

Chapter 2

Deformation Quantization and Group Actions

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Abstract This set of notes corresponds to a mini-course given in Villa de Leyva in July 2015. It does not contain any new result and is meant to be an elementary first introduction to formal Deformation Quantization, hoping it will be an incentive to learn more advanced topics in the subject. Quantization of a classical system is a way to pass from classical to quantum results. There exist several mathematical attempts to formulate possible quantization methods. Formal deformation quantization was introduced in the seventies by Flato et al. and understands quantization as a deformation (called a star product) of the structure of the algebra of classical observables. After an introduction to the concept of quantization in Sect. 2.1, we introduce formal deformation quantization in Sect. 2.2, the description of Fedosov's construction of a star product on a symplectic manifold in Sect. 2.3, an introduction to classifications of star products in Sect. 2.4 and a brief introduction to the notion of formality and its link with star products on a Poisson manifold in Sect. 2.5. Various notions of group actions in the context of deformation quantization are given in Sect. 2.6, along with the study of the invariance of a Fedosov's star product, and classifications of invariant star products on a manifold endowed with an invariant connection. We present in Sect. 2.7 the concept of reduction in the formal deformation quantization setting, and show how quantization commutes with reduction, considering here only the simplest form of reduction and following a simplified version of Bordemann–Waldmann's approach. We conclude by briefly mentioning in Sect. 2.8 convergence issues in the deformation quantization programme.

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Foreword A rather large, though not complete, bibliography is included for those who want to go beyond this introduction. Some references are directly linked to this introduction:

- the book [167] by S. Waldmann is an introduction to deformation quantization (in German);
 - concerning Sect. 2.3, the book [82] by B.V. Fedosov presents his construction of star products along with many of their properties, and introduces index theorems for deformation quantization on symplectic manifolds;
 - extending Sect. 2.4, the expository paper [104], joint with J. Rawnsley, gives an introduction to Deligne's Čech cohomology classes, associated to star products on a symplectic manifold;
 - to develop Sect. 2.5, the expository paper [59] by A. Cattaneo and D. Indelicato introduces formality and star products and the paper [64] by A. Cattaneo, G. Felder and L. Tomassini gives the globalization of a star product on a Poisson manifold; see also the original paper by Kontsevich [122];
 - Section 2.7 is taken from [106]; the reduction presented is a special case of a reduction procedure introduced by M. Bordemann, C. Herbig and S. Waldmann in [40].
- There are many important aspects of deformation quantization which are not addressed in these notes; some of them are mentioned with corresponding references.

Possible connexions with other classes given at the school appear in the text. There is in Sect. 2.1 a mention of the lectures of Abhay Ashtekar when quantum field theory is alluded to, and the lectures of Nathan Berkovits are referred to concerning superstring theory. In Sect. 2.6, one mentions the lectures of Christian Kassel when one speaks about quantum groups, and in Sect. 2.7 there is again a link to the lectures of Nathan Berkovits concerning BRST formalism.

2.1 What Do We Mean by Quantization?

Quantum theory provides a description of nature which is more fundamental than classical theory. It is necessary to describe atomic or subatomic physics (and it is also needed to describe some macroscopical phenomena such as superconductors and superfluids). It incorporates phenomena which can not be accounted for by classical physics like the quantization of certain physical properties, the uncertainty principle, etc.

We shall consider here quantum mechanics which provides a non relativistic description (i.e. the speed is far less than 3×10^8 m/s) of a finite number of particles with a finite number of degrees of freedom.

Remark 2.1 To go beyond this, quantum field theory provides a description which incorporates higher velocities, for instance to describe a system including photons, or a system with a varying number of particles; it merges quantum principles and special relativity. In this realm quantum electrodynamics provides a description of

electromagnetic interactions, quantum chromodynamics of strong nuclear forces, electroweak theory of electromagnetic force and weak nuclear forces, and the standard model of particles unifies the three type of interactions. However, it has been proven difficult to build quantum theories of gravity (the remaining fundamental force); string theory is a candidate for such a theory. Quantum field theory on curved space time is the object of the lectures given by Abhay Ashtekar and an introduction to superstring theory is given in the lectures of Nathan Berkovits.

By quantization of a classical system, we mean a way to pass from classical to quantum results. One could wonder why we are interested in quantization, since it could appear to be an artificial problem, nature being quantum. A first motivation lies in the difficulty of directly providing a quantum description of a physical system, and the classical description is often easier to obtain; hence one often uses the classical description as a starting point to find a quantum description. Furthermore, a given physical theory remaining valid within a range of measurements, any modified theory should give the same results in the initial range. The description of a system by classical mechanics is adequate in the macroscopic non relativistic world, for size much larger than 10^{-9} m and speed far less than 3×10^8 m/s.

Guidelines as how to pass from a classical description to a quantum one are based on the precepts that there exists a classical limit, and that to any classical observable there corresponds a quantum one.

2.1.1 *Classical Mechanics*

Classical mechanics, in its Hamiltonian formulation on the motion space, can be described in the framework of symplectic manifolds (or more generally Poisson manifolds). The motion space is in general the quotient of the evolution space by the motion; it can often be identified with a space of possible initial values for positions and momenta. Observables are families $f_\bullet = \{f_t \mid t \in \mathbb{R}\}$ of smooth functions on that manifold M . The dynamics is defined in terms of a Hamiltonian $H \in C^\infty(M)$ and the time evolution of an observable f_\bullet is governed by the equation:

$$\frac{d}{dt} f_t = -\{H, f_t\}.$$

For instance a particle of mass m moving in \mathbb{R}^3 subject to a force F which is the gradient of a potential $F = -\nabla V$, has a position determined by the 3 coordinates q^1, q^2, q^3 whose evolution in time is governed by Newton's equations

$$m \frac{d^2 q^i}{dt^2} = -\frac{\partial V}{\partial q^i}.$$

Introducing the momenta $p_i = m \frac{dq^i}{dt}$, and the Hamiltonian $H' := \frac{p^2}{2m} + V$ with $p^2 := \sum_{j=1}^3 p_j^2$, the motion of the system in the evolution space $\mathcal{E} := \mathbb{R} \times T^*\mathbb{R}^3$ with coordinates $(t, q^1, q^2, q^3, p_1, p_2, p_3)$ is given by the flow of the vector field

$$\frac{\partial}{\partial t} + \sum_{i=1}^3 \left(\frac{\partial H'}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H'}{\partial q^i} \frac{\partial}{\partial p_i} \right).$$

Let $\pi = \mathcal{E} \rightarrow \mathcal{E}/\sim =: M$ be the projection on the space of motions (two points in \mathcal{E} being equivalent when they belong to the same orbit under the flow). The 2-form $\Omega := \sum_{j=1}^3 dp_j \wedge dq^j - dH' \wedge dt$ on \mathcal{E} is the pullback under π of a symplectic form ω on M . The Hamiltonian H' is the pullback of a function H on M . An observable given by a time independent function f on \mathcal{E} (i.e. the pullback by the projection on the second factor of a function on the phase space $T^*\mathbb{R}^3$, that is a function of the positions and momenta), is now represented by a collection of functions $\{f_t\}$ on the motion space M , the function f_t evaluated at a point $m \in M$ being the value of f at time t in the corresponding motion. Then

$$\frac{d}{dt} f_t = \sum_{i=1}^3 \left(\frac{\partial H'}{\partial p_i} \frac{\partial f}{\partial q^i} - \frac{\partial H'}{\partial q^i} \frac{\partial f}{\partial p_i} \right) (t) = -\{H, f_t\}.$$

An Incursion in Poisson and Symplectic Manifolds

Definition 2.1 A **Poisson bracket**, defined on the space of complex valued smooth functions on a manifold M , is a \mathbb{C} -bilinear map $(u, v) \mapsto \{u, v\}$ on $C^\infty(M) := C^\infty(M, \mathbb{C})$, such that, for any $u, v, w \in C^\infty(M)$:

1. $\overline{\{u, v\}} = \{\overline{u}, \overline{v}\}$ (reality)
2. $\{u, v\} = -\{v, u\}$ (skew-symmetry);
3. $\{\{u, v\}, w\} + \{\{v, w\}, u\} + \{\{w, u\}, v\} = 0$ (Jacobi's identity);
4. $\{u, vw\} = \{u, v\}w + \{u, w\}v$ (Leibniz rule).

Exercise 2.1 The Leibniz rule is equivalent to the bracketing with a function u being a derivation of the associative algebra of smooth functions on M .

Bracketing with a function u is therefore given by a vector field X_u on M , which is called the **Hamiltonian vector field** associated to the function u :

$$\{u, v\} = X_u v \tag{2.1}$$

By skew-symmetry, a Poisson bracket is thus given in terms of a contravariant skew-symmetric 2-tensor P on M , called the **Poisson tensor**, by

$$\{u, v\} = P(du \wedge dv) \quad \left(\text{in local coordinates} \quad \{u, v\} = \sum_{i,j=1}^{m=\dim M} P^{ij} \frac{\partial u}{\partial y_i} \frac{\partial v}{\partial y_j} \right). \tag{2.2}$$

Exercise 2.2 The Jacobi identity for the Poisson bracket is equivalent to the vanishing of the Schouten bracket:

$$[P, P] = 0 \quad \left(\text{locally} \sum_{r=1}^{\dim M} \left(P^{ir} \frac{\partial}{\partial y_r} P^{jk} + P^{jr} \frac{\partial}{\partial y_r} P^{ki} + P^{kr} \frac{\partial}{\partial y_r} P^{ij} \right) = 0 \right).$$

The Schouten bracket is the extension -as a graded derivation for the exterior product- of the bracket of vector fields to skew-symmetric contravariant tensor fields; it will be developed in Sect. 2.5.2.

A **Poisson manifold**, denoted (M, P) , is a manifold M with a Poisson bracket defined by the Poisson tensor P .

A **first example** is \mathbb{R}^{2n} with coordinates $\{q^i, p_i; 1 \leq i \leq n\}$ and the **canonical Poisson bracket**

$$\{u, v\} = \sum_{j=1}^n \left(\frac{\partial u}{\partial q^j} \frac{\partial v}{\partial p_j} - \frac{\partial u}{\partial p_j} \frac{\partial v}{\partial q^j} \right). \quad (2.3)$$

More generally, on \mathbb{R}^m with coordinates $\{x^i; 1 \leq i \leq m\}$, any constant real skew-symmetric contravariant 2-tensor P defines a Poisson structure with Poisson bracket

$$\{u, v\} = \sum_{j,k=1}^m P^{jk} \frac{\partial u}{\partial x^j} \frac{\partial v}{\partial x^k}.$$

A particular class of Poisson manifolds, essential in classical mechanics, is the class of **symplectic manifolds**. If (M, ω) is a symplectic manifold (i.e. ω is a closed nondegenerate 2-form on M) and if $u, v \in C^\infty(M)$, the Poisson bracket of u and v is defined by

$$\{u, v\} := X_u(v) = \omega(X_v, X_u),$$

where X_u denotes the Hamiltonian vector field corresponding to the function u , given by $i(X_u)\omega = du$.

Exercise 2.3 In coordinates the components of the corresponding Poisson tensor P^{ij} form the inverse matrix of the components ω_{ij} of ω . Symplectic manifolds are exactly Poisson manifolds for which the Poisson tensor is non degenerate at each point.

Amongst the symplectic manifolds, there is the cotangent bundle $T^*N \xrightarrow{\pi} N$ to a manifold N , endowed with the symplectic form $d\Theta$ where Θ is the Liouville 1-form on T^*N :

$$\Theta_\eta(X) := \eta(\pi_*X) \quad \text{for any } \eta \in T^*N \text{ and } X \in T_\eta T^*N.$$

This appears as the phase space of a classical system with configuration space N .

Duals of Lie algebras form the class of linear Poisson manifolds. If \mathfrak{g} is a Lie algebra then its dual \mathfrak{g}^* is endowed with the Poisson tensor P defined by

$$P_\xi(X, Y) := \xi([X, Y])$$

for $X, Y \in \mathfrak{g} = (\mathfrak{g}^*)^* \sim (T_\xi \mathfrak{g}^*)^*$. If $\{X^1, \dots, X^m\}$ is a basis of \mathfrak{g} and x^j the corresponding linear coordinates on \mathfrak{g}^*

$$x^j : \mathfrak{g}^* \rightarrow \mathbb{R} : \xi \mapsto \xi(X^j),$$

and if c_k^{ij} denote the structure constants $[X^i, X^j] = \sum_k c_k^{ij} X^k$, this bracket writes

$$\{u, v\} = \sum_{i,j,k} c_k^{ij} x^k \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j}. \quad (2.4)$$

2.1.2 Quantum Mechanics

Quantum mechanics, in its usual Heisenberg formulation, takes place in the framework of Hilbert spaces (states are rays in such a space). Observables are families $A_\bullet = \{A_t, t \in \mathbb{R}\}$ of self-adjoint operators on the Hilbert space. The dynamics is defined in terms of a Hamiltonian H , which is a self-adjoint operator, and the time evolution of an observable A_\bullet is governed by the equation:

$$\frac{dA_t}{dt} = \frac{i}{\hbar} [H, A_t]$$

where \hbar is the reduced Planck constant $\hbar = \frac{h}{2\pi} \simeq 10^{-34}$ J.s.

A natural suggestion for quantization is a correspondence $\mathcal{Q}: f \mapsto \mathcal{Q}(f)$ mapping a function f to a self-adjoint operator $\mathcal{Q}(f)$ on a Hilbert space \mathcal{H} in such a way that $\mathcal{Q}(1) = \text{Id}$ and

$$[\mathcal{Q}(f), \mathcal{Q}(g)] = i\hbar \mathcal{Q}(\{f, g\}) + O(\hbar^2).$$

There is no correspondence defined on all smooth functions on M so that

$$[\mathcal{Q}(f), \mathcal{Q}(g)] = i\hbar \mathcal{Q}(\{f, g\}),$$

when one puts an irreducibility requirement which is necessary not to violate Heisenberg's uncertainty principle. More precisely, Van Hove [165] proved that there is no irreducible representation of the Heisenberg algebra, viewed as the algebra of constants and linear functions on \mathbb{R}^{2n} endowed with the Poisson bracket, which extends to a representation of the algebra of polynomials on \mathbb{R}^{2n} .

We shall now describe commonly used quantizations of \mathbb{R}^{2n} endowed with its **canonical Poisson bracket** as defined in Eq. (2.3):

in coordinates $\{q^i, p_i; 1 \leq i \leq n\}$ $\{u, v\} = \sum_{j=1}^n \left(\frac{\partial u}{\partial q^j} \frac{\partial v}{\partial p_j} - \frac{\partial u}{\partial p_j} \frac{\partial v}{\partial q^j} \right)$.

1. Standard Ordering

The standard ordering yields a bijection \mathcal{Q}_{st} between (complex valued) polynomials on \mathbb{R}^{2n} , $\mathbb{C}[p_i, q^j]$, and the space of differential operators on \mathbb{R}^n with polynomial coefficients $D_{(polyn)}(\mathbb{R}^n)$. It assigns to the constant function 1, the operator $\mathcal{Q}_{st}(1) = \text{Id}$, to the classical observables q^i the quantum operators of multiplication by q^i , $\mathcal{Q}_{st}(q^i) := Q^i := q^i \cdot$, and to p_i the differential operators of order 1 involving derivation with respect to q^i , $\mathcal{Q}_{st}(p_i) := P_i := -i\hbar \frac{\partial}{\partial q^i}$. One has to specify what is associated to other classical observables given by polynomials in q^i and p_j since Q^j and P_j no longer commute. For the standard ordering, one defines

$$\mathcal{Q}_{st}(q^{i_1} \dots q^{i_n} p_1^{j_1} \dots p_n^{j_n}) := Q^{i_1} \dots Q^{i_n} P_1^{j_1} \dots P_n^{j_n}.$$

Equivalently, for any $f, g \in \mathbb{C}[p_i, q^j]$ and any $\phi \in C^\infty(\mathbb{R}^n, \mathbb{C})$:

$$\mathcal{Q}_{st}(f)\phi = \sum_{r=0}^{\infty} \sum_{r_1+\dots+r_n=r} \frac{(\hbar/i)^r}{r!} \frac{\partial^r f}{\partial p_1^{r_1} \dots \partial p_n^{r_n}} \Big|_{p=0} \frac{\partial^r \phi}{\partial q^{1r_1} \dots \partial q^{nr_n}},$$

so that the deformed product on $\mathbb{C}[p_i, q^j]$ corresponding to the composition of operators in $D_{(polyn)}(\mathbb{R}^n)$ via the bijection \mathcal{Q}_{st} is given by

$$\begin{aligned} f *_{Std} g &:= \mathcal{Q}_{st}^{-1}(\mathcal{Q}_{st}(f) \circ \mathcal{Q}_{st}(g)) \\ &= \sum_{r=0}^{\infty} \sum_{r_1+\dots+r_n=r} \frac{(\hbar/i)^r}{r!} \frac{\partial^r f}{\partial p_1^{r_1} \dots \partial p_n^{r_n}} \frac{\partial^r g}{\partial q^{1r_1} \dots \partial q^{nr_n}}. \end{aligned} \quad (2.5)$$

Its classical limit is: $f *_{Std} g = fg + (\hbar/i) \sum_j \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q^j} + O(\hbar^2)$.

Remark 2.2 Consider the space $C_c(\mathbb{R}^n, \mathbb{C})$ of compactly supported smooth functions endowed with the Hermitian scalar product

$$\langle \phi, \psi \rangle := \int \overline{\phi(q^1, \dots, q^n)} \psi(q^1, \dots, q^n) dq^1 \dots dq^n.$$

Then $\langle \phi, A\psi \rangle = \langle A\phi, \psi \rangle$ for $A = P_i$ or $A = Q^j$ but the property is not true for all polynomials; for instance, it is not true for the operator associated to the real function $p_i q^i$; indeed, the adjoint of $Q^i P_i$ is $P_i Q^i = Q^i P_i - i\hbar \text{Id}$. Hence the operator $\mathcal{Q}_{st}(p_i q^i)$ cannot be extended to a self-adjoint operator in the Hilbert completion $L^2(\mathbb{R}^n, dq)$ of $(C_c(\mathbb{R}^n, \mathbb{C}), \langle \cdot, \cdot \rangle)$.

2. Weyl Ordering

The Weyl ordering is again a bijection \mathcal{Q}_{Weyl} between the polynomials $\mathbb{C}[p_i, q^j]$ and the space of differential operators $D_{(polyn)}(\mathbb{R}^n)$. It assigns to the constant function 1, the operator $\mathcal{Q}_{Weyl}(1) = \text{Id}$, to the classical observables q^i the quantum operators $\mathcal{Q}_{Weyl}(q^i) := Q^i := q^i \cdot$ of multiplication by q^i , and to p_i the differential operators of order 1 $\mathcal{Q}_{Weyl}(p_i) := P_i$ and to a polynomial in p 's and q 's the corresponding totally symmetrized polynomial in Q^i and P_j , e.g.

$$\mathcal{Q}_{Weyl}(q^1(p^1)^2) = \frac{1}{3}(Q^1(P^1)^2 + P^1 Q^1 P^1 + (P^1)^2 Q^1).$$

Exercise 2.4 $\mathcal{Q}_{Weyl}(\exp(aq + bp)) = \exp(aQ + bP)$ and $\mathcal{Q}_{st}(\exp(aq + bp)) = \exp aQ \exp bP$ for a, b formal parameters (i.e. when one expands in powers of a and b the equality is true for any power of a and b); now $\exp(aQ + bP) = e^{\frac{\hbar ab}{2i}} \exp aQ \exp bP$, so that

$$\mathcal{Q}_{Weyl}(f) = \mathcal{Q}_{st}(\tilde{T}f)$$

for $\tilde{T} = e^{\frac{\hbar}{2i} \sum_j \frac{\partial^2}{\partial q^j \partial p_j}}$. Then the deformed product on $\mathbb{C}[p_i, q^j]$ corresponding to the composition of operators in $D_{(polyn)}(\mathbb{R}^n)$ via the bijection \mathcal{Q}_{Weyl} is

$$\begin{aligned} f *_{Weyl} g &:= \mathcal{Q}_{Weyl}^{-1} (\mathcal{Q}_{Weyl}(f) \circ \mathcal{Q}_{Weyl}(g)) = \tilde{T}^{-1} ((\tilde{T}f) *_{Std} (\tilde{T}g)) \quad (2.6) \\ &= \sum_{r=0}^{\infty} \sum_{s=0}^r \frac{r!}{s!(r-s)!} (-1)^{r-s} \frac{(\hbar/i)^r}{r!} \frac{\partial^r f}{\partial q^s \partial p^{r-s}} \frac{\partial^r g}{\partial p^s \partial q^{r-s}} \\ &= f \cdot g + \frac{\hbar}{2i} \{f, g\} + O(\hbar^2) \end{aligned}$$

(using multi indices or working in dimension 1).

3. Wick Ordering

Set $z = q + ip$ (we present here the complex dimension 1 case; the formulas are analogous in dimension n with multiindices) and let $\mathcal{O}(\mathbb{C})$ be the set of antiholomorphic functions on \mathbb{C} with hermitian scalar product defined by $\langle \phi, \psi \rangle := \frac{1}{2\pi\hbar} \int \overline{\phi(\bar{z})} \psi(\bar{z}) e^{\frac{-|z|^2}{2\hbar}} dz d\bar{z}$ which may diverge. Let

$$\mathcal{H} := \{ \phi \in \mathcal{O}(\mathbb{C}) \mid \langle \phi, \phi \rangle < \infty \}.$$

The set of polynomials in \bar{z} is dense in \mathcal{H} . The Wick ordering assigns to the constant function 1, the operator $\mathcal{Q}_{Wick}(1) = \text{Id}$, to the function z the quantum operators $\mathcal{Q}_{Wick}(z) := 2\hbar \frac{\partial}{\partial \bar{z}}$, to \bar{z} the multiplication by \bar{z} , $\mathcal{Q}_{Wick}(\bar{z}) := \bar{z} \cdot$ and to any polynomial

$$\mathcal{Q}_{Wick}(\bar{z}^n z^m) := (2\hbar)^m \bar{z}^n \left(\frac{\partial}{\partial \bar{z}} \right)^m \text{ i.e.}$$

$$\mathcal{Q}_{Wick}(f)\phi = \sum_{r=0}^{\infty} \frac{(2\hbar)^r}{r!} \frac{\partial^r f}{\partial z^r} \Big|_{z=0} \frac{\partial^r \phi}{\partial \bar{z}^r} \quad \forall f \in \mathbb{C}[p, q], \phi \in \mathbb{C}[\bar{z}].$$

Exercise 2.5 The deformed product corresponding to the composition of operators is given by

$$\begin{aligned} f *_{Wick} g &:= \mathcal{Q}_{Wick}^{-1} (\mathcal{Q}_{Wick}(f) \circ \mathcal{Q}_{Wick}(g)) \\ &= \sum_{r=0}^{\infty} \frac{(2\hbar)^r}{r!} \frac{\partial^r f}{\partial z^r} \frac{\partial^r g}{\partial \bar{z}^r} = fg + 2\hbar \frac{\partial f}{\partial z} \frac{\partial g}{\partial \bar{z}} + O(\hbar^2). \end{aligned} \quad (2.7)$$

This Wick product satisfies the hermitian property: $\overline{f *_{Wick} g} = \bar{g} *_{Wick} \bar{f}$. Setting $\Delta' := \frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial p^2}$ and $\tilde{T}' := \exp \frac{\hbar}{4} \Delta'$, one gets

$$\tilde{T}' (f *_{Weyl} g) = \tilde{T}' f *_{Wick} \tilde{T}' g \quad \forall f, g \in \mathbb{C}[p, q]. \quad (2.8)$$

Remark 2.3 Formulas (2.5), (2.6) and (2.7) do not converge in general if we replace polynomials by smooth functions. To make them well defined, a way to proceed is to replace the purely imaginary complex number $\frac{\hbar}{i}$ by a formal parameter ν and to consider formal power series in that parameter. This will lead to the definition of formal deformation quantization (see next section).

Remark 2.4 Other mathematical formulations of quantization exist, such as

- Geometric Quantisation of Kostant and Souriau [158] which proceeds in two steps. Prequantization of a symplectic manifold (M, ω) where one builds, if it exists, a pre-quantum bundle which is a Hermitian line bundle with a connection $(L \rightarrow M, h, \nabla)$ such that the curvature is $\frac{\omega}{i\hbar}$; if \mathcal{H} denotes the Hilbert space of L^2 sections of the bundle L , one defines a correspondence $Q : C^\infty(M) \rightarrow (Op)(\mathcal{H})$, with values in operators acting on \mathcal{H} , by $Q(f) := i\hbar \nabla_{X_f} + f$. Clearly $[Q(f), Q(g)] = i\hbar Q(\{f, g\})$ and $Q(1) = \text{id}$ but there is no irreducibility. In a second step, one introduces the concept of polarization to “cut down the number of variables”.

- In the case where the symplectic manifold is compact Kähler and admits a prequantization line bundle, one can use the framework of geometric quantization to define the Toeplitz quantization (see, for instance, [38]) which acts on holomorphic sections of this line bundle. A function f acts on a holomorphic section s by projecting fs on the space of holomorphic sections.

- Closely related is Berezin’s quantisation [19, 20] where one builds on a particular class of Kähler manifolds a family of associative algebras using a symbolic calculus. Examples of deformation quantization have been constructed using asymptotic expansions of these quantizations (see, for instance, [38, 51, 117, 147]).

2.2 Deformation Quantization

Observe that the two mathematical frameworks for classical and quantum mechanics are very different. This makes it difficult to see classical mechanics as a limit of quantum mechanics. Deformation Quantization was introduced by Flato, Lichnerowicz and Sternheimer in [93], and developed in [15]: they

suggest that quantization be understood as a deformation of the structure of the algebra of classical observables rather than a radical change in the nature of the observables.

This deformation approach to quantization is part of a “deformation approach” to the developments of physics which was one of the seminal ideas stressed by Moshe Flato: one looks at some (new) level of a theory in physics as a deformation of a former one [92].

One stresses here the fundamental aspect of the space of observables rather than the set of states; observables behave indeed in a nice way when one deals with composite systems: both in the classical and in the quantum picture, the space of observables for combined systems is the tensor product of the spaces of observables.

The algebraic structure of classical observables that one deforms is the algebraic structure of the space of smooth functions on a Poisson manifold: the associative structure given by the usual product of functions and the Lie structure given by the Poisson bracket. Formal deformation quantisation is defined in terms of a star product which is a formal deformation of that structure.

2.2.1 Definition and Examples of Star Products

Definition 2.2 (Bayen *et al.* [15]) A **star product** on a Poisson manifold (M, P) is a bilinear map

$$N \times N \rightarrow N[[\hbar]], \quad (u, v) \mapsto u * v = u *_v v := \sum_{r \geq 0} \hbar^r C_r(u, v)$$

where $N = C^\infty(M)$ [we consider in general complex valued functions] such that

1. when the map is extended \hbar -linearly (and continuously in the \hbar -adic topology) to $N[[\hbar]] \times N[[\hbar]]$ it is formally associative $(u * v) * w = u * (v * w)$;
2. (a) $C_0(u, v) = uv$, (b) $C_1(u, v) - C_1(v, u) = \{u, v\}$ (c) $1 * u = u * 1 = u$;
3. the C_r 's are bidifferential operators on M , i.e. given in any local chart (U, φ) with local coordinates $\{x^i : 1 \leq i \leq m = \dim M\}$ by

$$C_r(u, v)|_U = \sum_{k \leq K, k' \leq K'} \sum_{i_1, \dots, i_k; j_1, \dots, j_{k'}} P^{i_1, \dots, i_k; j_1, \dots, j_{k'}} \frac{\partial^{[k]} u}{\partial x^{i_1} \dots \partial x^{i_k}} \frac{\partial^{[k']} v}{\partial x^{j_1} \dots \partial x^{j_{k'}}}$$

(it is then more precisely a **differential star product**).

When each C_r is of order maximum r in each argument, one speaks of a **natural star product**.

If $\overline{f * g} = \overline{g} * \overline{f}$ for any $v = i\lambda$, $\lambda \in /R$, the star product is called **Hermitian**.

If there were a quantization in the usual sense, i.e. a correspondence between functions on the Poisson manifold (M, P) and algebras A_h of operators on a Hilbert space (depending on a parameter h related to the Plank's constant), one could look at the deformed products $*_h$ of two functions as corresponding to the composition of the corresponding operators in A_h . One can think of a star product as the expansion in the parameter h of such deformed products. In particular, one can define the star products on \mathbb{R}^{2n} (with its canonical Poisson structure) coming from the quantization of polynomial functions given by the standard, the Weyl and the Wick orderings.

Exercise 2.6 The standard ordering (see Eq. (2.5)) yields:

$$f *_\text{st} g := \sum_{r=0}^{\infty} \frac{v^r}{r!} \sum_{r_1+\dots+r_n=r} \frac{\partial^r f}{\partial p_1^{r_1} \dots \partial p_n^{r_n}} \frac{\partial^r g}{\partial q^{1r_1} \dots \partial q^{nr_n}}, \quad (2.9)$$

the Weyl ordering (see Eq. (2.6)) yields in coordinates $\{x^1 = p_1, \dots, x^n = p_n, x^{n+1} = q^1, \dots, x^{2n} = q^n\}$

$$f *_\text{weyl} g = \sum_{r=0}^{\infty} \frac{v^r}{r!} \sum_{i_1, \dots, i_r, j_1, \dots, j_r} P_0^{i_1 j_1} \dots P_0^{i_r j_r} \frac{\partial^r f}{\partial x^{i_1} \dots \partial x^{i_r}} \frac{\partial^r g}{\partial x^{j_1} \dots \partial x^{j_r}} \quad (2.10)$$

with $P_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, and the Wick ordering (see Eq. (2.7)) yields

$$f *_\text{wick} g := \sum_{r=0}^{\infty} v^r \sum_{i_1, \dots, i_r} \frac{(2i)^r}{r!} \frac{\partial^r f}{\partial z^{i_1} \dots \partial z^{i_r}} \frac{\partial^r g}{\partial \bar{z}^{i_1} \dots \partial \bar{z}^{i_r}} \quad (2.11)$$

Those three star products are natural; the ones corresponding to Weyl and Wick orderings are Hermitian.

Remark 2.5 A star product can also be defined not on the whole of $C^\infty(M)$ but on any subspace N of it which is stable under pointwise multiplication and Poisson bracket.

In (b) we require the skew-symmetric part of C_1 to be $\frac{1}{2}\{, \}$; one finds in the literature other normalisations; originally it was $\{, \}$ and often it is $\frac{i}{2}\{, \}$; all these amount to a rescaling of the parameter.

By (b) the centre of the deformed algebra $(C^\infty(M)[[v]], *)$ consists of series whose terms Poisson commute with all functions, so elements of $\mathbb{R}[[v]]$ when M is symplectic and connected.

Properties (a) and (b) imply that the **star commutator** defined by $[u, v]_* = u * v - v * u$, which obviously makes $C^\infty(M)[[v]]$ into a Lie algebra, has the form $[u, v]_* =$

$v\{u, v\} + \dots$ so that repeated bracketing leads to higher and higher order terms. We denote $ad_* u(v) := [u, v]_*$.

Example 2.1 (The Moyal star product) The simplest example of a deformation quantisation is the Moyal product for a constant Poisson structure P on a vector space $V = \mathbb{R}^m$:

$$P = \sum_{i,j} P^{ij} \partial_i \wedge \partial_j, \quad P^{ij} = -P^{ji} \in \mathbb{R}$$

where $\partial_i = \partial/\partial x^i$ is the partial derivative in the direction of the coordinate x^i , $i = 1, \dots, n$. The formula for **the (formal) Moyal product associated to P** is

$$(u *_{M(P)} v)(z) = \exp\left(\frac{\nu}{2} P^{rs} \partial_{x^r} \partial_{y^s}\right) (u(x)v(y)) \Big|_{x=y=z}. \quad (2.12)$$

Associativity of $*_{M(P)}$ follows from the fact that

$$\partial_{t^k} (u *_{M(P)} v)(t) = (\partial_{x^k} + \partial_{y^k}) \exp\left(\frac{\nu}{2} P^{rs} \partial_{x^r} \partial_{y^s}\right) (u(x)v(y)) \Big|_{x=y=t}.$$

Indeed,

$$\begin{aligned} ((u *_{M(P)} v) *_{M(P)} w)(x') &= \exp\left(\frac{\nu}{2} P^{rs} \partial_{t^r} \partial_{z^s}\right) ((u *_{M(P)} v)(t)w(z)) \Big|_{t=z=x'} \\ &= \exp\left(\frac{\nu}{2} P^{rs} (\partial_{x^r} + \partial_{y^r}) \partial_{z^s}\right) \exp\left(\frac{\nu}{2} P^{r's'} \partial_{x^{r'}} \partial_{y^{s'}}\right) ((u(x)v(y))w(z)) \Big|_{x=y=z=x'} \\ &= \exp\left(\frac{\nu}{2} P^{rs} (\partial_{x^r} \partial_{z^s} + \partial_{y^r} \partial_{z^s} + \partial_{x^r} \partial_{y^s})\right) ((u(x)v(y))w(z)) \Big|_{x=y=z=x'} \\ &= (u *_{M(P)} (v *_{M(P)} w))(x'). \end{aligned}$$

The **(formal) Moyal product** $*_M$ is the one associated to a non degenerate P on \mathbb{R}^{2n} .

Exercise 2.7 Writing $P_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, and using (see Eq. (2.10)), show that

$$f *_M g = f *_{\text{weyl}} g. \quad (2.13)$$

Definition 2.3 When P is non degenerate (i.e. $V = \mathbb{R}^{2n}$, $P_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$), the space of polynomials in ν whose coefficients are polynomials on V with Moyal product is called **the Weyl algebra** $(S(V^*)[\nu], *_M)$.

Remark 2.6 Moyal star product is the star product (see Eq. (2.13)) coming from the quantization of polynomials on \mathbb{R}^{2n} with Weyl's ordering. Moyal used in 1949 the deformed bracket which corresponds to the commutator of operators to study

quantum statistical mechanics and the Moyal product first appeared in Groenewold [99]. Weyl quantization can be extended beyond polynomials; heuristically one would like to write

$$“\mathcal{Q}_{Weyl}(F)” = \left(\frac{1}{2\pi}\right)^{2n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{F}(u, v) e^{i(uQ+vP)} du dv,$$

where \hat{F} is the Fourier transform $\hat{F}(u, v) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F(q, p) e^{-i(uq+vp)} dq dp$.

Exercise 2.8 If one develops the above formally, using the fact that on a nice test function ϕ , $(e^{iuQ}\phi)(x) = e^{iu \cdot x} \phi(x)$, $(e^{ivP}\phi)(x) = \phi(x + \hbar v)$ and $e^{i(uQ+vP)} = e^{-\frac{i}{2}\hbar u \cdot v} e^{iuQ} \circ e^{ivP}$, one gets the formula

$$(\mathcal{Q}_{Weyl}(F)(\phi))(x) := \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} F\left(\frac{x+y}{2}, 2\pi\hbar\xi\right) e^{-2\pi i(y-x)\xi} \phi(y) dy \right) d\xi;$$

which one takes as a definition of $\mathcal{Q}_{Weyl}(F)$; it is well defined for a test function ϕ in the Schwartz space when F satisfies weak regularity bounds (there exists a constant $C > 0$ and constants $C_{i,j} > 0$, such that $\forall i, j \geq 0$ and for all x, p , one has $|\nabla_x^i \nabla_p^j F(x, p)| \leq C_{i,j} (1 + |x| + |p|)^C$).

The above formula coincides with the previous one when F is a polynomial. The map \mathcal{Q}_{Weyl} gives an isometry between the space $L^2(\mathbb{R}^{2n})$ and the space of Hilbert Schmidt operators on $L^2(\mathbb{R}^n)$, associating a self-adjoint operator to a real function.

Exercise 2.9 If F and G are two Schwartz functions, then the composition

$$\mathcal{Q}_{Weyl}(F) \circ \mathcal{Q}_{Weyl}(G)$$

is equal to $\mathcal{Q}_{Weyl}(F \times_{\hbar} G)$ where $F \times_{\hbar} G$ is the function defined by

$$(F \times_{\hbar} G)(u) := \left(\frac{1}{\pi\hbar}\right)^{2n} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{2i}{\hbar}\Omega(v,w)} F(u+v) G(u+w) dv dw \quad (2.14)$$

$$= \left(\frac{1}{\pi\hbar}\right)^{2n} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{\frac{2i}{\hbar}(\Omega(u,v)+\Omega(v,w)+\Omega(w,u))} F(v) G(w) dv dw. \quad (2.15)$$

with $\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

The result is a Schwartz function; hence \times_{\hbar} gives an associative product on the space of Schwartz functions, called the convergent Moyal star product. The (formal) Moyal star product introduced before can be seen as an asymptotic expansion in $\nu = \hbar/i$ of this composition law.

Example 2.2 (The standard $*$ -product on \mathfrak{g}^*) Let \mathfrak{g}^* be the dual of a Lie algebra \mathfrak{g} . The algebra of polynomials on \mathfrak{g}^* is identified with the symmetric algebra $S(\mathfrak{g})$. One

defines a new associative law on this algebra by a transfer of the product \circ in the universal enveloping algebra $U(\mathfrak{g})$, via the bijection between $S(\mathfrak{g})$ and $U(\mathfrak{g})$ given by the total symmetrization σ :

$$\sigma : S(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \quad X_1 \dots X_k \mapsto \frac{1}{k!} \sum_{\rho \in S_k} X_{\rho(1)} \circ \dots \circ X_{\rho(k)}.$$

Then $U(\mathfrak{g}) = \bigoplus_{n \geq 0} U_n$ where $U_n := \sigma(S^n(\mathfrak{g}))$ and we decompose an element $u \in U(\mathfrak{g})$ accordingly $u = \sum u_n$. We define for $P \in S^p(\mathfrak{g})$ and $Q \in S^q(\mathfrak{g})$

$$P * Q = \sum_{n \geq 0} (v)^n \sigma^{-1}((\sigma(P) \circ \sigma(Q))_{p+q-n}). \quad (2.16)$$

This yields a differential star product on \mathfrak{g}^* [102]; it is characterised by

$$X * (X_1 \dots X_k) = X X_1 \dots X_k + \sum_{j=1}^k \frac{(-1)^j}{j!} v^j B_j[[X, X_{r_1}], \dots, X_{r_j}] X_1 \dots \widehat{X_{r_1}} \dots \widehat{X_{r_j}} \dots X_k$$

where B_j are the Bernoulli numbers. For $v = 2\pi i$, this star product writes [79]:

$$u * v(\xi) = \int_{\mathfrak{g} \times \mathfrak{g}} \hat{u}(X) \hat{v}(Y) e^{2i\pi \langle \xi, CBH(X, Y) \rangle} dX dY$$

where $\hat{u}(X) = \int_{\mathfrak{g}^*} u(\eta) e^{-2i\pi \langle \eta, X \rangle}$ and where CBH denotes Campbell-Baker-Hausdorff formula for the product of elements in the group in a logarithmic chart ($\exp X \exp Y = \exp CBH(X, Y) \quad \forall X, Y \in \mathfrak{g}$).

Remark 2.7 The standard star product on \mathfrak{g}^* does not always restrict to orbits (except for the Heisenberg group) so other algebraic constructions of star products on $S(\mathfrak{g})$ were considered (for instance in [9, 10, 50, 90]). When \mathfrak{g} is semisimple, if \mathcal{H} is the space of harmonic polynomials and if I_1, \dots, I_r are generators of the space of invariant polynomials, then any polynomial $P \in S(\mathfrak{g})$ writes uniquely as a sum $P = \sum_{a_1 \dots a_r} I_1^{a_1} \dots I_r^{a_r} h_{a_1 \dots a_r}$ where $h_{a_1 \dots a_r} \in \mathcal{H}$. One considers the linear isomorphism σ' between $S(\mathfrak{g})$ and $U(\mathfrak{g})$ induced by this decomposition

$$\sigma'(P) = \sum_{a_1 \dots a_r} (\sigma(I_1) \circ)^{a_1} \dots (\sigma(I_r) \circ)^{a_r} \circ \sigma(h_{a_1 \dots a_r}).$$

The associative composition law in $U(\mathfrak{g})$, pulled back by this isomorphism σ' , gives a star product on $S(\mathfrak{g})$ which is not defined by differential operators. With Cahen and Rawnsley, we proved [56] that if \mathfrak{g} is semisimple, there is no differential star product on any neighbourhood of 0 in \mathfrak{g}^* such that $C * u = Cu$ for the quadratic invariant polynomial $C \in S(\mathfrak{g})$ and all $u \in S(\mathfrak{g})$ (thus no differential star product which is tangential to the orbits).

2.2.2 Existence of Star Products

In 1983, De Wilde and Lecomte proved [68] that on any symplectic manifold there exists a differential star product. Their technique works to prove the existence of a differential star product on a regular Poisson manifold [129]. In 1985, but appearing only in the West in the nineties [83], Fedosov gave a recursive construction of a star product on a symplectic manifold (M, ω) . In 1994, he extended this result to give a recursive construction in the context of regular Poisson manifold [82]. Independently, also using the framework of Weyl bundles, Omori, Maeda and Yoshioka [140] gave an other proof of existence of a differential star product on a symplectic manifold, gluing local Moyal star products.

In 1997, Kontsevich [122] gave a proof of the existence of a star product on any Poisson manifold and gave an explicit formula for a star product for any Poisson structure on $V = \mathbb{R}^m$. This appeared as a consequence of the proof of his formality theorem. Tamarkin [162] gave a version of the proof in the framework of the theory of operads.

2.2.3 The Notion of States

The star product model gives a quantization model for the algebra of observables, so here an algebra over formal power series $\mathbb{C}[[\hbar]]$. In the usual presentation of quantum mechanics, observables are operators on a Hilbert and states are rays in that Hilbert space. Model algebras of quantum observables are complex algebras of bounded linear operators on a complex Hilbert space. These are prototypes of C^* -algebras. Recall that a C^* -algebra is a Banach algebra over \mathbb{C} endowed with a $*$ involution (i.e. an involutive semilinear antiautomorphism) such that $\|a\| = \|a^*\|$ and $\|aa^*\| = \|a\|^2$ for each element a in the algebra. Recall that if $\mathcal{A} = \mathcal{B}(\mathcal{H})$ is the algebra of bounded linear operators on a Hilbert space \mathcal{H} and if ψ is a non vanishing element of \mathcal{H} , the ray it generates defines the linear functional

$$\omega : \mathcal{A} \rightarrow \mathbb{C} : A \mapsto \omega(A) := \frac{\langle \psi, A\psi \rangle}{\langle \psi, \psi \rangle}$$

which is positive in the sense that $\omega(A^*A) \geq 0$. This lead to define a state in the theory of C^* algebras as a positive linear functional. Bordemann, Römer and Waldmann [39] give the following intrinsic description of the notion of states for formal star products, generalizing the notion of a state to the framework of $*$ -algebras.

Definition 2.4 (1) An associative commutative unital ring R is said to be **ordered** with positive elements P if the product of two elements in P is in P , the sum of two elements in P is in P , and R is the disjoint union $R = P \cup \{0\} \cup -P$. (Examples are given by $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{R}[[\lambda]]$; in the case of $\mathbb{R}[[\lambda]]$, a series $a = \sum_{r=r_0}^{\infty} a_r \lambda^r$ is positive if its lowest order non vanishing term is positive ($a_{r_0} > 0$).)

Let $C = R(i)$ be the ring extension by a square root i of -1 of an ordered ring. For instance $C = \mathbb{C}$ for $R = \mathbb{R}$ or, for our use here in deformation quantization, $C = \mathbb{C}[[\lambda]]$ for $R = \mathbb{R}[[\lambda]]$ with $v = i\lambda$.

(2) An associative algebra \mathcal{A} over C is called a ***-algebra** if it has an involutive antilinear antiautomorphism $*$: $\mathcal{A} \rightarrow \mathcal{A}$ called the ***-involution**. (Examples: any C^* algebra is a *-algebra over \mathbb{C} , in particular the *-algebra over \mathbb{C} of bounded linear operators on a Hilbert space with the involution given by taking the adjoint; also the deformed algebra $(C^\infty(M)[[v = i\lambda]], *)$ with a Hermitian star product and conjugaison is a *-algebra over $\mathbb{C}[[\lambda]]$).

(3) A linear functional $\omega : \mathcal{A} \rightarrow C$ over a *-algebra over C is called **positive** if

$$\omega(A^*A) \geq 0 \quad \text{for any } A \in \mathcal{A}.$$

(4) A **state** for a *-algebra \mathcal{A} with unit over C is a positive linear functional which satisfies $\omega(1) = 1$.

Remark 2.8 The positive linear functionals on $C^\infty(M)$ are the compactly supported Borel measures.

The δ -functional on \mathbb{R}^{2n} is not positive with respect to the Moyal star product: if $H := \frac{1}{2m}p^2 + kq^2$, $(H *_{\text{Moyal}} H)(0, 0) = \frac{kv^2}{2m} = \frac{-k\lambda^2}{2m} < 0$.

Bursztyn and Waldmann prove in [45] that for a Hermitian star product, any classical state ω_0 on $C^\infty(M)$ can be deformed into a state for the deformed algebra, $\omega = \sum_{r=0}^{\infty} \lambda^r \omega_r$.

2.3 Fedosov's Star Products on a Symplectic Manifold

Fedosov gives a construction [83] of a star product on a symplectic manifold (M, ω) , when one has chosen a symplectic connection and a sequence of closed 2-forms on M . One obtains the star product by identifying the space $C^\infty(M)[[v]]$ with an algebra of flat sections of an associative algebra bundle, the so-called Weyl bundle, endowed with a flat connection.

2.3.1 The Weyl Bundle

Let (V, Ω) be a symplectic vector space and consider the space of polynomials in v whose coefficients are polynomials on V with Moyal star product; this is the Weyl algebra $S(V^*)[[v]]$.

Exercise 2.10 Show that the Weyl algebra $S(V^*)[[v]]$ is isomorphic to the universal enveloping algebra $U(\mathfrak{h})$ of the Heisenberg Lie algebra $\mathfrak{h} = V^* \oplus \mathbb{R}v$ with Lie bracket

$$[y^i, y^j] = (\Omega^{-1})^{ij} v.$$

Indeed both are associative algebras generated by V^* and ν and the map sending an element of $V^* \subset \mathfrak{h}$ to the corresponding element in $V^* \subset S(V^*)$ viewed as a linear function on V and mapping $\nu \in \mathfrak{h}$ on $\nu \in \mathbb{R}[\nu] \subset S(V^*)[\nu]$ satisfies: $\xi *_M \xi' - \xi' *_M \xi = [\xi, \xi']$ for all $\xi, \xi' \in \mathfrak{h}$ so extends to a morphism of associative algebras.

There is a grading on $U(\mathfrak{h})$ assigning the degree 1 to the y^i 's and the degree 2 to the element ν . The **formal Weyl algebra** W is the completion in that grading of the above algebra. An element of the formal Weyl algebra is of the form

$$a(y, \nu) = \sum_{m=0}^{\infty} \left(\sum_{2k+l=m} a_{k,i_1,\dots,i_l} \nu^k y^{i_1} \dots y^{i_l} \right).$$

The product in $U(\mathfrak{h})$ is given by the Moyal star product

$$(a \circ b)(y, \nu) = \left(\exp \left(\frac{\nu}{2} (\Omega^{-1})^{ij} \frac{\partial}{\partial y^i} \frac{\partial}{\partial z^j} \right) a(y, \nu) b(z, \nu) \right) \Big|_{y=z}$$

and the same formula also defines the product in W .

Definition 2.5 The symplectic group $Sp(V, \Omega)$ of the symplectic vector space (V, Ω) consists of all invertible linear transformations A of V with $\Omega(Au, Av) = \Omega(u, v)$, for all $u, v \in V$. $Sp(V, \Omega)$ acts as automorphisms of \mathfrak{h} by $A \cdot f = f \circ A^{-1}$ for $f \in V^*$ and $A \cdot \nu = 0$. This action extends to both $U(\mathfrak{h})$ and W and on the latter is denoted by ρ . It satisfies $\rho(A)(a \circ b) = \rho(A)(a) \circ \rho(A)(b)$. Explicitly: $\rho(A)(\sum_{2k+l=m} a_{k,i_1,\dots,i_l} \nu^k y^{i_1} \dots y^{i_l}) = \sum_{2k+l=m} a_{k,i_1,\dots,i_l} \nu^k (A^{-1})_{j_1}^{i_1} \dots (A^{-1})_{j_l}^{i_l} y^{j_1} \dots y^{j_l}$.

To any element B in the Lie algebra $sp(V, \Omega)$ of the symplectic group, we associate the quadratic element \overline{B} in W defined by

$$\overline{B} = \frac{1}{2} \sum_{ijr} \Omega_{ri} B_j^r y^i y^j.$$

This is an identification since the condition to be in $sp(V, \Omega)$ is that $\sum_r \Omega_{ri} B_j^r$ is symmetric in i and j .

Exercise 2.11 Show that the natural action $\rho_*(B)$ is given by:

$$\rho_*(B)y^l = \frac{-1}{\nu} [\overline{B}, y^l]$$

where $[a, b] := (a \circ b) - (b \circ a)$ for any $a, b \in W$.

Since both sides act as derivations this extends to all of W as

$$\rho_*(B)a = \frac{-1}{\nu} [\overline{B}, a]. \quad (2.17)$$

Definition 2.6 If (M, ω) is a symplectic manifold, we can form its bundle $F(M)$ of symplectic frames. A symplectic frame at the point $x \in M$ is a linear symplectic isomorphism $\xi_x : (V, \Omega) \rightarrow (T_x M, \omega_x)$. The bundle $F(M)$ is a principal $Sp(V, \Omega)$ -bundle over M (the action on the right of an element $A \in Sp(V, \Omega)$ on a frame ξ_x is given by $\xi_x \circ A$).

The associated bundle $\mathscr{W} = F(M) \times_{Sp(V, \Omega), \rho} W$ is a bundle of algebras on M called the bundle of formal Weyl algebras, or, more simply, **the Weyl bundle**. Its **sections** are formal series

$$a(x, y, v) = \sum_{2k+l \geq 0} v^k a_{k, i_1, \dots, i_l}(x) y^{i_1} \dots y^{i_l} \quad (2.18)$$

where the coefficients a_{k, i_1, \dots, i_l} define (in the i 's) symmetric covariant l -tensor fields on M . So $\mathscr{W} \simeq \oplus_p \mathbb{C} \otimes S^p(T^*M)[[v]]$. We denote by $\Gamma(\mathscr{W})$ the space of those sections. The pointwise product of two sections makes $\Gamma(\mathscr{W})$ into an algebra, and **the multiplication** has the form

$$(a \circ b)(x, y, v) = \left(\exp \left(\frac{v}{2} P^{ij} \frac{\partial}{\partial y^i} \frac{\partial}{\partial z^j} \right) a(x, y, v) b(x, z, v) \right) \Big|_{y=z}, \quad (2.19)$$

where P is the Poisson tensor associated to the symplectic structure (thus $\sum_j P^{ij} \omega_{jk} = \delta_k^i$). The center of this algebra coincide with $C^\infty(M)[[v]]$.

2.3.2 Flat Connections on the Weyl Bundle

Let (M, ω) be a symplectic manifold. A **symplectic connection** on M is a connection ∇ on TM which is torsion-free and satisfies $\nabla_X \omega = 0$. Such connections always exist but, unlike the Riemannian case, are not unique.

Exercise 2.12 To see the existence, take any torsion-free connection ∇' and define S by $\omega(S(X, Y), Z) = \frac{1}{3}((\nabla'_X \omega)(Y, Z) + (\nabla'_Y \omega)(X, Z))$. Check that $\nabla'_X Y = \nabla'_X Y + S(X, Y)$ defines a symplectic connection.

Remark 2.9 ([105]) Any natural star product $* = \sum_{r \geq 0} C_r$ on a symplectic manifold defines a unique symplectic connection ∇ such that

$$C_1(u, v) = \frac{1}{2}\{u, v\} + [\{u, Ev\} + \{Eu, v\} - E(\{u, v\})]$$

with E a differential operator of order 2 and

$$C_2(u, v) + C_2(v, u) = \frac{1}{4} P^{ij} P^{i'j'} \nabla_{ii'}^2 u \nabla_{jj'}^2 v + ((\text{ad } E)^2 m)(u, v) + A_2(u, v)$$

with $((\text{ad } E)^2 m)(u, v) = E^2(uv) + 2Eu.Ev - E^2u.v - u.E^2v - 2E(Eu.v + u.Ev)$ and A_2 a differential operator of order 1 in each argument.

A symplectic connection defines a connection 1-form in the symplectic frame bundle and so a connection in all associated bundles (i.e. a covariant derivative of sections); we denote by ∂ the connection in \mathscr{W} . For any vector field X on M , the covariant

derivative ∂_X is a derivation of the algebra $\Gamma(\mathcal{W})$. We consider \mathcal{W} -valued q -forms on M to express the connection and its curvature; these are sections of the bundle $\mathcal{W} \otimes \Lambda^q T^*M \simeq \mathbb{C} \otimes (\oplus_p S^p(T^*M) \otimes \Lambda^q(T^*M))[[v]]$ and locally have the form

$$\sum_{2k+p \geq 0} v^k a_{k, i_1, \dots, i_l, j_1, \dots, j_q}(x) y^{i_1} \dots y^{i_p} dx^{j_1} \wedge \dots \wedge dx^{j_q}$$

where the coefficients, symmetric in i_1, \dots, i_p and anti-symmetric in j_1, \dots, j_q , are covariant tensors. Such sections can be multiplied using the product in \mathcal{W} and simultaneously exterior multiplication $a \otimes \omega \circ b \otimes \omega' = (a \circ b) \otimes (\omega \wedge \omega')$. The space of \mathcal{W} -valued forms $\Gamma(\mathcal{W} \otimes \Lambda^*)$ is then a graded Lie algebra with respect to the bracket

$$[s, s'] = s \circ s' - (-1)^{q_1 q_2} s' \circ s \quad \text{for } s_i \in \Gamma(\mathcal{W} \otimes \Lambda^{q_i}).$$

The connection ∂ in \mathcal{W} is given by

$$\partial: \Gamma(\mathcal{W}) \rightarrow \Gamma(\mathcal{W} \otimes \Lambda^1) \quad \partial a = da - \frac{1}{v} [\bar{\Gamma}, a] \quad \text{with } \bar{\Gamma} = \frac{1}{2} \sum_{ijk} \omega_{ki} \Gamma_{rj}^k y^i y^j dx^r,$$

where Γ_{kl}^i are the Christoffel symbols of ∇ in TM (which define an element of the symplectic Lie algebra with respect to the il indices). As usual, the connection ∂ in \mathcal{W} extends to a covariant exterior derivative on $\Gamma(\mathcal{W} \otimes \Lambda^*)$, also denoted by ∂ , by using the Leibnitz rule:

$$\partial(a \otimes \omega) = \partial(a) \wedge \omega + a \otimes d\omega.$$

The curvature of ∂ is then given by $\partial \circ \partial$ which is a 2-form with values in $\text{End}(\mathcal{W})$.

Exercise 2.13 The curvature of ∂ admits a simple expression in terms of the curvature R of the symplectic connection ∇ :

$$\partial \circ \partial a = \frac{1}{v} [\bar{R}, a] \quad \text{where } \bar{R} = \frac{1}{4} \sum_{ijklr} \omega_{rl} R_{ijk}^l y^r y^k dx^i \wedge dx^j. \quad (2.20)$$

The idea is to try to modify ∂ to have zero curvature. In order to do so we use a further technical tool, coming from Koszul's long exact sequence. Given any finite dimensional vector space V' , the Koszul long exact sequence is:

$$0 \rightarrow S^q(V') \xrightarrow{\delta'} V' \otimes S^{q-1}(V') \xrightarrow{\delta'} \Lambda^2 V' \otimes S^{q-2}(V') \xrightarrow{\delta'} \dots \xrightarrow{\delta'} \Lambda^{q-1}(V') \otimes V' \xrightarrow{\delta'} \Lambda^q(V') \rightarrow 0$$

where δ' is the skew-symmetrisation operator:

$$\delta'(v^1 \wedge \dots \wedge v^q \otimes w^1 \dots w^p) = \sum_{i=1}^p v^1 \wedge \dots \wedge v^q \wedge w^i \otimes w^1 \dots w^{i-1} w^{i+1} \dots w^p.$$

The symmetrisation operator reads:

$$s(v^1 \wedge \dots \wedge v^q \otimes w^1 \dots w^p) \sum_{i=1}^q (-1)^{q-i} v^1 \wedge \dots \wedge v^{i-1} \wedge v^{i+1} \wedge \dots \wedge v^q \otimes v^i \cdot w^1 \dots w^p.$$

They satisfy $(\delta')^2 = 0$, $s^2 = 0$, $(\delta' \circ s + s \circ \delta')|_{\Lambda^q V' \otimes S^p(V')} = (p+q) \text{Id}$.

For any $a \in \Gamma(\mathcal{W} \otimes \Lambda^q)$, we write

$$a = \sum_{p \geq 0, q \geq 0} a_{pq} = \sum_{2k+p \geq 0, q \geq 0} v^k a_{k, i_1, \dots, i_p, j_1, \dots, j_q} y^{i_1} \dots y^{i_p} dx^{j_1} \wedge \dots \wedge dx^{j_q};$$

in particular $a_{00} = \sum_k v^k a_k$ with $a_k \in C^\infty(M)$; and we define

$$\delta(a) := \sum_k dx^k \wedge \frac{\partial a}{\partial y^k}, \quad \delta^{\sim 1}(a_{pq}) := \begin{cases} \frac{1}{p+q} \sum_k y^k i(\frac{\partial}{\partial x^k}) a_{pq} & \text{if } p+q > 0; \\ 0 & \text{if } p+q = 0. \end{cases} \quad (2.21)$$

Exercise 2.14 Show that

$$\delta^2 = 0, \quad (\delta^{\sim 1})^2 = 0, \quad (\delta \circ \delta^{\sim 1} + \delta^{\sim 1} \circ \delta)(a) = a - a_{00};$$

and that δ can be written in terms of the algebra structure by

$$\delta(a) = \frac{1}{v} \left[\sum_{ij} -\omega_{ij} y^i dx^j, a \right],$$

hence δ is a graded derivation of $\Gamma(\mathcal{W} \otimes \Lambda^*)$. Verify that $\partial\delta + \delta\partial = 0$.

We now look for a connection D on \mathcal{W} , so that D_X is a derivation of the algebra $\Gamma(\mathcal{W})$ for any vectorfield X on M , and so that D is flat in the sense that $D \circ D = 0$. Such a connection can be written as a sum of ∂ and a $\text{End}(\mathcal{W})$ -valued 1-form. The latter is taken in a particular form:

$$Da = \partial a - \delta(a) - \frac{1}{v} [r, a]. \quad (2.22)$$

Exercise 2.15 Show that

$$D \circ Da = \frac{1}{v} \left[\bar{R} - \partial r + \delta r + \frac{1}{2v} [r, r], a \right]$$

with \bar{R} defined by (2.20), and that $[r, r] = 2r \circ r$.

The connection D is flat provided the first term in the bracket is a central 2-form.

Theorem 2.1 (Fedosov [83]) *For any given series of closed 2-forms on M , $\tilde{\Omega} = \sum_{i \geq 1} v^i \omega_i$, the equation*

$$\delta r = -\bar{R} + \partial r - \frac{1}{v} r^2 + \tilde{\Omega} \quad (2.23)$$

has a unique solution $r \in \Gamma(\mathscr{W} \otimes \Lambda^1)$ satisfying the normalization condition $\delta^{-1} r = 0$ and such that the \mathscr{W} -degree of the leading term of r is at least 3.

Proof We apply δ^{-1} to the Eq. (2.23) using the fact that r is a 1-form and thus $r_{00} = 0$. Then r , if it exists, must satisfy

$$r = \delta^{-1} \delta r = -\delta^{-1} \bar{R} + \delta^{-1} \partial r - \frac{1}{v} \delta^{-1} r^2 + \delta^{-1} \tilde{\Omega}. \quad (2.24)$$

Two solutions of this equation will have a difference which satisfies the same equation but without the \bar{R} term and the $\tilde{\Omega}$ term. If the first non-zero term of the difference has finite degree m , then the leading term of $\delta^{-1} \partial r$ has degree $m + 1$ and of $\delta^{-1} (r^2/h)$ has degree $2m - 1$. Since both of these are larger than m for $m \geq 2$, such a term cannot exist so the difference must be zero. Hence the solution is unique. Existence is very similar. We observe that the above argument shows that the Eq. (2.24) for r determines the homogeneous components of r recursively. So it is enough to show that such a solution satisfies both conditions of the theorem. Obviously $\delta^{-1} r = 0$. Let $A = \delta r + \bar{R} - \partial r + \frac{1}{v} r^2 - \tilde{\Omega} \in \Gamma(\mathscr{W} \otimes \Lambda^2)$. Then $\delta^{-1} A = \delta^{-1} \delta r + \delta^{-1} (\bar{R} - \partial r + \frac{1}{v} r^2 - \tilde{\Omega}) = r - r = 0$. Also $DA = \partial A - \delta A - \frac{1}{v} [r, A] = 0$. We can now apply a similar argument to that which proved uniqueness. Since $A_{00} = 0$, $\delta^{-1} A = 0$ and $DA = 0$ we have $A = \delta^{-1} \delta A = \delta^{-1} (\partial A - \frac{1}{v} [r, A])$ and recursively we can see that each homogeneous component of A must vanish, which shows that (2.23) holds and the theorem is proved. \square

Carrying out the recursion (2.24) to determine r explicitly, one easily sees [21] that: r_m only depends on ω_i for $2i + 1 \leq m$ and the first term in r which involves ω_k is:

$$r_{2k+1} = \delta^{-1} (v^k \omega_k) + \tilde{r}_{2k+1} \quad (2.25)$$

where the last term does not involve ω_k .

2.3.3 Fedosov's Star Products

Given a series of closed 2-forms on M , $\tilde{\Omega} = \sum_{i \geq 1} h^i \omega_i$, we consider the flat connection D on the Weyl bundle constructed as above, corresponding to r in $\Gamma(\mathscr{W} \otimes \Lambda^1)$ given inductively by (2.24). Since D_X acts as a derivation of the pointwise multiplication of sections, the space \mathscr{W}_D of flat sections is a subalgebra of the space of sections of \mathscr{W} :

$$\mathscr{W}_D = \{a \in \Gamma(\mathscr{W}) | Da = 0\}.$$

Theorem 2.2 ([83]) *For any $a_o \in C^\infty(M)[[v]]$ there is a unique $a \in \mathscr{W}_D$ such that $a(x, 0, v) = a_o(x, v)$.*

Proof This is very much like the above argument. We have $Da = 0 \Leftrightarrow \delta a = \partial a - \frac{1}{v}[r, a]$. Since a is a 0-form, $\delta^{-1}a = 0$; we apply δ^{-1} and get:

$$a = \delta^{-1}\delta a + a_o = \delta^{-1}\left(\partial a - \frac{1}{v}[r, a]\right) + a_o. \quad (2.26)$$

We solve this equation recursively for a , so $a(x, 0, v) = a_o(x, v)$. The fact that $A = Da$ vanishes follows as before by showing that $\delta^{-1}A = 0$ and $DA = D^2a = 0$. The uniqueness of the element a follows by an induction argument for the difference of two solutions.

Definition 2.7 Define the symbol map $\sigma : \Gamma(\mathscr{W}) \rightarrow C^\infty(M)[[v]]$, by

$$\sigma(a) = a(x, 0, v). \quad (2.27)$$

Theorem 2.2 tells us that σ is a linear isomorphism when restricted to \mathscr{W}_D ; it is used to transport the algebra structure of \mathscr{W}_D to $C^\infty(M)[[v]]$.

$$a * b := \sigma(\sigma^{-1}(a) \circ \sigma^{-1}(b)), \quad a, b \in C^\infty(M)[[v]]. \quad (2.28)$$

Exercise 2.16 Check that this defines a $*$ -product on $C^\infty(M)$. If the curvature of ∇ vanishes and $\tilde{\Omega} = 0$, show that one gets back the Moyal $*$ -product.

This $*$ -product is called **the Fedosov star product**; its construction depends only on the choice of a symplectic connection ∇ and the choice of a series $\tilde{\Omega}$ of closed 2-forms on M so can be denoted $*_{\nabla, \tilde{\Omega}}$. The Fedosov star product $*_{\nabla, \tilde{\Omega}}$ is natural and the connection associated to it (see Remark 2.9) is ∇ . Writing $u *_{\nabla, \tilde{\Omega}} v = \sum_{i \geq 0} v^i C_r^{\nabla, \tilde{\Omega}}(u, v)$, we have [21] that, for any r , $C_r^{\nabla, \tilde{\Omega}}$ only depends on ω_i for $i < r$ and

$$C_{r+1}^{\nabla, \tilde{\Omega}}(u, v) = \omega_r(X_u, X_v) + \tilde{C}_{r+1}(u, v) \quad (2.29)$$

where the last term does not depend on ω_r .

2.4 Classification of Poisson Deformations and Star Products

2.4.1 Hochschild Cohomology

Star products on a manifold M are examples of deformations -in the sense of Gerstenhaber [98]- of associative algebras. Their study uses the Hochschild coho-

mology [111] of the algebra, here $C^\infty(M)$, where p -cochains are p -linear maps from $(C^\infty(M))^p$ to $C^\infty(M)$ and where the **Hochschild coboundary operator** maps the p -cochain C to the $p + 1$ -cochain

$$(\partial C)(u_0, \dots, u_p) = u_0 C(u_1, \dots, u_p) + \sum_{r=1}^p (-1)^r C(u_0, \dots, u_{r-1} u_r, \dots, u_p) \\ + (-1)^{p+1} C(u_0, \dots, u_{p-1}) u_p.$$

For differential star products, we consider differential cochains, i.e. given by differential operators on each argument.

Exercise 2.17 The associativity condition for a star product at order k in the parameter ν reads

$$(\partial C_k)(u, v, w) = \sum_{r+s=k, r, s > 0} (C_r(C_s(u, v), w) - C_r(u, C_s(v, w))).$$

If one has cochains C_j , $j < k$ such that the star product they define is associative to order $k - 1$, then the right hand side above is a cocycle ($\partial(\text{RHS}) = 0$) and one can extend the star product to order k if it is a coboundary ($\text{RHS} = \partial(C_k)$).

Theorem 2.3 (Vey [166]) *Every differential p -cocycle C on a manifold M is the sum of the coboundary of a differential $(p-1)$ -cochain and a 1-differential skew-symmetric p -cocycle A :*

$$C = \partial B + A. \quad (2.30)$$

In particular, a cocycle is a coboundary if and only if its total skew-symmetrization, which is automatically 1-differential in each argument, vanishes. Also

$$H_{\text{diff}}^p(C^\infty(M), C^\infty(M)) = \Gamma(\Lambda^p T M).$$

Furthermore [53], given a connection ∇ on M , B can be defined from C by universal formulas.

By universal, we mean the following: any p -differential operator D of order maximum k in each argument can be written

$$D(u_1, \dots, u_p) = \sum_{|\alpha_1| < k, \dots, |\alpha_p| < k} D_{|\alpha_1|, \dots, |\alpha_p|}^{\alpha_1 \dots \alpha_p} \nabla_{\alpha_1} u_1 \dots \nabla_{\alpha_p} u_p \quad (2.31)$$

where α 's are multiindices, $D_{|\alpha_1|, \dots, |\alpha_p|}$ are tensors (symmetric in each of the p groups of indices) and $\nabla_\alpha u = (\nabla \dots (\nabla u)) \left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_q}} \right)$ when $\alpha = (i_1, \dots, i_q)$. We claim that there is a B such that the tensors defining B are universally defined as linear combinations of the tensors defining C , universally meaning in a way which is independent of the form of C . An elementary proof of the above theorem can be

found in [104]. Note that requiring differentiability of the cochains is essentially the same as requiring them to be local [54], local meaning that $C_r(u, v)(x) = 0$ as soon as u (or v) vanishes in a neighborhood of x .

2.4.2 Equivalence of Star Products

Definition 2.8 Two star products $*$ and $*'$ on (M, P) are said to be **equivalent** if there is a series $T = \text{Id} + \sum_{r=1}^{\infty} v^r T_r$ of linear operators T_r on $C^\infty(M)$, such that

$$T(f * g) = Tf *' Tg. \quad \text{We then write} \quad *' = T \cdot *. \quad (2.32)$$

One can write $T = \exp A$ where A is a series of linear operators on $C^\infty(M)$.

The T_r automatically vanish on constants since 1 is a unit for $*$ and for $*'$. Using linear operators which do not necessarily vanish on constants, one can pass from any associative deformation of the product of functions on a Poisson manifold (M, P) to another such deformation with 1 being a unit.

Exercise 2.18 Show that, on \mathbb{R}^{2n} , the Wick, the Standard and Moyal star products are all equivalent, in view of Eqs. (2.6) and (2.8); for instance, $*_{\text{Wick}} = T' \cdot *_{\text{Weyl}}$ for $T' = \exp \frac{iv}{4} \left(\frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial p^2} \right)$.

Proposition 2.1 (Lichnerowicz [126], Deligne [72]) *If $*$ and $*'$ are differential star products and $T(u) = u + \sum_{r \geq 1} v^r T_r(u)$ is an equivalence so that $*' = T \cdot *$, then the T_r are differential operators.*

Proof If $T = \text{Id} + v^k T_k + \dots$, then $\partial T_k = C'_k - C_k$ is differential, so $C'_k - C_k$ is a differential 2-cocycle with vanishing skew-symmetric part. Thus, using Vey's formula, it is the coboundary of a differential 1-cochain E and $T_k - E$, being a 1-cocycle, is a vector field, hence T_k is differential. One then proceeds by induction, considering $T' = (\text{Id} + v^k T_k)^{-1} \circ T = \text{Id} + v^{k+1} T'_{k+1} + \dots$ and the two differential star products $*$ and $*''$, where $*'' = (\text{Id} + v^k T_k)^{-1} \cdot *'$, which are differential and equivalent through $T' (*'' = T' \cdot *)$.

A differential star product is equivalent to one with linear term in v given by $\frac{1}{2}\{u, v\}$. Indeed $C_1(u, v)$ is a Hochschild cocycle with antisymmetric part given by $\frac{1}{2}\{u, v\}$ so $C_1 = \frac{1}{2}P + \partial B$ for a differential 1-cochain B . If $T(u) := u + vB(u)$ then $*' = T \cdot *$ has the required form.

Proposition 2.2 ([128]) *Let $*$ and $*'$ be two differential star products on (M, ω) and suppose that $H^2(M; \mathbb{R}) = 0$. Then there exists a series $T = \text{Id} + \sum_{k \geq 1} v^k T_k$ on $C^\infty(M)[[v]]$ such that $*' = T \cdot *$.*

Proof Let us suppose that, modulo some equivalence, the two star products $*$ and $*'$ coincide up to order k . Then associativity at order k shows that $C_k - C'_k$

is a Hochschild 2-cocycle and so by (2.3) can be written as $(C_k - C'_k)(u, v) = (\partial B)(u, v) + A(X_u, X_v)$ for a 2-form A . The total skew-symmetrization of the associativity relation at order $k + 1$ shows that A is a closed 2-form. Since the second cohomology vanishes, A is exact, $A = dF$. Transforming by the equivalence defined by $Tu = u + v^{k-1}2F(X_u)$, we can assume that the skew-symmetric part of $C_k - C'_k$ vanishes. Then $C_k - C'_k = \partial B$ where B is a differential operator. Using the equivalence defined by $T = I + v^k B$ we can assume that the star products coincide, modulo an equivalence, up to order $k + 1$ and the result follows from induction.

In 1994, Fedosov proved the recursive construction explained in Sect. 2.3 and showed that two star products constructed with cohomologous series of 2-forms are equivalent. Following an induction reasoning as above, and using formula (2.29), it is easy [21] to show that any differential star product on a symplectic manifold (M, ω) is equivalent to a Fedosov star product. Hence the equivalence classes of star products on a symplectic manifold are parametrised by elements in $H^2(M; \mathbb{R})[[v]]$. This parametrization is also proven by Nest and Tsygan [135], and Deligne [72].

Definition 2.9 A **Poisson deformation** of the Poisson bracket on a Poisson manifold (M, P) is a Lie algebra deformation of $(C^\infty(M), \{, \})$ which is a derivation in each argument, i.e. of the form

$$\{u, v\}_v = P_v(du, dv) \quad (2.33)$$

where $P_v = P + \sum v^k P_k$ is a series of skew-symmetric contravariant 2-tensors on M (such that $[P_v, P_v] = 0$). Two Poisson deformations P_v and P'_v of the Poisson bracket P on a Poisson manifold (M, P) are **equivalent** if there exists a formal path in the diffeomorphism group of M , starting at the identity, i.e. a series

$$T = \exp D = \text{Id} + \sum_j \frac{1}{j!} D^j \text{ for } D = \sum_{r \geq 1} v^r D_r, \quad (2.34)$$

where the D_r are vector fields on M , such that

$$T\{u, v\}_v = \{Tu, Tv\}'_v \quad (2.35)$$

where $\{u, v\}_v = P_v(du, dv)$ and $\{u, v\}'_v = P'_v(du, dv)$.

Flato, Lichnerowicz and Sternheimer studied in [93] 1-differential deformations of the Poisson bracket on symplectic manifolds; one gets.

Proposition 2.3 *On a symplectic manifold (M, ω) , the equivalence classes of Poisson deformations of the Poisson bracket P are parametrised by $H^2(M; \mathbb{R})[[v]]$.*

One first shows by induction that any Poisson deformation P_v of the Poisson bracket P on a symplectic manifold (M, ω) is of the form P^Ω for a series $\Omega = \omega + \sum_{k \geq 1} v^k \omega_k$ where the ω_k are closed 2-forms, and $P^\Omega(du, dv) = -\Omega(X_u^\Omega, X_v^\Omega)$ where $X_u^\Omega = X_u + v(\dots) \in \Gamma(TM)[[v]]$ is the element defined by $i(X_u^\Omega)\Omega = du$.

One then shows that two Poisson deformations P^{Ω} and $P^{\Omega'}$ are equivalent if and only if ω_k and ω'_k are cohomologous for all $k \geq 1$. In fact

$$TP^{\Omega}(du, dv) = P^{\Omega'}(d(Tu), d(Tv))$$

with $T = \exp D$ for $D = \sum_{r \geq 1} v^r D_r$ iff $\Omega' = \exp(\mathcal{L}_D)\Omega$ so iff $\Omega' - \Omega = d\alpha$ for $\alpha = \sum_{k \geq 0} v^k \alpha_k$ with

$$d\alpha = (\exp(\mathcal{L}_D) - \text{Id})\Omega = d\left(\sum_{k \geq 0} \frac{1}{(k+1)!} i(D)(\mathcal{L}_D)^k \Omega\right).$$

In 1997, Kontsevich proved that the coincidence of the set of equivalence classes of star products and Poisson deformations is true for general Poisson manifolds:

Theorem 2.4 ([122]) *The set of equivalence classes of differential star products on a Poisson manifold (M, P) can be naturally identified with the set of equivalence classes of Poisson deformations of P .*

Parametrization of equivalence classes of special star products are known; in particular for star products on pseudo Kähler manifolds with “separation of variables” (i.e. such that $f * u = fu$ and $u * g = u$ whenever f is holomorphic or g antiholomorphic), Karabegov [115] showed that one has even a parametrization of all such star products by series of closed $(1, 1)$ -forms.

Remark 2.10 Although the definition of equivalence is mathematically beautiful, it has drawbacks; a given classical polynomial function on \mathbb{R}^{2n} , when quantized relatively to two different orderings, does not lead to operators with the same spectrum. Hence equivalence is too broad to give isospectrality for a given classical observable (provided one could define a good notion of spectrum!). On the other hand, if one considers the whole deformed algebras, one likes to know when two deformed algebras have equivalent sets of representations. This enters the realm of Morita equivalence. The theory of representations of $*$ -algebras (in the sense of Definition 2.4) is introduced by Bordemann and Waldmann in [37, 168] extending classical constructions existing for C^* -algebras; the Morita equivalence of star products is studied in [47–49].

Remark 2.11 Deligne [72] defines cohomological classes associated to differential star products on a symplectic manifold and this leads to an intrinsic way to define a characteristic class $c(*)$ of a star product $*$, which parametrizes its equivalence class; the methods do not extend to general Poisson manifolds. A selfcontained presentation of these classes is given in [104]. This allows to characterize isomorphisms in the following way: two differential star products $*$ on (M, ω) and $*'$ on (M', ω') are isomorphic if and only if there exist $f(v) = \sum_{r \geq 1} v^r f_r \in \mathbb{R}[[v]]$ with $f_1 \neq 0$ and $\psi: M' \rightarrow M$, a symplectomorphism, such that $(\psi^{-1})^* c(*')(f(v)) = c(*) (v)$. In particular [101]: if $H^2(M; \mathbb{R}) = \mathbb{R}[\omega]$ then there is only one star product up to

equivalence and change of parameter. A symplectomorphism ψ of a symplectic manifold can be extended to a ν -linear automorphism of a given differential star product on (M, ω) if and only if $(\psi)^*c(*) = c(*)$. Notice that this is always the case if ψ can be connected to the identity by a path of symplectomorphisms (and this result is in Fedosov [82]). Homomorphisms of star products have been studied by Bordemann in [36].

2.5 Star Products on Poisson Manifolds and Formality

Kontsevich proved that the set of equivalence classes of star products is in bijection with the set of equivalence classes of formal Poisson structures on a general Poisson manifold in [122], as a consequence of his formality theorem. A differential star product on M is defined by a series of bidifferential operators satisfying some identities; a formal Poisson structure on a manifold M is defined by a series of bivector fields (i.e. contravariant skew-symmetric 2-tensors) P satisfying certain properties; to describe a correspondence between these objects, one considers algebras they belong to.

Definition 2.10 A **graded Lie algebra** is a \mathbb{Z} -graded vector space $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}^i$ endowed with a bilinear operation

$$[\ , \]: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$$

which is graded ($[a, b] \in \mathfrak{g}^{|a|+|b|}$), graded skew-symmetric,

$$[a, b] = -(-1)^{|a||b|}[b, a]$$

and satisfies the graded Jacobi identity

$$[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]] \quad (a \in \mathfrak{g}^{|a|}, b \in \mathfrak{g}^{|b|}).$$

Any Lie algebra is a graded Lie algebra concentrated in degree 0, and the degree zero part \mathfrak{g}^0 and the even part $\mathfrak{g}^{even} := \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}^{2i}$ of any graded Lie algebra are Lie algebras in the usual sense.

Definition 2.11 A **differential graded Lie algebra** (briefly DGLA) is a graded Lie algebra \mathfrak{g} endowed with a differential, $d: \mathfrak{g} \rightarrow \mathfrak{g}$, i.e. a linear operator of degree 1 ($d: \mathfrak{g}^i \rightarrow \mathfrak{g}^{i+1}$) which squares to zero ($d \circ d = 0$) and satisfies the compatibility condition (Leibniz rule)

$$d[a, b] = [da, b] + (-1)^\alpha[a, db] \quad a \in \mathfrak{g}^\alpha, b \in \mathfrak{g}^\beta.$$

The natural notions of morphisms of graded and differential graded Lie algebras are graded linear maps which commute with the differentials and the brackets (a graded

linear map $\phi: \mathfrak{g} \rightarrow \mathfrak{h}$ of degree k is a linear map such that $\phi(\mathfrak{g}^i) \subset \mathfrak{h}^{i+k} \forall i \in \mathbb{Z}$. Remark that a morphism of DGLA's has to be a degree 0 in order to commute with the other structures.

Any DGLA has a cohomology complex defined by

$$\mathcal{H}^i(\mathfrak{g}) := \text{Ker}(d: \mathfrak{g}^i \rightarrow \mathfrak{g}^{i+1}) / \text{Im}(d: \mathfrak{g}^{i-1} \rightarrow \mathfrak{g}^i).$$

The set $\mathcal{H} := \bigoplus_i \mathcal{H}^i(\mathfrak{g})$ has a natural structure of graded vector space and inherits the structure of a graded Lie algebra, defined by $[|a|, |b|]_{\mathcal{H}} := |[a, b]_{\mathfrak{g}}|$ where $|a| \in \mathcal{H}$ denote the equivalence class of a closed element $a \in \mathfrak{g}$. The cohomology of a DGLA is itself a DGLA with zero differential.

Any morphism $\phi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ of DGLA's induces a morphism $(\phi): \mathcal{H}_1 \rightarrow \mathcal{H}_2$. A morphism of DGLA's inducing an isomorphism in cohomology is called a **quasi-isomorphism**.

2.5.1 The DGLA of Polydifferential Operators

Let A be an associative algebra with unit on a field \mathbb{K} ; consider the complex of multilinear maps from A to itself:

$$\mathcal{C} := \sum_{i=-1}^{\infty} \mathcal{C}^i \quad \mathcal{C}^i := \text{Hom}_{\mathbb{K}}(A^{\otimes(i+1)}, A)$$

(we shifted the degree by one; the degree $|A|$ of a $(p+1)$ -linear map A is equal to p). The Lie bracket of linear operators is the skew-symmetrization of the composition of linear operators. This notion is extended to multilinear operators: for $A_1 \in \mathcal{C}^{m_1}$, $A_2 \in \mathcal{C}^{m_2}$, one defines:

$$(A_1 \circ A_2)(f_1, \dots, f_{m_1+m_2+1}) := \sum_{j=1}^{m_1} (-1)^{(m_2)(j-1)} A_1(f_1, \dots, f_{j-1}, A_2(f_j, \dots, f_{j+m_2}), f_{j+m_2+1}, \dots, f_{m_1+m_2+1})$$

for any $(m_1 + m_2 + 1)$ -tuple of elements of A and the **Gerstenhaber bracket** is defined by

$$[A_1, A_2]_G := A_1 \circ A_2 - (-1)^{m_1 m_2} A_2 \circ A_1.$$

It gives \mathcal{C} the structure of a graded Lie algebra. The differential d_D is defined by

$$d_D A = -[\mu, A] = -\mu \circ A + (-1)^{|A|} A \circ \mu$$

where μ is the usual product in the algebra A . Hence $dA = (-1)^{|A|+1}\delta A$ where δ is the Hochschild coboundary. The graded Lie algebra \mathcal{C} with the differential d_D is a differential graded Lie algebra.

Here the algebra A is $C^\infty(M)$, and we consider the subalgebra of \mathcal{C} consisting of multidifferential operators $\mathcal{D}_{poly}(M) := \bigoplus \mathcal{D}_{poly}^i(M)$ with $\mathcal{D}_{poly}^i(M)$ the space of multi differential operators acting on $i + 1$ smooth functions on M and vanishing on constants. Clearly $\mathcal{D}_{poly}(M)$ is closed under the Gerstenhaber bracket and under the differential d_D , so that it is a DGLA.

Proposition 2.4 *An element C in $\nu\mathcal{D}_{poly}^1(M)[[v]]$, i.e. a series of bidifferential operators, yields a deformation of the usual associative pointwise product μ of functions, $* = \mu + C$, which defines a differential star product on M if and only if $d_DC - \frac{1}{2}[C, C]_G = 0$.*

2.5.2 The DGLA of Multivector Fields

A **k -multivector field** is a section of the k -th exterior power $\Lambda^k TM$ of the tangent space TM ; the **Schouten-Nijenhuis bracket** is the bracket of multivectorfields defined by extending the usual Lie bracket of vector fields

$$[X_1 \wedge \dots \wedge X_k, Y_1 \wedge \dots \wedge Y_l]_S = \sum_{r=1}^k \sum_{s=1}^l (-1)^{r+s} [X_r, X_s] X_1 \wedge \dots \wedge \widehat{X}_r \wedge \dots \wedge X_k \wedge Y_1 \wedge \dots \wedge \widehat{Y}_s \wedge \dots \wedge Y_l.$$

Since the bracket of an r - and an s - multivector fields on M is an $r + s - 1$ - multivector field, we define a structure of graded Lie algebra on the space $\mathcal{T}_{poly}(M)$ of multivector fields on M by setting $\mathcal{T}_{poly}^i(M)$ to be the set of skew-symmetric contravariant $i + 1$ -tensorfields on M (remark again a shift in the grading). The graded Lie algebra $\mathcal{T}_{poly}(M)$ is a differential graded Lie algebra choosing the differential d_T to be identically zero.

Proposition 2.5 *An element $P \in \nu\mathcal{T}_{poly}^1(M)[[v]]$ (i.e. a series of bivectorfields on the manifold M) defines a formal Poisson structure on M if and only if $d_T P - \frac{1}{2}[P, P]_S = 0$.*

If one could construct an isomorphism of DGLA between the algebra $\mathcal{T}_{poly}(M)$ of multivector fields and the algebra $\mathcal{D}_{poly}(M)$ of multidifferential operators, this would give a correspondence between a formal Poisson tensor on M and a formal differential star product on M . By Theorem 2.3 the cohomology of the algebra of multidifferential operators is given by multivector fields

$$\mathcal{H}^i(\mathcal{D}_{poly}(M)) \simeq \mathcal{T}_{poly}^i(M).$$

This bijection is induced by the natural map $U_1: \mathcal{T}_{poly}^i(M) \longrightarrow \mathcal{D}_{poly}^i(M)$ which extends the usual identification between vector fields and first order differential operators, and is defined by:

$$U_1(X_0 \wedge \dots \wedge X_n)(f_0, \dots, f_n) = \frac{1}{(n+1)!} \sum_{\sigma \in \mathcal{S}_{n+1}} \varepsilon(\sigma) X_0(f_{\sigma(0)}) \dots X_n(f_{\sigma(n)}). \quad (2.36)$$

Exercise 2.19 Compute at order 2 to show that this map fails to preserve the Lie structure.

One extends the notion of morphism between two DGLA's to construct a morphism whose first order approximation is this isomorphism (2.36). To do so one introduces the notion of L_∞ -morphism.

2.5.3 L_∞ -Algebras, L_∞ -Morphism and Formality

A toy picture of our situation (finding a correspondence between a formal Poisson tensor P on M and a formal differential star product $\ast = \mu + C$ on M) is the following. If C and P were elements in neighborhoods of zero of finite dimensional vector spaces V_1 and V_2 , one could consider analytic vector fields X_1 on V_1 , X_2 on V_2 , vanishing at zero, given by $(X_1)_C = d_D C - \frac{1}{2}[C, C]_G$, $(X_2)_P = d_T P - \frac{1}{2}[P, P]_S$ and one would be interested in finding a correspondence between zeros of X_2 and zeros of X_1 . An idea would be to construct an analytic map $\phi: V_2 \rightarrow V_1$ such that $\phi(0) = 0$ and $\phi_\ast X_2 = X_1$. Such a map can be viewed as an algebra morphism $\phi^\ast: A_1 \rightarrow A_2$ where A_i is the algebra of analytic functions on V_i vanishing at zero. The vector field X_i can be seen as a derivation of the algebra A_i . A real analytic function being determined by its Taylor expansion at zero, one can look at $C(V_i) := \sum_{n \geq 1} S^n(V_i)$ as the dual space to A_i ; it is a coalgebra. One views the derivation of A_i corresponding to the vector field X_i dually as a coderivation Q_i of $C(V_i)$. One is then looking for a coalgebra morphism $F: C(V_2) \rightarrow C(V_1)$ so that $F \circ Q_2 = Q_1 \circ F$. This is generalized to the framework of graded algebras with the notion of L_∞ -morphism between L_∞ -algebras.

Definition 2.12 A **graded coalgebra** on the base ring \mathbb{K} is a \mathbb{Z} -graded vector space $C = \bigoplus_{i \in \mathbb{Z}} C^i$ with a comultiplication, i.e. a graded linear map $\Delta: C \rightarrow C \otimes C$ such that $\Delta(C^i) \subset \bigoplus_{j+k=i} C^j \otimes C^k$ and such that one has coassociativity, i.e. $(\Delta \otimes \text{id})\Delta(x) = (\text{id} \otimes \Delta)\Delta(x)$ for every $x \in C$.

A **counit** (if it exists) is a morphism $e: C \rightarrow \mathbb{K}$ such that $e(C^i) = 0$ for any $i > 0$ and such that $(e \otimes \text{id})\Delta = (\text{id} \otimes e)\Delta = \text{id}$.

The coalgebra is **cocommutative** if $T \circ \Delta = \Delta$ where $T: C \otimes C \rightarrow C \otimes C$ is the twisting map: $T(x \otimes y) := (-1)^{|x||y|} y \otimes x$ for x, y homogeneous elements of degree respectively $|x|$ and $|y|$.

Additional structures that can be put on an algebra can be dualized to give dual versions on coalgebras.

Example 2.3 (The coalgebra $C(V)$) If $V = \bigoplus_{i \in \mathbb{Z}} V^i$ is a graded vector space over \mathbb{K} , one defines the tensor algebra $T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ with $V^{\otimes 0} = \mathbb{K}$, and the symmetric algebra $S(V) = T(V) / \langle x \otimes y - (-1)^{|x||y|} y \otimes x \rangle$ which is a naturally graded associative algebra. It has a structure of coalgebra with comultiplication Δ defined by $\Delta v := 1 \otimes v + v \otimes 1$ for a homogeneous element $v \in V$ and extended as algebra homomorphism.

The reduced symmetric space is $C(V) := S^+(V) := \bigoplus_{n \geq 0} S^n(V)$; it is the cofree cocommutative coalgebra without counit constructed on V . (Remark that $\Delta v = 0$ iff $v \in V$.)

Definition 2.13 A **coderivation** of degree d on a graded coalgebra C is a graded linear map $\delta: C^i \rightarrow C^{i+d}$ which satisfies the (co-)Leibniz identity $\Delta \delta(v) = \delta v' \otimes v'' + (-1)^{d|v'|} v' \otimes \delta v''$ if $\Delta v = \sum v' \otimes v''$. This can be rewritten with the usual Koszul sign conventions $\Delta \delta = (\delta \otimes \text{id} + \text{id} \otimes \delta) \Delta$.

Definition 2.14 A L_{∞} -**algebra** is a graded vector space V over \mathbb{K} and a degree 1 coderivation Q defined on the reduced symmetric space $C(V[1])$ so that $Q \circ Q = 0$. (Given any graded vector space V , we can obtain a new graded vector space $V[k]$ by shifting the grading of the elements of V by k , i.e. $V[k] = \bigoplus_{i \in \mathbb{Z}} V[k]^i$ where $V[k]^i := V^{i+k}$.)

Definition 2.15 A L_{∞} -**morphism** between two L_{∞} -algebras, $F: (V, Q) \rightarrow (V', Q')$, is a morphism $F: C(V[1]) \rightarrow C(V'[1])$ of graded coalgebras, so that $F \circ Q = Q' \circ F$.

Any algebra morphism from $S^+(V)$ to $S^+(V')$ is uniquely determined by its restriction to V and any derivation of $S^+(V)$ is determined by its restriction to V . In a dual way, a coalgebra-morphism F from the coalgebra $C(V)$ to the coalgebra $C(V')$ is uniquely determined by the composition of F and the projection on $\pi': C(V') \rightarrow V'$. Similarly, any coderivation Q of $C(V)$ is determined by the composition $F \circ \pi$ where π is the projection of $C(V)$ on V .

Definition 2.16 We call **Taylor coefficients of a coalgebra-morphism** $F: C(V) \rightarrow C(V')$ the sequence of maps $F_n: S^n(V) \rightarrow V'$ and **Taylor coefficients of a coderivation** Q of $C(V)$ the sequence of maps $Q_n: S^n(V) \rightarrow V$.

Proposition 2.6 Given V and V' two graded vector spaces, any sequence of linear maps $F_n: S^n(V) \rightarrow V'$ of degree zero determines a unique coalgebra morphism $F: C(V) \rightarrow C(V')$ for which the F_n are the Taylor coefficients. Similarly, if V is a graded vector space, any sequence $Q_n: S^n(V) \rightarrow V, n \geq 1$ of linear maps of degree i determines a unique coderivation Q of $C(V)$ of degree i whose Taylor coefficients are the Q_n .

The Taylor coefficients of a coderivation Q of $C(V[1])$ of degree 1 are the linear maps

$$Q_n: S^n(V[1]) \rightarrow V[2].$$

Proposition 2.7 Any L_∞ -algebra (V, Q) such that all the Taylor coefficients Q_n of Q vanish for $n > 2$ yields a differential graded Lie algebra and vice versa.

A morphism of graded coalgebras between $C(V[1])$ and $C(V'[1])$ is equivalent to a sequence of linear maps (the Taylor coefficients)

$$F_n : S^n(V[1]) \rightarrow V'[1];$$

it defines a L_∞ -morphism between two L_∞ -algebras (V, Q) and (V', Q') iff $F \circ Q = Q' \circ F$. For DGLA's, there exist L_∞ -morphisms between two DGLA's which are not DGLA-morphisms.

Definition 2.17 Given a L_∞ algebra (V, Q) over a field of characteristic zero, and given $\mathfrak{m} = \nu\mathbb{R}[[v]]$, a **m-point** is an element $p \in \nu C(V)[[v]]$ such that $\Delta p = p \otimes p$ or, equivalently, it is an element

$$p = e^v - 1 = v + \frac{v^2}{2} + \dots \quad (2.37)$$

where v is an even element in $V[1] \otimes \mathfrak{m} = \nu V[1][[v]]$.

A **solution of the generalized Maurer–Cartan equation** is a m-point p at which Q vanishes; equivalently, it is an odd element $v \in \nu V[[v]]$ such that $Q_1(v) + \frac{1}{2}Q_2(v \cdot v) + \dots = 0$. If \mathfrak{g} is a DGLA, it is thus an element $v \in \nu \mathfrak{g}[[v]]$ such that $dv - \frac{1}{2}[v, v] = 0$.

Exercise 2.20 The image under a L_∞ morphism of a solution of the generalised Maurer–Cartan equation is again such a solution. In particular, if one builds a L_∞ morphism $F : \mathcal{T}_{poly}(M) \rightarrow \mathcal{D}_{poly}(M)$ between the two DGLA's we have defined, the image under F of the point $e^\alpha - 1$ corresponding to a formal Poisson tensor,

$$\alpha \in \nu \mathcal{T}_{poly}^1(M)[[v]] \text{ such that } [\alpha, \alpha]_S = 0, \quad (2.38)$$

yields a star product on M ,

$$* = \mu + \sum_n F_n(\alpha^n). \quad (2.39)$$

Definition 2.18 Two L_∞ -algebras (V, Q) and (V', Q') are **quasi-isomorphic** if there is a L_∞ -morphism F such that $F_1 : V \rightarrow V'$ induces an isomorphism in cohomology. Such a F is called a quasi-isomorphism.

Kontsevich has proven that if F is a L_∞ -morphism between two L_∞ -algebras (V, Q) and (V', Q') so that $F_1 : V \rightarrow V'$ induces an isomorphism in cohomology, then there exists a L_∞ -morphism G between (V', Q') and (V, Q) so that $G_1 : V' \rightarrow V$ is a quasi inverse for F_1 .

Definition 2.19 Kontsevich's **formality** is a quasi isomorphism between the (L_∞ -algebra structure associated to the) DGLA of multidifferential operators, $\mathcal{D}_{poly}(M)$, and its cohomology, the DGLA of multivector fields $\mathcal{T}_{poly}(M)$.

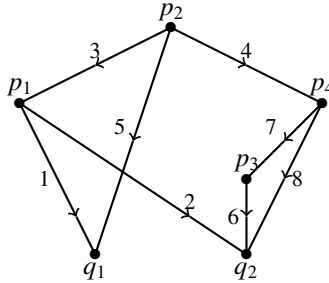
2.5.4 Formality for \mathbb{R}^d

Kontsevich [122] gave an explicit formula for the Taylor coefficients of a formality for \mathbb{R}^d , i.e. the Taylor coefficients F_n of an L_∞ -morphism $F : (\mathcal{T}_{poly}(\mathbb{R}^d), Q) \rightarrow (\mathcal{D}_{poly}(\mathbb{R}^d), Q')$ where Q corresponds to the DGLA $(\mathcal{T}_{poly}(\mathbb{R}^d), [\ , \]_S, D_T = 0)$ and Q' to the DGLA $(\mathcal{D}_{poly}(\mathbb{R}^d), [\ , \]_G, d_D)$ with $F_1 : \mathcal{T}_{poly}(\mathbb{R}^d) \rightarrow \mathcal{D}_{poly}(\mathbb{R}^d)$ given by U_1 as in Eq.(2.36). The formula writes

$$F_n = \sum_{m \geq 0} \sum_{\Gamma \in G_{n,m}} \mathcal{W}_\Gamma B_\Gamma$$

- where $G_{n,m}$ is a set of oriented admissible graphs;

An admissible graph $\Gamma \in G_{n,m}$ has n aerial vertices labelled p_1, \dots, p_n , has m ground vertices labelled q_1, \dots, q_m . From each aerial vertex p_i , a number k_i of arrows are issued; each of them can end on any vertex except p_i but there can not be multiple arrows. There are no arrows issued from the ground vertices. One gives an order to the vertices: $(p_1, \dots, p_n, q_1, \dots, q_m)$ and one gives a compatible order to the arrows, labeling those issued from p_i with $(k_1 + \dots + k_{i-1} + 1, \dots, k_1 + \dots + k_{i-1} + k_i)$. The arrows issued from p_i are named $\text{Star}(p_i) = \{\overrightarrow{p_i a_1}, \dots, \overrightarrow{p_i a_{k_i}}\}$ with $v_{k_1 + \dots + k_{i-1} + j} = \overrightarrow{p_i a_j}$.



An example of graph Γ_1 in $G_{3,2}$

- where B_Γ associates a m -differential operator to an n -tuple of multivectorfields; Given a graph $\Gamma \in G_{n,m}$ and given n multivectorfields $(\alpha_1, \dots, \alpha_n)$ on \mathbb{R}^d , one defines a m -differential operator $B_\Gamma(\alpha_1 \dots \alpha_n)$; it vanishes unless α_1 is a k_1 -tensor, α_2 is a k_2 -tensor, ..., α_n is a k_n -tensor and in that case it is given by:

$$B_\Gamma(\alpha_1 \dots \alpha_n)(f_1, \dots, f_n) = \sum_{i_1, \dots, i_K} D_{p_1} \alpha_1^{i_1 \dots i_{k_1}} D_{p_2} \alpha_2^{i_{k_1+1} \dots i_{k_1+k_2}} \dots D_{p_n} \alpha_n^{i_{k_1+\dots+k_{n-1}+1} \dots i_K} D_{q_1} f_1 \dots D_{q_m} f_m$$

where $K := k_1 + \dots + k_n$ and where

$$D_a := \prod_{j|\vec{v_j}=\vec{a}} \partial_{i_j}.$$

For the graph Γ_1 as above, B_{Γ_1} associates a bidifferential operator [since $n = 2$] of order 2 in the first variable [since two arrows arrive at the first ground vertex q_1] and of order 3 in the second variable [since three arrows arrive at the second ground vertex q_2] to a quadruple of multivectorfields $(\alpha_1, \dots, \alpha_4)$ on \mathbb{R}^d [since $m = 4$]. For this operator not to vanish, α_1 is a 2-tensor [since two arrows start from the first aerial vertex p_1], α_2 is a 3-tensor [since three arrows start from the second aerial vertex p_2], α_3 is a 1-tensor (a vector field) and α_4 is a 2-tensor; we have then

$$B_{\Gamma_1}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)(f, g) = \sum_{i_1, \dots, i_8} \frac{\partial \alpha_1^{i_1 i_2}}{\partial x^{i_3}} \alpha_2^{i_3 i_4 i_5} \frac{\partial \alpha_3^{i_6}}{\partial x^{i_7}} \frac{\partial \alpha_4^{i_7 i_8}}{\partial x^{i_4}} \frac{\partial^2 f}{\partial x^{i_1} \partial x^{i_5}} \frac{\partial^3 g}{\partial x^{i_2} \partial x^{i_6} \partial x^{i_8}}.$$

- where \mathcal{W}_{Γ} is the integral of a form ω_{Γ} over the compactification of a configuration space $C_{\{p_1, \dots, p_n\}\{q_1, \dots, q_m\}}^+$.

Let \mathcal{H} denote the upper half plane $\mathcal{H} = \{z \in \mathbb{C} | \text{Im}(z) > 0\}$. We define

$$Conf_{\{z_1, \dots, z_n\}\{t_1, \dots, t_m\}}^+ := \{z_1, \dots, z_n, t_1, \dots, t_m \mid \begin{array}{l} z_j \in \mathcal{H}; z_i \neq z_j \text{ for } i \neq j; \\ t_j \in \mathbb{R}; t_1 < t_2 < \dots < t_m \end{array} \}$$

and $C_{\{p_1, \dots, p_n\}\{q_1, \dots, q_m\}}^+$ to be the quotient of this space by the action of the 2-dimensional group G of all transformations of the form $z_j \mapsto az_j + b \quad t_i \mapsto at_i + b \quad a > 0, b \in \mathbb{R}$. The configuration space $C_{\{p_1, \dots, p_n\}\{q_1, \dots, q_m\}}^+$ has dimension $2n + m - 2$ and has an orientation induced on the quotient by $\Omega_{\{z_1, \dots, z_n; t_1, \dots, t_m\}} = dx_1 \wedge dy_1 \wedge \dots \wedge dx_n \wedge dy_n \wedge dt_1 \wedge \dots \wedge dt_m$ if $z_j = x_j + iy_j$.

The compactification $\overline{C_{\{p_1, \dots, p_n\}\{q_1, \dots, q_m\}}^+}$ is defined as the closure of the image of the configuration space $C_{\{p_1, \dots, p_n\}\{q_1, \dots, q_m\}}^+$ into the product of a torus and the product of real projective spaces $P^2(\mathbb{R})$ under the map Ψ induced from a map ψ defined on $Conf_{\{z_1, \dots, z_n\}\{t_1, \dots, t_m\}}^+$ in the following way: to any pair of distinct points A, B taken amongst the $\{z_j, \bar{z}_j, t_k\}$, ψ associates the angle $\arg(B - A)$ and to any triple of distinct points A, B, C in that set, ψ associates the element of $P^2(\mathbb{R})$ which is the equivalence class of the triple of real numbers $(|A - B|, |B - C|, |C - A|)$.

Given a graph $\Gamma \in G_{n,m}$, one defines a form on $\overline{C_{\{p_1, \dots, p_n\}\{q_1, \dots, q_m\}}^+}$ induced by

$$\omega_{\Gamma} = \frac{1}{(2\pi)^{k_1 + \dots + k_n} (k_1)! \dots (k_n)!} d\Phi_{\vec{v}_1} \wedge \dots \wedge d\Phi_{\vec{v}_k}$$

where $\Phi_{\vec{p}_j \vec{a}} = \text{Arg}(\frac{a - p_j}{a - \bar{p}_j})$.

For a detailed proof of this formality, we refer the reader to [12, 44]. This formality for \mathbb{R}^d associates a star product on $C^\infty(\mathbb{R}^d)$ to a formal Poisson tensor on \mathbb{R}^d and gives:

Theorem 2.5 ([122]) *Let α be a Poisson tensor on \mathbb{R}^d (thus $\alpha \in \mathcal{T}_{poly}^1(\mathbb{R}^d)$ and $[\alpha, \alpha]_S = 0$), let X be a vector field on \mathbb{R}^d , let $f, g \in C^\infty(\mathbb{R}^d)$. Then the series of*

bidifferential operators

$$P(\alpha) := \mu + C(\alpha) := \mu + \sum_{j=1}^{\infty} \frac{v^j}{j!} F_j(\alpha \cdot \cdot \alpha) \quad (2.40)$$

defines a star product $*$ on \mathbb{R}^d and $A(X, \alpha) = \sum_{j=0}^{\infty} \frac{v^j}{j!} F_{j+1}(X \cdot \alpha \cdot \cdot \alpha)$ is a series of differential operators yielding the relation

$$A(X, \alpha) f * g + f * A(X, \alpha) g - A(X, \alpha)(f * g) = \frac{d}{dt} \Big|_0 P(\Phi_{t*}^X \alpha)(f, g) \quad (2.41)$$

where Φ_t^X is the flow of X .

Kontsevich builds a formality for any manifold M . Cattaneo, Felder and Tomassini give in [64] a globalization on a Poisson manifold of Kontsevich local formula for a star product given above. Using similar techniques, Dolgushev [77] gave a globalisation of Kontsevich's formality, using a torsion free connection on the manifold. In particular this proves the existence of a universal star product when one has chosen a torsion free connection ∇ (universal meaning whose corresponding tensors -see formula (2.31)- are polynomials in the Poisson tensor, the curvature tensor and their covariant derivatives).

Remark 2.12 Tamarkin [162] gave another formulation to the quantization of Poisson manifolds, in the language of operads and Drinfeld's associators. Starting with Kontsevich's and Tamarkin's approaches, formality theory has rapidly evolved and now enters into many fields of research in mathematics (see, for instance [42, 118, 157]). In particular, the general pattern in non commutative geometries is that commutative rings of functions on classical spaces are replaced by more general non-commutative variants, regardless of whether there is still an actual space of points corresponding to this. Deformation theoretic ideas have been important to give classes of examples (see, for instance, [164]). A nice description of formality and its links with representation theory is given in the book [44]. In that area, recent results give new associators built using the formality, and a new proof of the Kashiwara Vergne conjecture by Alekseev and Torossian [2, 3].

2.6 Group Actions in Deformation Quantization

2.6.1 In a Classical Setting

Definition 2.20 Let (M, P) be a Poisson manifold and consider a smooth left action of a Lie group G on the manifold M ,

$$G \times M \rightarrow M : (g, p) \mapsto \rho(g)p = g \cdot p.$$

The group acts by **Poisson diffeomorphisms** if and only if

$$\{\rho(g)^*u, \rho(g)^*v\} = \rho(g)^*({u, v}) \quad \forall u, v \in C^\infty(M), \forall g \in G, \quad (2.42)$$

or, equivalently, if and only if $\rho(g)_*P = P$ for all $g \in G$.

Exercise 2.21 When the Poisson structure is associated to a symplectic structure (M, ω) , condition (2.42) is equivalent to $\rho(g)^*\omega = \omega$ for all $g \in G$.

When G acts by Poisson diffeomorphisms, it is a symmetry group for our classical system. Any element X in the Lie algebra \mathfrak{g} of G gives rise to a fundamental vector field X^{*M} defined by

$$X_p^{*M} = \frac{d}{dt}\bigg|_0 \rho(\exp -tX)p$$

(the minus sign is used to have a Lie algebra homomorphism, $\mathfrak{g} \rightarrow \chi(M)$ into the Lie algebra of smooth vector fields $[X^{*M}, Y^{*M}] = [X, Y]^{*M}$, $\forall X, Y \in \mathfrak{g}$) and we have an infinitesimal Poisson action of the Lie algebra \mathfrak{g}

$$\mathcal{L}_{X^{*M}}\{u, v\} = \{\mathcal{L}_{X^{*M}}u, v\} + \{u, \mathcal{L}_{X^{*M}}v\} \quad (2.43)$$

or equivalently $\mathcal{L}_{X^{*M}}P = 0$; or, in the symplectic case, $\mathcal{L}_{X^{*M}}\omega = 0$ i.e. $\iota(X^{*M})\omega$ is a closed 1-form.

The action of the Lie group is completely determined by the action of its Lie algebra when the Lie group G is connected.

Of particular importance in physics is the case of a so called **(almost) Hamiltonian action** where each fundamental vector field is Hamiltonian, i.e. when for each $X \in \mathfrak{g}$ there exists a function f_X on M such that

$$X^{*M}u = \{f_X, u\} \quad \forall u \in C^\infty(M). \quad (2.44)$$

In the symplectic case this amounts to say that $\iota(X^{*M})\omega = df_X$.

Indeed, when the Hamiltonian governing the dynamics on (M, P) is invariant under the action of G , any of those functions f_X is a constant of the motion. One can always assume, when all the fundamental vector fields are Hamiltonian, that $X \rightarrow f_X$ is linear.

A further assumption is to require that the fundamental vector fields are Hamiltonian by means of a G equivariant map from M into the dual of the Lie algebra (G acting on \mathfrak{g}^* by Ad^*)

$$J : M \rightarrow \mathfrak{g}^* \quad (2.45)$$

i.e. $X^{*M}u = \{J(X), u\}$, $\forall u \in C^\infty(M)$ with $J(X) \in C^\infty(M)$ defined by $J(X)(p) := \langle J(p), X \rangle$, $\langle \cdot, \cdot \rangle$ denoting the pairing between \mathfrak{g} and its dual. One says then that the action possesses a **G equivariant moment map J** . Equivariance means that the Hamiltonian functions satisfy

$$J(X)\rho((g)p) = (J(\text{Ad}g^{-1}X))(p) \text{ and thus } \{J(X), J(Y)\} = J([X, Y]) \quad (2.46)$$

An action so that each fundamental vector field is Hamiltonian and so that the correspondence $X \mapsto f_X$ can be chosen to be a homomorphism of Lie algebras is also called a **strongly Hamiltonian action**. When the group G acting on M is connected, it is equivalent to the existence of a G equivariant moment map.

2.6.2 In the Deformation Quantization Setting

The action of a Lie group on the classical Hilbert space framework of quantum mechanics is described by a unitary representation of the group on the Hilbert space.

In the setting of deformation quantization, the classical action of a group G on a Poisson manifold extends to the algebra of observables $C^\infty(M)[[\hbar]]$ and one can define in this way different notions of invariance of the deformation quantization under the action of a Lie group.

Definition 2.21 Assume (M, P) is a Poisson manifold and G is a Lie group acting on M ; as before $G \times M \rightarrow M : (g, p) \mapsto \rho(g)p = g \cdot p$. Let $(C^\infty(M)[[\hbar]], *)$ be a deformation quantization of (M, P) . The star product is said to be **geometrically invariant** if, for any $g \in G$ and all $u, v \in C^\infty(M)$, one has

$$\rho(g)^*(u * v) = \rho(g)^*u * \rho(g)^*v. \quad (2.47)$$

Exercise 2.22 Show that geometric invariance implies (looking at the skew symmetric part of order 1 in the parameter \hbar) that

$$\rho(g)^* (\{u, v\}) = \{\rho(g)^*u, \rho(g)^*v\}$$

so that G acts by Poisson diffeomorphisms. Any fundamental vector field X^{*M} is then a derivation of the star product

$$X^{*M} (u * v) = (X^{*M}u) * v + u * (X^{*M}v). \quad (2.48)$$

More generally, symmetries in quantum theories are automorphisms of the algebra of observables. Thus we define a symmetry σ of a star product $* = \sum_r \hbar^r C_r$ as an automorphism of the $\mathbb{C}[[\hbar]]$ -algebra $C^\infty(M)[[\hbar]]$ with multiplication given by $*$:

$$\sigma(u * v) = \sigma(u) * \sigma(v), \quad \sigma(1) = 1,$$

where σ is determined by a formal series $\sigma(u) = \sum_{r \geq 0} \hbar^r \sigma_r(u)$ of linear maps. Any such automorphism σ of a star product on a Poisson manifold (M, P) can be written $\sigma(u) = T(u \circ \tau)$ where τ is a Poisson diffeomorphism of (M, P) and $T = \text{Id} + \sum_{r \geq 1} \hbar^r T_r$ is a formal series of differential maps.

A Lie group G acts as **symmetries of a deformed algebra** $(C^\infty(M)[[\nu]], *)$ if there is a homomorphism

$$\sigma : G \rightarrow \text{Aut}(M, *).$$

In that case, one can write

$$\sigma(g)u = T(g)(\tau(g)^*u) \quad \text{for any } u \in C^\infty(M)$$

and $\tau : G \times M \rightarrow M$ defines a Poisson action of G on (M, P) .

At the level of the Lie algebra, **an action of the Lie algebra \mathfrak{g} on the deformed algebra**, is a homomorphism

$$D : \mathfrak{g} \rightarrow \text{Der}(M, *)$$

into the space of derivations of the star product.

Now a derivation D of the star product is said to be **essentially inner** or **Hamiltonian** if $D = \frac{1}{\nu} \text{ad}_* u$ for some $u \in C^\infty(M)[[\nu]]$. We denote by $\text{Inn}(M, *)$ the essentially inner derivations of $*$. It is a linear subspace of $\text{Der}(M, *)$ and is the quantum analogue of the Hamiltonian vector fields. By analogy with the classical case, we call an action of a Lie algebra (or of a Lie group) on a deformed algebra **almost $*$ -Hamiltonian** if each $D(X)$, for any $X \in \mathfrak{g}$, is essentially inner, and we call (quantum) Hamiltonian a linear choice of functions \tilde{f}_X satisfying

$$D(X) = \frac{1}{\nu} \text{ad}_* \tilde{f}_X, \quad X \in \mathfrak{g}.$$

We say the action is **$*$ -Hamiltonian** if u_X can be chosen to make the map

$$\mathfrak{g} \rightarrow C^\infty(M)[[\nu]] : X \mapsto \tilde{f}_X$$

a homomorphism of Lie algebras.

When the deformed algebra is invariant by a classical (undeformed) Poisson action of a Lie group G on M , if the action of the Lie algebra \mathfrak{g} defined by the fundamental vector fields ($D(X) = X^{*M}$) is $*$ -Hamiltonian, a map $\tilde{J} : \mathfrak{g} \rightarrow C^\infty(M)[[\nu]]$ is called **a quantum moment map** [172]. Thus it is a homomorphism of algebras

$$\mathfrak{g} \rightarrow C^\infty(M)[[\nu]] : X \mapsto \tilde{J}_X \quad \text{such that} \quad X^{*M} = \frac{1}{\nu} \text{ad}_* \tilde{J}_X \quad \forall X \in \mathfrak{g}.$$

In [8], they called quantization such a homomorphism of Lie algebras.

When there is a strongly Hamiltonian action of Lie group G on a Poisson manifold (M, P) a star product is said to be **covariant** under G if

$$f_X * f_Y - f_Y * f_X = \nu f_{[X, Y]} \quad \forall X, Y \in \mathfrak{g}$$

where $f : \mathfrak{g} \rightarrow \mathbb{C}^\infty(M)$ is the homomorphism of Lie algebras describing the fundamental vector fields as Hamiltonian vector fields ($X^{*M}u = \{f_X, u\}$) and it is called **strongly invariant** if it is both geometrically invariant and covariant. In that case, f is a quantum moment map.

Exercise 2.23 Check that the Moyal star product on \mathbb{R}^{2n} endowed with the canonical Poisson bracket $P_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ is strongly invariant under the natural action of the symplectic group on \mathbb{R}^{2n} .

Remark 2.13 One can go further and look at actions of a deformed group on a star product [81]. One way to deform a group is to deform in the Hopf category a Hopf algebra associated to the group. One enters into the realm of quantum groups [79]; these are introduced in the lectures of Christian Kassel. Links between quantum groups and deformation quantization appear in [23, 30, 31].

2.6.3 Classification of Invariant Star Products

When a Lie group G acts on the symplectic manifold (M, ω) and is a group of symmetries of a natural $*$ product, then (see Remark 2.9) there is a symplectic connection on (M, ω) which is invariant under G . We shall say that two star products which are invariant under G are G -invariantly equivalent if there is an equivalence $T = \text{id} + \sum_{j=1}^{\infty} v^j T_j$ between them which commutes with the action of G . Using the results stated before, one can prove.

Proposition 2.8 [22] *Let G be a Lie group which acts symplectically on (M, ω) . Suppose $*$ is a star product which is invariant under G and assume there is a symplectic connection which is invariant under G . Then, there exists a series of G -invariant closed 2-form $\Omega \in Z^2(M; \mathbb{R})^{G-\text{inv}}[[v]]$ such that $*$ is G -invariantly equivalent to the Fedosov star product constructed from the invariant connection ∇ and Ω , i.e. there exists a series $T = \text{id} + \sum_{j=1}^{\infty} v^j T_j$ of G -invariant differential operators such that $*$ = $T \cdot *_{\nabla, \Omega}$.*

*Furthermore, two G -invariant star products $*_{\nabla, \Omega}$ and $*_{\nabla, \Omega'}$ are G -invariantly equivalent if and only if $\Omega - \Omega'$ is the boundary of a series of G -invariant 1-forms on M .*

Hence there is a bijection between the G -invariant equivalence classes of G -invariant $$ -products on (M, ω) and the space of formal series of elements in the second space of invariant cohomology of M , $H^2(M, \mathbb{R})^{G-\text{inv}}[[v]]$.*

Remark 2.14 On a Poisson manifold (MP) endowed with a \mathfrak{g} -action, if there exists a \mathfrak{g} -invariant connection, one can use Dolgushev's formality [77] to build also in this case a correspondence between \mathfrak{g} -invariant equivalence classes of \mathfrak{g} -invariant Poisson deformations of P and \mathfrak{g} -invariant equivalence classes of \mathfrak{g} invariant star products.

2.6.4 Invariance of Fedosov's Star Product

We shall denote by $*_{\nabla, \Omega}$ the star product on a symplectic manifold (M, ω) obtained by Fedosov's construction using the symplectic connection ∇ and the series of closed 2-forms Ω .

Lemma 2.1 *Any diffeomorphism ϕ of (M, ω) is a symmetry of $*_{\nabla, \Omega}$ iff it preserves the symplectic 2-form ω , the connection ∇ and the series of closed 2-forms Ω . A vector field X is a derivation of $*_{\nabla, \Omega}$ if and only if $\mathcal{L}_X \omega = 0$, $\mathcal{L}_X \Omega = 0$, and $\mathcal{L}_X \nabla = 0$.*

Exercise 2.24 Prove this Lemma, using the fact that the star product $*_{\nabla, \Omega}$ is natural and the associated connection (see Remark 2.9) is ∇ . Hence invariance of ∇ is a necessary condition for the invariance of $*_{\nabla, \Omega}$ by a diffeomorphism of M . Use also the characterization, given in Eq. 2.29, of the 2-forms appearing in Ω .

Many authors have studied whether such a derivation is Hamiltonian for the star product (see, for instance [105, 123, 132]). We give here the proof obtained with J. Rawnsley.

Theorem 2.6 [105] *A vector field X is an inner derivation of $* = *_{\nabla, \Omega}$ if and only if $\mathcal{L}_X \nabla = 0$ and there exists a series of functions λ_X such that*

$$i(X)\omega - i(X)\Omega = d\lambda_X.$$

In that case $X(u) = \frac{1}{v}(\text{ad}_ \lambda_X)(u)$.*

Proof With the same notation as above, for any smooth vector field X on M , one has:

$$\delta \circ i(X) + i(X) \circ \delta = \frac{1}{v} \text{ad}_*(\omega_{ij} X^i y^j) \quad \text{ad}_* r \circ i(X) + i(X) \circ \text{ad}_* r = \text{ad}_*(i(X)r)$$

and $\partial \circ i(X) + i(X) \circ \partial = \mathcal{L}_X - (\nabla_i X)^j y^i \partial_{y^j}$ which can be rewritten as

$$\partial \circ i(X) + i(X) \circ \partial = \mathcal{L}_X + \frac{1}{v} \text{ad}_* \left(-\frac{1}{2} (\nabla_i (i(X)\omega))_j y^i y^j \right) + \frac{1}{2} (di(X)\omega)_{ip} y^i P^{jp} \partial_{y^j}.$$

This gives the generalised Cartan formula first given by Neumaier:

$$\begin{aligned} \mathcal{L}_X &= D \circ i(X) + i(X) \circ D + \frac{1}{v} \text{ad}_*(\omega_{ij} X^i y^j) + \frac{1}{v} \text{ad}_*(i(X)r) \\ &\quad + \frac{1}{v} \text{ad}_* \left(\frac{1}{2} (\nabla_i (i(X)\omega))_j y^i y^j \right) - \frac{1}{2} (di(X)\omega)_{ip} y^i P^{jp} \partial_{y^j}. \end{aligned} \quad (2.49)$$

The last term obviously drops out when X is a symplectic vector field.

We now assume that X is a symplectic vector field preserving the connection and preserving the series of 2-forms Ω , then $\mathcal{L}_X r = 0$ so

$$-Di(X)r = i(X)Dr + \frac{1}{v} \left[\omega_{ij} X^i y^j + \frac{1}{2} (\nabla_i (i(X)\omega))_j y^i y^j + i(X)r, r \right].$$

Using the definition of r , this gives $-Di(X)r = i(X)\bar{R} - i(X)\Omega + \frac{1}{\nu} [\omega_{ij}X^i y^j + \frac{1}{2}(\nabla_i(i(X)\omega))_j y^i y^j, r]$. On the other hand, using the fact that $Da = \partial a - \delta(a) - \frac{1}{\nu}[r, a]$ one has

$$D(\omega_{ij}X^i y^j) = -i(X)\omega + \partial(\omega_{ij}X^i y^j) + \frac{1}{\nu}[\omega_{ij}X^i y^j, r],$$

$D(\frac{1}{2}(\nabla_i(i(X)\omega))_j y^i y^j) = -\nabla_i(i(X)\omega)_j dx^i y^j + \partial(\frac{1}{2}(\nabla_i(i(X)\omega))_j y^i y^j) + \frac{1}{\nu}[\frac{1}{2}(\nabla_i(i(X)\omega))_j y^i y^j, r]$. Since X is an affine vector field, one has $(i(X)R)(Y)Z = (\nabla^2 X)(Y, Z)$ so that

$$\partial\left(\frac{1}{2}(\nabla_i(i(X)\omega))_j y^i y^j\right) = -\frac{1}{2}((\nabla^2 X)_{ki}^p \omega)_{jp} y^i y^j dx^k = i(X)\bar{R}.$$

Hence $D(-i(X)r - \omega_{ij}X^i y^j - \frac{1}{2}(\nabla_i(i(X)\omega))_j y^i y^j) = i(X)\omega - i(X)\Omega$. So, for any vector field X so that $\mathcal{L}_X \omega = 0$, $\mathcal{L}_X \Omega = 0$ and $\mathcal{L}_X \nabla = 0$, one has

$$\mathcal{L}_X = D \circ i(X) + i(X) \circ D + \frac{1}{\nu} \text{ad}_*(T(X))$$

with $T(X) = i(X)r + \omega_{ij}X^i y^j + \frac{1}{2}(\nabla_i(i(X)\omega))_j y^i y^j$ and $DT(X) = -i(X)\omega + i(X)\Omega$.

In particular, if there exists a series of smooth functions λ_X such that

$$i(X)\omega - i(X)\Omega = d\lambda_X \quad (2.50)$$

one can write $\mathcal{L}_X = D \circ i(X) + i(X) \circ D + \frac{1}{\nu} \text{ad}_*(\lambda_X + T(X))$ with $D(\lambda_X + T(X)) = 0$.

Thus $\lambda_X + T(X)$ is the flat section associated to the series of smooth function on M obtained by taking the part of $\lambda_X + T(X)$ with no y terms hence λ_X (notice that $i(X)r$ has no terms without a y from the construction of r). If Q denotes the quantisation map associating a flat section to a series in ν of smooth functions, the above yields

$$\mathcal{L}_X = D \circ i(X) + i(X) \circ D + \frac{1}{\nu} \text{ad}_*(Q(\lambda_X)).$$

Since in those assumptions the map Q commutes with \mathcal{L}_X one has

$$Q(Xf) = \mathcal{L}_X Q(f) = \frac{1}{\nu} [Q(\lambda_X), Q(f)]$$

so that for any smooth function f , one has

$$Xf = \frac{1}{\nu} (\text{ad}_* \lambda_X)(f).$$

This was first stated by Kravchenko (Proposition 4.3 of [123]).

We have seen above that such a vector field X is an inner derivation if $i(X)(\omega - \Omega)$ is exact. We shall show now that this is also a necessary condition.

Assume X is a vector field on M such that there exists a series of smooth functions λ_X with

$$X(u) = \frac{1}{v}(\text{ad}_* \lambda_X)(u) \quad (2.51)$$

for every smooth function u on M . Then X is a derivation of $*$ so $\mathcal{L}_X \omega = 0$, $\mathcal{L}_X \Omega = 0$, $\mathcal{L}_X \nabla = 0$ and

$$Q(Xf) = \mathcal{L}_X Q(f) = \frac{1}{v}[T(X), Q(f)]$$

with $T(X) = i(X)r + \omega_{ij}X^i y^j + \frac{1}{2}(\nabla_i(i(X)\omega))_j y^i y^j$ and $DT(X) = -i(X)\omega + i(X)\Omega$.

Taking a contractible open set U in M , there exists a series of smooth locally defined functions λ_X^U on U so that $(i(X)\omega - i(X)\Omega)|_U = d\lambda_X^U$ and, everything being local, we have on U

$$D(\lambda_X^U + T(X))|_U = 0,$$

thus $\lambda_X^U + T(X)$ is the flat section on U associated to the series of smooth functions on U obtained by taking the part of $\lambda_X^U + T(X)$ with no y terms (which is λ_X^U) and

$$Q(X(u))|_U = \mathcal{L}_X Q(u)|_U = \frac{1}{v}[Q(\lambda_X^U), Q(u)]|_U$$

so that $X(u)|_U = \frac{1}{v}(\text{ad}_{*\nabla, \Omega} \lambda_X^U)(u)|_U$ for any smooth function u . Comparing this with Eq. (2.51) shows that $\lambda_X^U - \lambda_X$ is a constant on U , hence $i(X)\omega - i(X)\Omega = d\lambda_X$.

A direct corollary of the above theorem tells us whether a Fedosov star product which is invariant under the action of a Lie algebra admits a quantum moment map:

Proposition 2.9 *A \mathfrak{g} -invariant Fedosov star product for (M, ω) is obtained from a \mathfrak{g} -invariant connexion and a \mathfrak{g} -invariant series of closed 2-forms Ω . It admits a quantum Hamiltonian if and only if there is a linear map*

$$\hat{J} : \mathfrak{g} \rightarrow C^\infty(M)[[v]]$$

such that

$$d(\hat{J}(X)) = \iota(X^{*M})\omega - \iota(X^{*M})\Omega \quad \forall X \in \mathfrak{g}.$$

We then have $X^{*M}u = \frac{1}{v}\text{ad}_* \hat{J}(X)u$. It admits a quantum moment map if and only if it is Hamiltonian and the linear map $\tilde{J} : \mathfrak{g} \rightarrow C^\infty(M)[[v]]$ such that $d(\tilde{J}(X)) = \iota(X^{*M})\omega - \iota(X^{*M})\Omega$ can be chosen so that

$$\tilde{J}([X, Y]) = -\omega(X^{*M}, Y^{*M}) + \Omega(X^{*M}, Y^{*M}) \quad \forall X, Y \in \mathfrak{g}.$$

In a recent preprint [148], Reichert and Waldmann give a characterization of equivalence classes of \mathfrak{g} invariant star products admitting a quantum moment map J , for \mathfrak{g} -invariant equivalences intertwining the quantum moments maps, by series in the second \mathfrak{g} -equivariant cohomology.

2.7 Reduction in Deformation Quantization

An important classical tool to “reduce the number of variables”, i.e. to start from a “big” Poisson manifold M with real Poisson tensor P and construct a smaller one M_{red} , is given by reduction: one considers an embedded coisotropic submanifold in the Poisson manifold,

$$\iota : C \hookrightarrow M.$$

Recall that a submanifold of a Poisson manifold is called **coisotropic** iff the vanishing ideal

$$\mathcal{I}_C = \{f \in C^\infty(M) \mid \iota^* f = 0\} = \ker \iota^*.$$

is closed under Poisson bracket. This is equivalent to say that

$$P^\sharp(N^*C) \subset TC \quad \text{with} \quad N^*C(x) = \{\alpha_x \in T_x^*M \mid \alpha_x(X) = 0 \forall X \in T_x C\},$$

where $P^\sharp : T^*M \rightarrow TM$ is induced by P through $\beta(P^\sharp(\alpha) := P(\alpha, \beta)$.

In the symplectic case $P^\sharp(N^*C) = TC^\perp$ is the orthogonal with respect to the symplectic 2-form ω of the tangent space to C , so that coisotropy means

$$TC_x^\perp := \{Y \in T_x M \mid \omega_x(X, Y) = 0 \quad \forall X \in T_x C\} \subset TC.$$

The distribution defined by $P^\sharp(N^*C)$, called the **characteristic distribution**, is involutive. It is spanned at each point by the Hamiltonian vector fields corresponding to functions which are locally in \mathcal{I}_C .

We assume that the canonical foliation has a nice leaf space M_{red} (a structure of smooth manifold such that the canonical projection $\pi : C \rightarrow M_{\text{red}}$ is a submersion). In this case one can show that M_{red} is a Poisson manifold in a canonical way: one defines the normalizer of the vanishing ideal

$$\mathcal{B}_C = \{f \in C^\infty(M) \mid \{f, \mathcal{I}_C\} \subseteq \mathcal{I}_C\},$$

and

$$\mathcal{B}_C / \mathcal{I}_C \ni [f] \mapsto \iota^* f \in \pi^* C^\infty(M_{\text{red}}) = \mathcal{A}_{\text{red}} \quad (2.52)$$

induces an isomorphism of Poisson algebras. We prove this in a simple context in Sect. 2.7.1.

Passing to a deformation quantized version of phase space reduction, one starts with a formal star product \star on M . The associative algebra $\mathcal{A} = (C^\infty(\mathbf{M})[[\hbar]], \star)$ plays the role of the quantized observables of the big system. A good analog of the vanishing ideal \mathcal{I}_C will be a left ideal $\mathcal{J}_C \subseteq C^\infty(\mathbf{M})[[\hbar]]$ such that the quotient $C^\infty(\mathbf{M})[[\hbar]] / \mathcal{J}_C$ is in $\mathbb{C}[[\hbar]]$ -linear bijection to the functions $C^\infty(C)[[\hbar]]$ on C . We then define the normalizer of \mathcal{J}_C with respect to the commutator Lie bracket of \mathcal{A} ,

$$\mathcal{B}_C = \{\mathbf{a} \in \mathcal{A} \mid [\mathbf{a}, \mathcal{I}_C] \subseteq \mathcal{I}_C\},$$

and consider the associative algebra $\mathcal{B}_C / \mathcal{I}_C$ as the reduced algebra \mathcal{A}_{red} .

Of course, we need then to show that $\mathcal{B}_C / \mathcal{I}_C$ is in $\mathbb{C}[[v]]$ -linear bijection to $C^\infty(M_{\text{red}})[[v]]$ in such a way, that the isomorphism induces a star product \star_{red} on M_{red} . Starting from a strongly invariant star product on M , we describe below the method used in [106] with S. Waldmann to construct a good left ideal inspired by the BRST approach in [40] but simpler as we only need the deformation of the Koszul part of the BRST complex. Other approaches to reduction in deformation quantization appear in Fedosov [82] and in Cattaneo–Felder [62].

Remark 2.15 BRST formalism is a differential geometric approach to quantize a field theory with a gauge symmetry. We refer to the lectures of Nathan Berkovits. The mathematical background of BRST construction describes the space of functions on some reduced spaces as the 0-cohomology space of a complex.

We present only the particular case of the Marsden–Weinstein reduction : consider a smooth left action $G \times M \longrightarrow M : (g, p) \mapsto \rho(g)p$ of a connected Lie group G on M by Poisson diffeomorphisms and assume we have an ad^* -equivariant momentum map J . The constraint manifold C is chosen to be the level surface of J for momentum $0 \in \mathfrak{g}^*$ (thus we assume, for simplicity, that 0 is a regular value). Then $C = J^{-1}(\{0\})$ is an embedded submanifold which is coisotropic. The group G acts on C and the reduced space is the orbit space of this group action of G on C . To guarantee a good quotient we assume that G acts freely and properly and we assume that G acts properly not only on C but on all of M . In this case there exists an open neighbourhood $M_{\text{nice}} \subseteq M$ of C with a G -equivariant diffeomorphism

$$\Phi : M_{\text{nice}} \longrightarrow U_{\text{nice}} \subseteq C \times \mathfrak{g}^* \quad (2.53)$$

onto an open neighbourhood U_{nice} of $C \times \{0\}$, where the G -action on $C \times \mathfrak{g}^*$ is the product action of the one on C and Ad^* , such that for each $p \in C$ the subset $U_{\text{nice}} \cap (\{p\} \times \mathfrak{g}^*)$ is star-shaped around the origin $\{p\} \times \{0\}$, and the momentum map J is given by the projection onto the second factor, i.e. $J|_{M_{\text{nice}}} = \text{pr}_2 \circ \Phi$. For a proof of this see for instance [40, Lemma 3].

2.7.1 The Classical Koszul Resolution

We consider $C^\infty(M, \Lambda_{\mathbb{C}}^\bullet \mathfrak{g}) = C^\infty(M) \otimes \Lambda_{\mathbb{C}}^\bullet \mathfrak{g}$ with the canonical free $C^\infty(M)$ -module structure. The group G acts on $C^\infty(M, \Lambda_{\mathbb{C}}^\bullet \mathfrak{g})$ by the combined action of G on the manifold and the adjoint action on \mathfrak{g} extended to the exterior algebra $\Lambda_{\mathbb{C}}^\bullet \mathfrak{g}$ by automorphisms of the exterior product. We denote this G -action and the corresponding \mathfrak{g} -action by ρ . The **Koszul differential** is defined to be

$$\partial x = \iota(J)x, \quad (2.54)$$

where $x \in C^\infty(M, \Lambda_{\mathbb{C}}^\bullet \mathfrak{g})$ and $\iota(J)$ denotes the insertion of J at the first position in the $\Lambda_{\mathbb{C}}^\bullet \mathfrak{g}$ -part of x . If $e_1, \dots, e_N \in \mathfrak{g}$ denotes a basis with dual basis $e^1, \dots, e^N \in \mathfrak{g}^*$ then we can write $J = \sum_a J_a e^a$ with scalar functions $J_a \in C^\infty(M)$ and $\partial x = J_a \iota(e^a)x$. The map ∂ is a graded derivation of the standard wedge product on $C^\infty(M, \Lambda_{\mathbb{C}}^\bullet \mathfrak{g})$ of degree -1 , it is $C^\infty(M)$ -linear and $\partial^2 = 0$; we have thus a complex of free $C^\infty(M)$ -modules. We write ∂_k for the restriction of ∂ to the antisymmetric degree $k \geq 1$.

We use the particular tubular neighbourhood M_{nice} of C to define a **prolongation map**

$$\text{prol} : C^\infty(C) \ni \phi \mapsto \text{prol}(\phi) = (\text{pr}_1 \circ \Phi)^* \phi \in C^\infty(M_{\text{nice}}). \quad (2.55)$$

This prolongation is G -equivariant: $\rho(g)^* \text{prol}(\phi) = \text{prol}(\rho(g)^* \phi)$. It deserves its name since, for all $\phi \in C^\infty(C)$, we have $\iota^* \text{prol}(\phi) = \phi$.

We define a **homotopy**, on M_{nice} for convenience: let $x \in C^\infty(M_{\text{nice}}, \Lambda_{\mathbb{C}}^k \mathfrak{g})$; since U_{nice} is star-shaped, we set

$$(h_k x)(p) = e_a \wedge \int_0^1 t^k \frac{\partial(x \circ \Phi^{-1})}{\partial \mu_a}(c, t\mu) dt, \quad (2.56)$$

where $\Phi(p) = (c, \mu)$ for $p \in M_{\text{nice}}$ and μ_a denote the linear coordinates on \mathfrak{g}^* with respect to the basis e^1, \dots, e^N . The collection of all these maps h_k gives a map $h : C^\infty(M_{\text{nice}}, \Lambda_{\mathbb{C}}^\bullet \mathfrak{g}) \longrightarrow C^\infty(M_{\text{nice}}, \Lambda_{\mathbb{C}}^{\bullet+1} \mathfrak{g})$.

Proposition 2.10 [40, Lemmas 5 and 6] *The Koszul complex $(C^\infty(M_{\text{nice}}, \Lambda_{\mathbb{C}}^\bullet \mathfrak{g}), \partial)$ is acyclic with homotopy h and homology $C^\infty(C)$ in degree 0: we have*

$$h_{k-1} \partial_k + \partial_{k+1} h_k = \text{id}_{C^\infty(M_{\text{nice}}, \Lambda_{\mathbb{C}}^k \mathfrak{g})} \quad \text{for } k \geq 1 \quad (2.57)$$

$$\text{prol } \iota^* + \partial_1 h_0 = \text{id}_{C^\infty(M_{\text{nice}})} \quad (2.58)$$

as well as $\iota^* \partial_1 = 0$. Thus the Koszul complex is a free resolution of $C^\infty(C)$ as $C^\infty(M_{\text{nice}})$ -module. We have

$$h_0 \text{prol} = 0, \quad (2.59)$$

and all the homotopies h_k are G -equivariant.

Here resolution means that the homology at $k = 0$ is isomorphic to $C^\infty(C)$ as a $C^\infty(M_{\text{nice}})$ -module.

Exercise 2.25 Show that the image of ∂_1 is just

$$\ker \iota^* \cap C^\infty(M_{\text{nice}}) = \mathcal{I}_C \cap C^\infty(M_{\text{nice}})$$

using formula (2.58). This gives immediately

$$\begin{aligned} \ker \partial_0 / \text{Im } \partial_1 &= \ker \partial_0 / (\mathcal{I}_C \cap C^\infty(M_{\text{nice}})) \\ &= C^\infty(M_{\text{nice}}) / (\mathcal{I}_C \cap C^\infty(M_{\text{nice}})) \cong C^\infty(C), \end{aligned} \quad (2.60)$$

Use the Koszul complex to prove (2.52): f is in \mathcal{B}_C iff $0 = \iota^*\{J_X, f\} = \iota^*(\mathcal{L}_{X^*M} f) = \mathcal{L}_{X^*C}(\iota^* f) \forall X \in \mathfrak{g}$ iff $\iota^* f \in \pi^* C^\infty(M_{\text{red}})$. For $u \in C^\infty(M_{\text{red}})$ show that $\text{prol}(\pi^* u) \in \mathcal{B}_C$ whence (2.52) is surjective. The injectivity of (2.52) is clear by definition.

The Poisson bracket on M_{red} can then be defined through (2.52) and gives explicitly

$$\pi^*\{u, v\}_{\text{red}} = \iota^*\{\text{prol}(\pi^* u), \text{prol}(\pi^* v)\} \quad (2.61)$$

for $u, v \in C^\infty(M_{\text{red}})$, since the left hand side of (2.52) is canonically a Poisson algebra.

Since for the phase space reduction in deformation quantization we will only need a very small neighbourhood of C , the neighbourhood M_{nice} is sufficient; the geometry of M far away from C plays no role and we may assume without restriction $M_{\text{nice}} = M$ in the following.

2.7.2 The Quantized Koszul Complex

Before defining the deformed Koszul operator we make some further assumptions on the star product \star on M : we assume it to be strongly invariant, i.e. \mathfrak{g} -covariant,

$$J_X \star J_Y - J_Y \star J_X = \nu J_{[X, Y]} \quad \forall X, Y \in \mathfrak{g} \quad (2.62)$$

and G -invariant

$$\rho(g)^*(f \star h) = (\rho(g)^* f) \star (\rho(g)^* h) \quad (2.63)$$

for all $g \in G$ and $f, h \in C^\infty(M)[[\lambda]]$.

Using the exterior (\wedge) product for $\Lambda_{\mathbb{C}}^\bullet \mathfrak{g}$ we extend \star to $C^\infty(M, \Lambda_{\mathbb{C}}^\bullet \mathfrak{g})$ in the canonical way.

Definition 2.22 (*Quantized Koszul operator*) Let $\kappa \in \mathbb{C}[[\nu]]$. The quantized Koszul operator $\partial^{(\kappa)} : C^\infty(M, \Lambda_{\mathbb{C}}^\bullet \mathfrak{g})[[\nu]] \longrightarrow C^\infty(M, \Lambda_{\mathbb{C}}^{\bullet+1} \mathfrak{g})[[\nu]]$ is defined to be

$$\partial^{(\kappa)} x = \iota(e^a) x \star J_a + \frac{\nu}{2} C_{ab}^c e_c \wedge \iota(e^a) \iota(e^b) x + \nu \kappa \iota(\Delta) x, \quad (2.64)$$

(with summation over repeated indices) where $C_{ab}^c = e^c([e_a, e_b])$ are the structure constants of \mathfrak{g} and $\Delta(X) = \text{Tr ad}(X)$ for $X \in \mathfrak{g}$ is the modular one-form, $\Delta \in \mathfrak{g}^*$, of \mathfrak{g} (with respect to the chosen basis we have $\Delta = C_{ab}^b e^a$).

Lemma 2.2 ([106, Lemma 3.4]) *Let \star be a strongly invariant \star -product and $\kappa \in \mathbb{C}[[\nu]]$. Then $\partial^{(0)} \iota(\Delta) + \iota(\Delta) \partial^{(0)} = 0$, $\partial^{(\kappa)}$ is left \star -linear; the classical limit of $\partial^{(\kappa)}$ is ∂ , $\partial^{(\kappa)}$ is G -equivariant, and $\partial^{(\kappa)} \circ \partial^{(\kappa)} = 0$.*

The element κ can be arbitrary; in particular, $\kappa = 0$ gives a very simple choice; however, we set $\partial = \partial^{(\kappa=\frac{1}{2})}$. The following constructions will depend on κ ; if we omit the reference to κ in our notation, we always mean the particular value $\kappa = \frac{1}{2}$.

Following [40] we define a deformation of the restriction map ι^* and the homotopy:

$$\iota_\kappa^* = \iota^* \left(\text{id} + \left(\partial_1^{(\kappa)} - \partial_1 \right) h_0 \right)^{-1} : C^\infty(M)[[v]] \longrightarrow C^\infty(C)[[v]] \quad (2.65)$$

and

$$\mathbf{h}_0^{(\kappa)} = h_0 \left(\text{id} + \left(\partial_1^{(\kappa)} - \partial_1 \right) h_0 \right)^{-1} : C^\infty(M)[[v]] \longrightarrow C^\infty(M, \mathfrak{g})[[v]], \quad (2.66)$$

which are both well-defined since $\partial^{(\kappa)}$ is a deformation of ∂ . From [40, Proposition 25]

$$\mathbf{h}_0^{(\kappa)} \text{prol} = 0, \quad \iota_\kappa^* \partial_1^{(\kappa)} = 0, \quad \text{and} \quad \iota_\kappa^* \text{prol} = \text{id}_{C^\infty(C)[[v]]}. \quad (2.67)$$

The homotopy equation becomes

$$\text{prol} \iota_\kappa^* + \partial_1^{(\kappa)} \mathbf{h}_0^{(\kappa)} = \text{id}_{C^\infty(M)[[v]]}. \quad (2.68)$$

2.7.3 The Reduced Star Product

We now give an explicit description of the quotient $\mathcal{B}_C / \mathcal{I}_C$ where

$$\mathcal{B}_C = \{ f \in C^\infty(M)[[v]] \mid [f, \mathcal{I}_C]_\star \subseteq \mathcal{I}_C \}. \quad (2.69)$$

Proposition 2.11 [40, Theorems 29 and 32] *Let $f \in C^\infty(M)[[v]]$ and $u, v \in C^\infty(M_{\text{red}})[[v]]$.*

- *We have $f \in \mathcal{B}_C$ iff $\mathcal{L}_{X^*c} \iota_\kappa^* f = 0$ for all $X \in \mathfrak{g}$ iff $\iota_\kappa^* f \in \pi^* C^\infty(M_{\text{red}})[[v]]$.*
- *The quotient algebra $\mathcal{B}_C / \mathcal{I}_C$ is isomorphic to $C^\infty(M_{\text{red}})[[v]]$ via the mutually inverse maps*

$$\mathcal{B}_C / \mathcal{I}_C \ni [f] \mapsto \iota_\kappa^* f \in \pi^* C^\infty(M_{\text{red}})[[v]] \quad (2.70)$$

and

$$C^\infty(M_{\text{red}})[[v]] \ni u \mapsto [\text{prol}(\pi^* u)] \in \mathcal{B}_C / \mathcal{I}_C. \quad (2.71)$$

- *The induced associative product $\star_{\text{red}\kappa}$ on $C^\infty(M_{\text{red}})[[v]]$ from $\mathcal{B}_C / \mathcal{I}_C$ is explicitly given by*

$$\pi^*(u \star_{\text{red}\kappa} v) = \iota_\kappa^* (\text{prol}(\pi^* u) \star \text{prol}(\pi^* v)). \quad (2.72)$$

This is a bidifferential star product quantizing the Poisson bracket (2.61).

Proof We give a sketch of the proof. For the first part note that $\mathcal{I}_C = \ker \iota_\kappa^*$ according to (2.68). Now let $g = g^a \star J_a + \nu \kappa C_{ba}^a g^b$ with $g^a \in C^\infty(M)[[v]]$ be in the image of $\partial_1^{(\kappa)}$. For $f \in C^\infty(M)[[v]]$ we have by a straightforward computation

$$[f, g]_\star = \partial_1^{(\kappa)} h + \nu g^a \star \mathcal{L}_{(e_a)^*M} f$$

with some $h \in C^\infty(M, \mathfrak{g})$ using the strong invariance of \star . Thus $[f, g]_\star$ is in \mathcal{I}_C iff $g^a \star \mathcal{L}_{(e_a)^*M} f$ is in the image of $\partial_1^{(\kappa)}$ for all g^a . This shows that $f \in \mathcal{B}_C$ iff $\mathcal{L}_{X^*M} f \in \text{Im} \partial_1^{(\kappa)} = \ker \iota_\kappa^*$. Since ι_κ^* is G -invariant the first part follows. The second part is then clear from the first part, and (2.72) is a straightforward translation using the isomorphisms (2.70) and (2.71). One can show that $\star_{\text{red}\kappa}$ is bidifferential and that it is indeed a star product on M_{red} .

Remark 2.16 The algebra of quantum observables is not only an associative algebra but it has a \star -involution (see Definition 2.4); in the usual picture, where observables are represented by operators, this \star -involution corresponds to the passage to the adjoint operator. In the framework of deformation quantization, complex conjugation is a \star -involution on $\mathcal{A} = (C^\infty(\mathbf{M})[[\lambda]], \star)$ if the star product is Hermitian. We study in [106] the existence of natural \star -involutions on the reduced quantum algebra assuming that \star is Hermitian: the choice of a formal series of smooth densities on the embedded coisotropic submanifold $C = J^{-1}(0)$, with some equivariance property, defines a \star -involution for \star_{red} on the reduced space. Whether the corresponding \star -involution is the complex conjugation (which is a \star -involution in the Marsden–Weinstein context) yields to define a new notion of quantized unimodular class. We study representations (in the sense of [37]) of the reduced algebra with the \star -involution given by complex conjugation, relating the categories of modules of the big and of the reduced algebras.

2.8 Some Remarks About Convergence

A formal deformation is not enough for physics; \hbar is a constant of nature and not a formal parameter. Although a nice representation theory has been introduced for \star -algebras [37], there is no reasonable general notion of spectra for formal star product algebras (except for a few examples with convergence as in [16]); thus formal deformation quantization can not predict in general values of measurements, and hence is not a complete answer to the quantization problem.

Many examples of star products, like the global symbolic calculus on cotangent bundles or like Berezin or Toeplitz quantization of Kähler manifolds, are obtained as asymptotic expansions for $\hbar \rightarrow 0$ of some convergent counterpart in usual quantization (see for instance [38, 51]). Whether the asymptotics can be used to recover the convergent quantization is still unknown. Some partial convergence results in this context were obtained, for instance in [51] for the product of two given functions and in [33] for subalgebras where the product converges.

The framework of C^* -algebras provides the background for a good notion of spectra (the spectrum of an element a in a unital C^* -algebra is the set of $\lambda \in \mathbb{C}$ such that $a - \lambda 1$ is not invertible); an ideal situation would be to construct a C^* -algebra with a physical interpretation of some elements. It is not enough to know a

C^* -algebra of observables of a system, one still needs a rule stating which algebra element corresponds to which physical observable. A problem is that, except for some simple situations, it is hard to write down a C^* -algebra corresponding to a quantum system of which one knows the classical counterpart, although that is the aim of quantization (build a quantum description, given a classical physical system). Formal deformation quantization is not a solution but might be a first step: one can try to use the powerful results for that theory (in particular concerning existence, classification, invariance and constructions) to build, in a second step, a C^* -algebraic framework.

Rieffel introduced the notion of strict deformation quantization (see [150–152]): A **strict deformation quantization** [150] of a dense $*$ -subalgebra \mathbb{A}' of a C^* -algebra, in the direction of a Poisson bracket $\{.,.\}$ defined on \mathbb{A}' , is an open interval $I \subset \mathbb{R}$ containing 0, and the assignment, for each $\hbar \in I$, of an associative product \times_{\hbar} , an involution $*_{\hbar}$ and a C^* -norm $\|\cdot\|_{\hbar}$ (for \times_{\hbar} and $*_{\hbar}$) on \mathbb{A}' , which coincide for $\hbar = 0$ to the original product, involution and C^* -norm on \mathbb{A}' , such that the corresponding field of C^* -algebras, with continuity structure given by the elements of \mathbb{A}' as constant fields, is a continuous field of C^* -algebras, and such that for all $a, b \in \mathbb{A}'$, $\|\frac{(a \times_{\hbar} b - ab)}{i\hbar} - \{a, b\}\|_{\hbar} \rightarrow 0$ as $\hbar \rightarrow 0$.

Group actions appear here in an essential way: Rieffel introduced a general way to construct such C^* -algebraic deformations based on a strongly continuous isometrical action of \mathbb{R}^d on a C^* -algebra \mathbb{A}

$$\alpha : \mathbb{R}^d \times \mathbb{A} \rightarrow \mathbb{A} : (x, a) \mapsto \alpha_x a.$$

The product formula for the smooth vectors \mathbb{A}^{∞} with respect to this action is defined, using an oscillatory integral, choosing a fixed element θ in the orthogonal Lie algebra $so(d)$, by

$$a \times_{\hbar} b := a *_{\theta}^{\alpha} b := \left(\frac{1}{\pi \hbar}\right)^d \int_{\mathbb{R}^d \times \mathbb{R}^d} \alpha_x(a) \alpha_y(b) \exp\left(\frac{2i}{\hbar} x \cdot \theta y\right) dx dy$$

and it gives a pre C^* associative algebra structure on \mathbb{A}^{∞} . This generalizes the Weyl quantization of \mathbb{R}^{2n} . Indeed formula (2.14) can be rewritten

$$F \times_{\hbar} G = \left(\frac{1}{\pi \hbar}\right)^{2n} \int_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} \tau_v(F) \tau_w(G) e^{\frac{2i}{\hbar} \Omega(v, w)} dv dw$$

where τ denotes the action of \mathbb{R}^{2n} on functions on \mathbb{R}^{2n} by translation.

Beliaevsky et al. generalize the construction to actions of Lie groups that admit negatively curved left-invariant Kähler structure. An important observation due to Weinstein is the relevance in the phase appearing in the product kernel (see Eq. (2.15)) of the symplectic flux $S(x, y, z) = \Omega(x, y) + \Omega(y, z) + \Omega(z, x)$ through a geodesic triangle that admits the points x, y and z as mid-points of its geodesic edges. This lead to the study of symmetric symplectic spaces, and, more precisely here to sym-

plectic groups which have a structure of symmetric symplectic spaces. Bieliavsky et al. build analogues of Weyl's quantization which give universal deformation formulas for those groups and obtain new examples of strict deformation quantization [24–26, 28].

A possible drawback of considering “convergent star products” given by integral formulas (like the convergent star product defined on the space of Schwartz functions on \mathbb{R}^{2n} given by formula (2.14)) is the difficulty to extend the construction to infinite dimensional cases, which are unavoidable when dealing with quantum field theory.

Another approach to the convergence problem is the following. Taking the formal power series defining the star product, one can ask for convergence in a mathematically meaningful way. This has been achieved by Waldmann et al. in a growing number of examples [17, 39, 80, 169]. They build seminorms which guarantee the convergence of the deformed multiplication. In this way, they construct topological non-commutative algebras, essentially of Fréchet type. It does not yet reach the C^* -framework but it already gives an algebra over \mathbb{C} and not just over $C[[\nu]]$. One can then study Hilbert space representations of this algebra by (still a priori unbounded) operators. Convergence of the Moyal star product on a Fréchet algebra has also been studied by Omori et al. in [138].

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