m(\mu) = \frac{(2n)!}{2^n n!} (1 - \mu^2)^{n/2} = 1 \cdot 3 \dots (2n - 1) (1 - \mu^2)^{n/2}$$

and

$$P_{n+1}^n(\mu) = \mu (2n + 1) P_n^n(\mu)$$

For example, $P_1^1(\mu) = (1 - \mu^2)^{1/2}$; $P_2^1(\mu) = 3\mu (1 - \mu^2)^{1/2}$; $P_2^2(\mu) = 3(1 - \mu^2)$. For $m < 0$, the relation (2.1.18) should be used.

11. In practice, the normalized Legendre functions (2.1.6) with integer m and n can be calculated with the following recurrence formulas:

$$D_{mn} Q_n^m(\mu) = \mu Q_{n-1}^m(\mu) - D_{m,n-1} Q_{n-2}^m(\mu) \quad (2.1.36)$$

$$(1 - \mu^2) \frac{d}{d\mu} Q_n^m(\mu) = (2n + 1) D_{mn} Q_{n-1}^m - n\mu Q_n^m \quad (2.1.37)$$

or

$$\frac{d}{d\mu} Q_n^m(\mu) = -\frac{m\mu}{1 - \mu^2} Q_n^m + \sqrt{\frac{n(n+1) - m(m+1)}{1 - \mu^2}} Q_n^{m+1}$$

[214, 257], where

$$D_{mn} = \left(\frac{n^2 - m^2}{4n^2 - 1} \right)^{1/2} \quad (2.1.38)$$

In conclusion, we give without proof a frequently used addition theorem for spherical harmonics [287].

Theorem 2.1.1 *Let ω be an angle between two unit radius-vectors $\vec{x}_1, \vec{x}_2$ corresponding to points $x_1, x_2 \in S$. Then*

$$P_n(\vec{x}_1 \cdot \vec{x}_2) = \frac{4\pi}{2n+1} \sum_{m=-n}^n Y_n^m(x_1) \overline{Y_n^m}(x_2) \quad (2.1.39)$$

where $\vec{x}_1 \cdot \vec{x}_2 = \cos \omega$ is the scalar product of vectors $\vec{x}_1$ and $\vec{x}_2$.

In particular, if $x_1 = x_2 = x$ then $P_n(0) = 1$ and

$$\sum_{m=-n}^n |Y_n^m(x)|^2 = \sum_{m=-n}^n [Q_n^m(\mu)]^2 = \frac{2n+1}{4\pi} \quad (2.1.40)$$

Due to (2.1.40), the values of spherical harmonics and associated Legendre functions are limited:

$$\begin{aligned} |Y_n^m(\lambda, \mu)| &= |Q_n^m(\mu)| \leq \left(\frac{2n+1}{8\pi} \right)^{1/2} \\ |P_n^m(\mu)| &\leq \left[\frac{(n+m)!}{(n-m)!} \right]^{1/2} \end{aligned} \quad (2.1.41)$$

Evidently, $|P_n(\mu)| \leq 1$.

One more useful addition formula is obtained if we replace f and g in the identity

$$\Delta(fg) = f\Delta g + g\Delta f + 2\nabla f \cdot \nabla g$$

by $Y_n^m(\lambda, \mu)$ and $\overline{Y_n^m}(\lambda, \mu)$, and sum the result over m from $-n$ to n . Then (2.1.40) leads to

$$\sum_{m=-n}^n |\nabla Y_n^m(x)|^2 = \chi_n \left(\frac{2n+1}{4\pi} \right) \quad (2.1.42)$$

with $\chi_n = n(n+1)$ (see (2.1.12)).

2.2 Geographical Coordinates Maps

Let us introduce an inner metric on the unit sphere S . The sphere S is transformed to a metric space if we define the distance between two arbitrary points $x, y \in S$ by

$$\rho(x, y) = \arccos(\vec{x} \cdot \vec{y}) = \omega \quad (2.2.1)$$

(Fig. 2.5) where we use the notations of Theorem 2.1.1.

In the metric (2.2.1), the sphere S is a convex set [32, Theorem 18.4.2]. Note that there exists the only shortest way between two points $x, y \in S$ if $y \neq -x$. In the case $y = -x$ the shortest way is not unique because any semicircle connecting the points x and $-x$ represent the shortest way between these points. Any open (in the metric (2.2.1)) set Ω on S having a local coordinate system we will call the coordinate domain $\Omega \subset S$.

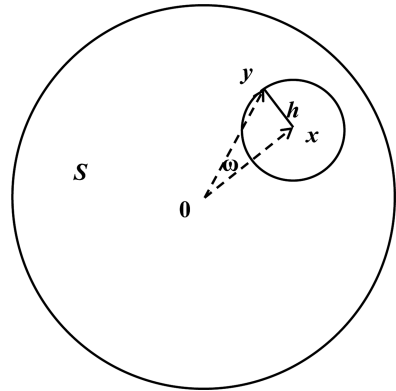
Definition 2.2.1 A couple (Ω, f) consisting of a coordinate domain $\Omega \subset S$ and injection $f : \Omega \mapsto \mathbb{R}^2$, $f(x) = (x_1, x_2)$, is called the map on the sphere S . Moreover, the mapping $f : x \mapsto f(x)$ is almost always a homeomorphism. We will denote the inverse mapping f^{-1} by

$$\tau : (x_1, x_2) \in \mathbb{R}^2 \mapsto x \in S \subset \mathbb{R}^3.$$

The geographical coordinates map is one of the most often used maps on a sphere. Let N and P be the northern and southern poles of sphere S in a coordinate system (λ, μ) . Geographic coordinates are valid on the whole sphere except for a closed set (large semicircle of sphere)

$$\Gamma = \{x \in S : |\lambda(x)| = \pi, \mu(x) \in [-1, 1]\}$$

Fig. 2.5 Metric $\rho(x, y)$ on the sphere



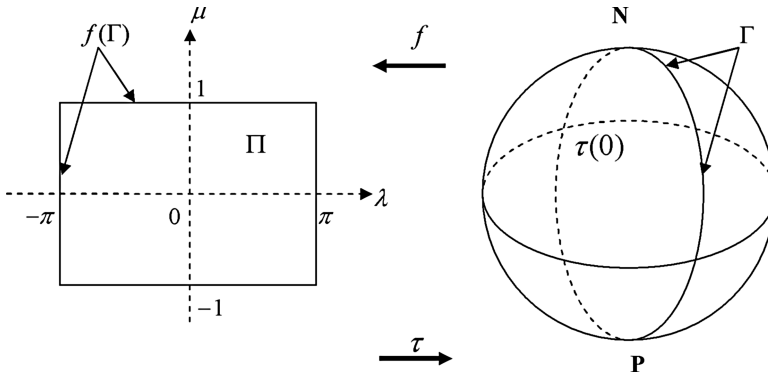


Fig. 2.6 Geographical coordinates map

called the Date line (Fig. 2.6). The mapping f is not single-valued on Γ . Indeed, the longitude λ can have two values $+\pi$ and $-\pi$ on Γ if $|\mu| < 1$. Besides, both poles (N and P) have no certain value of λ from interval $[-\pi, \pi]$.

Definition 2.2.2 ([32]) Let Ω be an open set $S \setminus \Gamma$. We say that the couple (Ω, f) is a geographical coordinates map, if $f : x \mapsto (\lambda(x), \mu(x))$ is the isomorphism between a spherical point $x \in \Omega$ as element of $\mathbb{R}^3$ and a point $(\lambda(x), \mu(x))$ of $\mathbb{R}^2$. As image $f(\Omega)$ we have the open set (rectangular) $\Pi = (-\pi, \pi) \times (-1, 1) \subset \mathbb{R}^2$. The inverse mapping is defined by

$$x = \tau(\lambda, \mu) = \left(\sqrt{1 - \mu^2} \cos \lambda, \sqrt{1 - \mu^2} \sin \lambda, \mu \right) \in \mathbb{R}^3$$

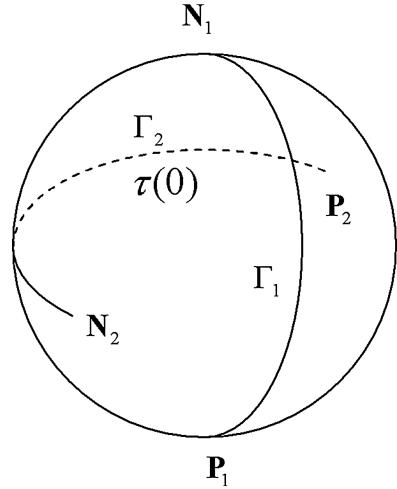
A plane element $d\lambda d\mu$ in $\mathbb{R}^2$ corresponds to a surface element dS in $\mathbb{R}^3$. The Jacobian of transformation τ is equal to unity [32]. Since the geographic coordinates are defined almost everywhere on S , the integral over S can be written as

$$\int_S \psi(x) dS(x) = \int_{-1}^1 \int_{-\pi}^{\pi} \psi(\tau(\lambda, \mu)) d\lambda d\mu \quad (2.2.2)$$

Let us consider now two geographical coordinates maps (Ω_1, f_1) and (Ω_2, f_2) corresponding to different northern poles N_1 and N_2 of sphere S , respectively. If we take pole N_2 so that it had coordinates $(\pi/2, 0)$ in the geographic system (λ_1, μ_1) then the Date line Γ_2 of system (λ_2, μ_2) is a part of the equator of system (λ_1, μ_1) (Fig. 2.7), that is, the lines Γ_1 and Γ_2 do not intersect, and the union $\Omega_1 \cap \Omega_2$ of open sets Ω_1 and Ω_2 is the open cover of sphere S .

Since sphere S is a differentiable compact manifold, the theorem on a partition of unity is valid for S [78].

Fig. 2.7 Open cover of the sphere by two coordinate domains



Theorem 2.2.1 Let $\{\Omega_i\}$ be an open cover of the sphere S by a finite number of coordinate domains Ω_i . There exists a partition of unity $\{\varphi_i(x)\}$ subordinate to the cover $\{\Omega_i\}$, that is, a finite collection of infinitely differentiable functions $\varphi_i(x) \in \mathbb{C}^\infty(S)$ with the following properties:

1. $\text{supp } \varphi_i(x) \subset \Omega_i$ for each i ,
2. $0 \leq \varphi_i(x) \leq 1$ for each i and any $x \in S$,
3. $\sum_i \varphi_i(x) = 1$ for any $x \in S$.

Here $\text{supp } f(x)$ means the support of $f(x)$, i.e. the closure of all points $x \in S$ such that $f(x) \neq 0$.

Due to Theorem 2.2.1, any continuous function $f(x)$ can be represented on S as

$$f(x) = \sum_i f_i(x) \quad (2.2.3)$$

where $f_i(x) = f(x) \cdot \varphi_i(x)$ and $\varphi_i(x)$ is the function from the partition of unity. Note that integral

$$\int_S f(x) dS(x) = \sum_i \int_{\Omega_i} f_i(x) dS(x) \quad (2.2.4)$$

does not depend on the choice of a concrete cover $\{\Omega_i\}$ of sphere S [131].

We now give useful relations between the coordinates (λ, μ) and (λ_1, μ_1) of any point on a sphere [376]:

$$\mu_1 = \mu \mu_0 + \sqrt{1 - \mu^2} \sqrt{1 - \mu_0^2} \cos(\lambda - \lambda_0), \quad (2.2.5)$$

$$\mu = \mu_1 \mu_0 - \sqrt{1 - \mu_1^2} \sqrt{1 - \mu_0^2} \cos \lambda_1, \quad (2.2.6)$$

$$\sqrt{1 - \mu_1^2} \sin \lambda_1 = \sqrt{1 - \mu^2} \sin (\lambda - \lambda_0). \quad (2.2.7)$$

Here (λ_0, μ_0) are the coordinates of pole N_1 of coordinate system (λ_1, μ_1) in the coordinate system (λ, μ) . Equation (2.2.5) gives the value of μ_1 in terms of λ, μ, λ_0 , and μ_0 , while (2.2.6) is the corresponding formula for the inverse transformation. Once μ_1 is calculated from λ, μ, λ_0 , and μ_0 by means of (2.2.5), λ_1 can be obtained from (2.2.6) and (2.2.7). Indeed, these equations give $\cos \lambda_1$ and $\sin \lambda_1$ respectively, which together determine λ_1 . Conversely, once μ is calculated from λ_1, μ_1 , and μ_0 by means of (2.2.6), $\lambda - \lambda_0$ can be obtained from (2.2.5) and (2.2.7).

Note that (2.2.5)–(2.2.7) can also be interpreted as the formulas of transformation of spherical harmonics $Y_1^m(\lambda, \mu)$ ($-1 \leq m \leq 1$) in the subspace $\mathbf{H}_1$ under a rotation of sphere (see Remark 2.1.1 and (2.1.14)).

2.3 Orthogonal Projections on $\mathbf{H}_n$ and Fractional Derivatives

We now introduce operators of projection and fractional differentiation (fractional derivatives) of functions on the unit sphere [3, 104, 151, 318].

Definition 2.3.1 We denote by Φ the space of sequences $\hat{\psi} = \{\psi_n^m\} = \{\psi_\alpha\}$ of Fourier coefficients

$$\psi_n^m = \langle \psi, Y_n^m \rangle \quad (2.3.1)$$

of functions $\psi \in \mathbb{C}^\infty(S)$.

Due to Parseval-Steklov identity [287], the mapping $\mathbb{C}^\infty(S) \mapsto \Phi$ is isometric isomorphism, conserving the inner product (2.1.1) and norm (2.1.2) of elements. Therefore, we will identify the spaces $\mathbb{C}^\infty(S)$ and Φ . Since $(-\Delta)^k \psi \in \mathbb{C}^\infty(S)$ for every $\psi \in \mathbb{C}^\infty(S)$ and any natural k , and

$$(-\Delta)^k \psi = \sum_{n=0}^{\infty} \sum_{m=-n}^n [n(n+1)]^k \psi_n^m Y_n^m, \quad (2.3.2)$$

the Fourier coefficients ψ_n^m of every sequence $\hat{\psi} \in \Phi$ tend to zero as $n \rightarrow \infty$ faster than the sequence $1/n^m$ for any m [117, 169].

Definition 2.3.2 We define Φ^* as dual to Φ space of formal Fourier series $h \sim \sum_{\alpha} h_{\alpha} Y_{\alpha}$ such that the sequence $\hat{h} = \{h_{\alpha}\}$ of Fourier coefficients h_{α} satisfies the condition

$$\sum_{\alpha(0)} \psi_{\alpha} \bar{h}_{\alpha} < \infty \quad (2.3.3)$$

for all $\hat{\psi} \in \Phi$. Note that symbol $\sim$ asserts nothing about the convergence of series. Thus, the space Φ^* of continuous linear functionals

$$h(\psi) \equiv \langle \hat{\psi}, \hat{h} \rangle = \sum_{\alpha} \psi_{\alpha} \bar{h}_{\alpha} \quad (2.3.4)$$

on Φ can be identified (due to isomorphism) with the space of such sequences $\hat{h} = \{h_n^m\}$ whose elements h_n^m increase with n not faster than some degree n [169].

Definition 2.3.3 The Hilbert space

$$\mathbb{L}^2(S) = \{\psi \in \Phi^* : \|\psi\| < \infty\} \quad (2.3.5)$$

is the closure of $\mathbb{C}^{\infty}(S)$ in the norm (2.1.2). It is the space of generalized functions on S with inner product (2.1.1). The space $\mathbb{L}^2(S)$ is the direct orthogonal sum of the subspaces $\mathbf{H}_n$ defined by (2.1.13) [131, 348]:

$$\mathbb{L}^2(S) = \oplus_{n=0}^{\infty} \mathbf{H}_n \quad (2.3.6)$$

Definition 2.3.4 Let ω be an angle between two unit radius-vectors $\vec{x}_1, \vec{x}_2$ corresponding to points $x_1, x_2 \in S$ (see (2.2.1)). A function $z(\vec{x} \cdot \vec{y})$ depending only on the distance $\rho(x, y) = \arccos(\vec{x} \cdot \vec{y}) = \omega$ between two points x, y of sphere is called the zonal function. It is known [131] that the convolution of a function $\psi \in \mathbb{L}^2(S)$ and a zonal function $Z(\vec{x} \cdot \vec{y}) \in \mathbb{L}^2(S)$ is defined by

$$(\psi * z)(x) = \frac{1}{4\pi} \int_S \psi(y) Z(\vec{x} \cdot \vec{y}) dS(y) \quad (2.3.7)$$

Definition 2.3.5 Let n be an integer, $n \geq 0$. The convolution

$$Y_n(\psi) = (2n + 1) (\psi * P_n) \quad (2.3.8)$$

is used to compute the orthogonal projection of any element $\psi \in \mathbb{L}^2(S)$ onto the subspace $\mathbf{H}_n$ of the homogeneous spherical polynomials of degree n [131, 287].

To prove that (2.3.8) is really the projection it is sufficient to show that $Y_n(Y_n(\psi)) = Y_n(\psi)$. First we show that

$$Y_n(\psi(x)) = \sum_{m=-n}^n \psi_n^m Y_n^m(x) \quad (2.3.9)$$

and hence $Y_n(\psi) \in \mathbf{H}_n$. In fact, due to Definition 2.3.5, Theorem 2.1.1, and formulas (2.3.1) and (2.3.7) we have

$$\begin{aligned} Y_n(\psi(x)) &= \frac{2n+1}{4\pi} \int_S \psi(y) P_n(\vec{x} \cdot \vec{y}) dS(y) = \int_S \psi(y) \sum_{m=-n}^n Y_n^m(x) \overline{Y_n^m(y)} dS(y) \\ &= \sum_{m=-n}^n Y_n^m(x) \langle \psi(y), Y_n^m(y) \rangle = \sum_{m=-n}^n \psi_n^m Y_n^m(x) \end{aligned} \quad (2.3.10)$$

Besides, (2.3.9) and Theorem 2.1.1 lead to

$$\begin{aligned} (2n+1)(Y_n(\psi) * P_n)(x) &= \frac{2n+1}{4\pi} \sum_{m=-n}^n \psi_n^m \int_S Y_n^m(y) P_n(\vec{x} \cdot \vec{y}) dS(y) \\ &= \sum_{m=-n}^n \psi_n^m \sum_{k=-n}^n Y_n^k(x) \int_S Y_n^m(y) \overline{Y_n^k(y)} dS(y) \\ &= Y_n(\psi(x)) \end{aligned}$$

and hence $Y_n(Y_n(\psi)) = Y_n(\psi)$ for any function $\psi \in \mathbb{L}^2(S)$.

Obviously, every function from the subspace $\mathbf{H}_0$ is a constant:

$$Y_0(\psi) = \frac{1}{4\pi} \int_S \psi(x) dS = \text{Const} \quad (2.3.11)$$

Definition 2.3.6 Let N be an integer, $N \geq 0$. The finite dimensional subspaces $\mathbb{P}^N$ and $\mathbb{P}_0^N$ of spherical polynomials of degree $n \leq N$ are defined as the direct orthogonal sums of subspaces $\mathbf{H}_n$:

$$\mathbb{P}^N = \oplus_{n=0}^N \mathbf{H}_n, \quad \mathbb{P}_0^N = \oplus_{n=1}^N \mathbf{H}_n = \{\psi \in \mathbb{P}^N : Y_0(\psi) = 0\}. \quad (2.3.12)$$

Thus, every function of $\mathbb{P}_0^N$ is orthogonal to any constant on the sphere.

Definition 2.3.7 The orthogonal projection of elements $\psi \in \mathbb{L}^2(S)$ onto $\mathbb{P}^N$ is defined by

$$T_N \psi(x) = \sum_{n=0}^N Y_n(\psi(x)) = (\psi * S_N)(x) \quad (2.3.13)$$

where

$$S_N(\vec{x} \cdot \vec{y}) = \sum_{n=0}^N (2n+1) P_n(\vec{x} \cdot \vec{y}) \quad (2.3.14)$$

is the convolution kernel [175]. Thus, for each function $\psi(x) \in \mathbb{L}^2(S)$ the element $T_N\psi$ is the triangular truncation of the Fourier series of spherical harmonics of function ψ . Note that Parseval-Steklov's identities

$$\|\psi\|^2 = \sum_{\alpha(0)} |\psi_\alpha|^2 = \sum_{n=0}^{\infty} \|Y_n(\psi)\|^2 \quad (2.3.15)$$

$$\langle \psi, h \rangle = \sum_{\alpha(0)} \psi_\alpha \bar{h}_\alpha = \sum_{n=0}^{\infty} \langle Y_n(\psi), Y_n(h) \rangle \quad (2.3.16)$$

hold for any functions $\psi, h \in \mathbb{L}^2(S)$ [287]. Due to (2.3.15), each function $\psi(x) \in \mathbb{L}^2(S)$ is represented by its own Fourier-Laplace series

$$\psi = \sum_{n=0}^{\infty} Y_n(\psi) \equiv \sum_{n=0}^{\infty} \sum_{m=-n}^n \psi_n^m Y_n^m \quad (2.3.17)$$

that is,

$$\|\psi - T_N \psi\| \rightarrow 0 \quad \text{as } N \rightarrow \infty \quad (2.3.18)$$

Definition 2.3.8 Let $\psi(x) \in \mathbb{C}^\infty(S)$. The spherical operator $\Lambda^s = (-\Delta)^{s/2}$ of real order $s > 0$ is defined by means of relations

$$Y_n(\Lambda^s \psi) = \chi_n^{s/2} Y_n(\psi) = [n(n+1)]^{s/2} Y_n(\psi) \quad (2.3.19)$$

which are valid for any natural number n .

Thus, Λ^s is a multiplier operator which is completely defined by infinite set of multipliers $\{\chi_n^{s/2}\}_{n=0}^{\infty}$ [47, 151, 160]. We will consider Λ^s as the derivative of real order s of functions on a sphere, besides,

$$\Lambda^s \psi(x) = \sum_{n=1}^{\infty} \chi_n^{s/2} Y_n(\psi(x)) \equiv \sum_{\alpha(1)} \chi_\alpha^{s/2} \psi_\alpha Y_\alpha(x) \quad (2.3.20)$$

In particular, if $s = 1$ then operator Λ can be interpreted as the square root of the nonnegative and symmetric Laplace operator (2.1.11).

Definition 2.3.9 Let s be a real number. For all $\psi(x) \in \mathbb{C}^\infty(S)$, we define a multiplier operator $D^s \equiv (E - \Delta)^{s/2}$ by means of relations

$$Y_n(D^s \psi) = d_n^{s/2} Y_n(\psi) \quad (2.3.21)$$

which are valid for any integer number $n \geq 0$, or

$$D^s \psi(x) = \sum_{n=0}^{\infty} d_n^{s/2} Y_n(\psi(x)) \equiv \sum_{\alpha(0)} d_\alpha^{s/2} \psi_\alpha(x) \quad (2.3.22)$$

where

$$d_\alpha \equiv d_n = 1 + \chi_n = 1 + n(n+1) \quad (2.3.23)$$

Remark 2.3.1 Note that unlike operator Λ^s defined only for positive real degree s , operator D^s is defined for any real degree s . However, if we consider only the functions which are orthogonal to the subspace $\mathbf{H}_0$ on the sphere S ($Y_0(\psi) = 0$, see (2.3.11)) then Λ^s can also be defined for any real degree s , and we need not introduce operators D^s .

It is well known that the main disadvantage of local derivatives $\partial^n / \partial \lambda^n$ and $\partial^n / \partial \mu^n$ is that they depend on the choice of a coordinate system, i.e. on a sphere rotation. The new derivatives Λ^s and D^s and projectors Y_n and T_N are invariant with respect to any element of the group $SO(3)$ of sphere rotations [356, 379], and hence they are free from this disadvantage. These operators can be considered as a generalization of the operators $(-\Delta)^n$ and $(E - \Delta)^n$ from integer degrees n to real degrees s . As we will see, the use of derivatives Λ^s and D^s allows specifying the degree of smoothness of functions on the sphere.

2.4 Hilbert Spaces $\mathbb{H}^s$ on a Sphere

In this section, we introduce a family of Hilbert spaces $\mathbb{H}^s$ of generalized functions (distributions) on the sphere, which depends on a real parameter s , besides, a function ψ belongs to $\mathbb{H}^s$ for some s if its s th fractional derivative belongs to the space $\mathbb{L}^2(S)$ [3, 104, 151, 318, 320].

Definition 2.4.1 We denote by $\mathbb{C}_0^\infty(S)$ the space of infinitely differentiable functions which are orthogonal to any constant on a sphere:

$$\mathbb{C}_0^\infty(S) = \{\psi \in \mathbb{C}^\infty(S) : Y_0(\psi) = 0\}$$

Remark 2.4.1 The operator Λ^s is defined on functions of $\mathbb{C}_0^\infty(S)$ by means of (2.3.20) for any real degree s (see Remark 2.3.1).

Definition 2.4.2 For any real s , we introduce in $\mathbb{C}^\infty(S)$ an inner product $\langle \cdot, \cdot \rangle_s$ and a norm $|\cdot|_s$ in the following way:

$$\begin{aligned} (\psi, h)_s &= (D^s \psi, D^s h) \\ &= \sum_{n=0}^{\infty} d_n^s \langle Y_n(\psi), Y_n(h) \rangle \equiv \sum_{\alpha(0)} d_\alpha^s \psi_\alpha \bar{h}_\alpha, \end{aligned} \quad (2.4.1)$$

$$\begin{aligned} |\psi|_s &= |D^s \psi|_0 = (\psi, \psi)_s^{1/2} \\ &= \left\{ \sum_{n=0}^{\infty} d_n^s \|Y_n(\psi)\|^2 \right\}^{1/2} \equiv \left\{ \sum_{\alpha(0)} d_\alpha^s |\psi_\alpha|^2 \right\}^{1/2} \end{aligned} \quad (2.4.2)$$

Nevertheless, in the space $\mathbb{C}_0^\infty(S)$, the inner products $\langle \cdot, \cdot \rangle_s$ and norms $\|\cdot\|_s$ are defined in the other way:

$$\langle \psi, h \rangle_s = \langle \Lambda^s \psi, \Lambda^s h \rangle = \sum_{\alpha(1)} \chi_\alpha^s \psi_\alpha \bar{h}_\alpha \quad (2.4.3)$$

$$\|\psi\|_s = \langle \psi, \psi \rangle_s^{1/2} = \left\{ \sum_{n=1}^{\infty} \chi_n^s \|Y_n(\psi)\|^2 \right\}^{1/2} = \left\{ \sum_{\alpha(1)} \chi_\alpha^s |\psi_\alpha|^2 \right\}^{1/2} \quad (2.4.4)$$

Definition 2.4.3 Let s be a real number. The Hilbert spaces obtained by closing the space $\mathbb{C}^\infty(S)$ in the norm (2.4.2), and the space $\mathbb{C}_0^\infty(S)$ in the norm (2.4.4) we denote as $\mathbb{H}^s$ and $\mathbb{H}_0^s$, respectively.

In particular, $\mathbb{H}^0 \equiv \mathbb{L}^2(S)$. For the sake of brevity we will keep the symbols $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ for the inner product and norm in $\mathbb{H}^0$, that is, $\langle \psi, h \rangle_0 \equiv \langle \psi, h \rangle$ and $\|\psi\|_0 \equiv \|\psi\|$ (see (2.1.1) and (2.1.2)).

It will be shown (see below Lemma 1.4.1) that the embeddings

$$\Phi \equiv \mathbb{C}^\infty(S) \subset \mathbb{H}^r \subset \mathbb{H}^s \subset \mathbb{H}^0 \subset \mathbb{H}^{-s} \subset \mathbb{H}^{-r} \subset \Phi^*$$

are continuous if $0 < s < r$ (see also [3]). Besides, it will be proved that the dual space $(\mathbb{H}^s)^*$ coincides with the space $\mathbb{H}^{-s}$ for all $s > 0$ (see below Lemma 2.4.5).

Let s and r be real numbers. The operator $D^r : \mathbb{C}^\infty(S) \mapsto \mathbb{C}^\infty(S)$ is symmetric:

$$(D^r \psi, h)_s = (\psi, D^r h)_s,$$

and therefore admits extension on the elements of $\mathbb{H}^s$.

Definition 2.4.4 A function (distribution) $z \in \mathbb{H}^s$ is called derivative $D^r \psi$ of a function $\psi \in \mathbb{H}^s$ if

$$(z, h)_s = (\psi, D^r h)_s$$

holds for all $h \in \mathbb{C}^\infty(S)$ where $D^r h$ is defined by (2.3.22).

Remark 2.4.2 Evidently that for any $\psi \in \mathbb{C}^\infty(S)$, the derivative $D^r \psi$ defined by (2.3.22) satisfies Definition 2.4.4. It is easy to show that the extension $D^r : \mathbb{H}^s \mapsto \mathbb{H}^s$ of operator $D^r : \mathbb{C}^\infty(S) \mapsto \mathbb{C}^\infty(S)$ is closed. Indeed, if there exists a sequence $\{\psi_n\} \subset \mathbb{C}^\infty(S)$ such that $|\psi_n - \psi|_s \rightarrow 0$ and $|D^r \psi_n - z|_s \rightarrow 0$ then

$$(z, h)_s = \lim_{n \rightarrow \infty} (D^r \psi_n, h)_s = \lim_{n \rightarrow \infty} (\psi_n, D^r h)_s = (\psi, D^r h)_s$$

and by Definition 2.4.4, $\psi \in \mathfrak{D}(D^r)$ and $z = D^r \psi$, where $\mathfrak{D}(D^r)$ is the domain of operator $D^r : \mathbb{H}^s \mapsto \mathbb{H}^s$. The closed symmetric operators $\Lambda^r : \mathbb{H}^s \mapsto \mathbb{H}^s$ (for $r > 0$) and $\Lambda^r : \mathbb{H}_0^s \mapsto \mathbb{H}_0^s$ (for any real r) are introduced in the similar way.

The following assertions establish the estimates for the embeddings $\mathbb{H}^{s+r} \subset \mathbb{H}^s$ and $\mathbb{H}_0^{s+r} \subset \mathbb{H}_0^s$ if $r > 0$ [320].

Lemma 2.4.1 *Let s be a real, $r > 0$, $\psi \in \mathbb{H}^{s+r}$ and $z \in \mathbb{H}_0^{s+r}$. Then the following assertions hold:*

1. $\psi \in \mathbb{H}^s$ and

$$|\psi|_s \leq |\psi|_{s+r} \quad (2.4.5)$$

$$|\psi|_{s+r} = |D^r \psi|_s \quad (2.4.6)$$

2. $z \in \mathbb{H}_0^s$ and

$$\|z\|_s \leq 2^{-r/2} \|z\|_{s+r} \quad \text{and} \quad \|\psi\|_{s+r} = \|\Lambda^r \psi\|_s \quad (2.4.7)$$

Proof First let $\psi \in \mathbb{C}^\infty(S)$ and $h = D^s \psi$. Then it is obvious that (2.4.6) is valid. By (2.4.2) and (2.3.23) we have

$$\begin{aligned} |\psi|_{s+r}^2 &= |D^{s+r} \psi|_0^2 = |D^r (D^s \psi)|_0^2 = |D^r h|_0^2 = \sum_{\alpha(0)} d_\alpha^r |h_\alpha|^2 \\ &\geq \min_{n \geq 0} \{d_n^r\} |h|_0^2 = |\psi|_s^2 \end{aligned}$$

Similarly, if $z \in \mathbb{C}_0^\infty(S)$ and $h = \Lambda^s z$ then (2.4.4) implies

$$\begin{aligned} \|z\|_{s+r}^2 &= \|\Lambda^{s+r} z\|^2 = \|\Lambda^r h\|^2 = \sum_{\alpha(1)} \chi_\alpha^r |h_\alpha|^2 \\ &\geq \min_{n \geq 1} \{n(n+1)\}^r \|h\|^2 = 2^r \|z\|_s^2 \end{aligned}$$

We now consider the general case when $\psi \in \mathbb{H}^{s+r}$. Then there exists a Cauchy sequence $\{\psi_n\}$ of functions $\psi_n \in \mathbb{C}^\infty(S)$ such that $|\psi_n - \psi|_{s+r} \rightarrow 0$. Since $\{\psi_n\}$ and $\{D^r \psi_n\}$ are Cauchy sequences in $\mathbb{H}^s$ and (2.4.6) is valid for each ψ_n , we have

$$|\psi|_{s+r} = \lim_{n \rightarrow \infty} |\psi_n|_{s+r} = \lim_{n \rightarrow \infty} |D^r \psi_n|_s = |D^r \psi|_s$$

We used here the fact that operator $D^r : \mathbb{H}^s \mapsto \mathbb{H}^s$ is closed. The assertions (2.4.5) and (2.4.7) for $\psi \in \mathbb{H}^{s+r}$ and $z \in \mathbb{H}_0^s$ are proved similarly. *Q.E.D.*

Corollary 2.4.1 *Let r and s be real numbers, and $r > 0$. Then operator $D^{-r} : \mathbb{H}^s \mapsto \mathbb{H}^s$ is bounded: $|D^{-r} \psi|_s \leq |\psi|_s$. The equality is realized at $\psi = \text{const}$, that is, the spectral norm of operator $D^{-r} : \mathbb{H}^s \mapsto \mathbb{H}^s$ equals unity and is independent of s and r . The spectral norm of operator $\Lambda^{-r} : \mathbb{H}_0^s \mapsto \mathbb{H}_0^s$ equals $2^{-r/2}$, and hence is independent of s .*

Remark 2.4.3 The equalities (2.4.6) and (2.4.7) are valid for any real r . In this book, we will study the dynamics of an incompressible barotropic fluid on a rotating sphere, and all the functions under consideration will be orthogonal to any constant on the sphere. Therefore, in what follows, we will mainly use the norms (2.4.4).

The next assertion follows from (2.4.6).

Corollary 2.4.2 *Let r and s be real numbers. The mapping $D^r : \mathbb{H}^{s+r} \mapsto \mathbb{H}^s$ is isometry and isomorphism. In particular, at $r = -2s$, the operator $D^{-2s} : \mathbb{H}^{-s} \mapsto \mathbb{H}^s$ is isometric isomorphism.*

The next lemma is the Poincaré inequality for functions on a sphere [46]:

Lemma 2.4.2 *If $\psi \in \mathbb{H}_0^1$ then*

$$\|\psi\| \leq 1/\sqrt{2} \|\nabla \psi\| \quad (2.4.8)$$

Proof Let $\psi \in \mathbb{C}_0^\infty(S)$ and let $\Omega_1 \cup \Omega_2$ be a cover of sphere S by coordinate domains $\Omega_i = S \setminus \Gamma_i$ (see Fig. 2.7). Due to Theorem 2.2.1, there exists a partition of unity $\{\varphi_i\}$ subordinate to the cover $\{\Omega_i\}$ ($i = 1, 2$). Using geographical coordinates maps (see Sect. 2.3) and Theorem 2.2.1, we obtain

$$\begin{aligned} \|\nabla \psi\|^2 &= \int_S \nabla \psi \cdot \nabla \bar{\psi} dS = \sum_{i=1}^2 \int_{\Omega_i} \nabla \psi(x) \cdot \nabla (\varphi_i(x) \bar{\psi}(x)) dS(x) \\ &= \sum_{i=1}^2 \int_{\Pi_i} \nabla \psi(\tau(\lambda_i, \mu_i)) \cdot \nabla [\varphi_i(\tau(\lambda_i, \mu_i)) \bar{\psi}(\tau(\lambda_i, \mu_i))] d\lambda_i d\mu_i \\ &= \sum_{i=1}^2 \int_{\Pi_i} [-\Delta \psi(\tau(\lambda_i, \mu_i))] \varphi_i(\tau(\lambda_i, \mu_i)) \bar{\psi}(\tau(\lambda_i, \mu_i)) d\lambda_i d\mu_i \\ &= \sum_{i=1}^2 \int_{\Omega_i} [-\Delta \psi(x)] \varphi_i(x) \bar{\psi}(x) dS(x) = \int_S \left(\sum_{i=1}^2 \varphi_i \right) [-\Delta \psi] \bar{\psi} dS \\ &= \langle -\Delta \psi, \psi \rangle = \langle \Lambda^2 \psi, \psi \rangle = \|\Lambda \psi\|^2 \equiv \sum_{\alpha(1)} \chi_\alpha |\psi_\alpha|^2 \geq 2 \|\psi\|^2 \end{aligned}$$

Let now $\psi \in \mathbb{H}_0^1$, and $\|\psi_n - \psi\|_1 \rightarrow 0$ where $\psi_n \in \mathbb{C}_0^\infty(S)$. Then standard passage to the limit concludes the proof. *Q.E.D.*

Thus, we also proved that

$$\|\nabla \psi\| = \|(-\Delta)^{1/2} \psi\| = \|\Lambda \psi\| = \|\psi\|_1 \quad (2.4.9)$$

for all $\psi \in \mathbb{H}_0^1$. The last equality means that Poincaré inequality (2.4.8) is the particular case of (2.4.7) for $s = 0$ and $r = 1$. In fact, a more general assertion than Lemma 2.4.2 is valid:

Lemma 2.4.3 *Let r, s , and t be real numbers, $r < t$, $a = \sqrt{2}$, and $\psi \in \mathbb{H}_0^{s+t}$. Then*

$$\|\Lambda^r \psi\|_s \leq a^{r-t} \|\Lambda^t \psi\|_s \quad (2.4.10)$$

Proof The proof of lemma at $s = 0$ is similar to that of (2.4.7). Indeed, let $\psi \in \mathbb{C}^\infty(S)$ and $h = \Lambda^r \psi$. Then

$$\|\Lambda^t \psi\|^2 = \|\Lambda^{t-r} h\|^2 = \sum_{\alpha(1)} \chi_\alpha^{t-r} |h_\alpha|^2 \geq \min_{n \geq 1} \{\chi_n^{t-r}\} \|h\|^2 = 2^{t-r} \|\Lambda^r \psi\|^2$$

In the case when $\psi \in \mathbb{H}_0^t$ we consider a sequence $\psi_n \in \mathbb{C}^\infty(S)$ such that $\psi_n \rightarrow \psi$ and then pass to the limit. *Q.E.D.*

Since the operators Λ^r and Λ^s commute, Lemma 2.4.3 is valid for any real number s :

$$\begin{aligned} \|\Lambda^r \psi\|_s &= \|\Lambda^s \Lambda^r \psi\| = \|\Lambda^r (\Lambda^s \psi)\| \leq a^{r-t} \|\Lambda^t (\Lambda^s \psi)\| \\ &= a^{r-t} \|\Lambda^s (\Lambda^t \psi)\| = a^{r-t} \|\Lambda^t \psi\|_s. \quad \text{Q.E.D.} \end{aligned}$$

The following proposition asserts the equivalence of the norms defined by means of operators Λ^r and D^s [318].

Lemma 2.4.4 *Let $s > 0$ and $b = \sqrt{3/2}$. Then the following inequalities are valid:*

$$\frac{1}{2} (|\psi|_0 + |\Lambda^s \psi|_0) \leq |\psi|_s \leq b^s (|\psi|_0 + |\Lambda^s \psi|_0) \quad \text{for any } \psi \in \mathbb{H}^s, \quad (2.4.11)$$

$$\|\Lambda^s \psi\| \leq \|D^s \psi\| \leq b^s \|\Lambda^s \psi\| \quad \text{for any } \psi \in \mathbb{H}_0^s, \quad (2.4.12)$$

$$b^{-s} \|\Lambda^{-s} \psi\| \leq \|D^{-s} \psi\| \leq \|\Lambda^{-s} \psi\| \quad \text{for any } \psi \in \mathbb{H}_0^{-s}. \quad (2.4.13)$$

Proof We prove only (2.4.11). The other assertions are proved in the same way. Let $\psi \in \mathbb{H}^s$. Then the right part of inequality (2.4.11) follows from the estimate

$$\begin{aligned} |\psi|_s^2 &= \sum_{\alpha(0)} d_\alpha^s |\psi_\alpha|^2 = |\psi|_0^2 + \sum_{\alpha(1)} \chi_\alpha^s (1 + \chi_\alpha^{-1})^s |\psi_\alpha|^2 \\ &\leq |\psi|_0^2 + \left(\frac{3}{2}\right)^s |\Lambda^s \psi|_0^2 \leq b^{2s} (|\psi|_0 + |\Lambda^s \psi|_0)^2 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 |\psi|_s^2 &= |\psi_0|^2 + \sum_{\alpha(1)} (1 + \chi_\alpha)^s |\psi_\alpha|^2 \\
 &\geq |\psi_0|^2 + \frac{1}{2} \sum_{\alpha(1)} |\psi_\alpha|^2 + \frac{1}{2} \sum_{\alpha(1)} \chi_\alpha^s |\psi_\alpha|^2 \\
 &\geq \frac{1}{2} \left(|\psi|_0^2 + |\Lambda^s \psi|_0^2 \right) \geq \frac{1}{4} (|\psi|_0 + |\Lambda^s \psi|_0)^2
 \end{aligned}$$

This demonstrates the validity of left part of (2.4.11). *Q.E.D.*

It is easy to prove some inequalities for functions from Hilbert spaces $\mathbb{H}^s$. Indeed, let s be a real and $r \geq 0$. Due to (2.4.1),

$$(\psi, h)_s = \sum_{\alpha(0)} (d_\alpha^{(s+r)/2} \psi_\alpha) (d_\alpha^{(s-r)/2} \overline{h_\alpha})$$

and therefore both the generalized Schwartz inequality

$$|(\psi, h)_s| \leq |\psi|_{s+r} |h|_{s-r} \quad (2.4.14)$$

and the estimate

$$|\psi|_s^2 \leq |\psi|_{s+r} |\psi|_{s-r} \quad (2.4.15)$$

hold for any $\psi \in \mathbb{H}^{s+r}$ and $h \in \mathbb{H}^{s-r}$. In the particular case that $s = 0$,

$$|(\psi, h)_0| \leq |\psi|_r |h|_{-r}$$

for all $\psi \in \mathbb{H}^r$ and $h \in \mathbb{H}^{-r}$, and

$$|\langle \psi, h \rangle| \leq \|\psi\|_r \|h\|_{-r} \quad (2.4.16)$$

for all $\psi \in \mathbb{H}_0^r$ and $h \in \mathbb{H}_0^{-r}$.

We now obtain two interpolating inequalities well known in the theory of periodic functions [33, 171, 303]. Let r, s , and t be real numbers, $t \leq r < s$, $\rho > 0$, and $\psi \in \mathbb{H}^s$. If we take $a = \rho^{1/(s-r)} d_\alpha$ in the inequality

$$1 \leq a^{s-r} + a^{t-r}, \quad a > 0$$

(d_α is the multiplier (2.3.23)) then we get

$$d_\alpha^r \leq \rho d_\alpha^s + \rho^{p/(p-1)} d_\alpha^t$$

where

$$p = \frac{r-t}{s-t} < 1 \quad (2.4.17)$$

Taking into account (2.4.2), we obtain the first interpolating inequality (ρ is a positive number; see also [3]):

$$|\psi|_r \leq \rho |\psi|_s + \rho^{p/(p-1)} |\psi|_t \quad (2.4.18)$$

If we now take

$$\rho = |\psi|_s^{p-1} |\psi|_t^{1-p}$$

in (2.4.18) then we derive the second interpolating inequality

$$|\psi|_r \leq 2 |\psi|_t^{1-p} |\psi|_s^p \quad (2.4.19)$$

We now give the common form of a continuous linear functional on $\mathbb{H}_0^s$ [318].

Lemma 2.4.5 *Let $s \geq 0$. Then for all $h \in \mathbb{H}_0^{-s}$ a sesquilinear form $G(\psi) = \langle \psi, h \rangle$ is a continuous linear functional on $\mathbb{H}_0^s$ with the norm*

$$|||G||| = \|h\|_{-s} = \sup_{\|\psi\|_s=1} \langle \psi, h \rangle$$

Conversely, every continuous linear functional on $\mathbb{H}_0^s$ has the form $\langle \psi, h \rangle$ where $h \in \mathbb{H}_0^{-s}$ is uniquely determined element.

Proof Let $G(\psi)$ be a continuous linear functional on $\mathbb{H}_0^s$. Then there exists an element $g \in \mathbb{H}_0^s$ such that $G(\psi) = \langle \psi, g \rangle_s$ and $|||G||| = \|g\|_s$. Let us take $h = \Lambda^{2s}g$. Due to Corollary 2.4.2, $h \in \mathbb{H}_0^{-s}$. It follows from (2.4.6) that $\|h\|_{-s} = \|g\|_s$ and

$$G(\psi) = \langle \psi, g \rangle_s = \langle \psi, \Lambda^{2s}g \rangle = \langle \psi, h \rangle$$

Let us show the uniqueness of such representation. Suppose that $G(\psi) = \langle \psi, h \rangle = \langle \psi, z \rangle$ where $h, z \in \mathbb{H}_0^{-s}$. Then we have $\langle \psi, h - z \rangle = 0$. Taking $\psi = \Lambda^{-2s}(h - z)$ we obtain $\|h - z\|_{-s} = 0$, and hence $h = z$. *Q.E.D.*

Remark 2.4.4 For each real s , the functions

$$W_{s,\alpha} = D^{-s}Y_\alpha = d_\alpha^{-s/2}Y_\alpha \quad (2.4.20)$$

form orthonormal basis in $\mathbb{H}^s$, and

$$\psi = \sum_{\alpha(0)}^N q_{s,\alpha} W_{s,\alpha} = \sum_{\alpha(0)}^N \psi_\alpha Y_\alpha, \quad q_{s,\alpha} = d_\alpha^{s/2} \psi_\alpha$$

for all $\psi \in \mathbb{H}^s$, where d_α is determined by (2.3.23) and Y_α is the spherical harmonic (2.1.3).

Definition 2.4.5 Let $\{\varphi_i\}$ and $\{\psi_j\}$ be dual orthonormal bases: $(\varphi_i, \psi_j) = \delta_{ij}$ where $(\cdot, \cdot)$ is an inner product. A compact operator A defined by

$$Ag = \sum_j (\varphi_j, g) \alpha_j \psi_j$$

is called k -operator if $\alpha_j > 0$ for all j and $\sum_j \alpha_j < \infty$ [287, Chap. 12].

Theorem 2.4.1 ([320]) Let s be a real and $r > 0$. Then the bounded operators $D^{-r} : \mathbb{H}^s \mapsto \mathbb{H}^s$ and $\Lambda^{-r} : \mathbb{H}_0^s \mapsto \mathbb{H}_0^s$ are compact operators at $r > 0$, Hilbert-Schmidt operators at $r > 1$, and k -operators at $r > 2$.

Proof The multipliers $\chi_\alpha^{-r/2}$ and $d_\alpha^{-r/2}$ in the canonical expansions (2.3.20) and (2.3.22) of operators Λ^{-r} and D^{-r} tend to zero if index α increases. Due to the theorem from Chap. 12 of work [287], the operators D^{-r} and Λ^{-r} are compact. Further, both the series

$$\sum_{\alpha(1)} \|\Lambda^{-r} Y_\alpha\|^2 = \sum_{\alpha(1)} \chi_\alpha^{-r} = \sum_{n=1}^{\infty} \frac{2n+1}{[n(n+1)]^r}$$

and

$$\sum_{\alpha(0)} |D^{-r} Y_\alpha|_0^2 = \sum_{\alpha(0)} d_\alpha^{-r} = \sum_{\alpha(0)} (1 + \chi_\alpha)^{-r} = 1 + \sum_{\alpha(1)} \|\Lambda^{-r} Y_\alpha\|^2$$

diverge if $r \leq 1$ and converge at $r > 1$. This follows immediately from the comparison of these series with the series $\sum_n n^{-r}$ [382]. Here we used the relation

$$\frac{2n+1}{[n(n+1)]^r} = \frac{1}{n^{r-1}(n+1)^r} + \frac{1}{n^r(n+1)^{r-1}}$$

Hence, both the series $\sum_{\alpha(1)} \chi_\alpha^{-r/2}$ and $\sum_{\alpha(0)} d_\alpha^{-r/2}$ converge at $r > 2$. As a result, the validity of the other statements of theorem follows from [287] (lemma of Chap. 12). *Q.E.D.*

Remark 2.4.5 Thus, if $r > 1$ then one can define the maximal (Hilbert-Schmidt) norms

$$\|D^{-r}\|_* = \left\{ \sum_{\alpha(0)} d_\alpha^{-r} \right\}^{1/2} \quad \text{and} \quad \|\Lambda^{-r}\|_* = \left\{ \sum_{\alpha(1)} \chi_\alpha^{-r} \right\}^{1/2}, \quad (2.4.21)$$

And if $r > 2$ then we can also introduce the traces

$$\text{Spur}(D^{-r}) = \sum_{\alpha(0)} d_\alpha^{-r/2}, \quad \text{Spur}(\Lambda^{-r}) = \sum_{\alpha(1)} \chi_\alpha^{-r/2} \quad (2.4.22)$$

of matrices of operators $D^{-r} : \mathbb{H}^s \mapsto \mathbb{H}^s$ and $\Lambda^{-r} : \mathbb{H}_0^s \mapsto \mathbb{H}_0^s$, respectively. Note that values (2.4.21) and (2.4.22) are independent of the choice of orthonormal basis in $\mathbb{H}^s$ and $\mathbb{H}_0^s$ [166].

The following statement is the corollary of Theorem 2.4.1.

Lemma 2.4.6 *Let s be a real and $r > 0$. Then each set M bounded in $\mathbb{H}^{s+r}$ is compact in $\mathbb{H}^s$.*

Proof Let $\{\psi_n\} \subset M$ be a sequence that weakly converges in $\mathbb{H}^{s+r}$. We now show that $\{\psi_n\}$ is the Cauchy sequence in $\mathbb{H}^s$. Indeed, due to (2.4.6),

$$|\psi_n - \psi_m|_s = |D^{-r}(\psi_n - \psi_m)|_{s+r}$$

But, due to Theorem 2.4.1, the sequence $\{D^{-r}\psi_n\}$ converges in $\mathbb{H}^{s+r}$ -norm. Therefore, $\{\psi_n\}$ is the Cauchy sequence in $\mathbb{H}^s$. *Q.E.D.*

Remark 2.4.6 In fact, it is sufficient to prove that the embedding $\mathbb{H}^s$ to $\mathbb{H}^t$ is compact for some fixed number $t < s$. Then the estimate (2.4.19) implies that the embedding $\mathbb{H}^s$ to $\mathbb{H}^r$ is compact for all $r \in (t, s)$. The Lemma 2.4.6 is a particular case of Rellich's lemma [40].

Lemma 2.4.7 ([3]) *Let $s > 1$. Then the space $\mathbb{H}^s$ is a normed ring, i.e. if $\psi, h \in \mathbb{H}^s$ then $\psi h \in \mathbb{H}^s$ and*

$$\|\psi h\|_s \leq C \|\psi\|_s \|h\|_s \quad (2.4.23)$$

where C is a positive constant that depends only on s .

The convergence of finite spherical sums (spherical polynomials) was analyzed in detail in [37, 369]. Due to (2.3.18), each function $\psi(x) \in \mathbb{L}^2(S)$ is approximated by finite sums $T_N \psi$ of the subspace $\mathbb{P}^N$ of spherical polynomials of degree N . However, the rate of convergence is not determined here. We now give a simple lemma about the order of approximation of a smooth function on a sphere by finite sums (2.3.13) [318].

Lemma 2.4.8 *Let s be a real, $r > 0$, and $\psi \in \mathbb{H}^{s+r}$. Then the best approximation in $\mathbb{H}^s$ -norm is given by the following estimate:*

$$|\psi - T_N \psi|_s \leq N^{-r} |\psi|_{s+r} \quad (2.4.24)$$

If $\psi \in \mathbb{H}_0^{s+r}$ then

$$\|\psi - T_N \psi\|_s \leq N^{-r} \|\psi\|_{s+r} \quad (2.4.25)$$

Proof We prove only (2.4.24). Since $\chi_{N+1} = (N+1)(N+2) > N^2$, then due to (2.3.23), $d_{N+1} = 1 + \chi_{N+1} \geq N^2$. Therefore, for all $\psi \in \mathbb{H}^{s+r}$, we have

$$\begin{aligned} |\psi - T_N \psi|_s^2 &= \sum_{\alpha(N+1)} d_\alpha^s |\psi_\alpha|^2 \\ &\leq d_{N+1}^{-r} \cdot \sum_{\alpha(N+1)} d_\alpha^{s+r} |\psi_\alpha|^2 \\ &\leq N^{-2r} |\psi|_{s+r}^2. \quad Q.E.D. \end{aligned}$$

We see that the application of real derivatives Λ^s and D^r allows one to specify the degree of smoothness of functions on a sphere. In fact, the degree of smoothness of a function ψ is completely determined by the rate of decrease of its Fourier coefficients ψ_n^m as n grows [238, 333, 409]. The theory of best approximation of functions on a sphere in Banach spaces $\mathbb{L}^p(S)$ ($1 \leq p \leq \infty$) is developed in [151, 161, 205, 258, 259, 295–297].

Remark 2.4.7 Let s be a real and $r > 0$. The Bernstein-Nikolskii and Favard inequalities

$$|\psi|_{s+r} = |D^r \psi|_s \leq 3^{r/2} N^r |\psi|_s \quad \text{and} \quad \|z\|_{s+r} = \|\Lambda^r z\|_s \leq N^r \|z\|_s \quad (2.4.26)$$

hold for any spherical polynomials $\psi \in \mathbb{P}^N$ and any $z \in \mathbb{P}_0^N$. The proof of estimates (2.4.26) is trivial. These estimates, together with (2.4.5) and (2.4.7), demonstrate the equivalence of norms $|\psi|_s$ and $|\psi|_{s+r}$ in the finite-dimensional space $\mathbb{P}^N$ and also the equivalence of norms $\|z\|_s$ and $\|z\|_{s+r}$ in $\mathbb{P}_0^N$ [118, 159, 266, 267].

2.5 Space $\mathbb{C}(S)$ of Continuous Functions on a Sphere

Let $\mathbb{C}(S)$ and $\mathbb{C}_0(S) = \{\psi \in \mathbb{C}(S) : Y_0(\psi) = 0\}$ be Banach spaces of continuous functions on the unit sphere S equipped with the norm

$$\|\psi\|_{\mathbb{C}(S)} = \max_{x \in S} |\psi(x)| \quad (2.5.1)$$

It is known [131] that if a function $\psi \in \mathbb{C}^\infty(S)$ then

$$\left\| \sum_{n=N}^{\infty} d_n^{s/2} |Y_n(\psi)| \right\|_{\mathbb{C}(S)} \rightarrow 0 \quad \text{as } N \rightarrow \infty \quad \text{for any } s > 0, \quad (2.5.2)$$

where $Y_n(\psi)$ is the homogeneous spherical polynomial of degree n defined by (2.3.8), that is, Fourier-Laplace series of ψ converges absolutely and uniformly for each $s > 0$.

Suppose now that ψ is a function of some Banach space $\mathbb{X}$ on a sphere, and consider the problem of approximation of ψ by linear sums

$$U_N(\psi) = \sum_{n=0}^N a_{nN} Y_n(\psi) \quad (2.5.3)$$

where $\{a_{nN}\}$ is a triangle matrix, besides,

$$a_{0N} = 1 \text{ for any } N, \text{ and } a_{nN} = 0 \text{ if } n > N$$

In other words, we are interested when the distance $\|U_N(\psi) - \psi\|_{\mathbb{X}}$ will tend to zero for each function $\psi \in \mathbb{X}$ as $N \rightarrow \infty$. It is known that the most common criterion of such convergence is based on the Banach-Steinhaus theorem about weak convergence of a sequence of continuous linear functionals on $\mathbb{X}$ [169]. In the particular case of functions on a sphere from a Hilbert space $\mathbb{H}^s$ the answer is positive if as the linear sums (2.5.3) we examine the finite sums (2.3.13):

$$U_N(\psi) = T_N \psi$$

In this case $a_{nN} = 1$ in (2.5.3) for any n and N (see Lemma 2.4.8). This follows immediately from the Parseval-Steklov identity (2.4.2). In a general Banach space such identities are not valid, and the convergence problem is substantially more complicated.

Now we consider the case $\mathbb{X} = \mathbb{C}(S)$ analyzed in detail in some works (see, for example, [369]). Each function (2.5.3) can be represented as the convolution

$$U_N(\psi) = (\psi * K_N) \quad (2.5.4)$$

where the zonal function K_N is given by

$$K_N(\vec{x} \cdot \vec{y}) = \sum_{n=0}^N (2n+1) a_{nN} P_n(\vec{x} \cdot \vec{y}) \quad (2.5.5)$$

Theorem 2.5.1 ([369]) *Let $\psi \in \mathbb{C}(S)$. Then a sequence $U_N(\psi(x))$ converges to $\psi(x)$ in a point $x \in S$, that is*

$$\lim_{N \rightarrow \infty} U_N(\psi(x)) = \psi(x), \quad (2.5.6)$$

if and only if the following two conditions are satisfied:

$$\lim_{N \rightarrow \infty} a_{nN} = 1, \quad (n = 1, 2, 3, \dots, N) \quad (2.5.7)$$

and

$$\int_{-1}^1 |K_N(\mu)| d\mu \leq C_0 \quad (2.5.8)$$

where C_0 is a constant not depending on N , $\mu = \vec{x} \cdot \vec{y} = \sin \varphi$, and φ is the latitude in the geographic coordinate system with the point x as its pole.

Proof If $x \in S$ is fixed then elements $U_N(\psi(x))$ and $\psi(x)$ can be considered as continuous linear functionals on $\mathbb{C}(S)$. According to Banach-Steinhaus criterion, (2.5.6) is valid if and only if it holds for every spherical harmonic $Y_\alpha(x)$ (the complete set of $Y_\alpha(x)$ forms a basis in $\mathbb{C}(S)$) and the norms of the functionals $U_N(\psi(x))$ are bounded uniformly for all N . But the validity of (2.5.6) for each spherical harmonic implies (2.5.7). Further, let us calculate the norm of the functional $U_N(\psi(x))$:

$$\begin{aligned} \|U_N\|_C &= \sup_{\|\psi\|_{\mathbb{C}(S)} \leq 1} |U_N(\psi(x))| \\ &= \frac{1}{4\pi} \int_S |K_N(\vec{x} \cdot \vec{y})| dS(y) \\ &= \frac{1}{2} \int_{-1}^1 |K_N(\mu)| d\mu \end{aligned} \quad (2.5.9)$$

Hence, the uniform boundedness of the norms of $U_N(\psi)$ with respect to N means that (2.5.8) is also satisfied.

It is easily seen that (2.5.6) implies the uniform convergence of $U_N(\psi(x))$ to $\psi(x)$ on the sphere S . Indeed, it is shown in [369] that for each function $\psi \in \mathbb{C}(S)$ and arbitrarily small $\epsilon > 0$ there exists a spherical polynomial

$$\psi_M = \sum_{\alpha(0)}^M b_\alpha Y_\alpha$$

of degree M from subspace $\mathbb{P}^M$ such that $\|\psi - \psi_M\|_{\mathbb{C}(S)} < \epsilon$. It follows from (2.5.8) and (2.5.9) that

$$\begin{aligned} \|\psi - U_N(\psi)\|_{\mathbb{C}(S)} &\leq \|\psi - \psi_M\|_{\mathbb{C}(S)} \\ &\quad + \|U_N(\psi - \psi_M)\|_{\mathbb{C}(S)} + \|\psi_M - U_N(\psi_M)\|_{\mathbb{C}(S)} \\ &\leq (1 + \tfrac{1}{2}C_0)\epsilon + \sum_{\alpha(0)}^M |b_\alpha| \|Y_\alpha - U_N(Y_\alpha)\|_{\mathbb{C}(S)} \end{aligned}$$

where due to (2.5.6) and (2.5.7), the last term in RHS can be made less than ϵ for sufficiently large number N .

Thus, the conditions (2.5.7) and (2.5.8) completely solve the problem of uniform convergence of linear sums (2.5.3) to continuous function ψ on a sphere. *Q.E.D.*

If we use in (2.5.3) Cesaro (C, α) sums of the series (2.3.17) ($\alpha > -1$; see [37]) then the kernel K_N in (2.5.5) will have the following form

$$K_N^\alpha(\vec{x} \cdot \vec{y}) = \frac{1}{C_{N+\alpha}^\alpha} \sum_{n=0}^N (2n+1) C_{N+\alpha-n}^\alpha P_n(\vec{x} \cdot \vec{y}) \quad (2.5.10)$$

Here C_k^α is the binomial coefficient. The next theorem estimates the kernel K_N^α for different values of α .

Theorem 2.5.2 ([123, p. 347]) *Let*

$$L_N^\alpha = \int_{-1}^1 |K_N^\alpha(\mu)| d\mu \quad (2.5.11)$$

Then

$$L_N^\alpha \sim N^{\frac{1}{2}-\alpha} \quad \text{at} \quad \alpha < \frac{1}{2},$$

$$L_N^\alpha \sim \ln N \quad \text{at} \quad \alpha = \frac{1}{2},$$

$$0 < L_N^\alpha < C \quad \text{at} \quad \alpha > \frac{1}{2}.$$

Here $\sim$ is the symbol of equivalence. The proof of Theorem 2.5.2 for the general case of a multidimensional sphere is given in [369]; Theorem 2.1.

In particular, it follows from Theorem 2.5.2 that the sequence of finite Fourier sums $T_N \psi$ converges uniformly to ψ not for each function ψ from space $\mathbb{C}(S)$. Indeed, in this case the kernel is

$$S_N(\vec{x} \cdot \vec{y}) \equiv K_N^0(\vec{x} \cdot \vec{y})$$

($\alpha = 0$) and, due to Theorem 2.5.2, condition (2.5.8) of Theorem 2.5.1 is not satisfied. Therefore, by Banach-Steinhaus theorem, there exist such functions ψ in $\mathbb{C}(S)$ for which the sums $T_N \psi$ do not converge to ψ in the norm (2.5.1). Moreover, the unboundedness of norms (2.5.9) of the functionals $U_N(\psi)$ implies that for each point $x \in S$ there exists a function $\psi \in \mathbb{C}(S)$ whose Fourier-Laplace series diverges in the point x . However, due to the same Theorem 2.5.2, the arithmetic mean sums, that is, Cesaro $(C, 1)$ sums

$$U_N(\psi(x)) = \frac{1}{N+1} \sum_{n=0}^N T_N \psi(x) = (\psi * K_N^1)(x) \quad (2.5.12)$$

converge uniformly to $\psi(x)$ for all $\psi(x) \in \mathbb{C}(S)$.

We now give three definitions and, in addition to Lemma 2.4.8, one more relation between the degree of smoothness of a function on the sphere and the order of its best approximation by spherical sums [45, 175].

Definition 2.5.1 For each function $\psi \in \mathbb{C}(S)$ the operator of spherical translation by an angle h is defined as

$$S_h \psi(x) = \frac{1}{2\pi \sin h} \oint_{\rho(x,y)=h} \psi(y) dy \quad (2.5.13)$$

where $\rho(x, y)$ is the inner metric (2.2.1) on a sphere, and the integration is carried out over all such points $y \in S$ that the angle between two unit radius vectors $\vec{x}$ and $\vec{y}$ is equal to h . In other words, the spherical translation of a function $\psi(x)$ by some angle h is the mean value of $\psi(x)$ on the closed contour, all points of which are removed from the center point x at the distance h (see Fig. 2.5).

Definition 2.5.2 The quantity

$$\omega_\psi(\gamma) = \sup_{h \leq \gamma} \|\psi(x) - S_h \psi(x)\|_{\mathbb{C}(S)} \quad (2.5.14)$$

is called modulus of continuity of a function $\psi(x) \in \mathbb{C}(S)$.

Concerning the properties of the modulus of continuity see [28, 87, 171].

Definition 2.5.3 ([175]) We say that a continuous function ψ belongs to a class $Lip(\alpha)$ of Lipschitz functions on a sphere ($0 < \alpha < 2$) if

$$\omega_\psi(\gamma) \leq M \gamma^\alpha \quad (2.5.15)$$

Besides, if $\alpha = 2$ then

$$\omega_\psi(\gamma) \leq M \sin^2\left(\frac{\gamma}{2}\right) \ln\left(1/\sin^2 \frac{\gamma}{2}\right)$$

And if $\alpha > 2$ then $\Delta \psi = 0$ and $\psi = \text{const.}$

Theorem 2.5.3 ([175]) Let $n \geq 0$ and $\psi(x) \in \mathbb{C}(S)$. Then $\Delta^n \psi \in Lip(\alpha)$ if and only if there exists a spherical sum $U_N(\psi(x))$ of $\mathbb{P}^N$ such that

$$\|\psi(x) - U_N(\psi(x))\|_{\mathbb{C}(S)} \leq M \cdot N^{-(2n+\alpha)}$$

The theory of best approximation of continuous functions on a sphere is developed in [45, 87, 160, 206].

In conclusion of this section, we establish a sufficient condition for a function to belong to $\mathbb{C}(S)$.

Theorem 2.5.4 Let $r > 1$, $\psi(x) \in \mathbb{H}^r$ and $z(x) \in \mathbb{H}_0^r$. Then $\psi(x)$ and $z(x)$ can be changed only on a set of measure zero so that to belong to $\mathbb{C}(S)$ and $\mathbb{C}_0(S)$, respectively. Moreover, we have

$$\|\psi\|_{\mathbb{C}(S)} \leq K_r \|\psi\|_r \quad (2.5.16)$$

and

$$\|z\|_{\mathbb{C}_0(S)} \leq M_r \|z\|_r \quad (2.5.17)$$

where

$$K_r = \|D^{-r}\|_* / \sqrt{4\pi}, \quad M_r = \|\Lambda^{-r}\|_* / \sqrt{4\pi} \quad (2.5.18)$$

and $\|D^{-r}\|_*$ and $\|\Lambda^{-r}\|_*$ are the Hilbert-Schmidt norms (2.4.21) of Hilbert-Schmidt operators D^{-r} and Λ^{-r} .

Proof Let $\psi \in \mathbb{H}^r$. Then

$$\begin{aligned} \sum_{n=0}^N |Y_n(\psi(x))| &\leq \sum_{\alpha(0)}^N \left(d_\alpha^{r/2} |\psi_\alpha| \right) \left(d_\alpha^{-r/2} |Y_\alpha(x)| \right) \\ &\leq \left\{ \sum_{\alpha(0)}^N d_\alpha^r |\psi_\alpha|^2 \right\}^{1/2} \left\{ \sum_{\alpha(0)}^N d_\alpha^{-r} |Y_\alpha(x)|^2 \right\}^{1/2} \leq K_r \|\psi\|_r \end{aligned}$$

where

$$K_r^2 = \sum_{n=0}^{\infty} d_n^{-r} \sum_{m=-n}^n |Y_n^m(x)|^2 \quad (2.5.19)$$

Due to the addition theorem (see (2.1.40)), the functional series (2.5.19) is actually the numerical series

$$K_r^2 = \frac{1}{4\pi} \sum_{n=0}^{\infty} (2n+1) d_n^{-r} \equiv \frac{1}{4\pi} \sum_{\alpha(0)} d_\alpha^{-r} = \frac{1}{4\pi} \|D^{-r}\|_*^2 \quad (2.5.20)$$

It follows from Theorem 2.4.1 and Remark 2.4.5 that series (2.5.19) is uniformly convergent on the sphere S at $r > 1$, and therefore the series $\sum_{n=0}^N Y_n(\psi(x))$ converges uniformly and absolutely. Besides, the estimate (2.5.16) is valid for the function $\psi(x)$. The estimate (2.5.17) is established similarly. *Q.E.D.*

Note that the estimate $\|\psi\|_{\mathbb{C}(D)} \leq C \|\psi\|_r$ is given in [3] for arbitrary domain D , but without specifying the constant C .

The formulas (2.5.18) give the exact value of this constant in the case when D is a two-dimensional sphere.

2.6 Some Estimates in the Norms of $\mathbb{L}^p(S)$ and $\mathbb{L}^p(0, T; \mathbb{X})$

In Chap. 3 we will use some estimates for functions $\psi(x)$ having fractional derivatives from $\mathbb{L}^p(S)$ and for functions $\psi(t, x)$ of $\mathbb{L}^p(0, T; \mathbb{X})$, where $\mathbb{X}$ is a Banach space of functions on the sphere S . We now introduce a few definitions. Let $1 \leq p \leq \infty$.

Definition 2.6.1 We denote by $\mathbb{L}^p(S)$ a completion of space $\mathbb{C}(S)$ in the norm

$$\|\psi\|_{\mathbb{L}^p(S)} = \left(\int_S |\psi(x)|^p dS \right)^{1/p} \quad (2.6.1)$$

if $p \neq \infty$, and in the norm

$$\|\psi\|_{\mathbb{L}^\infty(S)} = \sup_{x \in S} |\psi(x)| \quad (2.6.2)$$

if $p = \infty$. Also, we introduce subspaces $\mathbb{L}_0^p(S)$ by

$$\mathbb{L}_0^p(S) = \{\psi \in \mathbb{L}^p(S) : Y_0(\psi) = 0\}$$

Let $s > 0$. The operators Λ^{-s} and D^{-s} defined respectively on $\mathbb{C}_0^\infty(S)$ and $\mathbb{C}^\infty(S)$ (see Definitions 2.3.8, 2.3.9 and Remark 2.3.1) can be represented as the convolutions

$$\Lambda^{-s}\psi(x) = (\psi * G_s)(x), \quad \psi(x) \in \mathbb{C}_0^\infty(S) \quad (2.6.3)$$

$$D^{-s}\psi(x) = (\psi * B_s)(x), \quad \psi(x) \in \mathbb{C}^\infty(S) \quad (2.6.4)$$

with the zonal functions

$$G_s(\vec{x} \cdot \vec{y}) = \sum_{n=1}^{\infty} (2n+1) \chi_n^{-s/2} P_n(\vec{x} \cdot \vec{y}) \quad (2.6.5)$$

and

$$B_s(\vec{x} \cdot \vec{y}) = \sum_{n=0}^{\infty} (2n+1) d_n^{-s/2} P_n(\vec{x} \cdot \vec{y}) \quad (2.6.6)$$

where $\chi_n = n(n+1)$ and $d_n \equiv d_n = 1 + \chi_n$ (see (2.1.12) and (2.3.23)). The operators Λ^{-s} and D^{-s} are called spherical operators of fractional integration [161], or potential operators [302].

If $s > 2$ then the series (2.6.5) and (2.6.6) converge uniformly and absolutely (see Theorem 2.4.1 and Remark 2.4.5). The functions G_s and B_s are analyzed in $\mathbb{L}^p(S)$ spaces in [16].

It is known that the Young inequality

$$\|\psi * z\|_{\mathbb{L}^r(S)} \leq C \|\psi\|_{\mathbb{L}^p(S)} \|z\|_{\mathbb{L}^q(S)}$$

is valid for every function $\psi \in \mathbb{L}^p(S)$ and any zonal function $z \in \mathbb{L}^q(S)$, where $1 \leq p, q, r \leq \infty$ and $1/r = 1/p + 1/q - 1$ [302]. Thus, the convolution operator from $\mathbb{L}^p(S)$ to $\mathbb{L}^r(S)$ is bounded for all $z \in \mathbb{L}^q(S)$. For more explanations about the convolution operator, see [60, 109, 356].

Lemma 2.6.1 ([104]) *Let $1 < p < \infty$. For each function ψ such that $\Lambda\psi \in \mathbb{L}^p(S)$ the following norms are equivalent:*

$$C_1 \|\Lambda\psi\|_{\mathbb{L}^p(S)} \leq \|\nabla\psi\|_{\mathbb{L}^p(S)} \leq C_2 \|\Lambda\psi\|_{\mathbb{L}^p(S)} \quad (2.6.7)$$

We now give important results on the continuous embeddings for the spaces $\mathbb{L}^p(S)$ and $\mathbb{H}^s$.

Theorem 2.6.1 ([16]) *Let $p > 1$, $0 < s < 2/p$ and $\psi \in \mathbb{L}_0^p(S)$. Then $\Lambda^{-s}\psi = (\psi * G_s) \in \mathbb{L}_0^r(S)$ where s, p , and r satisfy the relation $s = 2(1/p - 1/r)$ and*

$$\|\Lambda^{-s}\psi\|_{\mathbb{L}^r(S)} \leq C \|\psi\|_{\mathbb{L}^p(S)}$$

As a particular case at $p = 2$ we obtain

Lemma 2.6.2 *Let $0 < s < 1$. If $r = 2/(1 - s)$ then each function ψ from $\mathbb{H}_0^s$ belongs to $\mathbb{L}^r(S)$ and*

$$\|\psi\|_{\mathbb{L}^r(S)} \leq C \|\psi\|_s \quad (2.6.8)$$

Corollary 2.6.1 *The operator of embedding $\mathbb{H}_0^{1/2}$ to $\mathbb{L}^4(S)$ is compact.*

Indeed, due to Lemma 2.6.2,

$$\|\psi\|_{\mathbb{L}^4(S)} \leq C_0 \|\psi\|_{1/2} \quad (2.6.9)$$

Since the sphere is a compact set, the continuous embedding $\mathbb{H}_0^{1/2}$ to $\mathbb{L}^4(S)$ is compact.

Remark 2.6.1 Lemma on the continuous embedding $\mathbb{H}_0^1$ to $\mathbb{L}^4(S)$ was established in [361] (see also [147]). A compact embedding $\mathbb{H}_0^1$ to $\mathbb{L}^p(S)$ was proved in [318] by using Theorem 7.1 from [178] and Theorem 2.2.1 about the partition of unity:

$$\|\psi\|_{\mathbb{L}^p(S)} \leq K_0 \|\psi\|^{1-\alpha} \|\Lambda\psi\|^\alpha \quad (2.6.10)$$

where

$$K_0 = \max\{2, p/2\}, \quad \alpha = 1 - 2/p, \quad p \in [2, \infty) \quad (2.6.11)$$

Thus, Lemma 2.6.2 and Corollary 2.6.1 specify these results.

Let $\mathbb{X}$ be a Banach space of functions on the sphere S , $t \in (0, T)$, and let $Q = (0, T) \times S$.

Definition 2.6.2 ([203]) Let $p \in [1, \infty)$. We denote by $\mathbb{L}^p(0, T; \mathbb{X})$ the Banach space of measurable functions $\psi(t, x)$ image of which is in $\mathbb{X}$ for any $t \in (0, T)$ and such that

$$\left(\int_0^T \|\psi\|_{\mathbb{X}}^p dt \right)^{1/p} = \|\psi\|_{\mathbb{L}^p(0, T; \mathbb{X})} < \infty \quad (2.6.12)$$

If $p = \infty$ then the norm in $\mathbb{L}^\infty(0, T; \mathbb{X})$ is defined as

$$\|\psi\|_{\mathbb{L}^\infty(0, T; \mathbb{X})} = \text{vrai sup}_{t \in (0, T)} \|\psi(t)\|_{\mathbb{X}} \quad (2.6.13)$$

Hereafter, we will use $\mathbb{X} = \mathbb{H}_0^s$ or $\mathbb{X} = \mathbb{L}^p(S)$, and the notation $\mathbb{L}^p(Q) = \mathbb{L}^p(0, T; \mathbb{L}^p(S))$.

Lemma 2.6.3 Let $1 \leq p \leq \infty$, $0 < s < 1$ and $s = 1 - 2/r$. Then each function ψ from $\mathbb{L}^p(0, T; \mathbb{H}_0^s)$ belongs to $\mathbb{L}^p(0, T; \mathbb{L}^r(S))$ and

$$\|\psi\|_{\mathbb{L}^p(0, T; \mathbb{L}^r(S))} \leq C \|\psi\|_{\mathbb{L}^p(0, T; \mathbb{H}_0^s)} \quad (2.6.14)$$

In order to prove (2.6.14) it is sufficient to consider the p -th power of inequality (2.6.8) and then integrate the result over interval $(0, T)$.

Lemma 2.6.4 ([318]) Let $\psi(t, x) \in \mathbb{L}^\infty(0, T; \mathbb{H}_0^0) \cap \mathbb{L}^{p\alpha}(0, T; \mathbb{H}_0^1)$ where p and α be the constants from conditions (2.6.11). Then $\psi \in \mathbb{L}^p(Q)$.

Indeed, due to (2.6.10) we have

$$\|\psi\|_{\mathbb{L}^p(S)}^p \leq K_0^p \|\psi\|^{p(1-\alpha)} \|\Lambda\psi\|^{p\alpha} \leq C \|\psi\|_1^{p\alpha}$$

Integrating the last inequality over the time interval $(0, T)$ and using Lemma 2.6.3 we obtain the necessary estimate

$$\|\psi\|_{\mathbb{L}^p(Q)} \leq C \|\psi\|_{\mathbb{L}^{p\alpha}(0, T; \mathbb{H}_0^1)} \quad (2.6.15)$$

In particular, if $p = 4$ then $\alpha = 1/2$ and $p\alpha = 2$.

Mathematical Problems of the Dynamics of
Incompressible Fluid on a Rotating Sphere

Skiba, Y.

2017, XII, 239 p. 34 illus., Hardcover

ISBN: 978-3-319-65411-9