Chapter 2
Q-Conditional Symmetries of Reaction-Diffusion Systems

Abstract A recently developed theoretical background for searching Q-conditional (nonclassical) symmetries of systems of evolution partial differential equations is presented. We generalize the standard definition of Q-conditional symmetry by introducing the notion of Q-conditional symmetry of the p-th type and show that different types of symmetry of a given system generate a hierarchy of conditional symmetry operators. It is shown that Q-conditional symmetry of the p-th type possesses some special properties, which distinguish them from the standard conditional symmetry. The general class of two-component nonlinear reaction-diffusion systems is examined in order to find the Q-conditional symmetry operators. The relevant systems of so-called determining equations are solved under additional restrictions. As a result, several reaction-diffusion systems possessing conditional symmetry are constructed. In particular, it is shown that the diffusive Lotka–Volterra system, the Belousov–Zhabotinskii system (with the correctly specified coefficients) and some of their generalizations admit Q-conditional symmetry.

2.1 Reaction-Diffusion Systems and Their Applications

In 1952, Turing published a remarkable paper [56], in which a revolutionary idea about the mechanism of morphogenesis (the development of structures in an organism during its life) has been proposed. From the mathematical point of view Turing's idea immediately leads to the construction of reaction-diffusion systems (RDSs) (not single equations!) exhibiting so-called Turing instability (see, e.g., [41, Sect. 14.3]). Nowadays nonlinear RDSs are basic equations for many well-known nonlinear models used to describe a wide range of processes in physics, biology, medicine, chemistry, ecology, etc.

This chapter is mostly devoted to the investigation of two-component RDSs of the form

$$\begin{aligned} U_t &= \big(D^1(U)U_x\big)_x + F(U,V), \\ V_t &= \big(D^2(V)V_x\big)_x + G(U,V), \end{aligned} \tag{2.1}$$

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where $U = U(t,x)$ and $V = V(t,x)$ are two unknown functions representing the densities of populations (cells, tumours, chemicals), $F(U,V)$ and $G(U,V)$ are the given smooth functions describing the interaction between them and the environment, the functions $D^1(U)$ and $D^2(V)$ are the relevant diffusivities (hereafter they are positive smooth functions) and the subscripts t and x denote differentiation with respect to (w.r.t.) these variables. The class of RDSs (2.1) generalizes a number of nonlinear models describing various processes in biology, medicine and ecology (see, e.g., the well-known books [10, 41, 43, 45]).

Usually the diffusivities D^k $(k = 1, 2)$ are taken to be positive constants d_k. An important subclass of RDSs of the form (2.1) consists of those, which satisfy the well-known requirements leading to Turing instability, hence they can be used for description of the chemical basis of morphogenesis [10, Chap. 7], [43, Chap. 2]. The classical examples of such RDSs are the Gierer–Meinhardt system, the Schnakenberg system, etc.

The two-component diffusive Lotka–Volterra system (DLVS)

$$\begin{aligned} U_t &= d_1 U_{xx} + U(a_1 + b_1 U + c_1 V), \\ V_t &= d_2 V_{xx} + V(a_2 + b_2 U + c_2 V) \end{aligned} \tag{2.2}$$

is another common RDS of the form (2.1) [10, 41, 43, 45]. System (2.2) is the standard generalization of the classical Lotka–Volterra system that takes into account the diffusion process for interacting species (see the terms $d_1 U_{xx}$ and $d_2 V_{xx}$). Although the classical Lotka–Volterra system was independently introduced by Lotka and Volterra about 90 years ago, its different generalizations are widely studied at present because of their importance for mathematical modelling of a wide range of processes in biology, ecology, economics, etc.

It is well known that DLVS (2.2) models several types of interaction between two populations of species. Three common types are predator–prey interaction, the competition (for food, space, etc.) of species and mutualism. Each type of species interaction is defined by the signs of coefficients in DLVS (2.2). For example, the coefficients

$$a_k > 0, \ b_k \le 0, \ c_k \le 0, k = 1, 2$$

are used in order to describe the competition, while the cases $c_1 b_2 < 0$ and $c_1 > 0$, $b_2 > 0$ model predator–prey interaction and mutualism, respectively (see [42, Chap. 3] for details).

A separate subclass of the RDS class (2.1) is formed by so-called $\lambda - \omega$ systems, which possess spiral wave solutions. Spiral waves occur naturally in a wide variety of biological, physiological and chemical effects (see [41, Chap. 12] and the references therein). Typically the $\lambda - \omega$ systems have a complicated structure involving the nonlinearities $\lambda(U^2 + V^2)$ and $\omega(U^2 + V^2)$, hence their analysis is rather complicated. The most widely known are spiral waves occurring in the

Belousov–Zhabotinskii reaction [41, Sect. 13.3]. In contrast to the $\lambda - \omega$ systems, the corresponding mathematical model is simple and it is the Belousov–Zhabotinskii system

$$\begin{aligned} U_t &= d_1 U_{xx} + U(a_1 - b_1 U - c_1 V) + rV, \\ V_t &= d_2 V_{xx} - V(a_2 + b_2 U), \end{aligned} \tag{2.3}$$

where all the parameters should be nonnegative.

As noted above, typically the diffusivities D^k $(k = 1, 2)$ in RDSs of the form (2.1) are taken to be positive constant, however, in certain insect dispersal models they depend on the densities U and V. For example, a power dependence is adopted in diffusion models, when there is an increase in diffusion due to population pressure [29], [41, Sect. 11.4], [42, Sect. 13.4] (see also the application to modelling flows of thin films of viscous fluid [23]). Probably the simplest nonlinear RDS with variable diffusivities follows as a particular case from the seminal work [54] and takes the form

$$\begin{aligned} u_t &= ((d_1 + d_{11} u)u)_{xx} + u(a_1 - b_1 u - c_1 v), \\ v_t &= ((d_2 + d_{22} v)v)_{xx} + v(a_2 - b_2 u - c_2 v), \end{aligned} \tag{2.4}$$

where the diffusivities $D^1 = d_1 + 2d_{11}u$ and $D^2 = d_2 + 2d_{22}v$ are linear functions. Systems of the form (2.4) are used in order to model the competition in a heterogeneous environment (see, e.g., [44] and references cited therein).

Nonlinear multi-component RDSs are an important tool for mathematical modelling of a wide range of processes involving several kinds of species (cells, chemicals, etc.). Such systems can possess some properties that are not common for the relevant two-component systems. Thus, it is time to extend the results obtained for two-component RDSs to the multi-component systems. It turns out that this is a highly nontrivial problem and there are not many rigorous studies in this direction. To the best of our knowledge, these studies are mostly focused on investigation of the multi-component DLVS (see [39, 57, 58] and the papers cited therein). It should be noted that the multi-component systems describe much more complicated interactions between n populations than DLVS (2.2). The case $n = 3$ is studied in Chap. 3.

During recent decades nonlinear RDSs have been extensively studied by means of different mathematical methods and techniques. In this chapter and Chaps. 3 and 4, we restrict ourselves to the application of symmetry-based methods (another terminology is group-theoretical methods) in order to construct subclasses of RDSs with nontrivial conditional symmetry, to identify and to study in detail those, which are used in biological applications.

2.2 Q-Conditional Symmetry for Systems of Partial Differential Equations

Although finding Lie symmetries of two-component RD systems of the form (2.1) was initiated about 30 years ago [6, 31, 32, 61], the complete Lie symmetry classification problem was completed only in the 2000s in papers [12, 21, 22] (for constant diffusivities) and [23, 24, 36] (for nonconstant diffusivities).

The time is therefore ripe for a complete description of non-Lie symmetries for the class of RDSs (2.1). However, it seems to be an extremely difficult task because, firstly, several definitions of non-Lie symmetries have been introduced (nonclassical symmetry [3, 8], conditional symmetry [20, 34, 35], generalized conditional symmetry [30, 37, 59], etc.), secondly, an exhaustive description of non-Lie symmetries needs to solve the corresponding systems of determining equations (DEs), which are *nonlinear* and can fully be solved only in exceptional cases.

Hereafter we use the most common notion among non-Lie symmetries, nonclassical symmetry, which we call Q-conditional symmetry following the well-known book [35] and our previous papers. It is well known that the notion of Q-conditional symmetry plays an important role in investigation of nonlinear partial differential equations (PDEs) because, having such symmetries in an explicit form, one may construct new exact solutions, which are not obtainable by the classical Lie algorithm. However, for an exhaustive description of such symmetries, one needs to solve the corresponding nonlinear systems of DEs and this is a very difficult task. To the best of our knowledge, only a few papers devoted to the search for Q-conditional symmetries for systems of PDEs were published before 2010 [2, 5, 25, 28, 40]. The majority of such papers were published during the current decade [4, 13, 15–19, 55].

Generally speaking, in order to solve the Q-conditional symmetry classification problem for the RDS class (2.1), one should look for new constructive approaches allowing to solve the relevant nonlinear system of DEs. A possible approach based on new definitions of Q-conditional symmetry was proposed in [13] and is presented in this section.

Consider a system of m evolution equations ($m \geq 2$) with two independent (t, x) and m dependent $u = (u^1, u^2, \ldots, u^m)$ variables. Let us assume that the kth-order ($k \geq 2$) equations of evolution type

$$u_t^i = F^i\left(t, x, u, u_x, \ldots, u_x^{(k)}\right),\ i = 1, 2, \ldots, m \tag{2.5}$$

are defined in a domain $\Omega \subset \mathbf{R}^2$ of the variables t and x. Hereafter, F^i are the smooth functions of the corresponding variables, the subscripts t and x denote differentiation w.r.t. these variables, $u_t^i = \frac{\partial u^i}{\partial t}$ and $u_x^{(j)} \equiv \frac{\partial^j u}{\partial x^j} = \left(\frac{\partial^j u^1}{\partial x^j}, \ldots, \frac{\partial^j u^m}{\partial x^j}\right), j = 1, 2, \ldots, k$.

It is well known (see, e.g., [7, Sect. 4.3]) that in order to find Lie invariance operators, one needs to consider system (2.5) as the manifold

$$\mathscr{M} = \{S_1 = 0, S_2 = 0, \ldots, S_m = 0\},$$

where

$$S_i \equiv u_t^i - F^i\left(t, x, u, u_x, \dots, u_x^{(k)}\right) \ (i = 1, 2, \dots, m),$$

in the prolonged space of the variables

$$t,\ x,\ u,\ \underset{1}{u}, \dots, \underset{k}{u}.$$

Here, the symbol $\underset{j}{u}$ $(j = 1, 2, \dots, k)$ denotes totalities of the jth-order derivatives w.r.t. the variables t and x.

According to the definition, system (2.5) is invariant (in the Lie sense!) under the transformations generated by the infinitesimal operator

$$Q = \xi^0(t, x, u)\partial_t + \xi^1(t, x, u)\partial_x + \eta^1(t, x, u)\partial_{u^1} + \cdots + \eta^m(t, x, u)\partial_{u^m}, \qquad (2.6)$$

if the following invariance criterion is satisfied:

$$\underset{k}{Q}(S_i)\Big|_{\mathscr{M}} = 0 \ (i = 1, 2, \dots, m). \qquad (2.7)$$

The operator $\underset{k}{Q}$ is the kth-order prolongation of the operator Q and its coefficients are expressed via the functions $\xi^0,\ \xi^1,\ \eta^1, \dots, \eta^m$ by well-known formulae (see, e.g., [46, 49]), which will be specified in the next section for $k = 2$.

The crucial idea, which is used for introducing the notion of Q-conditional symmetry (nonclassical symmetry) is to change the manifold $\mathscr{M}$; in particular, the operator Q is used for such a purpose. It was noted only recently [13] that there are several different possibilities to realize this idea in the case of PDE systems.

Definition 2.1 ([13]) Operator (2.6) is called Q-conditional symmetry (nonclassical symmetry) for an evolution system of the form (2.5) if the following invariance criterion is satisfied:

$$\underset{k}{Q}(S_i)\Big|_{\mathscr{M}_m} = 0,\ i = 1, 2, \dots, m, \qquad (2.8)$$

where the manifold $\mathscr{M}_m$ is

$$\left\{S_1 = 0, S_2 = 0, \dots, S_m = 0, Q(u^1) = 0, \dots, Q(u^m) = 0\right\},$$

where $Q(u^i) = \xi^0 u_t^i + \xi^1 u_x^i - \eta^i \ (i = 1, 2, \dots, m)$.

Definition 2.2 ([13]) Operator (2.6) is called Q-conditional symmetry of the first type for an evolution system of the form (2.5) if the following invariance criterion is satisfied:

$$\underset{k}{Q}(S_i)\Big|_{\mathscr{M}_1} = 0,\ i = 1, 2, \dots, m,$$

where the manifold $\mathscr{M}_1$ is

$$\left\{S_1 = 0, S_2 = 0, \ldots, S_m = 0, Q(u^{i_1}) = 0\right\}$$

with a fixed number i_1 $(1 \leq i_1 \leq m)$.

Definition 2.3 ([13]) Operator (2.6) is called Q-conditional symmetry of the p-th type for an evolution system of the form (2.5) if the following invariance criterion is satisfied:

$$\underset{k}{Q}\,(S_i)\Big|_{\mathscr{M}_p} = 0, \; i = 1, 2, \ldots, m,$$

where the manifold $\mathscr{M}_p$ is

$$\left\{S_1 = 0, S_2 = 0, \ldots, S_m = 0, Q(u^{i_1}) = 0, \ldots, Q(u^{i_p}) = 0\right\}$$

with any given numbers $i_1, \ldots, i_p$ $(1 \leq p \leq i_p \leq m)$.

Obviously, these three definitions coincide in the case of $m = 1$, i.e., a single evolution equation. If $m > 1$, then one obtains a hierarchy of conditional symmetry operators. It can easily be seen that

$$\mathscr{M}_m \subset \mathscr{M}_p \subset \mathscr{M}_1 \subset \mathscr{M},$$

hence, each Lie symmetry is automatically a Q-conditional symmetry of the first and p-th types, while Q-conditional symmetry of the first type is that of the p-th type. From the formal point of view, it is enough to find all the Q-conditional symmetry (nonclassical symmetry) operators. Having the full list of Q-conditional symmetries one may simply check, which of them is the Lie symmetry or/and Q-conditional symmetry of the p-th type. On the other hand, to construct Q-conditional symmetry of the p-th type for a system of PDEs, one needs to solve another nonlinear system, the system of DEs. It turns out that the system of DEs in the case $p = m$ (i.e., for searching Q-conditional symmetry) is much more complicated than in the case $p < m$, in particular $p = 1$ (i.e., for search of Q-conditional symmetry of the first type). As a result, examples of Q-conditional symmetry can be only found using particular solutions of the relevant system of DEs, while a complete classification of Q-conditional symmetries of the first type can be derived for many classes of PDE systems, including the RDS class (2.1).

Hereafter we assume that $\xi^0 \neq 0$, i.e., the so-called no-go case when $\xi^0 = 0$ is not taken into account. The natural reason to avoid examination of the no-go case follows the well-known statement (firstly proved in [60]) that exhaustive description of Q-conditional symmetries with $\xi^0 = 0$ for scalar evolution equations is equivalent to solving the equation in question. This statement can be easily extended to systems of evolution equations using Definition 2.1. However, very recently (see [19] for details) we have shown that the no-go case can be completely examined (at least for some subclasses of class (2.1)) using the notion of Q-conditional symmetry of the first type.

Using the definition of Q-conditional symmetry of the p-th type, one may prove that Properties 1.2 and 1.3 (see Chap. 1) are still valid, however Property 1.1 is no longer valid provided $p < m$ [13]. On the other hand, a new property can be formulated as follows.

Theorem 2.1 *Let us assume that* $X = \sum\limits_{l \neq i_1,\, l=1}^{m} \eta^l(t,x,u)\partial_{u^l}$ *(with a fixed number* i_1, $1 \leq i_1 \leq m$*) is a Lie symmetry operator of system (2.5) while* Q *is a* Q*-conditional symmetry of the first type, which was found using the manifold*

$$\mathcal{M}_1 = \left\{S_1 = 0,\ S_2 = 0, \ldots, S_m = 0,\ Q(u^{i_1}) = 0\right\}.$$

Then any linear combination $C_1X + C_2Q$ *(hereafter* C_1 *and* C_2 *are arbitrary constants) produces another* Q*-conditional symmetry of the first type.*

Proof In order to prove this theorem, one needs to show that the operator $Z = C_1X + C_2Q$ satisfies Definition 2.2 on the manifold

$$\mathcal{M}_1^* = \left\{S_1 = 0, S_2 = 0, \ldots, S_m = 0, Z(u^{i_1}) = 0\right\}.$$

This means that we need to prove that

$$\underset{k}{Z}(S_i)\Big|_{\mathcal{M}_1^*} = 0,\ i = 1, 2, \ldots, m. \tag{2.9}$$

Firstly we note that the manifold $\mathcal{M}_1$ coincides with $\mathcal{M}_1^*$ in this case because $Z(u^{i_1}) = (C_1X + C_2Q)(u^{i_1}) = C_2Q(u^{i_1})$. So one may write the following equalities

$$\begin{aligned} \underset{k}{Z}(S_i)\Big|_{\mathcal{M}_1^*} &= \underset{k}{Z}(S_i)\Big|_{\mathcal{M}_1} = \left(C_1\underset{k}{X} + C_2\underset{k}{Q}\right)(S_i)\Big|_{\mathcal{M}_1} \\ &= C_1\underset{k}{X}(S_i)\Big|_{\mathcal{M}_1} + C_2\underset{k}{Q}(S_i)\Big|_{\mathcal{M}_1} = C_1\underset{k}{X}(S_i)\Big|_{\mathcal{M}_1},\ i = 1, 2, \ldots, m. \end{aligned} \tag{2.10}$$

On the other hand

$$\underset{k}{X}(S_i)\Big|_{\mathcal{M}} = 0,\ i = 1, 2, \ldots, m$$

(here $\mathcal{M} = \{S_1 = 0, S_2 = 0, \ldots, S_m = 0\}$) because the Lie symmetry X of system (2.5) must satisfy criterion (2.7).

Finally, one easily realizes that $\mathcal{M}_1 \subset \mathcal{M}$, so that the above equalities produce

$$\underset{k}{X}(S_i)\Big|_{\mathcal{M}_1} = 0,\ i = 1, 2, \ldots, m. \tag{2.11}$$

This means that Z is a Q-conditional symmetry of the first type because (2.9) immediately follows from (2.10) and (2.11).

The proof is now complete. □

It should be stressed that Theorem 2.1 is not valid for arbitrary given Q-conditional symmetry but only for that of the first type. However, this theorem can be easily generalized on Q-conditional symmetry of the p-th type ($p < m$) using the relevant modification of the Lie symmetry operator.

Theorem 2.2 *Let us assume that* $X = \sum\limits_{l \in A,\, l=1}^{m} \eta^l(t,x,u)\partial_{u^l}$ *(with the fixed set of numbers* $A = \left\{ i_1, \ldots, i_p \mid 1 \le i_p \le m \right\}$, $p < m$*) is a Lie symmetry operator of system (2.5) while* Q *is a* Q*-conditional symmetry of the* p*-th type, which was found using the manifold* $\left\{S_1 = 0, S_2 = 0, \ldots, S_m = 0, Q(u^{i_1}) = 0, \ldots, Q(u^{i_p}) = 0\right\}$. *Then any linear combination* $C_1X + C_2Q$ *produces another* Q*-conditional symmetry of the* p*-th type of the evolution system (2.5).*

2.3 Systems of Determining Equations

In this section, we construct the system of DEs for finding Q-conditional symmetries for the class of RDSs (2.1) and present its preliminary analysis. From the very beginning, one notes that RDS (2.1) can be simplified by applying the Kirchhoff substitution

$$u = \int D^1(U)dU, \quad v = \int D^2(V)dV, \tag{2.12}$$

where $u(t,x)$ and $v(t,x)$ are new unknown functions and $D^k \neq 0,\ k = 1,2$ (in the case of nonconstant diffusivities, we assume that they have a finite number of roots). We remind the reader that the diffusivity coefficients must be nonnegative, hence there exist unique inverse functions to those arising in the right-hand sides of (2.12). Substituting (2.12) into (2.1), one obtains

$$\begin{aligned} u_{xx} &= d^1(u)u_t + C^1(u,v), \\ v_{xx} &= d^2(v)v_t + C^2(u,v), \end{aligned} \tag{2.13}$$

where the functions $d^1,\ d^2$ and $C^1,\ C^2$ are uniquely defined via $D^1,\ D^2$ and $F,\ G$, respectively. In fact, (2.1) and (2.13) are related by the formulae

$$\begin{aligned} d^1(u) = \tfrac{1}{D^1(U)},\ d^2(v) = \tfrac{1}{D^2(V)}, \\ C^1(u,v) = -F(U,V),\ C^2(u,v) = -G(U,V), \end{aligned} \tag{2.14}$$

where $U = D^1_*(u) \equiv \left(\int D^1(u)du\right)^{-1}$, $V = D^2_*(v) \equiv \left(\int D^2(v)dv\right)^{-1}$ (the superscripts -1 mean inverse functions). Hereafter we construct conditional symmetries for the class of RDSs (2.13) instead of (2.1). Having a conditional symmetry operator and a system of the form (2.13), one may easily transform them into the relevant operator and RDS of the form (2.1) provided the inverse functions in (2.14) are known.

Let us apply Definition 2.1 to construct the system of DEs for finding the Q-conditional symmetry operator of the form

$$Q = \partial_t + \xi(t,x,u,v)\partial_x + \eta^1(t,x,u,v)\partial_u + \eta^2(t,x,u,v)\partial_v. \tag{2.15}$$

Conditions (2.8) for system (2.13) take the form

$$\begin{aligned} \underset{2}{Q}\left(u_{xx} - d^1(u)u_t - C^1(u,v)\right)\Big|_{\mathscr{M}_2} &= 0,\\ \underset{2}{Q}\left(v_{xx} - d^2(v)v_t - C^2(u,v)\right)\Big|_{\mathscr{M}_2} &= 0, \end{aligned} \tag{2.16}$$

where the manifold $\mathscr{M}_2$ is

$$\left\{u_{xx} = d^1(u)u_t - C^1(u,v),\ v_{xx} = d^2(v)v_t + C^2(u,v),\ Q(u) = 0,\ Q(v) = 0\right\}.$$

One can apply the second prolongation of the operator Q

$$\begin{aligned} \underset{2}{Q} = Q &+ \rho^1_t\frac{\partial}{\partial u_t} + \rho^2_t\frac{\partial}{\partial v_t} + \rho^1_x\frac{\partial}{\partial u_x} + \rho^2_x\frac{\partial}{\partial v_x}\\ &+\sigma^1_{tx}\frac{\partial}{\partial u_{tx}} + \sigma^2_{tx}\frac{\partial}{\partial v_{tx}} + \sigma^1_{tt}\frac{\partial}{\partial u_{tt}} + \sigma^2_{tt}\frac{\partial}{\partial v_{tt}} + \sigma^1_{xx}\frac{\partial}{\partial u_{xx}} + \sigma^2_{xx}\frac{\partial}{\partial v_{xx}} \end{aligned}$$

to each equations of (2.13). Here the coefficients ρ^k and σ^k with the relevant indices are calculated by the well-known formulae (see, e.g., [46, 49]) and are presented below.

Since system (2.13) should be considered as a manifold in the prolonged space of independent variables

$$t,\ x,\ u,\ v,\ u_t,\ v_t,\ u_x,\ v_x,\ u_{tx},\ v_{xt},\ u_{tt},\ v_{tt},\ u_{xx},\ v_{xx},$$

we arrive at the second-order PDEs

$$\begin{aligned} \eta^1 d^1_u u_t + \eta^1 C^1_u + \eta^2 C^1_v + \rho^1_t d^1 &= \sigma^1_{xx},\\ \eta^2 d^2_v v_t + \eta^2 C^2_v + \eta^1 C^2_u + \rho^2_t d^2 &= \sigma^2_{xx} \end{aligned} \tag{2.17}$$

with $\rho_t^1,\ \rho_t^2,\ \sigma_{xx}^1$ and σ_{xx}^2 defined in (2.19). To obtain the system of DEs in an explicit form, one needs to take into account not only system (2.13) (it will lead only to the system of DEs for Lie symmetry operators!) but also two additional conditions

$$u_t + \xi u_x = \eta^1,\ v_t + \xi v_x = \eta^2 \tag{2.18}$$

generated by operator (2.15). Thus, inserting into (2.17) the explicit expression for ρ^k and σ^k:

$$\begin{aligned}
\rho_t^1 &= \eta_t^1 + \eta_u^1 u_t + \eta_v^1 v_t - u_x\left(\xi_t + \xi_u u_t + \xi_v v_t\right),\\
\rho_t^2 &= \eta_t^2 + \eta_u^2 u_t + \eta_v^2 v_t - v_x\left(\xi_t + \xi_u u_t + \xi_v v_t\right),\\
\sigma_{xx}^1 &= \eta_{xx}^1 + 2\eta_{xu}^1 u_x + 2\eta_{xv}^1 v_x + \eta_{uu}^1 u_x^2 + \eta_{vv}^1 v_x^2 + 2\eta_{uv}^1 u_x v_x + \eta_u^1 u_{xx}\\
&\quad + \eta_v^1 v_{xx} - u_x\left(\xi_{xx} + 2\xi_{xu}u_x + 2\xi_{xv}v_x + \xi_{uu}u_x^2 + \xi_{vv}v_x^2 + 2\xi_{uv}u_x v_x\right)\\
&\quad - u_x\left(\xi_u u_{xx} + \xi_v v_{xx}\right) - 2u_{xx}\left(\xi_x + \xi_u u_x + \xi_v v_x\right),\\
\sigma_{xx}^2 &= \eta_{xx}^2 + 2\eta_{xu}^2 u_x + 2\eta_{xv}^2 v_x + \eta_{uu}^2 u_x^2 + \eta_{vv}^2 v_x^2 + 2\eta_{uv}^2 u_x v_x + \eta_u^2 u_{xx}\\
&\quad + \eta_v^2 v_{xx} - u_x\left(\xi_{xx} + 2\xi_{xu}u_x + 2\xi_{xv}v_x + \xi_{uu}u_x^2 + \xi_{vv}v_x^2 + 2\xi_{uv}u_x v_x\right)\\
&\quad - u_x\left(\xi_u u_{xx} + \xi_v v_{xx}\right) - 2v_{xx}\left(\xi_x + \xi_u u_x + \xi_v v_x\right)
\end{aligned} \tag{2.19}$$

and excluding four derivatives $u_t,\ v_t,\ u_{xx},\ v_{xx}$ using (2.13) and (2.18), one arrives at two cumbersome equations of the form

$$\begin{aligned}
& d^1\left(\eta_t^1 + \eta_u^1(\eta^1 - \xi u_x) + \eta_v^1(\eta^2 - \xi v_x) - u_x\left(\xi_t + \xi_u(\eta^1 - \xi u_x)\right.\right.\\
&\quad \left.\left. + \xi_v(\eta^2 - \xi v_x)\right)\right) + \eta^1 d_u^1(\eta^1 - \xi u_x) + \eta^1 C_u^1 + \eta^2 C_v^1\\
&= \eta_{xx}^1 + 2\eta_{xu}^1 u_x + 2\eta_{xv}^1 v_x + \eta_{uu}^1 u_x^2 + \eta_{vv}^1 v_x^2 + 2\eta_{uv}^1 u_x v_x\\
&\quad - u_x\left(\xi_{xx} + 2\xi_{xu}u_x + 2\xi_{xv}v_x + \xi_{uu}u_x^2 + \xi_{vv}v_x^2 + 2\xi_{uv}u_x v_x\right)\\
&\quad + \left((\eta^1 - \xi u_x)d^1 + C^1\right)\left(\eta_u^1 - 2\xi_x - 3\xi_u u_x - 2\xi_v v_x\right)\\
&\quad + \left((\eta^2 - \xi v_x)d^2 + C^2\right)\left(\eta_v^1 - \xi_v u_x\right),\\
& d^2\left(\eta_t^2 + \eta_u^2(\eta^1 - \xi u_x) + \eta_v^2(\eta^2 - \xi v_x) - v_x\left(\xi_t + \xi_u(\eta^1 - \xi u_x)\right.\right.\\
&\quad \left.\left. + \xi_v(\eta^2 - \xi v_x)\right)\right) + \eta^2 d_v^2(\eta^2 - \xi v_x) + \eta^1 C_u^2 + \eta^2 C_v^2\\
&= \eta_{xx}^2 + 2\eta_{xu}^2 u_x + 2\eta_{xv}^2 v_x + \eta_{uu}^2 u_x^2 + \eta_{vv}^2 v_x^2 + 2\eta_{uv}^2 u_x v_x\\
&\quad - v_x(\xi_{xx} + 2\xi_{xu}u_x + 2\xi_{xv}v_x + \xi_{uu}u_x^2 + \xi_{vv}v_x^2 + 2\xi_{uv}u_x v_x)\\
&\quad + \left((\eta^2 - \xi v_x)d^2 + C^2\right)\left(\eta_u^2 - 2\xi_x - 3\xi_v v_x - 2\xi_u u_x\right)\\
&\quad + \left((\eta^1 - \xi u_x)d^1 + C^1\right)\left(\eta_u^2 - \xi_u v_x\right).
\end{aligned} \tag{2.20}$$

The next step is to take into account that the unknown functions η^1, η^2 and ξ do not depend on the derivatives u_x and v_x and therefore we split two equations arising in (2.20) w.r.t. $u_x^3,\ u_x v_x^2,\ v_x u_x^2,\ u_x v_x,\ u_x^2,\ v_x^2,\ v_x,\ u_x$ and $v_x^3,\ v_x u_x^2,\ u_x v_x^2,\ u_x v_x,\ u_x^2,\ v_x^2,\ v_x,\ u_x$, respectively. As a result, we obtain the nonlinear system of DEs

$$
\begin{aligned}
&(1)\ \xi_{uu} = \xi_{vv} = \xi_{uv} = 0,\\
&(2)\ \eta^1_{vv} = 0,\\
&(3)\ \eta^2_{uu} = 0,\\
&(4)\ 2\xi\xi_u d^1 + \eta^1_{uu} - 2\xi_{xu} = 0,\\
&(5)\ 2\xi\xi_v d^2 + \eta^2_{vv} - 2\xi_{xv} = 0,\\
&(6)\ \xi\xi_v(d^1 + d^2) + 2\eta^1_{uv} - 2\xi_{xv} = 0,\\
&(7)\ \xi\xi_u(d^1 + d^2) + 2\eta^2_{uv} - 2\xi_{xu} = 0,\\
&(8)\ \xi\eta^1_v(d^1 - d^2) + 2\eta^1_{xv} - 2\xi_v C^1 - 2\xi_v\eta^1 d^1 = 0,\\
&(9)\ \xi\eta^2_u(d^2 - d^1) + 2\eta^2_{xu} - 2\xi_u C^2 - 2\xi_u\eta^2 d^2 = 0,\\
&(10)\ -\xi\eta^1 d^1_u + (2\xi_u\eta^1 - \xi_t - \xi_v\eta^2 - 2\xi\xi_x)d^1\\
&\quad +\xi_v\eta^2 d^2 + 3\xi_u C^1 + \xi_v C^2 - 2\eta^1_{xu} + \xi_{xx} = 0,\\
&(11)\ -\xi\eta^2 d^2_v + (2\xi_v\eta^2 - \xi_t - \xi_u\eta^1 - 2\xi\xi_x)d^2\\
&\quad +\xi_u\eta^1 d^1 + 3\xi_v C^2 + \xi_u C^1 - 2\eta^2_{xv} + \xi_{xx} = 0,\\
&(12)\ (\eta^1)^2 d^1_u + (\eta^1_t + \eta^2\eta^1_v + 2\xi_x\eta^1)d^1 - \eta^2\eta^1_v d^2\\
&\quad +\eta^1 C^1_u + \eta^2 C^1_v - \eta^1_u C^1 + 2\xi_x C^1 - \eta^1_v C^2 - \eta^1_{xx} = 0,\\
&(13)\ (\eta^2)^2 d^2_v + (\eta^2_t + \eta^1\eta^2_u + 2\xi_x\eta^2)d^2 - \eta^1\eta^2_u d^1\\
&\quad +\eta^1 C^2_u + \eta^2 C^2_v - \eta^2_u C^1 + 2\xi_x C^2 - \eta^2_v C^2 - \eta^2_{xx} = 0.
\end{aligned}
\tag{2.21}
$$

The system of DEs (2.21) is very complicated and it seems to be unrealistic that its general solution can be derived for arbitrary given functions $d^1(u),\ C^1(u,v),\ d^2(v)$ and $C^2(u,v)$. This means that the conditional symmetry classification can be done only under additional restrictions (see Sects. 2.4 and 2.5). However, if one applies Definition 2.2 to search for Q-conditional symmetries of the first type, then the system of DEs obtained is simpler and the conditional symmetry classification problem can be completely solved (Chap. 4 is devoted to this topic).

From the point of view of qualitative PDE theory, system (2.21) is an overdetermined nonlinear system of PDEs with seven unknown functions $\xi,\ \eta^1,\ \eta^2,\ d^1(u),\ C^1(u,v),\ d^2(v)$ and $C^2(u,v)$. An overview of possible approaches in an attempt to create a general algorithm of integrating overdetermined systems is presented in [53] (see also discussion in [11]). However, to the best of our knowledge, there is no constructive algorithm of integration of such systems at present. In order to solve a given nonlinear overdetermined system, one should develop a separate algorithm, adapted to the system in question. We study system (2.21) in Sects. 2.4 and 2.5. In order to construct its solutions, some additional restrictions will be used.

In conclusion of this section, which (together with Sect. 2.2) contains a theoretical background for Chaps. 2–4, we present the following observation. According to the definition of conditional symmetry proposed in [9, Chap. 5], the differential consequences of (2.18) should be used, hence one may reformulate criterion (2.16) in a such way that the manifold

$$\mathscr{M}_2^* = \{u_{xx} = d^1(u)u_t - C^1(u,v),\ v_{xx} = d^2(v)v_t + C^2(u,v),\ Q(u) = 0, \\ Q(v) = 0,\ \tfrac{\partial}{\partial t}Q(u) = 0,\ \tfrac{\partial}{\partial x}Q(u) = 0,\ \tfrac{\partial}{\partial t}Q(v) = 0,\ \tfrac{\partial}{\partial x}Q(v) = 0\}$$

will be used instead of $\mathscr{M}_2$. It turns out that the definition obtained does not lead to any new conditional symmetries of system (2.13) because (2.13) is a system of *evolution equations* [13]. Here we present a sketch of the proof (the detailed proof is presented in [50]).

Let us calculate the differential consequences of the equations $Q(u) = 0$ and $Q(v) = 0$ (see (2.18)) w.r.t. the variables t and x:

$$u_{tt} = \eta^1_t + \eta^1_u u_t + \eta^1_v v_t - \xi_t u_x - \xi_u u_t u_x - \xi_v v_t u_x - \xi u_{xt}, \tag{2.22}$$

$$u_{tx} = \eta^1_x + \eta^1_u u_x + \eta^1_v v_x - \xi_x u_x - \xi_u u_x u_x - \xi_v v_x u_x - \xi u_{xx}, \tag{2.23}$$

$$v_{tt} = \eta^2_t + \eta^2_u u_t + \eta^2_v v_t - \xi_t v_x - \xi_u u_t v_x - \xi_v v_t v_x - \xi v_{xt}, \tag{2.24}$$

$$v_{tx} = \eta^2_x + \eta^2_u u_x + \eta^2_v v_x - \xi_x v_x - \xi_u u_x v_x - \xi_v v_x v_x - \xi v_{xx}. \tag{2.25}$$

Obviously, the derivatives $u_{tt},\ u_{tx},\ v_{tt}$ and v_{tx} can be easily found from (2.22)–(2.25). However, the expressions obtained do not affect the algorithm presented above because the governing equations (2.18) and (2.19) do not involve these derivatives, hence one again arrives at the system of DEs (2.21).

Other possibilities are to find the first-order derivatives from (2.22)–(2.25) and substitute into (2.18) and (2.19). However, the resulting system is again (2.21).

2.4 Conditional Symmetries of Reaction-Diffusion Systems with Constant Diffusivities

As we noted above, the diffusivity coefficients in RDSs are usually taken to be positive constants. Let us consider system (2.13) in the case $d^1 = \lambda_1$ and $d^2 = \lambda_2$:

$$\begin{aligned} u_{xx} &= \lambda_1 u_t + C^1(u,v), \\ v_{xx} &= \lambda_2 v_t + C^2(u,v), \end{aligned} \tag{2.26}$$

where λ_1 and λ_2 are positive constants. So, the system of DEs (2.21) for finding coefficients of operator (2.15) takes the form

$$
\begin{array}{l}
(1)\ \xi_{uu} = \xi_{vv} = \xi_{uv} = 0,\\
(2)\ \eta^1_{vv} = 0,\\
(3)\ \eta^2_{uu} = 0,\\
(4)\ 2\lambda_1\xi\xi_u + \eta^1_{uu} - 2\xi_{xu} = 0,\\
(5)\ 2\lambda_2\xi\xi_v + \eta^2_{vv} - 2\xi_{xv} = 0,\\
(6)\ (\lambda_1 + \lambda_2)\xi\xi_v + 2\eta^1_{uv} - 2\xi_{xv} = 0,\\
(7)\ (\lambda_1 + \lambda_2)\xi\xi_u + 2\eta^2_{uv} - 2\xi_{xu} = 0,\\
(8)\ (\lambda_1 - \lambda_2)\xi\eta^1_v + 2\eta^1_{xv} - 2\xi_v C^1 - 2\lambda_1\xi_v\eta^1 = 0,\\
(9)\ (\lambda_2 - \lambda_1)\xi\eta^2_u + 2\eta^2_{xu} - 2\xi_u C^2 - 2\lambda_2\xi_u\eta^2 = 0,\\
(10)\ \lambda_1(2\xi_u\eta^1 - \xi_t - \xi_v\eta^2 - 2\xi\xi_x) + \lambda_2\xi_v\eta^2\\
\quad +3\xi_u C^1 + \xi_v C^2 - 2\eta^1_{xu} + \xi_{xx} = 0,\\
(11)\ \lambda_2(2\xi_v\eta^2 - \xi_t - \xi_u\eta^1 - 2\xi\xi_x) + \lambda_1\xi_u\eta^1\\
\quad +3\xi_v C^2 + \xi_u C^1 - 2\eta^2_{xv} + \xi_{xx} = 0,\\
(12)\ \lambda_1(\eta^1_t + \eta^2\eta^1_v + 2\xi_x\eta^1) - \lambda_2\eta^2\eta^1_v + \eta^1 C^1_u + \eta^2 C^1_v\\
\quad -\eta^1_u C^1 + 2\xi_x C^1 - \eta^1_v C^2 - \eta^1_{xx} = 0,\\
(13)\ \lambda_2(\eta^2_t + \eta^1\eta^2_u + 2\xi_x\eta^2) - \lambda_1\eta^1\eta^2_u + \eta^1 C^2_u + \eta^2 C^2_v\\
\quad -\eta^2_u C^1 + 2\xi_x C^2 - \eta^2_v C^2 - \eta^2_{xx} = 0.
\end{array}
\tag{2.27}
$$

As pointed out in the previous section, the construction of the general solution of such systems is a difficult task. Here we solve system (2.27) under the additional restrictions

$$\xi = \xi(u, v),\ \eta^i = \eta^i(u, v),\ i = 1, 2 \tag{2.28}$$

in order to construct Q-conditionally invariant RDSs with constant diffusivities.

Solving Eqs. (1)–(3) of system (2.27), we obtain

$$\xi = au + bv + c,\ \eta^1 = p^1(u)v + q^1(u),\ \eta^2 = p^2(v)u + q^2(v), \tag{2.29}$$

where $a,\ b$ and c are arbitrary constants, and $p^1,\ p^2,\ q^1$ and q^2 are arbitrary smooth functions. Substituting (2.29) into Eqs. (6) and (7) from (2.27) and splitting the equations obtained w.r.t. the powers of u and v, we arrive at the system

$$
\begin{array}{c}
a^2(\lambda_1 + \lambda_2) = 0,\ b^2(\lambda_1 + \lambda_2) = 0,\\
(\lambda_1 + \lambda_2)a(bv + c) + 2p^2_v = 0,\ (\lambda_1 + \lambda_2)b(au + c) + 2p^1_u = 0.
\end{array}
\tag{2.30}
$$

Obviously, solutions of the first pair of Eq. (2.30) are $a = b = 0$ because λ_1 and λ_2 are positive, hence $\xi = c$. Solving Eqs. (4)–(7) of system (2.27), we obtain

$$p^i = \text{const} = \alpha_i\ (i = 1, 2),\ q^1 = \beta_1 u + \gamma_1,\ q^2 = \beta_2 v + \gamma_2,$$

where β_i and γ_i $(i = 1, 2)$ are arbitrary constants. Thus, expressions (2.29) take the form

$$\xi = c,\ \eta^1 = \alpha_1 v + \beta_1 u + \gamma_1,\ \eta^2 = \alpha_2 u + \beta_2 v + \gamma_2. \tag{2.31}$$

Substituting (2.31) into Eqs. (8) and (9) of system (2.27), we arrive at

$$c\alpha_1(\lambda_1 - \lambda_2) = 0,\ c\alpha_2(\lambda_1 - \lambda_2) = 0. \tag{2.32}$$

Solving the system of algebraic equations (2.32), we derive three solutions $\lambda_1 = \lambda_2$, $\alpha_1 = \alpha_2 = 0$ and $c = 0$. The general solution of the remaining Eqs. (10)–(13) essentially depends on the above solutions. Examination of the first two solutions leads to the following theorem.

Theorem 2.3 ([51]) *The system of DEs for finding Q-conditional symmetry operators of the form (2.15) (under restrictions (2.28)) for system (2.26) coincide with the system of DEs for finding of Lie symmetry operators provided* $\lambda_1 = \lambda_2$ *or* $\eta^1_v = \eta^2_u = 0$.

Proof Substituting (2.31), with $\lambda_1 = \lambda_2$, into system (2.27), we obtain that Eqs. (1)–(11) are transformed into identities, while Eqs. (12) and (13) take the form

$$\begin{aligned} \eta^1 C^1_u + \eta^2 C^1_v - \eta^1_u C^1 - \eta^1_v C^2 &= 0, \\ \eta^1 C^2_u + \eta^2 C^2_v - \eta^2_u C^1 - \eta^2_v C^2 &= 0. \end{aligned} \tag{2.33}$$

In [22] the DEs for finding Lie symmetries with condition $\lambda_1 = \lambda_2$ were written down in the explicit form. Substituting conditions (2.28) into these equations, we see that the equations obtained in this way are identical to Eq. (2.33).

Substituting (2.31), with $\alpha_1 = \alpha_2 = 0$, into system (2.27), we see, that Eqs. (1)–(11) also transform into identities, and Eqs. (12) and (13) take the form

$$\begin{aligned} \eta^1 C^1_u + \eta^2 C^1_v - \eta^1_u C^1 &= 0, \\ \eta^1 C^2_u + \eta^2 C^2_v - \eta^2_v C^2 &= 0. \end{aligned} \tag{2.34}$$

Comparing Eq. (2.34) with equations, which are obtained for finding Lie symmetries of system (2.26) with conditions (2.28) from [21], we see that they are identical.

The proof is now complete. □

Thus, to find Q-conditional symmetry operators, which are inequivalent to Lie symmetry operators, we must assume that $\lambda_1 \neq \lambda_2,\ \alpha_1^2 + \alpha_2^2 \neq 0$. Now one needs to set $c = 0$ (see Eq. (2.32)). In this case expressions (2.31) take the form

$$\xi = 0,\ \eta^1 = \alpha_1 v + \beta_1 u + \gamma_1,\ \eta^2 = \alpha_2 u + \beta_2 v + \gamma_2. \tag{2.35}$$

So, Eqs. (1)–(11) of system (2.27) are satisfied identically by expressions (2.35), while Eqs. (12) and (13) take the form

$$\begin{aligned}
&(\alpha_1 v + \beta_1 u + \gamma_1)C^1_u + (\alpha_2 u + \beta_2 v + \gamma_2)C^1_v - \beta_1 C^1 - \alpha_1 C^2 \\
&\quad = \alpha_1(\lambda_2 - \lambda_1)(\alpha_2 u + \beta_2 v + \gamma_2), \\
&(\alpha_1 v + \beta_1 u + \gamma_1)C^2_u + (\alpha_2 u + \beta_2 v + \gamma_2)C^2_v - \alpha_2 C^1 - \beta_2 C^2 \\
&\quad = \alpha_2(\lambda_1 - \lambda_2)(\alpha_1 v + \beta_1 u + \gamma_1).
\end{aligned} \tag{2.36}$$

Thus, we can formulate the following theorem.

Theorem 2.4 ([51]) *Nonlinear RDS (2.26) is Q-conditionally invariant under operator (2.15) with coefficients (2.35) if and only if (iff) the nonlinearities $C^1,\ C^2$ are the solutions of the linear first-order system (2.36).*

To find the general solution of system (2.36) one needs to analyse the two cases $\alpha_2 = 0$ and $\alpha_2 \neq 0$. The case $\alpha_2 \neq 0$, i.e., $\eta^2_u \neq 0$ (then automatically $\eta^1_v \neq 0$) is much more complicated and needs a separate examination (see a particular result in [51]).

In the case $\alpha_2 = 0$, system (2.36) contains an autonomous equation and has the form

$$\begin{aligned}
(\alpha_1 v + \beta_1 u + \gamma_1)C^1_u + (\beta_2 v + \gamma_2)C^1_v &= \beta_1 C^1 + \alpha_1 C^2 + \alpha_1(\lambda_2 - \lambda_1)(\beta_2 v + \gamma_2), \\
(\alpha_1 v + \beta_1 u + \gamma_1)C^2_u + (\beta_2 v + \gamma_2)C^2_v &= \beta_2 C^2.
\end{aligned} \tag{2.37}$$

Since $\alpha_1 \neq 0$, renaming $C^1 \to \alpha_1 C^1,\ u \to \alpha_1 u,\ \gamma_1 \to \alpha_1\gamma_1$, and taking into account that we can get rid of the parameter γ_1 using linear substitutions w.r.t. u and v, system (2.37) can be reduced to the form

$$\begin{aligned}
(v + \beta_1 u)C^1_u + (\beta_2 v + \gamma_2)C^1_v &= \beta_1 C^1 + C^2 + (\lambda_2 - \lambda_1)(\beta_2 v + \gamma_2), \\
(v + \beta_1 u)C^2_u + (\beta_2 v + \gamma_2)C^2_v &= \beta_2 C^2.
\end{aligned} \tag{2.38}$$

One notes the particular solution of system (2.38)

$$C^1_{part} = \frac{1}{2}(\lambda_2 - \lambda_1)(\alpha_1 v + \beta_1 u + \gamma_1),\ C^2_{part} = \frac{1}{2}(\lambda_1 - \lambda_2)(\beta_2 v + \gamma_2).$$

In order to construct the general solution of (2.38), we need to solve the corresponding homogeneous system, that is

$$\begin{aligned}
(v + \beta_1 u)C^1_u + (\beta_2 v + \gamma_2)C^1_v &= \beta_1 C^1 + C^2, \\
(v + \beta_1 u)C^2_u + (\beta_2 v + \gamma_2)C^2_v &= \beta_2 C^2.
\end{aligned} \tag{2.39}$$

The general solution of (2.39) depends essentially on the parameters $\beta_1,\ \beta_2$ and γ_2. As a result, the following theorem is proved.

Table 2.1 Q-conditional symmetry operators of RDS (2.26) with $\lambda_1 \neq \lambda_2$

	$C^1(u,v)$	$C^2(u,v)$	Q
1	$f(v) + ug(v)$	$vg(v)$	$\partial_t + v\partial_u$
2	$(v+u)f(v) - g(v)$	$g(v)$	$\partial_t + \beta_1(v+u)\partial_u,\ \beta_1 \neq 0$
3	$f(\omega) + g(\omega)v + \frac{1}{2}(\lambda_2 - \lambda_1)v$	$g(\omega) + \frac{1}{2}(\lambda_1 - \lambda_2)$ $\omega = 2u - v^2$	$\partial_t + v\partial_u + \partial_v$
4	$f(\omega) + g(\omega)v + \frac{1}{2}(\lambda_2 - \lambda_1)v$	$\beta_2 g(\omega)\,(v+\gamma_2) + \frac{1}{2}\beta_2(\lambda_1 - \lambda_2)(v+\gamma_2)$ $\omega = \beta_2 u - v + \gamma_2 \ln(v+\gamma_2)$	$\partial_t + v\partial_u + \beta_2(v+\gamma_2)\partial_v,$ $\beta_2 \neq 0$
5	$f(\omega)v + g(\omega)v\ln(v) + \frac{1}{2}(\lambda_2 - \lambda_1)(v+\beta_1 u)$	$\beta_1 g(\omega)v + \frac{1}{2}(\lambda_1-\lambda_2)\beta_1 v$ $\omega = v^{-1}\exp\left(\frac{\beta_1 u}{v}\right)$	$\partial_t + (v+\beta_1 u)\partial_u + \beta_1 v\partial_v,$ $\beta_1 \neq 0$
6	$f(\omega)v^{\frac{\beta_1}{\beta_2}} + g(\omega)v + \frac{1}{2}(\lambda_2 - \lambda_1)(v+\beta_1 u)$	$(\beta_2 - \beta_1)g(\omega)v + \frac{1}{2}(\lambda_1 - \lambda_2)\beta_2 v$ $\omega = v^{-\frac{\beta_1}{\beta_2}}((\beta_1-\beta_2)u + v)$	$\partial_t + (v+\beta_1 u)\partial_u + \beta_2 v\partial_v,$ $\beta_1\beta_2(\beta_1-\beta_2) \neq 0$
7	$f(\omega)\exp(\beta_1 v) - \gamma_2 g(\omega) + \frac{1}{2}(\lambda_2-\lambda_1)\beta_1(u+\gamma_2 v)$	$g(\omega) + \frac{1}{2}(\lambda_1 - \lambda_2)$ $\omega = \exp(-\beta_1 v)\times \left(u + \gamma_2 v + \frac{\gamma_2}{\beta_1}\right)$	$\partial_t + \beta_1(u+\gamma_2 v)\partial_u + \partial_v,\ \beta_1\gamma_2 \neq 0$

Theorem 2.5 ([51]) *RDS (2.26) with $\lambda_1 \neq \lambda_2$ is Q-conditionally invariant under operator (2.15) under restrictions (2.28) and $\eta^2_u = 0$ iff the system and corresponding operator have one of the seven forms listed in Table 2.1. Any other system of the form (2.26) admitting operator (2.15) with the above restrictions is reduced to one of those from Table 2.1 by the linear transformation $u \to c_1 u + c_2,\ v \to c_3 v + c_4$ with correctly specified constants $c_1 \neq 0,\ c_2 \neq 0,\ c_3$ and c_4.*

Table 2.1 presents seven subclasses of RDSs with the constant diffusivities, which admit Q-conditional symmetry. Each subclass involves arbitrary smooth functions f and g of the relevant arguments. Depending on the form of f and g, one may extract RDSs arising in applications and construct exact solutions for them using the symmetry operators obtained. This approach is realized for several nonlinear RDSs in Chaps. 3 and 4. Here we present an interesting example only. Let us consider case 4 from Table 2.1. Assuming that $f = c_1\omega^2 - a_1\omega$ ($c_1 \neq 0$) and $g = c_1\omega - a_2 + \frac{\lambda_2 - \lambda_1}{2}$, one may extract the nonlinear RDS

$$\begin{aligned} u_{xx} &= \lambda_1 u_t + \beta_2 u(-a_1 + \beta_2 c_1 u - c_1 v) + rv, \\ v_{xx} &= \lambda_2 v_t + \beta_2 v(-a_2 + \beta_2 c_1 u - c_1 v), \end{aligned} \tag{2.40}$$

where all the coefficients are arbitrary constants, while $a_2 = a_1 - r + \lambda_2 - \lambda_1$. Making the discrete transformation $v \to -v$ and setting $\beta_2 = 1$ (for simplicity), system (2.40) and its symmetry operator are transformed into

$$\begin{aligned} \lambda_1 u_t &= u_{xx} + u(a_1 - c_1 u - c_1 v) + rv, \\ \lambda_2 v_t &= v_{xx} + v(a_2 - c_1 u - c_1 v) \end{aligned} \tag{2.41}$$

and

$$\partial_t - v\partial_u + v\partial_v.$$

Now one realizes that system (2.41) with $r = 0$ is the DLVS describing, for example, the competition of two populations (provided all parameters are nonnegative), while (2.41) with $r \neq 0$ and $a_2 < 0$ is the Belousov–Zhabotinskii type system.

2.5 Conditional Symmetries of Reaction-Diffusion Systems with Power-Law Diffusivities

In this section, we find Q-conditional symmetry operators of the form

$$Q = \partial_t + \xi(t, x, U, V)\partial_x + \eta^1(t, x, U, V)\partial_U + \eta^2(t, x, U, V)\partial_V \tag{2.42}$$

of two-component RDSs with the power-law diffusivities

$$\begin{aligned} U_t &= (U^k U_x)_x + F(U, V), \\ V_t &= (V^l V_x)_x + G(U, V). \end{aligned} \tag{2.43}$$

As pointed out in Sect. 2.1, a power dependence of the diffusion coefficients $D^1(U)$ and $D^2(V)$ is typically adopted in models with variable diffusivities. Hence, RDSs of the form (2.43) form the most important class of such systems, if one wants to apply the results obtained for some real-world models. Note that RDSs with the diffusivities $D^1 = d_1 U^k$ and $D^2 = d_2 V^l$ (d_1 and d_2 are arbitrary positive constants) are reduced to the form (2.43) via scale transformations [23].

First of all we apply the local substitution

$$\begin{aligned} u &= U^{k+1},\ k \neq -1, \\ v &= V^{l+1},\ l \neq -1 \end{aligned} \tag{2.44}$$

(this is a particular case of the Kirchhoff substitution (2.12)) in order to simplify the further computations. Of course, the cases $k = l = -1$ and $k = -1,\ l \neq -1$ ($l = -1,\ k \neq -1$ is symmetric) are special and need separate investigation. Substitution (2.44) reduces operator (2.42) to the form (2.15) with $\partial_u = \frac{1}{k+1} U^{-k}\partial_U$, $\partial_v = \frac{1}{l+1} V^{-l}\partial_V$, while system (2.43) takes the form

$$\begin{aligned} u_{xx} &= u^m u_t + C^1(u, v), \\ v_{xx} &= v^n v_t + C^2(u, v), \end{aligned} \tag{2.45}$$

where $m = -\frac{k}{k+1} \neq -1,\ n = -\frac{l}{l+1} \neq -1,\ C^1(u, v) = -(k+1)F\left(u^{\frac{1}{k+1}}, v^{\frac{1}{l+1}}\right)$ and $C^2(u, v) = -(l+1)G\left(u^{\frac{1}{k+1}}, v^{\frac{1}{l+1}}\right)$.

Thus, the system of DEs (2.21) corresponding to the RDS system (2.45) takes the form

$$
\begin{aligned}
&(1)\ \xi_{uu} = \xi_{vv} = \xi_{uv} = 0,\\
&(2)\ \eta^1_{vv} = 0,\\
&(3)\ \eta^2_{uu} = 0,\\
&(4)\ 2\xi\xi_u u^m + \eta^1_{uu} - 2\xi_{xu} = 0,\\
&(5)\ 2\xi\xi_v v^n + \eta^2_{vv} - 2\xi_{xv} = 0,\\
&(6)\ \xi\xi_v(u^m + v^n) + 2\eta^1_{uv} - 2\xi_{xv} = 0,\\
&(7)\ \xi\xi_u(u^m + v^n) + 2\eta^2_{uv} - 2\xi_{xu} = 0,\\
&(8)\ \xi\eta^1_v(u^m - v^n) + 2\eta^1_{xv} - 2\xi_v C^1 - 2\xi_v\eta^1 u^m = 0,\\
&(9)\ \xi\eta^2_u(v^n - u^m) + 2\eta^2_{xu} - 2\xi_u C^2 - 2\xi_u\eta^2 v^n = 0,\\
&(10)\ -m\xi\eta^1 u^{m-1} + (2\xi_u\eta^1 - \xi_t - \xi_v\eta^2 - 2\xi\xi_x)u^m\\
&\quad +\xi_v\eta^2 v^n + 3\xi_u C^1 + \xi_v C^2 - 2\eta^1_{xu} + \xi_{xx} = 0,\\
&(11)\ -n\xi\eta^2 v^{n-1} + (2\xi_v\eta^2 - \xi_t - \xi_u\eta^1 - 2\xi\xi_x)v^n\\
&\quad +\xi_u\eta^1 u^m + 3\xi_v C^2 + \xi_u C^1 - 2\eta^2_{xv} + \xi_{xx} = 0,\\
&(12)\ m(\eta^1)^2 u^{m-1} + (\eta^1_t + \eta^2\eta^1_v + 2\xi_x\eta^1)u^m - \eta^2\eta^1_v v^n\\
&\quad +\eta^1 C^1_u + \eta^2 C^1_v - \eta^1_u C^1 + 2\xi_x C^1 - \eta^1_v C^2 - \eta^1_{xx} = 0,\\
&(13)\ n(\eta^2)^2 v^{n-1} + (\eta^2_t + \eta^1\eta^2_u + 2\xi_x\eta^2)v^n - \eta^1\eta^2_u u^m\\
&\quad +\eta^1 C^2_u + \eta^2 C^2_v - \eta^2_u C^1 + 2\xi_x C^2 - \eta^2_v C^2 - \eta^2_{xx} = 0.
\end{aligned}
\tag{2.46}
$$

Equations (1) from system (2.46) are easily integrated and lead to

$$\xi = a(t,x)u + b(t,x)v + c(t,x), \tag{2.47}$$

where $a,\ b$ and c are arbitrary (at the moment) smooth functions. Substituting (2.47) into Eqs. (6) and (7) of (2.46) and taking into account the second and third equations of (2.46), one arrives at the requirement $a = b = 0$. Thus, Eqs. (2)–(7) of system (2.46) can be straightforwardly integrated and their general solution takes the form

$$
\begin{aligned}
\xi &= \xi(t,x),\\
\eta^1 &= q^1(t)v + r^1(t,x)u + p^1(t,x),\\
\eta^2 &= q^2(t)u + r^2(t,x)v + p^2(t,x),
\end{aligned}
\tag{2.48}
$$

where the functions in the right-hand sides are arbitrary.

The remaining Eqs. (8)–(13) of system (2.46) involving the functions C^1 and C^2 are the classification equations. To solve them one should consider three different cases depending on the functions $q^1(t),\ q^2(t)$ and $\xi(t,x)$ arising in (2.48):

(a) $q^1(t) = q^2(t) = 0,\ \xi(t,x) \neq 0;$

(b) $q^1(t) = q^2(t) = 0,\ \xi(t,x) = 0;$

(c) $q^1(t)^2 + q^2(t)^2 \neq 0,\ \xi(t,x) = 0.$

Remark 2.1 The fourth possible case $q^1(t)^2 + q^2(t)^2 \neq 0,\ \xi(t,x) \neq 0$ arises only under the restriction $m = n = 0$, which follows from Eqs. (8)–(9) of system (2.46). Hereafter we assume $m^2 + n^2 \neq 0$, because RDSs with constant diffusivities were examined in Sect. 2.4.

It turns out that case ***(a)*** does not lead to any Q-conditional symmetry operators of the form (2.42).

Theorem 2.6 ([50]) *RDS (2.43) with* $k^2 + l^2 \neq 0$ *and* $(k+1)(l+1) \neq 0$ *admits only such operators of the form (2.42) with* $\xi \neq 0$ *and* $\eta^1_V = \eta^2_U = 0$*, which are equivalent to the Lie symmetry operators.*

Proof Here we present only a sketch of the proof. In order to prove Theorem 2.6 one should solve the system of DEs (2.46) under conditions $q^1(t) = q^2(t) = 0$, $\xi(t,x) \neq 0$ and (2.48). In particular, DEs (10)–(13) from (2.46) take the form

$$
\begin{aligned}
&\left(\xi_t + 2\xi\xi_x + m\xi r^1\right) u^m + m\xi p^1 u^{m-1} + 2r^1_x - \xi_{xx} = 0,\\
&\left(\xi_t + 2\xi\xi_x + n\xi r^2\right) v^n + n\xi p^2 v^{n-1} + 2r^2_x - \xi_{xx} = 0,\\
&\left(r^1 u + p^1\right) C^1_u + \left(r^2 v + p^2\right) C^1_v + \left(2\xi_x - r^1\right) C^1 - p^1_{xx} - r^1_{xx} u + m\left(p^1\right)^2 u^{m-1}\\
&\quad + \left(p^1_t + 2mh^1 p^1 + 2\xi_x p^1\right) u^m + \left(r^1_t + m(r^1)^2 + 2\xi_x r^1\right) u^{m+1} = 0, \qquad (2.49)\\
&\left(r^1 u + p^1\right) C^2_u + \left(r^2 v + p^2\right) C^2_v + \left(2\xi_x - r^2\right) C^2 - p^2_{xx} - r^2_{xx} v + n\left(p^2\right)^2 v^{n-1}\\
&\quad + \left(p^2_t + 2nh^2 p^2 + 2\xi_x p^2\right) v^n + \left(r^2_t + n\left(r^2\right)^2 + 2\xi_x r^2\right) v^{n+1} = 0.
\end{aligned}
$$

Now one notes that the third and fourth equations of system (2.49) are linear first-order PDEs w.r.t. C^1 and C^2. According to the standard technique of solving such equations, one needs to find variable ω, using the following ordinary differential equation (ODE)

$$
\frac{du}{r^1 u + p^1} = \frac{dv}{r^2 u + p^2}.
$$

Obviously, its solution essentially depends on $r^1,\ p^1,\ r^2$ and p^2. As a result, one needs to examine six different cases

(1) $r^1 = p^1 = r^2 = p^2 = 0,$

(2) $r^1 = p^1 = r^2 = 0,\ p^2 \neq 0,$

(3) $r^1 = p^1 = 0,\ r^2 \neq 0,$

(4) $r^1 = r^2 = 0, p^1 \neq 0,\ p^2 \neq 0,$

(5) $r^1 = 0,\ p^1 \neq 0,\ r^2 \neq 0,$

(6) $r^1 \neq 0,\ r^2 \neq 0.$

Note that three additional cases

$$r^2 = p^2 = r^1 = 0,\ p^1 \neq 0,$$
$$r^2 = p^2 = 0,\ r^1 \neq 0,$$
$$r^1 \neq 0,\ r^2 = 0,\ p^2 \neq 0$$

can be excluded from the examination because each of them can be obtained from those above by renaming $u \to v,\ v \to u$.

Here we consider in detail only case ***(1)***. In this case, the third and fourth equations of system (2.49) take the form

$$\xi_x C^1 = 0,\ \xi_x C^2 = 0,$$

hence two subcases, $\xi_x = 0$ and $C^1 = C^2 = 0$, arise. The latter simply means that $F^1(U, V) = F^2(U, V) = 0$, i.e., RDS (2.43) reduces to two independent diffusion equations.

The first subcase implies that $\xi_t = 0$ (see the first and second equations in (2.49)), hence $\xi = const$. Thus, we conclude that the nonlinear RDS

$$\begin{aligned} u_{xx} &= u^m u_t + C^1(u, v), \\ v_{xx} &= v^n v_t + C^2(u, v) \end{aligned} \tag{2.50}$$

admits only Q-conditional symmetry operators of the form

$$Q = \partial_t + \gamma \partial_x,\ \gamma = \text{const}.$$

On the other hand, system (2.50) is invariant w.r.t. the Lie symmetry operators $P_t = \partial_t$ and $P_x = \partial_x$, hence the above operator Q is nothing else but the Lie symmetry operator.

Cases ***(2)–(6)*** can be studied in a quite similar way. Finally, the detailed examination leads exactly to the Lie symmetry operators, which are listed in Table 1 [24], in each case.

The sketch is now complete. □

In contrast to case ***(a)***, examination of case ***(b)*** leads to new results.

Theorem 2.7 ([25, 50]) *RDS (2.43) with $k^2 + l^2 \neq 0$ and $(k+1)(l+1) \neq 0$ is Q-conditional invariant under the operator (2.42) with $\xi = 0$ and $\eta^1_V = \eta^2_U = 0$ iff it and the relevant operator have the forms listed in Table 2.2 (in the table, f and g are arbitrary smooth functions of the relevant argument, while λ_j ($j = 1, 2, 3, 4$) are arbitrary constants).*

Proof To prove the theorem one needs to construct the general solution of subsystem (8)–(13) of system (2.46) having the general solution (2.48) of subsystem (1)–(7) and applying the restrictions $\xi = 0$ and $\eta^1_v = \eta^2_u = 0$. Obviously, Eqs. (8)

Table 2.2 Q-conditional symmetries of RDS (2.43)

	RDSs of the form (2.43)	Q-conditional operators	Restrictions
1	$U_t = (U^k U_x)_x + f(U^{k+1})$ $V_t = \left(V^{-\frac{1}{2}} V_x\right)_x - 2\lambda V^{\frac{1}{2}} + g(U^{k+1})$	$\partial_t + 2p(x) V^{\frac{1}{2}} \partial_V$	$p_{xx} = p^2 + \lambda p, p \neq 0$
2	$U_t = (U^k U_x)_x + \lambda_1 U^{-k} +$ $f(U^{k+1} - \alpha V^{l+1})$ $V_t = (V^l V_x)_x + \lambda_2 V^{-l} +$ $g(U^{k+1} - \alpha V^{l+1})$	$\partial_t + \lambda_1 U^{-k} \partial_U +$ $\lambda_2 V^{-l} \partial_V$	$\alpha = \frac{\lambda_1 (k+1)}{\lambda_2 (l+1)}$, $\lambda_2 \neq 0$, $\lambda_1^2 + l^2 \neq 0$
3	$U_t = \left(U^{-\frac{1}{2}} U_x\right)_x - 2\lambda U^{\frac{1}{2}} +$ $f\left(U^{\frac{1}{2}} - V^{\frac{1}{2}}\right)$ $V_t = \left(V^{-\frac{1}{2}} V_x\right)_x - 2\lambda V^{\frac{1}{2}} +$ $g\left(U^{\frac{1}{2}} - V^{\frac{1}{2}}\right)$	$\partial_t + 2p(x) \left(U^{\frac{1}{2}} \partial_U +\right.$ $\left. V^{\frac{1}{2}} \partial_V\right)$	$p_{xx} = p^2 + \lambda p$, $p \neq 0$
4	$U_t = (U^k U_x)_x + \lambda_1 U^{-k} + f(\omega)$ $V_t = (V^l V_x)_x + (V^{l+1} -$ $\lambda_3)(g(\omega) + \lambda_2 V^{-l})$	$\partial_t + \lambda_1 U^{-k} \partial_U +$ $\lambda_2 (V - \lambda_3 V^{-l}) \partial_V$	$\omega = \frac{\exp\left(\lambda_2 (l+1) U^{k+1}\right)}{\left(V^{l+1} - \lambda_3\right)^{\lambda_1 (k+1)}}$, $\lambda_2 \neq 0$, either $\lambda_1^2 + \lambda_3^2 \neq 0$ or $\lambda_3^2 + k^2 \neq 0$ or $\lambda_1^2 + l^2 \neq 0$
5	$U_t = (U^k U_x)_x +$ $(U^{k+1} - \lambda_1)(f(\omega) + \lambda_2 U^{-k})$ $V_t = (V^l V_x)_x +$ $(V^{l+1} - \lambda_3)(g(\omega) + \lambda_4 V^{-l})$	$\partial_t + \lambda_2 (U - \lambda_1 U^{-k}) \partial_U +$ $\lambda_4 (V - \lambda_3 V^{-l}) \partial_V$	$\omega = \frac{\left(U^{k+1} - \lambda_1\right)^{\lambda_4 (l+1)}}{\left(V^{l+1} - \lambda_3\right)^{\lambda_2 (k+1)}}$, $\lambda_2 \lambda_4 \neq 0$, either $\lambda_1^2 + \lambda_3^2 \neq 0$, or $\lambda_3^2 + k^2 \neq 0$ or $\lambda_1^2 + l^2 \neq 0$

and (9) are automatically satisfied, while Eqs. (10) and (11) are reduced to the form $\eta^1_{xu} = 0$ and $\eta^2_{xv} = 0$, respectively, i.e.:

$$r^1 = r^1(t), \quad r^2 = r^2(t).$$

So, the remaining Eqs. (12) and (13) take the form

$$\begin{aligned}
&\left(r^1 u + p^1\right) C^1_u + \left(r^2 v + p^2\right) C^1_v - r^1 C^1 \\
&\quad + \left(r^1_t + m(r^1)^2\right) u^{m+1} + \left(p^1_t + 2mr^1 p^1\right) u^m + m \left(p^1\right)^2 u^{m-1} - p^1_{xx} = 0, \\
&\left(r^1 u + p^1\right) C^2_u + \left(r^2 v + p^2\right) C^2_v - r^2 C^2 \\
&\quad + \left(r^2_t + n(r^2)^2\right) v^{n+1} + \left(p^2_t + 2nr^2 p^2\right) v^n + n \left(p^2\right)^2 v^{n-1} - p^2_{xx} = 0.
\end{aligned} \tag{2.51}$$

System (2.51) consists of two independent first-order linear PDEs w.r.t. the unknown functions $C^1(u, v)$ and $C^2(u, v)$, therefore its general solution can be straightforwardly constructed, however we should remember that the coefficients in (2.51) are functions of t and x. To construct all possible solutions of (2.51) one needs to consider the cases ***(1)–(6)*** as above (up to renaming $u \to v$ and $v \to u$):

In case ***(1)***, operator (2.42) immediately takes the form $Q = \partial_t$, which is, of course, the Lie symmetry operator. A similar situation occurs in case ***(3)*** because all the operators obtained are equivalent to the relevant Lie symmetry operators listed in [23]. The most interesting cases are ***(2)*** and ***(4)–(6)***.

Consider case ***(2)*** in detail. In this case system (2.51) takes the form

$$\begin{aligned} &p^2C^1_v = 0,\\ &p^2C^2_v + p^2_t v^n + n(p^2)^2 v^{n-1} - p^2_{xx} = 0 \end{aligned} \tag{2.52}$$

and its formal integration leads to the solution

$$\begin{aligned} C^1 &= f(u)\\ C^2 &= \int\left(\frac{p^2_{xx}}{p^2} - \frac{p^2_t}{p^2}v^n - np^2v^{n-1}\right)dv + g(u), \end{aligned} \tag{2.53}$$

where f and g are arbitrary smooth functions. Since the function C^2 does not depend on t and x, three subcases should be separately examined: $n = 0, n = 1$ and $n \neq 0; 1$. The first subcase immediately gives $C^2 = \frac{p^2_{xx}-p^2_t}{p^2}v + g(u)$, so that

$$\frac{p^2_{xx} - p^2_t}{p^2} = \lambda,$$

where λ is an arbitrary constant. So, the system

$$\begin{aligned} u_{xx} &= u^m u_t + f(u),\\ v_{xx} &= v_t + \lambda v + g(u) \end{aligned} \tag{2.54}$$

admits the Q-conditional symmetry operator

$$Q = \partial_t + p^2(t,x)\partial_v, \tag{2.55}$$

where $p^2(t,x)$ is the general solution of the linear PDE $p^2_t = p^2_{xx} - \lambda p^2$. However, if one now applies substitution (2.44) to (2.54) and (2.55), then the RDS and the Lie symmetry listed in [23] (see case 5 in Table 1) are obtained. So, subcase $n = 0$ does not lead to any Q-conditional symmetries.

In the subcase $n = 1$ the general solution of (2.52) takes the form

$$C^1 = f(u), \quad C^2 = \lambda v + g(u),$$

where $\lambda = \frac{p^2_{xx}}{p^2} - p^2$. So, the system

$$\begin{aligned} u_{xx} &= u^m u_t + f(u),\\ v_{xx} &= vv_t + \lambda v + g(u) \end{aligned} \tag{2.56}$$

admits the Q-conditional symmetry operator

$$Q = \partial_t + p^2(x)\partial_v, \tag{2.57}$$

where the function $p^2(x)$ is the general solution of the nonlinear ODE

$$p^2_{xx} = (p^2)^2 + \lambda p^2. \tag{2.58}$$

Applying now substitution (2.44) to (2.56)–(2.57) and introducing the relevant notations, one arrives at the system and the Q-conditional symmetry operator listed in case 1 of Table 2.2.

Considering the subcase $n \neq 0; 1$, we immediately obtain $p^2 = \lambda = \text{const}$ (see (2.53)) and this leads to the system

$$\begin{aligned} u_{xx} &= u^m u_t + f(u), \\ v_{xx} &= v^n v_t - \lambda v^n + g(u) \end{aligned} \tag{2.59}$$

and the operator

$$Q = \partial_t + \lambda \partial_v. \tag{2.60}$$

Operator (2.60) is reduced to the form $Q = \partial_t + \frac{\lambda}{l+1} V^{-l} \partial_V$ with $\lambda \neq 0$ by using substitution (2.44). On the other hand, system (2.59) and operator (2.60) correspond to a particular case at $\lambda_1 = \alpha = 0$ of those listed in case 2 of Table 2.2. Thus, case ***(2)*** is completely investigated.

Case ***(4)*** can be examined in a quite similar way and the system

$$\begin{aligned} u_{xx} &= u^m u_t - \alpha\lambda u^m + f(u - \alpha v), \\ v_{xx} &= v^n v_t - \lambda v^n + g(u - \alpha v) \end{aligned} \tag{2.61}$$

and the operator

$$Q = \partial_t + \lambda(\alpha \partial_u + \partial_v) \tag{2.62}$$

are obtained, where $\alpha \neq 0$ is an arbitrary constant. It is easily seen that systems and operators (2.59)–(2.62) can be united, i.e., the restriction $\alpha \neq 0$ is not essential. Applying now substitution (2.44) to (2.61)–(2.62) and introducing the relevant notations, one arrives at the system and the Q-conditional symmetry operator listed in case 2 of Table 2.2. It turns out that the power $m = n = 1$ leads to an additional symmetry in this case. In fact, the system

$$\begin{aligned} u_{xx} &= u u_t + \lambda u + f(u - v), \\ v_{xx} &= v v_t + \lambda v + g(u - v) \end{aligned} \tag{2.63}$$

is conditionally invariant w.r.t. the operator

$$Q = \partial_t + p^2(x)(\partial_u + \partial_v), \tag{2.64}$$

where $p^2(x)$ is the general solution of the nonlinear ODE (2.58). Formulae (2.63)–(2.64) together with the substitution (2.44) generate the system and the operator listed in case 3 of Table 2.2. Those listed in cases 4 and 5 of the table can be similarly obtained by examination of cases ***(5)*** and ***(6)***.

Finally, we note that all operators arising in Table 2.2 are not Lie symmetry operators because any Lie symmetry operator of RDS (2.43) must be linear on U and V [23].

The proof is now complete. □

Remark 2.2 Restrictions on the coefficients λ_k arising in the last column of Table 2.2 guarantee that the relevant operators do not coincide with Lie symmetries. Of course, those operators are still Q-conditional symmetry operators if some λ_k vanish, however, they are equivalent to the relevant Lie symmetry operators obtained in [23].

Case ***(c)*** is the most complicated. In order to examine this case, one needs to consider separately three subcases

$$\textbf{\textit{(c1)}}\ p^1_x = 0,\ p^2_x \neq 0,$$
$$\textbf{\textit{(c2)}}\ p^1_x \neq 0,\ p^2_x \neq 0,$$
$$\textbf{\textit{(c3)}}\ p^1_x = p^2_x = 0.$$

We note that the subcase $p^2_x = 0,\ p^1_x \neq 0$ is reduced to ***(c1)***.

A complete analysis of subcase ***(c1)*** is done in [50] and the result can be formulated as follows.

Theorem 2.8 *RDS (2.43) with* $k^2 + l^2 \neq 0$ *and* $(k+1)(l+1) \neq 0$ *is Q-conditional invariant under the operator (2.42) with* $p^1_x = 0,\ p^2_x \neq 0$ *iff it and the relevant operator have 16 forms listed below (*$F(U)$ *is an arbitrary smooth function,* $\alpha,\ \beta,\ \gamma$ *and* λ_j $(j = 1, 2, 3, 4)$ *are arbitrary constants).*

$$U_t = (U^k U_x)_x + F^1,$$
$$V_t = V_{xx} + F^2,\ k \neq 0.$$

1. $F^1 = \lambda_1 U^{k+1} + \lambda_2,\ F^2 = \lambda_3 V + F(U),$
   $Q = \partial_t + \big(\gamma \exp\left((-\lambda_1(k+1) + \lambda_3)\, t\right) U^{k+1} + p\big)\, \partial_V,$
   $p_t = p_{xx} + \lambda_3 p + \gamma \lambda_2 \exp\left((-\lambda_1(k+1) + \lambda_3)\, t\right),\ \gamma \neq 0.$
2. $F^1 = F(U),\ F^2 = \lambda_3 V - \lambda_2(k+1)F(U) + \lambda_2\lambda_3 U^{k+1},$
   $Q = \partial_t + \big(\gamma \left(\lambda_2 U^{k+1} + V\right) + p\big)\, \partial_V,$
   $p_t = p_{xx} + \lambda_3 p,\ \gamma\lambda_2 \neq 0.$
3. $F^1 = \lambda_1 + \lambda_2 U^{-k},\ F^2 = \lambda_3 V + \lambda_4 U^{2(k+1)} + \lambda_5 U^{k+1},$
   $Q = \partial_t + \lambda_2 U^{-k}\partial_U + \left(\left(\gamma e^{\lambda_3 t} - \frac{2\lambda_2\lambda_4(k+1)}{\lambda_3}\right) U^{k+1} + p\right) \partial_V,$
   $p_t = p_{xx} + \lambda_3 p - (k+1)\left((\lambda_1 + \lambda_2)\left(\gamma e^{\lambda_3 t} + \frac{2\lambda_2(k+1)\lambda_4}{\lambda_3}\right) + \lambda_2\lambda_5\right),$
   $\lambda_2\lambda_3 \neq 0.$

4. $F^1 = \lambda_1 + \lambda_2 U^{-k},\ F^2 = \lambda_3 U^{2(k+1)} + \lambda_4 U^{k+1},$
   $Q = \partial_t + \lambda_2 U^{-k}\partial_U + \Big((2\lambda_2\lambda_3(k+1)t + \gamma)\, U^{k+1} + p\Big)\partial_V,$
   $p_t = p_{xx} - (2\lambda_2\lambda_3 t + \gamma)(\lambda_2 + \lambda_1)(k+1) + \lambda_2\lambda_4, \lambda_2 \neq 0.$
5. $F^1 = \lambda_1 + \lambda_2 U^{-k},\ F^2 = \lambda_3 V + \lambda_4 U^{k+1} + \lambda_5 \exp\left(\alpha U^{k+1}\right),$
   $Q = \partial_t + \lambda_2 U^{-k}\partial_U + \left(\left(\gamma e^{\lambda_3 t} + \frac{\alpha\lambda_2\lambda_4(k+1)}{\lambda_3}\right) U^{k+1} + \alpha\lambda_2(k+1)V + p\right)\partial_V,$
   $p_t = p_{xx} - \lambda_3 p - (\lambda_1 + \lambda_2)(k+1)\left(\gamma e^{\lambda_3 t} + \frac{\alpha\lambda_2\lambda_4(k+1)}{\lambda_3}\right) + \lambda_2\lambda_4(k+1),$
   $\alpha\lambda_2\lambda_3 \neq 0.$
6. $F^1 = \lambda_1 + \lambda_2 U^{-k},\ F^2 = \lambda_3 U^{k+1} + \lambda_4 \exp\left(\alpha U^{k+1}\right),$
   $Q = \partial_t + \lambda_2 U^{-k}\partial_U + \Big((\gamma - \alpha\lambda_2\lambda_3(k+1)t)\, U^{k+1} + \alpha\lambda_2(k+1)V + p\Big)\partial_V,$
   $p_t = p_{xx} - (\lambda_1 + \lambda_2)(k+1)(\gamma - \alpha\lambda_2\lambda_3 t) + \lambda_2\lambda_3(k+1),\ \alpha\lambda_2 \neq 0.$
7. $F^1 = \left(\lambda_1 - \frac{\lambda_2}{k+1}\right) U^{k+1} + \lambda_2 U,$
   $F^2 = \lambda_1(k+1)V - \lambda_3(k+1)U + \lambda_3 U^{k+1},$
   $Q = \partial_t + \frac{\lambda_2}{1-\alpha\exp(\lambda_2 kt)} U\partial_U$
   $+ \left(\frac{\lambda_3((\lambda_2-\beta)\exp(\lambda_2 kt) + \alpha\beta)}{\lambda_2\alpha(k+1)} U^{k+1} + \beta V + p\right)\partial_V,$
   $p_t = p_{xx} + \lambda_1(k+1)p,\ \lambda_2\alpha \neq 0.$
8. $F^1 = (\lambda_1 - \lambda_2)U^{k+1} + \lambda_2(k+1)U,$
   $F^2 = \lambda_1(k+1)V - \lambda_3(k+1)U + \lambda_3 U^{k+1} + \lambda_4(k+1)\ln U,$
   $Q = \partial_t + \frac{\lambda_2(k+1)}{1-\gamma\exp(\lambda_2 k(k+1)t)} U\partial_U + \left(\lambda_3\gamma^{-1}\exp(-\lambda_2 k(k+1)t)U^{k+1} + p\right)\partial_V,$
   $p_t = p_{xx} + \lambda_1(k+1)p - \frac{\lambda_2\lambda_4(k+1)^2}{1-\gamma\exp(\lambda_2 k(k+1)t)},\ \lambda_2\gamma \neq 0.$
9. $F^1 = \left(U^{k+1} + \alpha\right)\left(\lambda_1 + \lambda_2 U^{-k}\right),$
   $F^2 = (\lambda_1 + \lambda_2)(k+1)V + \lambda_3\left(U^{k+1} + \alpha\right)^\beta + \lambda_4\left(U^{k+1} + \alpha\right),$
   $Q = \partial_t + \lambda_2\left(U + \alpha U^{-k}\right)\partial_U$
   $+\Big((\gamma - \lambda_2\lambda_4(\beta - 1)(k+1)t)\, U^{k+1} + \beta\lambda_2(k+1)V + p\Big)\partial_V,$
   $p_t = p_{xx} + (\lambda_1 + \lambda_2)(k+1)p - \alpha\lambda_2\lambda_4(k+1)(\beta - 1)$
   $-\alpha(\lambda_1 + \lambda_2)(k+1)(\lambda_2\lambda_4(k+1)(\beta - 1)t + \gamma),\ \lambda_2\beta(\beta - 1) \neq 0.$
10. $F^1 = \left(U^{k+1} + \alpha\right)\left(\lambda_1 + \lambda_2 U^{-k}\right),$
   $F^2 = \lambda_3 V + \lambda_4\left(U^{k+1} + \alpha\right)^\beta + \lambda_5\left(U^{k+1} + \alpha\right),$
   $Q = \partial_t + \lambda_2(U + \alpha U^{-k})\partial_U + \Big(\Big(\gamma\exp((\lambda_1 - \lambda_2(k+1) + \lambda_3)t)$
   $+ \frac{\lambda_2\lambda_5(\beta-1)(k+1)}{\lambda_1-\lambda_2(k+1)+\lambda_3}\Big)U^{k+1} + \beta\lambda_2(k+1)V + p\Big)\partial_V,$
   $p_t = p_{xx} + \lambda_3 p + \alpha\Big((\lambda_1 - \lambda_2(k+1))\,\gamma\exp\left((\lambda_1 - \lambda_2(k+1) + \lambda_3)t\right)$
   $- \frac{\lambda_2\lambda_3\lambda_5(\beta-1)(k+1)}{\lambda_1-\lambda_2(k+1)+\lambda_3}\Big),\ \lambda_2(\lambda_1 - \lambda_2(k+1) + \lambda_3)\beta(\beta - 1) \neq 0.$
11. $F^1 = \left(U^{k+1} + \alpha\right)\left(\lambda_1 + \lambda_2 U^{-k}\right),$
   $F^2 = \lambda_3 V + \left(U^{k+1} + \alpha\right)\left(\lambda_4\ln(U^{k+1} + \alpha) + \lambda_5\right),$
   $Q = \partial_t + \lambda_2\left(U + \alpha U^{-k}\right)\partial_U + \Big(\Big(\gamma\exp\left((\lambda_1 - \lambda_2(k+1) + \lambda_3)t\right)$
   $- \frac{\lambda_2\lambda_4(k+1)}{\lambda_1-\lambda_2(k+1)+\lambda_3}\Big)U^{k+1} + \lambda_2(k+1)V + p\Big)\partial_V,$

$$p_t = p_{xx} + \lambda_3 p - \alpha(\lambda_1 + \lambda_2)(k+1)\gamma \exp\left((\lambda_1 - \lambda_2(k+1) + \lambda_3)t\right)$$
$$+\tfrac{\alpha\lambda_2\lambda_3\lambda_4(k+1)}{\lambda_1-\lambda_2(k+1)-\lambda_3},\ \lambda_1 - \lambda_2(k+1) + \lambda_3 \neq 0.$$

12. $F^1 = \left(U^{k+1} + \alpha\right)\left(\lambda_1 + \lambda_2 U^{-k}\right),$
$$F^2 = (\lambda_1 + \lambda_2)(k+1)V + \left(U^{k+1} + \alpha\right)\left(\lambda_3 \ln(U^{k+1} + \alpha) + \lambda_4\right),$$
$$Q = \partial_t + \lambda_2(U + \alpha U^{-k})\partial_U + \Big((\lambda_2\lambda_3(k+1)t + \gamma)\, U^{k+1}$$
$$+\lambda_2(k+1)V + p\Big)\partial_V,$$
$$p_t = p_{xx} + (\lambda_1 + \lambda_2)(k+1)\Big(p - \alpha(\lambda_2\lambda_3(k+1)t + \gamma)\Big) + \alpha\lambda_2\lambda_3(k+1).$$

13. $F^1 = \left(U^{k+1} + \alpha\right)\left(\lambda_1 + \lambda_2 U^{-k}\right),$
$$F^2 = (\lambda_1 + \lambda_2)(k+1)V + \lambda_3 \ln\left(U^{k+1} + \alpha\right) + \lambda_4\left(U^{k+1} + \alpha\right),$$
$$Q = \partial_t + \lambda_2(U + \alpha U^{-k})\partial_U + \Big((\lambda_2\lambda_4(k+1)t + \gamma)\, U^{k+1} + p\Big)\partial_V,$$
$$p_t = p_{xx} + (k+1)\Big((\lambda_1 + \lambda_2)\,(p - \alpha(\lambda_2\lambda_4(k+1)t + \gamma))$$
$$+\lambda_2(\alpha\lambda_4 + \lambda_3)\Big).$$

14. $F^1 = \left(U^{k+1} + \alpha\right)\left(\lambda_1 + \lambda_2 U^{-k}\right),$
$$F^2 = \lambda_3 V + \lambda_4 \ln\left(U^{k+1} + \alpha\right) + \lambda_5\left(U^{k+1} + \alpha\right),$$
$$Q = \partial_t + \lambda_2\left(U + \alpha U^{-k}\right)\partial_U$$
$$+\left(\left(\gamma \exp\left((\lambda_3 - (\lambda_1 + \lambda_2)(k+1))t\right) + \tfrac{\lambda_2\lambda_5}{(\lambda_1+\lambda_2)(k+1)-\lambda_3}\right)U^{k+1} + p\right)\partial_V,$$
$$p_t = p_{xx} + \lambda_3 p - \alpha(\lambda_1 + \lambda_2)(k+1)\gamma \exp\left((\lambda_3 - (\lambda_1 + \lambda_2)(k+1))\right)$$
$$+\lambda_2\left(\tfrac{\alpha\lambda_3\lambda_5}{\lambda_3-(\lambda_1+\lambda_2)(k+1)} + \lambda_4\right),\ \lambda_3 - (\lambda_1 + \lambda_2)(k+1) \neq 0.$$

15. $F^1 = \lambda_1 U + \lambda_2 U^{-k} + \lambda_3 U^{-k}V - \lambda_3 V + \lambda_4 U^{k+1} + \lambda_5,$
$$F^2 = (\lambda_4 - \lambda_1)(k+1)V,$$
$$Q = \partial_t + \left(\lambda_3 V - \lambda_1 U^{k+1} + \lambda_2\right)U^{-k}\partial_U + \left(\alpha U^{k+1} + \beta V + p\right)\partial_V,$$
$$p_t = p_{xx} + (k+1)\left((\lambda_4 - \lambda_1)p - \alpha(\lambda_5 + \lambda_2)\right),\ \lambda_3 \neq 0.$$

$$U_t = (U^k U_x)_x + F^1,$$
$$V_t = (V^{-\frac{1}{2}} V_x)_x + F^2.$$

16. $F^1 = \lambda_1 V - 2\lambda_1\left(V^{\frac{1}{2}} + \alpha\right)U^{-k} + \lambda_3 U^{k+1} - 2\lambda_1\lambda_2 U + \lambda_4,$
$$F^2 = 2\lambda_3(k+1)\left(V^{\frac{1}{2}} + \alpha\right) - 2\lambda_2(k+1)(\lambda_1 V + \lambda_4),$$
$$Q = \partial_t - 2\lambda_1\left(V^{\frac{1}{2}} + \lambda_2 U^{k+1} + \alpha\right)U^{-k}\partial_U + 2\left(p(x) - \lambda_1\lambda_2(k+1)V^{\frac{1}{2}}\right)V^{\frac{1}{2}}\partial_V,$$
$$p'' = p^2 - \lambda_3(k+1)p + \lambda_1\lambda_2(k+1)^2(\lambda_2\lambda_4 - \alpha\lambda_3),\ \lambda_1 \neq 0.$$

The detailed proof of Theorem 2.8 is based on the construction of the general solution of the system of DEs (2.46) provided the given restrictions on the operator Q take place. We omit here the relevant cumbersome calculations.

Remark 2.3 Each Q-conditional symmetry presented in cases 1–15 involves the function $p(t,x)$, which is an arbitrary solution of the linear diffusion equation $p_t = p_{xx} + pR_1(t,x) + R_0(t,x)$, where $R_1(t,x)$ and $R_0(t,x)$ are the correctly specified functions. The Q-conditional symmetry arising in case 16 contains the function $p(x)$, which is an arbitrary solution of the integrable ODE.

Remark 2.4 In case 16, we have corrected inexactnesses arising in [50].

Remark 2.5 Each system arising in Theorem 2.8 is semi-coupled, i.e., contains autonomous equations. It is unlikely that such systems can reflect any general physical or biological laws. However, they may be governing equations for some specific models describing real-world processes. For example, case 1 with $\lambda_1 = \lambda_2 = \lambda_3 = F(U) = 0$ generates a system of two autonomous diffusion equations admitting the Q-conditional symmetry $Q = \partial_t + (\gamma U^{k+1} + p)\partial_V$. These are governing equations for the classical Stefan type problem modelling melting and evaporation of metals (see, e.g., [1, 14]).

In contrast to ***(c1)***, examination of case ***(c2)*** leads to a trivial result: RDS (2.43) is not invariant under the operator (2.42) provided $p^1_x \neq 0$ and $p^2_x \neq 0$. The proof of this statement can be derived by solving the system of DEs (2.46) under the restrictions

$$q^1(t,x)^2 + q^2(t,x)^2 \neq 0,\ \xi = 0,\ p^1_x \neq 0,\ p^2_x \neq 0.$$

Finally, examination of case ***(c3)*** leads to the system

$$\begin{array}{l}(q^1v + r^1u + p^1)C^1_u + (q^2u + r^2v + p^2)C^1_v - r^1C^1 - q^1C^2 \\ \quad +(q^1(q^2u + r^2v + p^2) + q^1_tv + r^1_tu + p^1_t)u^m \\ \quad +m(q^1v + r^1u + p^1)^2u^{m-1} - q^1(q^2u + r^2v + p^2)v^n = 0, \\ (q^1v + r^1u + p^1)C^2_u + (q^2u + r^2v + p^2)C^2_v - r^2C^2 - q^2C^1 \\ \quad +(q^2(q^1v + r^1u + p^1) + q^2_tu + r^2_tv + p^2_t)v^n \\ \quad +n(q^2u + r^2v + p^2)^2v^{n-1} - q^2(q^1v + r^1u + p^1)u^m = 0\end{array} \tag{2.65}$$

(here $q^i = q^i(t),\ r^i = r^i(t),\ p^i = p^i(t),\ i = 1,2$), which should be solved w.r.t. the functions $C^i = C^i(u,v),\ i = 1,2$. Although system (2.65) is linear, the algorithm for the construction of its general solution is cumbersome because the functions $q^i,\ r^i$ and p^i $(i = 1,2)$ are not specified. Of course, the general solution can be easily constructed in the case of correctly specified coefficients. For example, if $q^1 = 0$, then the first equation is autonomous and system (2.65) can be solved in a similar way as it was done in [51].

As examples, we examined the systems

$$\begin{aligned} uu_t &= u_{xx} + u(a_1 + b_1u + c_1v), \\ vv_t &= v_{xx} + v(a_2 + b_2u + c_2v) \end{aligned} \tag{2.66}$$

and

$$\begin{aligned} uu_t &= u_{xx} + u(a_1 - b_1u - c_1v) + rv, \\ vv_t &= v_{xx} - v(a_2 + b_2u). \end{aligned} \tag{2.67}$$

Obviously, systems (2.66) and (2.67) are natural generalizations of DLVS (2.2) and of the Belousov–Zhabotinskii system (2.3), respectively. It turns out that both systems admit Q-conditional symmetry operators, which can be constructed using particular solutions of system (2.65) with $m = n = 1$ and the functions C^k $(k = 1, 2)$ taken from (2.66) and (2.67). Finally, we conclude that the generalized DLVS (2.66) admits Q-conditional symmetry of the form

$$Q = \partial_t + (a_1 + b_1 u + c_1 v)\partial_u + (a_2 + b_2 u + c_2 v)\partial_v$$

(the restriction $c_1^2 + b_2^2 \neq 0$ guarantees that it is a non-Lie symmetry) and the generalized Belousov–Zhabotinskii system (2.67) with

$$c_1 = r(2b_1 - b_2)(r + a_1 + a_2)(a_1 + a_2)^{-2} \text{ and } a_1 + a_2 \neq 0$$

is conditionally invariant w.r.t. the operator

$$Q = (a_1 + a_2)^2\partial_t - r^2(2b_1 - b_2)v\partial_u + r(a_1 + a_2)(2b_1 - b_2)v\partial_v$$

(the restriction $r(2b_1 - b_2) \neq 0$ guarantees that it is a non-Lie symmetry).

2.6 Concluding Remarks

A novel way to find new type of symmetries for PDEs was proposed in 1969 [8]. In the same paper, the idea was realized in the form of an algorithm for finding new symmetries of the linear heat (diffusion) equation. Although the algorithm is based on the classical Lie scheme [46, 49], the resulting symmetries can be non-Lie symmetries of the equation in question, therefore they were called nonclassical symmetries. Following [27, 35], we call them Q-conditional symmetries in order to distinguish from other types of symmetries (weak symmetry [47, 48, 52], conditional symmetry [20, 34, 35], generalized conditional symmetry [30, 59]) because each non-Lie symmetry can be called nonclassical. From the applicability point of view, the algorithm for finding Q-conditional symmetry of a given PDE is highly nontrivial (each time a nonlinear system of PDEs must be integrated) and this was a reason why nontrivial examples of Q-conditional symmetries were not found for a long time. In 1987 the Bluman–Cole algorithm was rediscovered in [33, 47] and later successfully applied to a wide range of nonlinear PDEs (see, e.g., [9, 26] and the references therein), especially reaction-diffusion-convection equations (see Chap. 1 for details).

It turns out that the problem of finding Q-conditional symmetry becomes much more complicated in the case of (multi-component) nonlinear systems of PDEs. To the best of our knowledge, there are very few papers devoted to the search for Q-conditional (nonclassical) symmetries of systems of evolution equations, which were published before 2005 [5, 28, 38]. A majority of such papers were published during the last 10 years [4, 13, 15–19, 25, 40, 55].

In this chapter, the recently developed theoretical background for searching for Q-conditional symmetries of evolution systems of PDEs is presented. We generalize the standard notation of Q-conditional (nonclassical) symmetry by introducing the notion of Q-conditional symmetry of the p-th type and show that different types of Q-conditional symmetry of a given system generate a hierarchy of conditional symmetry operators. It is shown that Q-conditional symmetry of the p-th type possesses some properties, which distinguish it from nonclassical symmetry.

The class of two-component nonlinear RDSs (2.1) is examined in order to find Q-conditional symmetry operators. The relevant system of DEs was derived and solved under additional restrictions, so that several RDSs of the form (2.1) possessing conditional symmetry were obtained. In particular, it was shown that the DLVS and the Belousov–Zhabotinskii system (with correctly specified coefficients) and some of their generalizations admit Q-conditional symmetry operators.

Finally, it is worth highlighting the following remarks about Q-conditional symmetry of the class of RDSs (2.1).

1. The system of DEs (2.21) is very complicated and we believe that its general solution cannot be derived without additional restrictions on the symmetry operators in question. In the case of scalar reaction-diffusion equations (RDEs), the relevant system of DEs is essentially simpler, hence its general solution can be constructed (see Chap. 1).
2. Particular solutions of system (2.21) usually lead to RDSs involving arbitrary function(s) in reaction terms (see Tables 2.1 and 2.2). Scalar RDEs with Q-conditional symmetry do not involve any arbitrary functions, but arbitrary constants only (see Chap. 1).
3. The definition of Q-conditional symmetry of the first type should be applied in order to obtain a system of DEs, which can be integrated without any restrictions (in contrast to (2.21)). Thus, a complete classification of such symmetries could be derived (see Chap. 4).

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