

Preface

Nowadays it is widely known that nonlinear reaction-diffusion systems (RDSs) are governing equations for many very important nonlinear models used to describe various processes in physics, biology, ecology, chemistry, etc. Since the 1970s they have been studied extensively by means of different mathematical methods. Construction of particular exact solutions for nonlinear partial differential equations (PDEs), especially for nonlinear RDSs, is an important problem. Finding exact solutions that have a physical, chemical or biological interpretation is of fundamental importance. The well-known principle of linear superposition cannot be applied to generate new exact solutions of nonlinear PDEs and systems of PDEs. Thus, the classical methods are not applicable for solving these equations. Of course, a change of variables can sometimes be found that transforms a given nonlinear PDE into a linear equation, but finding exact solutions of most nonlinear PDEs generally requires new methods.

The most powerful methods for construction of exact solutions of nonlinear PDEs are symmetry-based methods. These methods originate from the Lie method, which was created by the prominent Norwegian mathematician Sophus Lie at the end of the nineteenth century. The method was essentially developed using modern mathematical language by L.V. Ovsiannikov, G. Bluman, N. Ibragimov, W.F. Ames and some other researchers in the 1960s and 1970s. Although the technique of the Lie method is well known, the method still attracts the attention of researchers and new results are published on a regular basis.

However, it is well known that some nonlinear PDEs and systems of PDEs arising in applications have poor Lie symmetry. For example, the Fisher and Fitzhugh–Nagumo equations, which are widely used in mathematical biology, are invariant only under time and space translations. The Lie method is not efficient for such equations since it enables only those exact solutions to be constructed, which can be easily obtained without using this method. Taking this fact into account, other symmetry-based methods were developed. The best known among them is the method of nonclassical symmetries proposed by G. Bluman and J. Cole in 1969. Even though this approach was suggested almost 50 years ago, its successful applications for solving nonlinear equations were accomplished only in the 1990s

owing to D.J. Arrigo, P. Broadbridge, P. Clarkson, J.M. Hill, E.L. Mansfield, M.C. Nucci, P. Olver, E. Pucci, G. Saccomandi, E.M. Vorob'ev, P. Winternitz and others. A prominent role in applications and further development of the nonclassical symmetry method belongs to the Ukrainian school of symmetry analysis, which was created in the early 1980s and led by W.I. Fushchych (V.I. Fyshchich) until 1997 when he passed away. In particular, a concept of conditional symmetry was worked out and its applications to a wide range of nonlinear PDEs were realized by M. Serov, I. Tsyfra, R. Zhdanov, R. Popovych, R. Cherniha and others. Notably, following Fushchych's proposal dating back to 1988, we use the terminology ' Q -conditional symmetry' instead of 'nonclassical symmetry'.

It turns out that the problem of finding Q -conditional symmetry gets to be much more complicated in the case of the two- and multi-component nonlinear systems of PDEs. To the best of our knowledge, the pioneering papers devoted to the search for Q -conditional symmetries of systems of reaction-diffusion equations appeared only in the early 2000s, i.e., about 30 years later than the seminal Bluman and Cole work was published. Moreover, the majority of such papers were published during the last decade.

This book is devoted to searching for conditional symmetries of nonlinear RDSs and their application for constructing exact solutions. Properties of exact solutions obtained for several nonlinear systems (especially the diffusive Lotka–Volterra system (DLVS)) arising in real-world applications are studied in order to provide their biological, ecological and/or physical interpretation. The book is mostly based on authors' papers published during the last 10 years. Notably, several misprints and inexactnesses arising in those papers were corrected during the book preparation. To the best of our knowledge, this is the first monograph devoted to search and application of conditional symmetries in the case of systems of nonlinear PDEs (not scalar PDEs!). The reader does not need to study symmetry-based methods in detail in order to use the exact solutions obtained in an explicit form and to construct new solutions using conditional symmetry operators derived in the book. Each chapter contains both ideas for further theoretical generalizations and examples of real-world applications.

In Chap. 1, all the main results on Q -conditional symmetry (nonclassical symmetry) of the general class of nonlinear reaction-diffusion-convection equations are summarized. Although some of them were published about 25 years ago, and others were derived in the 2000s, this is the first attempt to present an extensive review of this matter. It is shown that several well-known equations arising in applications and their direct generalizations possess conditional symmetry. Notably, the Murray, Fitzhugh–Nagumo, and Huxley equations and their natural generalizations are identified. Moreover, several exact solutions (including travelling fronts) are constructed using the conditional symmetries obtained in order to find exact solutions with a biological interpretation.

In Chap. 2, the recently developed theoretical background for searching Q -conditional symmetries of systems of evolution PDEs is presented. We generalize the standard notation of Q -conditional symmetry by introducing the notion of Q -conditional symmetry of the p -th type and show that different types of symmetry

of a given system generate a hierarchy of conditional symmetry operators. It is shown that Q -conditional symmetry of the p -th type possesses some special properties, which distinguish it from the standard conditional symmetry. The general class of two-component nonlinear RDSs is examined in order to find the Q -conditional symmetry operators. The relevant systems of so-called determining equations are solved under additional restrictions. As a result, several RDSs possessing conditional symmetry are constructed. In particular, it is shown that DLVS, the Belousov–Zhabotinskii system (with the correctly specified coefficients) and some of their generalizations admit Q -conditional symmetry.

In Chap. 3, two- and three-component DLVSs are examined in order to find Q -conditional symmetries, to construct exact solutions and to provide their biological interpretation. A complete description of Q -conditional symmetries of the first type (a special subset of nonclassical symmetries) of these nonlinear systems is derived. An essential part of this chapter is devoted to the construction of exact solutions of the systems in question using the symmetries obtained. Starting from examples of travelling fronts (finding such solutions is important from the applicability point of view), we concentrate mostly on finding exact solutions with a more complicated structure. As a result, a wide range of such exact solutions are constructed for the two-component DLVS and some examples are presented for the three-component DLVS. Moreover, a realistic interpretation for two and three competing species is provided for some exact solutions.

In Chap. 4, two classes of two-component nonlinear RDSs are studied in order to find Q -conditional symmetries of the first type, to construct exact solutions and to show their applicability. The first class involves systems with the constant coefficient of diffusivity, while the second one contains systems with variable diffusivities only. The main theoretical results are given in the form of two theorems presenting exhaustive lists (up to the given sets of point transformations) of the RDSs belonging to the above classes and admitting Q -conditional symmetries of the first type. The systems obtained allow us to extract specific systems occurring in real-world models. A few examples are presented, including a modification of the classical prey–predator system with diffusivity and a system modelling the gravity-driven flow of thin films of viscous fluid. Exact solutions with attractive properties are found for these nonlinear systems and their possible biological and physical interpretations are presented.

The book is a monograph. Its academic level is suitable for graduate students and higher. Some parts of the book may be used in ‘Mathematical Biology’ and ‘Nonlinear Partial Differential Equations’ courses for master students and in the final year of undergraduate studies. Nowadays such courses are common in all leading universities all over the world.

The book was typeset in LaTeX using the Springer templates; the figures were drawn using the computer algebra package MAPLE.

The authors thank their Ukrainian colleagues for fruitful discussions, valuable critique and helpful suggestions, which helped us to write this modest work. We are especially grateful to Sergii Kovalenko, Anatoly Nikitin, Oleksii Pliukhin and Mykola Serov.

R. Ch. is indebted to John R. King for valuable discussions in 2013–2015 when he was Marie Curie Fellow at the University of Nottingham. R. Ch. is also grateful to George Bluman, Phil Broadbridge, Malte Henkel, Changzheng Qu, and Jacek Waniewski for fruitful discussions and valuable comments, which inspired us to do further research and generated new ideas.

Last but not least, we are grateful to Eva Hiripi, who invited us to write this book, and for her continuous support during the preparation of the monograph.

Finally, we express our thanks to our mothers, Stefaniya and Valentyna, for their love and support! R. Ch. reaffirms his love and thankfulness to his wife, Nataliya, for her love, patience and understanding throughout the process of writing the book.

Kyiv, Ukraine
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May 2017

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Nonlinear Reaction-Diffusion Systems

Conditional Symmetry, Exact Solutions and their
Applications in Biology

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2017, XIII, 160 p. 13 illus., 10 illus. in color., Softcover

ISBN: 978-3-319-65465-2