Chapter 3
Nearly Pseudo-Kähler and Nearly Para-Kähler Manifolds

3.1 Nearly Pseudo-Kähler and Nearly Para-Kähler Manifolds

3.1.1 General Properties

In this subsection we collect some information on almost ε-Hermitian manifolds with a special emphasis on the nearly ε-Kähler case.

Definition 3.1.1 An almost ε-Hermitian manifold $(M^{2m}, g, J^\varepsilon, \omega)$ is called nearly ε-Kähler manifold, provided that its Levi-Civita connection ∇ satisfies the nearly ε-Kähler condition

$$(\nabla_X J^\varepsilon) X = 0, \quad \forall X \in TM.$$

A nearly ε-Kähler manifold is called strict if $\nabla_X J^\varepsilon \neq 0$ for all non-trivial vector fields X.
A tensor field $B \in \Gamma((TM^*)^{\otimes 2} \otimes TM)$ on a pseudo-Riemannian manifold (M, g) is called (totally) skew-symmetric if the tensor $g(B(X, Y), Z)$ is a three-form. The following characterisation of a nearly ε-Kähler manifold is well-known in the Riemannian context and we refer to Proposition 3.2 of [110] for the complete proof in the pseudo-Riemannian setting.

Proposition 3.1.2 *An almost ε-Hermitian manifold $(M^{2m}, g, J^\varepsilon, \omega)$ satisfies the nearly ε-Kähler condition if and only if $d\omega$ is of real type $(3,0) + (0,3)$ and the Nijenhuis tensor is totally skew-symmetric.*

Remark 3.1.3 The notion of nearly ε-Kähler manifold corresponds to the generalised Gray-Hervella class $\mathcal{W}_1$ in [86]. However, in the para-Hermitian case, there

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are two subclasses, see [59]. Indeed, we already observed that

$$\mathcal{A} = -\nabla\omega \in [\![\Omega^{3,0}]\!] \overset{(2.36)}{=} \Gamma(\Lambda^3\mathcal{V}^* \oplus \Lambda^3\mathcal{H}^*)$$

for a nearly para-Kähler manifold.

Definition 3.1.4 A connection $\bar{\nabla}$ on an ε-Hermitian manifold $(M^{2m}, g, J^\varepsilon, \omega)$ is called ε-Hermitian provided, that it satisfies $\bar{\nabla}g = 0$ and $\bar{\nabla}J^\varepsilon = 0$.
The Riemannian case of the next result is due to [57], the para-complex case is shown in [78]. In fact, the sketched proof in [57] holds literally for the almost pseudo-Hermitian case with indefinite signature as well. A direct and simultaneous proof of all cases can be found in Proposition 3.4 of [110].

Theorem 3.1.5 *An ε-Hermitian manifold $(M^{2m}, g, J^\varepsilon, \omega)$ admits an ε-Hermitian connection with totally skew-symmetric torsion if and only if the Nijenhuis tensor is totally skew-symmetric. If this is the case, the connection $\bar{\nabla}$ and its torsion T are uniquely defined by*

$$g(\bar{\nabla}_X Y, Z) = g(\nabla_X Y, Z) + \frac{1}{2} g(T(X,Y), Z),$$
$$g(T(X,Y), Z) = \varepsilon g(N(X,Y), Z) - d\omega(J^\varepsilon X, J^\varepsilon Y, J^\varepsilon Z),$$

and we call $\bar{\nabla}$ the characteristic ε-Hermitian connection (with skew-symmetric torsion).
This connection can be seen as a natural generalisation of the Chern- or Bismut-connection. Another name for the characteristic connection is canonical connection.

Remark 3.1.6 An almost Hermitian manifold is said to be of type $\mathcal{G}_1$ if it admits a Hermitian connection with skew-symmetric torsion. In terms of the Gray-Hervella list [68], this means, that it is of type $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, i.e. the missing part is the almost Kähler component $\mathcal{W}_2$.

More generally, the proposition justifies to say that an almost ε-Hermitian manifold is of type $\mathcal{G}_1$ if it admits an ε-Hermitian connection with skew-symmetric torsion.

In particular, the proposition applies to nearly ε-Kähler manifolds $(M, g, J^\varepsilon, \omega)$. In fact, comparing the identities (2.35) and (2.41), we see that the real three-form $\mathcal{A}$ is of type $(3,0) + (0,3)$. Since $d\omega$ is the alternation of $\nabla\omega$, we have

$$d\omega = 3\nabla\omega = -3\mathcal{A} \in [\![\Omega^{3,0}]\!], \tag{3.1}$$

where $\mathcal{A}$ is defined in (2.39). Furthermore, if we apply the nearly ε-Kähler condition to the expression (2.37), the Nijenhuis tensor of a nearly ε-Kähler structure simplifies to

$$N(X,Y) = 4J^\varepsilon(\nabla_X J^\varepsilon)Y. \tag{3.2}$$

We conclude that the Nijenhuis tensor is skew-symmetric since

$$g(N(X,Y),Z) = -4\mathcal{A}(X,Y,J^\varepsilon Z) \overset{(2.41)}{=} -4\varepsilon J^{\varepsilon *}\mathcal{A}(X,Y,Z). \tag{3.3}$$

Explicitly the connection $\bar{\nabla}$ is then given by

$$\bar{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\varepsilon J^\varepsilon(\nabla_X J^\varepsilon)Y, \text{ for } X, Y \in \Gamma(TM). \tag{3.4}$$

In this case, the skew-symmetric torsion T of the characteristic ε-Hermitian connection simplifies to

$$T(X,Y) = \varepsilon J^\varepsilon(\nabla_X J^\varepsilon)Y = \frac{1}{4}\varepsilon N(X,Y)$$

due to the identities (3.1)–(3.3).

For a proof of the next result we may refer to Lemma 2.4 of [16] for nearly Kähler manifolds, Theorem 5.3 of [78] for nearly para-Kähler manifolds and Proposition 3.2 of [108] for the remaining case. As the attentive reader observes, the proof relies on the curvature identity (3.20), even though we list it already in this section as one of the very useful properties of the characteristic connection.

Proposition 3.1.7 *The characteristic ε-Hermitian connection $\bar{\nabla}$ of a nearly ε-Kähler manifold $(M^{2m}, J^\varepsilon, g, \omega)$ satisfies*

$$\bar{\nabla}(\nabla J^\varepsilon) = 0 \quad \textit{and} \quad \bar{\nabla}(T) = 0.$$

A direct consequence is the following Corollary.

Corollary 3.1.8 *On a nearly ε-Kähler manifold $(M^{2m}, J^\varepsilon, g, \omega)$ the tensors ∇J^ε and $N = 4\varepsilon T$ have constant length.*

Remark 3.1.9 In dimension 6, the fact that ∇J^ε has constant length is usually expressed by the equivalent assertion that a nearly ε-Kähler six-manifold is of constant type, i. e. there is a constant $\alpha \in \mathbb{R}$ such that

$$g((\nabla_X J^\varepsilon)Y, (\nabla_X J^\varepsilon)Y) = \alpha\,\{\, g(X,X)g(Y,Y) - g(X,Y)^2 + \varepsilon g(J^\varepsilon X, Y)^2 \,\}. \tag{3.5}$$

In fact, the constant is $\alpha = \frac{1}{4}\|\nabla J^\varepsilon\|^2$. Furthermore, it is well-known in the Riemannian case that strict nearly Kähler six-manifolds are Einstein manifolds with Einstein constant 5α [67]. The same is true in the para-Hermitian case [78] and in the pseudo-Hermitian case [108] or Theorem 3.2.8 of this chapter. The sign of the type constant depends on the signature $(2p, 2q)$ of g by $sign(p-q)$, see for example [82]. In particular, in the Riemannian case it follows $\alpha > 0$ and as a consequence a strict nearly Kähler manifold cannot be Ricci-flat.

The case $\|\nabla J^{\varepsilon}\|^2 = 0$ for a strict nearly ε-Kähler six-manifold can only occur in the para-complex world. We give different characterisations of such structures which provide an obvious break in the analogy of nearly para-Kähler and nearly pseudo-Kähler manifolds. To emphasise that we are only considering the nearly para-Kähler case we write τ for J^{ε} with $\varepsilon = 1$.

Proposition 3.1.10 *For a six-dimensional strict nearly para-Kähler manifold* (M^6, g, τ, ω) *the following properties are equivalent:*

(i) $\|\nabla\tau\|^2 = \|\mathcal{A}\|^2 = 0$.
(ii) *The three-form* $\mathcal{A} = -\nabla\omega \in [\![\Omega^{3,0}]\!]$ *is either in* $\Gamma(\Lambda^3\mathcal{V}^*)$ *or in* $\Gamma(\Lambda^3\mathcal{H}^*)$.
(iii) *The three-form* $\mathcal{A} = -\nabla\omega \in [\![\Omega^{3,0}]\!]$ *is not stable.*
(iv) *The metric g is Ricci-flat.*

In consequence for a Ricci-flat nearly para-Kähler manifold the 3-forms $D\omega(\cdot,\cdot,\cdot)$ and $N(\cdot,\cdot,\cdot)$ are not stable in the sense of Hitchin [75, 76], cf. Sect. 2.1 of Chap. 2 for details on stable forms and hence the powerful methods of stable forms are not available. The following observation is used later in this text to construct examples of non-flat Ricci-flat nearly para-Kähler six-manifolds.

Corollary 3.1.11

(a) On a Ricci-flat nearly para-Kähler six-manifold (M^6, τ, g) *the 3-forms* $\nabla\omega$ *and* N *have isotropic support.*
(b) Let (M, τ, g) *be a nearly para-Kähler manifold such that the Nijenhuis tensor* N *has isotropic support, then one has* $N_X \circ N_Y = 0$.

Proof The identity (3.5) combined with $\alpha = 0$ yields $g((\nabla_X\tau)Y, (\nabla_X\tau)Y) = 0$ and further

$$g(\tau(\nabla_X\tau)Y, \tau(\nabla_X\tau)Y) = 0. \tag{3.6}$$

This shows that the two 3-forms $g((\nabla_X\tau)Y, Z)$ and $g(\tau(\nabla_X\tau)Y, Z)$ have isotropic support. Finally we obtain after polarisation of (3.6), that one has

$$g(N_X N_Y Z, W) = -16 g(\tau(\nabla_Y\tau)Z, \tau(\nabla_X\tau)W) = 0$$

for all $X, Y, Z, W \in \Gamma(TM)$. This yields the last statement. □

Remark 3.1.12 Let us consider $\mathbb{R}^{2m}$ with its standard para-Hermitian structure (P_0, g_0) and isotropic basis $(e_1, \ldots, e_m, f_1, \ldots, f_m)$ with dual isotropic basis $(e^1, \ldots, e^m, f^1, \ldots, f^m)$, compare Eq. (2.33). Then the m-forms $e^1 \wedge \ldots \wedge e^m$ and $f^1 \wedge \ldots \wedge f^m$ are invariant under $SU(P_0, g_0)$, which follows from Eq. (2.34) and have isotropic support in the above sense for $m = 3$.

As the (restricted) holonomy of a Ricci-flat para-Kähler six-manifold (M^6, P, g) lies in $SU(P_0, g_0)$, it follows that a Ricci-flat para-Kähler six-manifold admits a family of (non-vanishing) parallel 3-forms with isotropic support.

Lemma 3.1.13 *Let (M, τ, g) be a nearly para-Kähler manifold such that the Nijenhuis tensor $N(X, Y, Z)$ has isotropic support, then it holds*

$$(\nabla_X N)(Y, Z) = 0, \tag{3.7}$$

$$(\nabla_X N)(Y, Z, W) = 0. \tag{3.8}$$

In particular, these identities are satisfied for a Ricci-flat nearly para-Kähler six-manifold.

Proof We directly compute Eq. (3.7) using Theorem 3.1.5 and Corollary 3.1.11 (b)

$$\begin{aligned}(\nabla_X N)(Y, Z) &= \nabla_X(N(Y, Z)) - N(\nabla_X Y, Z) - N(Y, \nabla_X Z)\\ &= \left(\bar{\nabla}_X - \frac{1}{8}N_X\right)(N(Y, Z)) - N\left(\left(\bar{\nabla}_X - \frac{1}{8}N_X\right)Y, Z\right)\\ &\quad -N\left(Y, \left(\bar{\nabla}_X - \frac{1}{8}N_X\right)Z\right) = (\bar{\nabla}_X N)(Y, Z) = 0.\end{aligned}$$

Combining Eq. (3.7) with $\nabla g = 0$ and $N(X, Y, Z) = g(N(X, Y), Z)$ we obtain Eq. (3.8). The last statement follows from Corollary 3.1.11 (a). □

Flat strict nearly para-Kähler manifolds (M, g, J, ω) are classified in work with V. Cortés, see Sect. 3.6 of this chapter. It turns out that these always satisfy $\|\nabla J^\varepsilon\|^2 = 0$. In [59], almost para-Hermitian structures on tangent bundles TN of real three-dimensional manifolds N^3 are discussed. It is shown that the existence of nearly para-Kähler manifolds satisfying the second condition of Proposition 3.1.10 is equivalent to the existence of a certain connection on N^3 without constructing an example. However, to our best knowledge, there was no reference for an example of a Ricci-flat non-flat strict nearly para-Kähler structure until the author's paper [109] discussed in Sect. 3.7 of this chapter.

3.1.2 Characterisations by Exterior Differential Systems in Dimension 6

The following lemma explicitly relates the Nijenhuis tensor to the exterior differential. For $\varepsilon = -1$, it gives a characterisation of Bryant's notion of a quasi-integrable $\mathrm{U}(p, q)$-structure, $p + q = 3$, in dimension 6 [24].

Let $(M^6, g, J^\varepsilon, \omega)$ be a six-dimensional almost ε-Hermitian manifold. If $\{e_1, \ldots, e_6 = Je_3\}$ is a local ε-unitary frame, we define a local frame $\{E^1, E^2, E^3\}$ of $(TM^{1,0})^*$ by

$$E^i := (e^i + \mathrm{i}_\varepsilon \varepsilon J^\varepsilon e^i) = (e^i + \mathrm{i}_\varepsilon e^{i+m})$$

for $i = 1, 2, 3$ and call it a local ε-unitary frame of $(1,0)$-forms. The dual vector fields of the $(1,0)$-forms are

$$E_i = e_i^{1,0} = \frac{1}{2}(e_i + \mathrm{i}_\varepsilon \varepsilon J^\varepsilon e_i) = \frac{1}{2}(e_i + \mathrm{i}_\varepsilon \varepsilon e_{i+m}),$$

such that the $\mathbb{C}_\varepsilon$-bilinearly extended metric in this kind of frame satisfies

$$g(E_i, \bar{E}_j) = \frac{1}{2}\sigma_i \delta_{ij} \qquad \text{and} \qquad g(E_i, E_j) = 0.$$

Lemma 3.1.14 *The Nijenhuis tensor of an almost ε-Hermitian six-manifold $(M^6, g, J^\varepsilon, \omega)$ is totally skew-symmetric if and only if for every local ε-unitary frame of $(1,0)$-forms, there exists a local $\mathbb{C}_\varepsilon$-valued function λ such that*

$$(dE^{s(1)})^{0,2} = \lambda\, \sigma_{s(1)}\, E^{\overline{s(2)}\,\overline{s(3)}} \tag{3.9}$$

for all even permutations s of $\{1, 2, 3\}$.

Proof First of all, the identities

$$N(\bar{V}, \bar{W}) = -4\varepsilon[\bar{V}, \bar{W}]^{1,0} \quad \text{and} \quad N(V, \bar{W}) = 0$$

for any vector fields $V = V^{1,0}$, $W = W^{1,0}$ in $TM^{1,0}$ follow immediately from the definition of N. Using the first identity, we compute in an arbitrary local ε-unitary frame

$$\begin{aligned} dE^i(\bar{E}_j, \bar{E}_k) &= -E^i([\bar{E}_j, \bar{E}_k]) = -2\sigma_i\, g([\bar{E}_j, \bar{E}_k], \bar{E}_i) \\ &= -2\sigma_i\, g([\bar{E}_j, \bar{E}_k]^{1,0}, \bar{E}_i) = \frac{1}{2}\varepsilon\, \sigma_i\, g(N(\bar{E}_j, \bar{E}_k), \bar{E}_i) \end{aligned}$$

for all possible indices $1 \leq i, j, k \leq 3$. If the Nijenhuis tensor is totally skew-symmetric, Eq. (3.9) follows by setting

$$\lambda = \frac{1}{2}\varepsilon\, g(N(\bar{E}_1, \bar{E}_2), \bar{E}_3). \tag{3.10}$$

Conversely, the assumption (3.9) for every local ε-unitary frame implies that the Nijenhuis tensor is everywhere a three-form when considering the same computation and $N(V, \bar{W}) = 0$. □

From the last Lemma we get the following Corollary.

Corollary 3.1.15 *For an almost ε-Hermitian six-manifold $(M^6, g, J^\varepsilon, \omega)$ with totally skew-symmetric Nijenhuis tensor, there exists a function $f \in C^\infty(M)$ such that one has*

$$g_p(X, Y) = f(p)\, \mathrm{tr}(N_X \circ N_Y),\ p \in M,\ X, Y \in T_pM.$$

In particular, if this function f does not vanish, i.e. if the almost complex structure is quasi-integrable, the almost ε-complex structure fixes the conformal class of g.

If there is an $SU^{\varepsilon}(p,q)$-reduction (cf. Sect. 2.4 of Chap. 2) with closed real part, this characterisation can be reformulated globally in the following sense.

Proposition 3.1.16 *Let (ω, ψ^+) be an $SU^{\varepsilon}(p,q)$-structure on a six-manifold M such that ψ^+ is closed. Then the Nijenhuis tensor is totally skew-symmetric if and only if*

$$d\psi^- = \nu\, \omega \wedge \omega \tag{3.11}$$

for a global real function ν.

Proof It suffices to proof this locally. Let $\{E^i\}$ be an ε-unitary frame of $(1,0)$-forms with $\sigma_1 = \sigma_2$ which is adapted to the $SU^{\varepsilon}(p,q)$-reduction such that $\Psi = \psi^+ + i_{\varepsilon}\psi^- = aE^{123}$ for a real constant a as in (2.48). The fundamental two-form is

$$\omega = -\frac{1}{2} i_{\varepsilon} \sum_{k=1}^{m} \sigma_k\, E^{k\bar{k}}$$

in such a frame. Furthermore, as ψ^+ is closed, we have $d\Psi = i_{\varepsilon} d\psi^- = -d\bar{\Psi}$, which implies that $d\Psi \in \Lambda^{2,2}$. Considering this, we compute the real 4-form

$$d\psi^- = \varepsilon i_{\varepsilon}\, d\Psi = \varepsilon i_{\varepsilon} a\, \left((dE^1)^{0,2} \wedge E^{23} + (dE^2)^{0,2} \wedge E^{31} + (dE^3)^{0,2} \wedge E^{12}\right)$$

and compare this expression with

$$\begin{aligned} \omega \wedge \omega &= \frac{1}{2}\varepsilon(\sigma_2\sigma_3\, E^{2\bar{2}3\bar{3}} + \sigma_1\sigma_3\, E^{1\bar{1}3\bar{3}} + \sigma_1\sigma_2\, E^{1\bar{1}2\bar{2}}) \\ &= -\frac{1}{2}\varepsilon\sigma_3(\sigma_1\, E^{\bar{2}\bar{3}23} + \sigma_2\, E^{\bar{3}\bar{1}31} + \sigma_3\, E^{\bar{1}\bar{2}12}). \end{aligned}$$

Hence, by Lemma 3.1.14, the Nijenhuis tensor is totally skew-symmetric if and only if $d\psi^- = \nu\, \omega \wedge \omega$ holds true for a real function ν. More precisely, the two functions ν and λ are related by the formula

$$\nu = -2\sigma_3 i_{\varepsilon} a \lambda. \tag{3.12}$$

□

An $SU^{\varepsilon}(p,q)$-structure (ω, ψ) is called *half-flat* if

$$d\psi = 0, \quad d\omega^2 = 0,$$

and *nearly half-flat* if

$$d\psi = \nu\, \omega \wedge \omega$$

for a real constant ν. These notions are defined for the Riemannian signature in [32] respectively [53] and extended to all signatures in our paper [46] presented in Chap. 4 of this text.

Corollary 3.1.17 *Let (ω, ψ^+) be a half-flat $\mathrm{SU}^\varepsilon(p,q)$-structure on a six-manifold M. Then, the Nijenhuis tensor is totally skew-symmetric if and only if (ω, ψ^-) is nearly half-flat.*

Proof If (ω, ψ^-) is nearly half-flat, Eq. (3.11) is satisfied by definition and the Nijenhuis tensor is skew-symmetric by the previous proposition. In particular one has $d\omega^2 = 0$. Conversely, if the Nijenhuis tensor is skew, we know that (3.11) holds true for a real function ν, since we have $d\psi^+ = 0$. Differentiating this equation and using $d\omega^2 = 0$, we obtain $d\nu \wedge \omega^2 = 0$. The assertion follows as wedging by ω^2 is injective on one-forms. □

Remark 3.1.18 An interesting property of $\mathrm{SU}^\varepsilon(p,q)$-structures which are both half-flat and nearly half-flat in the sense of the corollary is the fact that, given that the manifold and the $\mathrm{SU}^\varepsilon(p,q)$-structure are analytic, the structure can be evolved to both a parallel G_2-structure and a nearly parallel G_2-structure via the Hitchin flow. For details, we refer to [76] and [114] for the compact Riemannian case and Chap. 4 of this text or our paper [46] for the non-compact case and indefinite signatures.

In [33], six-dimensional nilmanifolds N admitting an invariant half-flat SU(3)-structure (ω, ψ^+) such that (ω, ψ^-) is nearly half-flat are classified. As six nilmanifolds admit such a structure, we conclude that these structures are not as scarce as nearly Kähler manifolds. It is also shown in the same article, that these structures induce invariant G_2-structures with torsion on $N \times S^1$.

We give another example of a (normalised) left-invariant SU(3)-structure on $S^3 \times S^3$ which satisfies $d\psi^+ = 0,\ d\psi^- = \omega \wedge \omega$ such that $d\omega$ neither vanishes nor is of type (3,0) + (0,3). We choose a global frame of left-invariant vector fields $\{e_1, e_2, e_3, f_1, f_2, f_3\}$ on $S^3 \times S^3$ such that

$$de^1 = e^{23}, \quad de^2 = e^{31}, \quad de^3 = e^{12}; \quad df^1 = f^{23}, \quad df^2 = f^{31}, \quad df^3 = f^{12},$$

and set with $x = 2 + \sqrt{3}$

$$\begin{aligned}
\omega &= e^1f^1 + e^2f^2 + e^3f^3,\\
\psi^+ &= -\frac{1}{2}x^2e^{123} + 2xe^{12}f^3 - 2xe^{13}f^2 - 2xe^1f^{23} + 2xe^{23}f^1\\
&\quad + 2xe^2f^{13} - 2xe^3f^{12} + (4x-8)f^{123},\\
\psi^- &= \frac{1}{2}xe^{123} - 2e^1f^{23} + 2e^2f^{13} - 2e^3f^{12} + 4f^{123},\\
g &= x\,(e^1)^2 + x\,(e^2)^2 + x\,(e^3)^2 + 4\,(f^1)^2 + 4\,(f^2)^2 + 4\,(f^3)^2\\
&\quad - 2x\,e^1\cdot f^1 - 2x\,e^2\cdot f^2 - 2xe^3\cdot f^3.
\end{aligned}$$

Finally, we come to the characterisation of six-dimensional nearly ε-Kähler manifolds by an exterior differential system generalising the classical result of [103] which holds for $\varepsilon = -1$ and Riemannian metrics.

Theorem 3.1.19 *Let* $(M, g, J^{\varepsilon}, \omega)$ *be an almost* ε*-Hermitian six-manifold. Then* M *is a strict nearly* ε*-Kähler manifold with* $\|\nabla J^{\varepsilon}\|^2 \neq 0$ *if and only if there is a reduction* $\Psi = \psi^{+} + \mathrm{i}_{\varepsilon}\psi^{-}$ *to* $\mathrm{SU}^{\varepsilon}(p,q)$ *which satisfies*

$$d\omega = 3\,\psi^{+}, \tag{3.13}$$

$$d\psi^{-} = 2\,\alpha\,\omega \wedge \omega, \tag{3.14}$$

where $\alpha = \frac{1}{4}\|\nabla J^{\varepsilon}\|^2$ *is constant and non-zero.*

Remark 3.1.20 Due to our sign convention $\omega = g(.,J^{\varepsilon}.)$, the constant α is positive in the Riemannian case and the second equation differs from that of other authors. Furthermore, we will sometimes use the term nearly ε-Kähler manifold of non-zero type if $\|\nabla J^{\varepsilon}\|^2 \neq 0$.

Proof By Proposition 3.1.2, the manifold M is nearly ε-Kähler if and only if $d\omega$ is of type $(3,0) + (0,3)$ and the Nijenhuis tensor is totally skew-symmetric.

Therefore, when $(g, J^{\varepsilon}, \omega)$ is a strict nearly ε-Kähler structure such that $\|\mathcal{A}\|^2 = \|\nabla J^{\varepsilon}\|^2$ is constant (by Corollary 3.1.8) and not zero (by assumption), we can define the reduction $\Psi = \psi^{+} + \mathrm{i}_{\varepsilon}\psi^{-}$ by $\psi^{+} = \frac{1}{3}d\omega = -\mathcal{A}$ and $\psi^{-} = J^{\varepsilon *}\psi^{+}$ such that the first equation is satisfied. Since ω is of type $(1,1)$ and therefore $d(\omega \wedge \omega) = 2d\omega \wedge \omega = 0$, this reduction is half-flat. Thus, Corollary 3.1.17 and the skew-symmetry of N imply that there is a constant $\nu \in \mathbb{R}$ such that $d\psi^{-} = \nu\,\omega \wedge \omega$.

According to (2.48), we can choose an ε-unitary local frame with $\sigma_1 = \sigma_2$, such that

$$\Psi = -\mathcal{A} - \mathrm{i}_{\varepsilon}J^{\varepsilon *}\mathcal{A} = aE^{123},$$

where a is constant and satisfies $4\alpha = \|\nabla J^{\varepsilon}\|^2 = \|\psi^{+}\|^2 = 4a^2\sigma_3$ by (2.52). Now, the functions defined in Lemma 3.1.14 and Proposition 3.1.16 evaluate as

$$\lambda \overset{(3.10)}{=} \frac{1}{2}\varepsilon g(N(\bar{E}_1,\bar{E}_2),\bar{E}_3) \overset{(3.3)}{=} -2J^{*}\mathcal{A}(\bar{E}_1,\bar{E}_2,\bar{E}_3) = -\varepsilon\,\mathrm{i}_{\varepsilon}a,$$

$$\nu \overset{(3.12)}{=} -2\sigma_3\mathrm{i}_{\varepsilon}a\lambda = 2\sigma_3 a^2 = 2\alpha.$$

Conversely, if a given $\mathrm{SU}^{\varepsilon}(p,q)$-structure satisfies the exterior system, the real three-form ψ^{+} is obviously closed and the Nijenhuis tensor is totally skew-symmetric by Corollary 3.1.17. Considering that $d\omega = 3\nabla\omega$ is of type $(3,0)+(0,3)$ by the first equation, the structure is nearly ε-Kähler. Since $\mathcal{A} = -\psi^{+}$ is stable, the structure is strict nearly ε-Kähler and $\|\nabla J^{\varepsilon}\| = \|\mathcal{A}\| \neq 0$ by Proposition 3.1.10. Now, the computation of the constants in the adapted ε-unitary frame shows that in fact $\|\nabla J^{\varepsilon}\| = 4\alpha$. □

3.1.3 Curvature Identities for Nearly ε-Kähler Manifolds

Most of these identities are here only used for the almost complex case. If we are only considering the complex case we write J and in case, that we consider the para-complex case we write τ for the ε-complex structure J^{ε}. The starting point of a series of curvature identities are

$$R(W,X,Y,Z) - R(W,X,JY,JZ) = g((\nabla_W J)X,(\nabla_Y J)Z), \tag{3.15}$$

$$\begin{aligned} R(W,X,W,Z) &+ R(W,JX,W,JZ) \\ &- R(W,JW,X,JZ) = 2g((\nabla_W J)X,(\nabla_W J)Z), \end{aligned} \tag{3.16}$$

$$R(W,X,Y,Z) = R(JW,JX,JY,JZ), \tag{3.17}$$

which were already proven for pseudo-Riemannian metrics by Gray [67]. In the para-complex case the analogue of the first identity, i.e. the relation

$$R(W,X,Y,Z) + R(W,X,\tau Y,\tau Z) = g((\nabla_W \tau)X,(\nabla_Y \tau)Z), \tag{3.18}$$

is shown in Proposition 5.2 of [78].

Let $\{e_i\}_{i=1}^{2n}$ be a local orthonormal frame field, then the Ricci- and the Ricci*-tensor are given by

$$g(Ric\,X,Y) = \sum_{i=1}^{2n} \epsilon_i\, R(X,e_i,Y,e_i), \quad g(Ric^*\,X,Y) = \frac{1}{2}\sum_{i=1}^{2n} \epsilon_i\, R(X,JY,e_i,Je_i)$$

with $\epsilon_i = g(e_i,e_i) = g(Je_i,Je_i)$ and $X,Y \in TM$. The frame $\{e_i\}_{i=1}^{2n}$ is called adapted if it holds $Je_i = e_{i+n}$ for $i = 1,\dots,n$. Then it follows using an adapted frame from Eqs. (3.16) and (3.17) that

$$g(rX,Y) := g((Ric - Ric^*)X,Y) = \sum_{i=1}^{2n} \epsilon_i\, g((\nabla_X J)e_i,(\nabla_Y J)e_i). \tag{3.19}$$

Using the right hand-side we see

$$[J,r] = 0.$$

For the second derivative of the complex structure one has the identity

$$2g(\nabla^2_{W,X}(J)Y,Z) = -\sigma_{X,Y,Z}\, g((\nabla_W J)X,(\nabla_Y J)JZ), \tag{3.20}$$

which was proven in [67] for Riemannian metrics and holds true in the pseudo-Riemannian setting, cf. [82, Proposition 7.1]. This identity implies

$$\sum_{i=1}^{2n} \epsilon_i \nabla^2_{e_i,e_i}(J)Y = -r(JY). \tag{3.21}$$

From Proposition 3.1.7 and the relation (3.4) of the connections ∇ and $\bar{\nabla}$ one obtains the following identities for the curvature tensor $\bar{R}$ of $\bar{\nabla}$ and the curvature tensor R of the Levi-Civita connection ∇

$$\begin{aligned} \bar{R}(W,X,Y,Z) &= R(W,X,Y,Z) - \frac{1}{2}g((\nabla_W J)X, (\nabla_Y J)Z) \\ &\quad + \frac{1}{4}\left[g((\nabla_W J)Y, (\nabla_X J)Z) - g((\nabla_W J)Z, (\nabla_X J)Y)\right] \qquad (3.22) \\ &= \frac{1}{4}[3R(W,X,Y,Z) + R(W,X,JY,JZ) \\ &\quad + \sigma_{XYZ} R(W,X,JY,JZ)], \\ \bar{R}(W,JW,Y,JZ) &= \frac{1}{4}[5R(W,JW,Y,JZ) \\ &\quad - R(W,Y,W,Z) - R(W,JY,W,JZ)]. \qquad (3.23) \end{aligned}$$

With the help of Eq. (3.22) it follows

$$\bar{R}(W,X,Y,Z) = \bar{R}(Y,Z,W,X) = -\bar{R}(X,W,Y,Z) = -\bar{R}(W,X,Z,Y). \tag{3.24}$$

Using $\bar{\nabla}J = 0$ and $\bar{\nabla}g = 0$ we obtain

$$\begin{aligned} \bar{R}(W,X,Y,Z) &= \bar{R}(W,X,JY,JZ) \qquad (3.25) \\ &= \bar{R}(JW,JX,Y,Z) = \bar{R}(JW,JX,JY,JZ). \end{aligned}$$

The general form of the first Bianchi identity (cf. Chapter III of [87]) for a connection with torsion yields in the case of parallel torsion:

$$\underset{XYZ}{\sigma} \bar{R}(W,X,Y,Z) = -\underset{XYZ}{\sigma} g((\nabla_W J)X, (\nabla_Y J)Z). \tag{3.26}$$

In a similar way we get from the second Bianchi identity (cf. Chapter III of [87]) for a connection with parallel torsion or from the second Bianchi identity for ∇

$$-\underset{VWX}{\sigma} \bar{\nabla}_V(\bar{R})(W,X,Y,Z) = \underset{VWX}{\sigma} \bar{R}((\nabla_V J)JW,X,Y,Z). \tag{3.27}$$

From deriving Eq. (3.22) and the second Bianchi identity of ∇ one gets after a direct computation

$$\underset{VWX}{\sigma} \nabla_V(\bar{R})(W,X,Y,Z) = \frac{1}{2} g((\nabla_Y J)Z, \underset{VWX}{\sigma} (\nabla_X J)(\nabla_V J)JW), \tag{3.28}$$

which implies

$$\underset{VWX}{\sigma} \nabla_V(\bar{R})(W,X,Y,JY) = 0. \tag{3.29}$$

Proposition 3.1.21 (Proposition 2.3 of [108]) *The tensor r on a nearly pseudo-Kähler manifold (M,J,g) is parallel with respect to the characteristic connection $\bar{\nabla}$.*

Theorem 3.1.22 (Theorem 2.4 of [108]) *Let (M,J,g) be a nearly pseudo-Kähler manifold and let W,X be vector fields on M then it holds*

$$\sum_{i,j=1}^{2n} \epsilon_i \epsilon_j g(re_i, e_j) \left[R(W,e_i,X,e_j) - 5R(W,e_i,JX,Je_j) \right] = 0. \tag{3.30}$$

Let us remark, that the Riemannian case is done in [67] and the para-Kähler case in [78].

3.2 Structure Results

As we have seen above, for a nearly pseudo-Kähler manifold $\nabla\omega$ is a differential form of type $(3,0)+(0,3)$. In consequence real two- or four-dimensional nearly pseudo-Kähler manifolds are automatically pseudo-Kähler. Six dimensional nearly pseudo-Kähler manifolds are either pseudo-Kähler manifolds or strict nearly pseudo-Kähler manifolds. Therefore we start this section in real dimension 8.[1]

3.2.1 Kähler Factors and the Structure in Dimension 8

The aim of this subsection is to split off the pseudo-Kähler factor of a nearly pseudo-Kähler manifold. This will be done by means of the kernel of ∇J and allows to reduce the (real) dimension from 8 to 6.

For $p \in M$ we set

$$\mathcal{K}_p = \ker(X \in T_pM \mapsto \nabla_X J).$$

[1]The reference for the section is the author's paper [108].

Theorem 3.2.1 *Let (M, J, g) be a nearly pseudo-Kähler manifold. Suppose, that the distribution $\mathcal{K}$ has constant dimension and admits an orthogonal complement,*

(i) *then M is locally a pseudo-Riemannian product $M = K \times M_1$ of a pseudo-Kähler manifold K and a strict nearly pseudo-Kähler manifold M_1.*
(ii) *if M is complete and simply connected then it is a pseudo-Riemannian product $M = K \times M_1$ of a pseudo-Kähler manifold K and a strict nearly pseudo-Kähler manifold M_1.*

Proof The distribution $\mathcal{K}$ is parallel for the characteristic connection $\bar{\nabla}$, since ∇J is $\bar{\nabla}$-parallel. By the formula (3.4) and the nearly Kähler condition it follows $\bar{\nabla}_X K = \nabla_X K$ for sections K in $\mathcal{K}$ and X in TM. This implies that $\mathcal{K}$ is parallel for the Levi-Civita connection and in consequence its orthogonal complement $(\mathcal{K})^{\perp}$ is Levi-Civita parallel. The proof of (i) finishes by the local version of the theorem of de Rham and the proof of (ii) by the global version. □

Remark 3.2.2 There exist nearly pseudo-Kähler manifolds (M, J, g) without pseudo-Kähler de Rham factor, such that $\mathcal{K}_\eta \neq \{0\}$ admits no orthogonal complement. In fact, we construct Levi-Civita flat nearly pseudo-Kähler manifolds in our paper [41], which is subject of Sect. 3.6 of this chapter, such that the three-form $\eta_p(X, Y, Z) = g_p(J(\nabla_X J)Y, Z)$, for $p \in M$, has a support $\Sigma_\eta \subset T_pM$ which is a maximally isotropic subspace (Here we identified T_pM and T_p^*M via the metric g.). Obviously, $J(\nabla_X J)Y$ and $J(\nabla_U J)V$ are elements of the support of η for arbitrary $X, Y, U, V \in T_pM$. It then follows $0 = g(J(\nabla_X J)Y, J(\nabla_U J)V) = g(J(\nabla_{J(\nabla_X J)Y} J)U, V)$ for all $V \in T_pM$. Hence it is $\Sigma_\eta \subset \mathcal{K}_\eta$. Moreover for general reasons we have shown before $\Sigma_\eta = \mathcal{K}_\eta^{\perp}$ which shows $\mathcal{K}_\eta \cap \mathcal{K}_\eta^{\perp} \neq \{0\}$ for the above examples. From these examples we learn, that the Theorem 3.2.1 does not hold true, if there is no orthogonal complement.

Definition 3.2.3 A nearly pseudo-Kähler manifold (M, J, g) is called nice if the three-form $g((\nabla J^\varepsilon)\cdot, \cdot)$ has non-zero length in each point $p \in M$.

Theorem 3.2.4 *Let (M^8, J, g) be a complete simply connected eight-dimensional nice nearly pseudo-Kähler manifold. Then $M = M_1 \times M_2$ where M_1 is a two-dimensional Kähler manifold and M_2 is a six-dimensional strict nearly pseudo-Kähler manifold.*

Proof Since (M, J, g) is a nice nearly pseudo-Kähler manifold we can use Lemma 2.3.6 of Chap. 2 to obtain an orthogonal splitting in the two-dimensional distribution $\mathcal{K}$ and its orthogonal complement, which coincides with Σ_η. Therefore we are in the situation of Theorem 3.2.1 (ii). □

3.2.2 Einstein Condition Versus Reducible Holonomy

In this part we study reducible $\bar{\nabla}$-holonomy and discuss the consequences in small dimensions.

Theorem 3.2.5 *Let (M, J, g) be a nearly pseudo-Kähler manifold.*

(i) *Suppose that r has more than one eigenvalue, then the characteristic Hermitian connection has reduced holonomy.*
(ii) *If the tensor field r has exactly one eigenvalue then M is a pseudo-Riemannian Einstein manifold.*

Proof

(i) Let μ_i for $i = 1, \ldots, l$ be the eigenvalues of r. Then the decomposition in the according eigenbundles $\mathrm{Eig}(\mu_i)$ is $\bar{\nabla}$-parallel and hence its holonomy is reducible.
(ii) From the identity of Theorem 3.1.22 and $r = \mu Id_{TM}$ we obtain

$$0 = \sum_{i=1}^{2n} \epsilon_i \left(R(W, e_i, X, e_i) - 5R(W, e_i, JX, Je_i)\right) = g((Ric - 5Ric^*)W, X),$$

where we used the Bianchi identity and an adapted frame to obtain the last equality. This shows comparing with $r = Ric - Ric^*$ that it holds $Ric = \frac{5}{4}\mu$.

□

Let us recall, that in the pseudo-Riemannian setting the decomposition into the eigenbundles is **not** automatically ensured to be an orthogonal direct decomposition. Therefore we introduce the following notion.

Definition 3.2.6 A nearly pseudo-Kähler manifold (M, J, g) is called decomposable if the above decomposition into the eigenbundles of the tensor r is orthogonal.

Lemma 3.2.7 *Let (M, J, g) be a decomposable nearly pseudo-Kähler manifold and denote by μ_i for $i = 1, \ldots, l$ the eigenvalues of r and by $E_i = \mathrm{Eig}(\mu_i)$, $i = 1, \ldots, l$, the corresponding eigenbundles.*

(i) For $X \in E_i$ and $Y \in E_j$ with $i \neq j$ one has $Ric(X, Y) = 0$.
(ii) For $X, Y \in E_i$ it is

$$Ric(X, Y) = \frac{\mu_i}{4} g(X, Y) + \frac{1}{\mu_i} \sum_{s=1}^{l} \mu_s \, g(r^s X, Y),$$

where the tensors $r^s \colon TM \to TM$, $1 \leq s \leq l$, are defined as

$$g(r^s X, Y) := -\mathrm{tr}_{E_s}\left((\nabla_X J) \circ (\nabla_Y J)\right).$$

Proof Let us first prove (i). We consider a basis of TM which gives a pseudo-orthonormal basis for the $E_i, i = 1, \ldots l$. The Ricci curvature decomposes w.r.t. the eigenbundles as

$$Ric(X, Y) = \sum_{s=1}^{l} \sum_{e_k \in E_s} \epsilon_k \, R(X, e_k, Y, e_k).$$

Using $\bar{R}(X, e_k, Y, e_k) = 0$ for $s \neq j$ one gets by Eq. (3.22)

$$R(X, e_k, Y, e_k) = \frac{1}{4} g((\nabla_X J)e_k, (\nabla_Y J)e_k), \text{ for } e_k \in E_s.$$

Further for $s = j$ one has $s \neq i$ and again it is

$$R(X, e_k, Y, e_k) = R(Y, e_k, X, e_k) = \frac{1}{4} g((\nabla_X J)e_k, (\nabla_Y J)e_k), \text{ for } e_k \in E_s.$$

In summary one obtains

$$Ric(X, Y) = \sum_{s=1}^{l} \sum_{e_k \in E_s} \epsilon_k R(X, e_k, Y, e_k) = \frac{1}{4} \sum_k \epsilon_k g((\nabla_X J)e_k, (\nabla_Y J)e_k) = \frac{1}{4} g(rX, Y) = 0.$$

This shows (i). Next we show part (ii). From the identity of Theorem 3.1.22 we conclude

$$0 = \sum_{s=1}^{l} \sum_{e_k \in E_s} \mu_s \epsilon_k \left(R(W, e_k, X, e_k) - 5R(W, e_k, JX, Je_k) \right).$$

As in part (i) we get for $s \neq i$ with help of Eq. (3.22)

$$R(X, e_k, JY, Je_k) = -3R(Y, e_k, X, e_k) = -\frac{3}{4} g((\nabla_X J)e_k, (\nabla_Y J)e_k), \text{ for } e_k \in E_s.$$

It follows, that

$$4 \sum_{s \neq i} \mu_s g(r^s X, Y) + \mu_i \left(\sum_{e_k \in E_i} \epsilon_k \left(R(W, e_k, X, e_k) - 5R(W, e_k, JX, Je_k) \right) \right) = 0$$

and another time using Eq. (3.22)

$$\mu_i g((Ric - 5Ric^*)X, Y) + 4 \sum_{s \neq i} (\mu_s - \mu_i) g(r^s X, Y) = 0,$$

which follows by

$$g((Ric - 5Ric^*)X, Y) = \sum_{s=1}^{l} \sum_{e_k \in E_s} \epsilon_k \left(R(X, e_k, Y, e_k) - 5R(X, e_k, JY, Je_k) \right).$$

The identity (ii) follows now from $Ric - Ric^* = r$ and $\sum_{s=1}^{l} r^s = r$. In fact, it is

$$g((Ric - 5Ric^*)X, Y) = -4g(Ric\,X, Y) + 5g(rX, Y) = \frac{4}{\mu_i}\sum_{s\neq i}(\mu_i - \mu_s)g(r^s X, Y)$$

and in consequence one obtains

$$4g(Ric\,X, Y) = 5g(rX, Y) - \frac{4}{\mu_i}\sum_{s\neq i}(\mu_i - \mu_s)g(r^s X, Y) = g(rX, Y) + \frac{4}{\mu_i}\sum_{s=1}^{l}\mu_s g(r^s X, Y),$$

which finishes the proof. □

Theorem 3.2.8 *A strict nearly pseudo-Kähler six-manifold (M^6, J, g) of constant type α is a pseudo-Riemannian Einstein manifold with Einstein constant 5α.*

Proof In an adapted basis we obtain from the symmetries of ∇J

$$g(rX, X) = 2\sum_{i=1}^{3}\epsilon_i\, g((\nabla_X J)e_i, (\nabla_X J)e_i) = -2\sum_{i=1}^{3}\epsilon_i\, g((\nabla_X J)^2 e_i, e_i).$$

This is exactly minus the trace of the operator $(\nabla_X J)^2$ which has a simple form in a cyclic frame. It follows after polarising $g(rX, Y) = 4\alpha g(X, Y)$. From Theorem 3.2.5 we compute the Einstein constant 5α where α is the type constant of the strict nearly pseudo-Kähler manifold M^6. □

Proposition 3.2.9 *Let (M^{10}, J, g) be a nice nearly pseudo-Kähler ten-manifold.*

(i) *Then the tensor r in a frame of the first type in Lemma 2.3.8 of Chap. 2 is given by*

$$\begin{aligned} re_1 &= 4(\alpha^2 + \beta^2)e_1, \\ re_2 &= 4\alpha^2 e_2, \quad re_3 = 4\alpha^2 e_3, \\ re_4 &= 4\beta^2 e_4, \quad re_5 = 4\beta^2 e_5, \\ r(Je_i) &= Jr(e_i), \quad i = 1, \ldots, 5, \end{aligned}$$

where α, β are constants.

(ii) *For a frame of the second type in Lemma 2.3.8 of Chap. 2 the tensor r is given by*

$$r\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = 4\begin{bmatrix} \alpha^2 + \beta^2\epsilon_4\epsilon_5 & 0 & \beta^2\epsilon_4\epsilon_5 \\ 0 & \alpha^2 & 0 \\ \beta^2\epsilon_4\epsilon_5 & 0 & \alpha^2 + \beta^2\epsilon_4\epsilon_5 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

$$
\begin{aligned}
re_4 &= 0,\\
re_5 &= 4\,\beta^2(2\epsilon_1\epsilon_4 - 1)e_5,\\
r(Je_i) &= Jr(e_i), \quad i = 1, \ldots, 5.
\end{aligned}
$$

The eigenvalues are $\{0; 4\alpha^2; 4\,\beta^2(2\epsilon_1\epsilon_4 - 1); 4(\alpha^2 + 2\beta^2\epsilon_4\epsilon_5)\}$*, where the eigenbundles are given as*

$$
\begin{aligned}
\mathrm{Ker}(r) &= \mathrm{span}\{e_4, Je_4\},\\
\mathrm{Eig}(r, 4\alpha^2) &= \mathrm{span}\{-e_1 + e_3, e_2, -Je_1 + Je_3, Je_2\},\\
\mathrm{Eig}(r, 4\,\beta^2(2\epsilon_1\epsilon_4 - 1)) &= \mathrm{span}\{e_5, Je_5\},\\
\mathrm{Eig}(r, 4(\alpha^2 + 2\beta^2\epsilon_4\epsilon_5)) &= \mathrm{span}\{e_1 + e_3, Je_1 + Je_3\},
\end{aligned}
$$

where α, β *are constants. For* $\beta^2 \neq 0$ *the second case is not decomposable.*

(iii) *Suppose* $\beta = 0$ *in the cases (i) and (ii). Then it follows*

$$
\begin{aligned}
\mathrm{Eig}(r, 4\alpha^2) &= \mathrm{span}\{e_1, e_2, e_3, Je_1, Je_2, Je_3\},\\
\mathrm{Ker}(r) &= \mathrm{span}\{e_4, e_5, Je_4, Je_5\}.
\end{aligned}
$$

Proof In an adapted frame we obtain from the symmetries of ∇J

$$
g(rX, Y) = 2\sum_{i=1}^{5} \epsilon_i\, g((\nabla_X J)e_i, (\nabla_Y J)e_i) = -2\sum_{i=1}^{5} \epsilon_i\, g((\nabla_Y J)(\nabla_X J)e_i, e_i).
$$

This is exactly minus the trace of the operator $(\nabla_Y J)(\nabla_X J)$. Using the form of Lemma 2.3.8 of Chap. 2 one can calculate r by hand or using computer algebra systems to obtain the claimed results □

Theorem 3.2.10 *Let* (M^{10}, J, g) *be a complete simply connected nice decomposable nearly pseudo-Kähler manifold of dimension 10. Then* M^{10} *is of one of the following types*

(i) *the tensor r has a kernel and* $M^{10} = K \times M^6$ *is a product of a four-dimensional pseudo-Kähler manifold K and a strict nearly pseudo-Kähler six-manifold* M^6*.*
(ii) *the tensor r has trivial kernel and r has eigenvalues* $4(\alpha^2+\beta^2)$ *with multiplicity* 2, $4\alpha^2, 4\beta^2$ *with multiplicity* 4 *for some* $\alpha, \beta \neq 0$.

A nice nearly pseudo-Kähler manifold (M^{10}, J, g) *is decomposable if the dimension of the kernel of r is not equal to two.*

Proof Since we suppose, that (M^{10}, J, g) is a nice and decomposable nearly pseudo-Kähler manifold, Proposition 3.2.9 implies that one has the two different cases:

(i) the distribution $\mathcal{K}$, which is the tangent space of the Kähler factor has dimension 4 and admits an orthogonal complement of dimension 6. This is part (iii) of Proposition 3.2.9. Part (i) of the Theorem now follows from Theorem 3.2.1.

(ii) the tensor r has trivial kernel and we are in the situation of Proposition 3.2.9 part (i) with $\alpha, \beta \neq 0$ and part (ii) follows.

□

Remark 3.2.11 Nearly pseudo-Kähler manifolds falling in the second case of the last theorem are related to twistor spaces in Sect. 3.4.5 of this chapter.

3.3 Twistor Spaces over Quaternionic and Para-Quaternionic Kähler Manifolds

In this section[2] we consider pseudo-Riemannian submersions $\pi : (M, g) \to (N, h)$ endowed with a complex structure J on M which is compatible with the decomposition (2.53).

Lemma 3.3.1 (Lemma 5.1 of [108]) *Let $\pi : (M, g) \to (N, h)$ be a pseudo-Riemannian submersion endowed with a complex structure J on M which is compatible with the decomposition* (2.53). *Then (M, g, J) is a pseudo-Kähler manifold if and only if the following equations*[3] *are satisfied*

$$\pi_{\mathcal{H}}((\nabla_X J)Y) = \pi_{\mathcal{H}}((\nabla_V J)X) = 0, \tag{3.31}$$

$$(\nabla^{\mathcal{V}}_U J)V = \pi_{\mathcal{V}}((\nabla_X J)V) = 0, \tag{3.32}$$

$$A_X(JY) - JA_X Y = 0, \quad A_X(JV) - JA_X V = 0, \tag{3.33}$$

$$T_V(JX) - JT_V X = 0, \quad T_U(JV) - JT_U V = 0, \tag{3.34}$$

where X, Y are vector fields in $\mathcal{H}$ and U, V are vector fields in $\mathcal{V}$.

Further we define a second complex structure by

$$\hat{J} := \begin{cases} J \text{ on } \mathcal{H}, \\ -J \text{ on } \mathcal{V}. \end{cases}$$

We observe that $\hat{\hat{J}} = J$. This construction was made in [98] for the Riemannian setting and imitates the construction on twistor spaces, which was first done in [50].

Proposition 3.3.2 *Suppose, that the foliation induced by the pseudo-Riemannian submersion π is totally geodesic and that (M, J, g) is a pseudo-Kähler manifold and J is compatible with the decomposition* (2.53)*, then the manifold $(M, \hat{g} = g_{\frac{1}{2}}, \hat{J})$ is a nearly pseudo-Kähler manifold. The distributions $\mathcal{H}$ and $\mathcal{V}$ are parallel with respect*

[2] The reference still is the author's paper [108].

[3] Please note, the small difference between the torsion $T(X, Y)$ of $\bar{\nabla}$ and the second fundamental form $T_U V$, which is not dangerous, as later we are considering totally geodesic fibrations.

to the characteristic Hermitian connection $\bar{\nabla}$ *of* $(M, \hat{g}, \hat{J})$. *In other words the nearly pseudo-Kähler manifold* $(M, \hat{g}, \hat{J})$ *has reducible* $\bar{\nabla}$*-holonomy.*

Proof Let U, V be vector fields in $\mathcal{V}$ and X, Y be vector fields in $\mathcal{H}$: In the following $\hat{\nabla}$ is the Levi-Civita connection of $\hat{g}$. Since the fibres are totally geodesic, i.e. $T \equiv 0$, we obtain from Eq. (2.54), that $\hat{\nabla}_U V = \hat{\nabla}^{\mathcal{V}}_U V + \hat{T}_U V = \nabla^{\mathcal{V}}_U V + T_U V = \nabla_U V$, which yields $(\hat{\nabla}_U \hat{J})V = -(\nabla_U J)V = 0$.

In the sequel we denote the O'Neill tensors of the pseudo-Riemannian foliations induced by $\mathcal{V}$ on (M, g) and on $(M, \hat{g})$ by A and $\hat{A}$, respectively. From Lemma 2.5.2 of Chap. 2 it follows $A_X Y = \hat{A}_X Y$ and consequently the same Lemma yields $\nabla_X Y = \hat{\nabla}_X Y$.

Since (M, g) is Kähler, Lemma 3.3.1 implies $A \circ J = J \circ A$ and we compute

$$\begin{aligned}
(\hat{\nabla}_X \hat{J})Y &= \hat{\nabla}_X(\hat{J}Y) - \hat{J}\hat{\nabla}_X Y \\
&= \pi_{\mathcal{H}}[\hat{\nabla}_X(JY)] + \pi_{\mathcal{V}}[\hat{\nabla}_X(JY)] - \hat{J}(\pi_{\mathcal{H}}(\hat{\nabla}_X Y) + \pi_{\mathcal{V}}(\hat{\nabla}_X Y)) \\
&= \pi_{\mathcal{H}}[\hat{\nabla}_X(JY) - J\hat{\nabla}_X Y] + \pi_{\mathcal{V}}[\hat{\nabla}_X(JY) + J\hat{\nabla}_X Y] \\
&= \pi_{\mathcal{H}}((\hat{\nabla}_X J)Y) + \hat{A}_X(JY) + J\hat{A}_X Y \\
&\overset{(2.60),(2.62),(3.33)}{=} \pi_{\mathcal{H}}((\nabla_X J)Y) + 2A_X(JY) \overset{(3.31)}{=} 2A_X(JY) = 2JA_X Y.
\end{aligned} \tag{3.35}$$

With the identity $A_X V = 2\hat{A}_X V$ of Lemma 2.5.2 of Chap. 2 we get

$$\begin{aligned}
(\hat{\nabla}_X \hat{J})V &= \hat{\nabla}_X(\hat{J}V) - \hat{J}\hat{\nabla}_X V \\
&= -\pi_{\mathcal{V}}(\hat{\nabla}_X(JV)) - \pi_{\mathcal{H}}(\hat{\nabla}_X(JV)) + J\pi_{\mathcal{V}}(\hat{\nabla}_X V) - J\pi_{\mathcal{H}}(\hat{\nabla}_X V) \\
&= -\pi_{\mathcal{V}}((\hat{\nabla}_X J)V) - \hat{A}_X JV - J\hat{A}_X V \\
&\overset{(2.60),(2.62),(3.33)}{=} -\pi_{\mathcal{V}}((\nabla_X J)V) - JA_X V = -A_X JV.
\end{aligned} \tag{3.36}$$

The vanishing of the second fundamental form T, Eq. (2.61) and a second time $A_X V = 2\hat{A}_X V$ show

$$\begin{aligned}
(\hat{\nabla}_V \hat{J})X &= \pi_{\mathcal{V}}(\hat{\nabla}_V(JX)) + \pi_{\mathcal{V}}(J\hat{\nabla}_V X) + \pi_{\mathcal{H}}(\hat{\nabla}_V(JX) - J\hat{\nabla}_V X) \\
&\overset{(2.63)}{=} \hat{T}_V(JX) + J(\hat{T}_V X) + \pi_{\mathcal{H}}((\nabla_V J)X) + \frac{1}{2}(JA_X V - A_{JX} V) = JA_X V,
\end{aligned} \tag{3.37}$$

where we used $A_{JX} V = -JA_X V$ which follows, since A_X is alternating (compare Eq. (2.59)) and commutes with J. The next Lemma finishes the proof. □

Lemma 3.3.3 (Lemma 5.3 of [108])

1) *Suppose, that* $(M, \hat{J}, \hat{g})$ *is a nearly pseudo-Kähler manifold and* $\hat{J}$ *is compatible with the decomposition* (2.53)*, then the following statements are equivalent:*

   (i) *the splitting* (2.53) *is* $\bar{\nabla}$*-parallel,*

(ii) *the fundamental tensors* $\hat{A}$ *and* $\hat{T}$ *satisfy:*

$$\hat{T}_V X = 0, \quad \hat{J}\hat{T}_V W = -\hat{T}_V \hat{J} W \Leftrightarrow \check{J}\hat{T}_V W = \hat{T}_V \check{J} W \text{ for } \check{J} = \hat{\hat{J}}, \tag{3.38}$$

$$\hat{A}_X V = \frac{1}{2}\hat{J}(\hat{\nabla}_X \hat{J})V, \quad \hat{A}_X Y = \frac{1}{2}\pi_{\mathcal{V}}\left(\hat{J}(\hat{\nabla}_X \hat{J})Y\right). \tag{3.39}$$

2) *If it holds* $(\hat{\nabla}_V \hat{J})W = 0$ *then* $\bar{\nabla}_V W \in \mathcal{V}$ *for* $V, W \in \mathcal{V}$ *is equivalent to* $T_V W = 0$. *Moreover it is* $(\hat{\nabla}^{\mathcal{V}}_V \hat{J})W = 0$.

We apply Proposition 3.3.2 to twistor spaces and obtain.

Corollary 3.3.4 *The twistor space* $\mathcal{Z}$ *of a quaternionic Kähler manifold of dimension* $4k$ *with negative scalar curvature admits a canonical nearly pseudo-Kähler structure of reducible holonomy contained in* $U(1) \times U(2k)$.

Proof We remark, that in negative scalar curvature the twistor space of a quaternionic Kähler manifold is the total space of a pseudo-Riemannian submersion with totally geodesic fibres. It admits a compatible pseudo-Kähler structure of signature $(2, 4k)$, cf. Besse [18, 14.86 b)]. The assumption of positive scalar curvature is often made to obtain a positive definite metric on $\mathcal{Z}$. Here we focus on pseudo-Riemannian metrics and consequently on negative scalar curvature. □

Proposition 3.3.5 *The twistor spaces* $\mathcal{Z}$ *of non-compact duals of Wolf spaces and of Alekseevskian spaces admit a nearly pseudo-Kähler structure.*

Proof Non-compact duals of Wolf spaces are known [117] to be quaternionic Kähler manifolds of negative scalar curvature. The same holds for Alekseevskian spaces [3, 38]. □

Studying the lists given in [3, 38, 117] we find examples of six-dimensional nearly pseudo-Kähler manifolds.

Corollary 3.3.6 *The twistor spaces* $\mathcal{Z}$ *of*

$$\tilde{\mathbb{H}}P^1 = \mathrm{Sp}(1,1)/\mathrm{Sp}(1)\mathrm{Sp}(1) \text{ and } SU(1,2)/S(U(1)U(2))$$

provide six-dimensional nearly pseudo-Kähler manifolds.

Remark 3.3.7 The situation in negative scalar curvature is more flexible than in the positive case. This is illustrated by the following results in this area: In the main theorem of [89] it is shown that the moduli space of complete quaternionic Kähler metrics on $\mathbb{R}^{4n}$ is infinite dimensional. A construction of super-string theory, called the *c-map* [54], yields continuous families of negatively curved quaternionic Kähler manifolds. Let us mention, that the c-map enjoys very recent interest [10, 73, 93] in differential geometry. These results show that Corollary 3.3.4 is a good source of examples.

Another source of examples is given by twistor spaces over *para-quaternionic Kähler manifolds*. Since these manifolds are less classical than quaternionic Kähler manifolds, we recall some definitions (cf. [5] and references therein).

Definition 3.3.8 Let $(\kappa_1, \kappa_2, \kappa_3) = (-1, 1, 1)$ or some permutation thereof. An almost para-quaternionic structure on a differentiable manifold M^{4k} is a rank 3 sub-bundle $Q \subset End(TM)$, which is locally generated by three anti-commuting endomorphism-fields $J_1, J_2, J_3 = J_1 J_2$. These satisfy $J_i^2 = \kappa_i Id$ for $i = 1, \dots, 3$. Such a triple is called standard local basis of Q. A linear torsion-free connection preserving Q is called para-quaternionic connection. An almost para-quaternionic structure is called a para-quaternionic structure if it admits a para-quaternionic connection. An almost para-quaternionic Hermitian structure (M, Q, g) is a pseudo-Riemannian manifold endowed with a para-quaternionic structure such that Q consists of skew-symmetric endomorphisms. For $n > 1$ (M^{4k}, Q, g) is a para-quaternionic Kähler manifold if Q is preserved by the Levi-Civita connection of g. In dimension 4 a para-quaternionic Kähler manifold M^4 is an anti-self-dual Einstein manifold.
We use the same notions omitting the word *para* for the quaternionic case. The condition that Q is preserved by the Levi-Civita connection is in a given standard local basis $\{J_i\}_{i=1}^3$ of Q equivalent to the equations

$$\nabla_X J_i = -\theta_k(X)\kappa_j J_j + \theta_j(X)\kappa_k J_k, \text{ for } X \in TM, \tag{3.40}$$

where i, j, k is a cyclic permutation of $1, 2, 3$ and $\{\theta_i\}_{i=1}^3$ are local one-forms. In the context of para-quaternionic manifolds one can define twistor spaces for $s = 1, 0, -1$

$$\mathcal{Z}^s := \{A \in Q \,|\, A^2 = sId, \text{ with } A \neq 0\}.$$

The case of interest in this text is $\mathcal{Z} = \mathcal{Z}^{-1}$, since this twistor space is a complex manifold, such that the conditions of Proposition 3.3.2 hold true (cf. [5]). Therefore we obtain the following examples of nearly pseudo-Kähler manifolds.

Corollary 3.3.9 *The twistor space $\mathcal{Z}$ of a para-quaternionic Kähler manifold with non-zero scalar curvature of dimension $4k$ admits a canonical nearly pseudo-Kähler structure of reducible holonomy contained in $U(k, k) \times U(1)$.*

Example 3.3.10 The para-quaternions $\widetilde{\mathbb{H}}$ are the $\mathbb{R}$-algebra generated by $\{1, i, j, k\}$ subject to the relations $i^2 = -1$, $j^2 = k^2 = 1$, $ij = -ji = k$. Like the quaternions, the para-quaternions are a real Clifford algebra which in the convention of [88] is $\widetilde{\mathbb{H}} = Cl_{1,1} \cong Cl_{0,2} \cong \mathbb{R}(2)$. One defines the para-quaternionic projective space $\widetilde{\mathbb{H}P}^n$ by the obvious equivalence relation on the para-quaternionic right-module $\widetilde{\mathbb{H}}^{n+1}$ of $(n+1)$-tuples of para-quaternions. The manifold $\widetilde{\mathbb{H}P}^n$ is a para-quaternionic Kähler manifold [21] in analogue to the quaternionic projective space $\mathbb{H}P^n$. This yields examples of the type described in the last Corollary.

3.4 Complex Reducible Nearly Pseudo-Kähler Manifolds

Motivation In this section we study the case of a nearly pseudo-Kähler manifold (M^{2n}, J, g), such that the holonomy of the characteristic connection $\bar{\nabla}$ is reducible, in the sense that the tangent bundle TM admits a splitting

$$TM = \mathcal{H} \oplus \mathcal{V}$$

into two $\bar{\nabla}$-parallel sub-bundles $\mathcal{H}, \mathcal{V}$, which are orthogonal and invariant with respect to the almost complex structure J. We refer to this situation as complex reducible. This is motivated by the examples on twistor spaces given in the last section. In Sect. 3.9.4 we see, that *real reducible* nearly Kähler manifolds are locally homogeneous.

3.4.1 General Properties

In this subsection we carefully check, generalising [99] to pseudo-Riemannian foliations, the information which follows from the decomposition into the J-invariant sub-bundles.

Lemma 3.4.1 (Lemma 6.1 of [108]) *In the situation of this section and for a vector field X in $\mathcal{H}$, a vector field Y in TM and vector fields U, V in $\mathcal{V}$ it is*

$$\bar{R}(X, Y, U, V) = g\left([\nabla_U J, \nabla_V J]X, Y\right) - g\left((\nabla_X J)Y, (\nabla_U J)V\right). \tag{3.41}$$

Corollary 3.4.2 *For vector fields X, Y in $\mathcal{H}$ and V, W in $\mathcal{V}$ one has*

(i) $(\nabla_X J)(\nabla_V J)W = 0$; $(\nabla_V J)(\nabla_X J)Y = 0$;
(ii) $(\nabla_X J)(\nabla_Y J)Z$ *belongs to $\mathcal{H}$ for all $Z \in \Gamma(\mathcal{H})$;*
(iii) $(\nabla_V J)(\nabla_W J)X$ *belongs to $\mathcal{H}$; and* $(\nabla_X J)(\nabla_Y J)V$ *belongs to $\mathcal{V}$.*

Proof

(i) follows from the fact, that $\bar{R}(JX, JY, V, W) = \bar{R}(X, Y, V, W)$ and that the first term of Eq. (3.41) has the same symmetry with respect to J. This yields on the one hand

$$g\left((\nabla_{JX} J)JY, (\nabla_V J)W\right) = g\left((\nabla_X J)Y, (\nabla_V J)W\right)$$

and on the other hand it is

$$g\left((\nabla_{JX} J)JY, (\nabla_V J)W\right) = -g\left((\nabla_X J)Y, (\nabla_V J)W\right).$$

Consequently one has $g((\nabla_X J)Y, (\nabla_V J)W) = 0$. Exchanging $\mathcal{H}$ and $\mathcal{V}$ finishes part (i).

(ii) From (i) one gets the vanishing of

$$\begin{aligned} g((\nabla_V J)(\nabla_Y J)Z, X) &= g(Z, (\nabla_Y J)(\nabla_V J)X) \\ &= -g(Z, (\nabla_Y J)(\nabla_X J)V) = -g((\nabla_X J)(\nabla_Y J)Z, V). \end{aligned}$$

(iii) From (i) it follows $0 = \bar{R}(X, U, V, W) = g([\nabla_V J, \nabla_W J]X, U)$. This yields $[\nabla_V J, \nabla_W J]X \in \mathcal{H}$ and by $[\nabla_V J, \nabla_{JW} J]JX = -\{\nabla_V J, \nabla_W J\}X \in \mathcal{H}$ we get the first part. The second part follows by replacing $\mathcal{H}$ and $\mathcal{V}$.

□

3.4.2 Co-dimension Two

Motivated by the above section on twistor spaces we suppose from now on that the real dimension of $\mathcal{V}$ is two.

Lemma 3.4.3 (Lemma 6.2 of [108]) *Let* $\dim_{\mathbb{R}}(\mathcal{V}) = 2$.

(i) *Then the restriction of the metric g is either of signature* $(2, 0)$ *or* $(0, 2)$.

(ii) a) $T(V, W) = 0$ *for all* $V, W \in \mathcal{V}$.
b) $T(X, U) \in \mathcal{H}$ *for all* $X \in \mathcal{H}$ *and* $U \in \mathcal{V}$.
c) *In dimension 6 it is* $T(X, Y) \in \mathcal{V}$ *for all* $X, Y \in \mathcal{H}$.
d) $\mathrm{Span}\{\pi_{\mathcal{V}}(T(X, Y)) \mid X, Y \in \mathcal{H}\} = \mathcal{V}$.

Corollary 3.4.4 *Let* $\dim_{\mathbb{R}}(\mathcal{V}) = 2$. *Then the foliation* $\mathcal{V}$ *has totally geodesic fibres and the O'Neill tensor is given by* $A_X Y = \frac{1}{2}\pi_{\mathcal{V}}(J(\nabla_X J)Y)$ *and* $A_X V = \frac{1}{2}J(\nabla_X J)V$. *Moreover it is* $\nabla^{\mathcal{V}} J = 0$.

Proof From Lemma 3.4.3 (ii) a) we obtain $(\nabla_V J)W = 0$ with $V, W \in \Gamma(\mathcal{V})$. By Lemma 3.3.3 part 2) it follows $T_V W = 0$ and $\nabla^{\mathcal{V}} J = 0$, since the decomposition $\mathcal{H} \oplus \mathcal{V}$ is $\bar{\nabla}$ parallel. Part 1) of Lemma 3.3.3 finishes the proof. □

Proposition 3.4.5 *Let* (M, J, g) *be a nearly pseudo-Kähler manifold such that the property of Lemma 3.4.3 (ii) c) is satisfied and such that* $\mathcal{V}$ *has dimension 2, then* $(M, \check{J} = \hat{J}, \check{g} = g_2)$ *is a pseudo-Kähler manifold.*[4]

It is natural to suppose the property of Lemma 3.4.3 (ii) c), since this holds true in the cases of *twistorial type* which are studied in the next sections.

Proof By the last Corollary the data of the submersion is $\check{T} = T \equiv 0$, $A_X Y = \check{A}_X Y = \frac{1}{2}\pi_{\mathcal{V}}(J(\nabla_X J)Y)$ and $\check{A}_X V = 2A_X V = J(\nabla_X J)V$. Since A anti-commutes with J it commutes with $\check{J}$. This yields the conditions (3.33) and (3.34) of

[4] Here we use $\check{\cdot}$ for the inverse construction of $\hat{\cdot}$.

Lemma 3.3.1 on the triple $\check{A}, \check{T}, \check{J}$. Further it holds $\nabla^{\mathcal{V}} J = 0$. From the reasoning of Eq. (3.35) we obtain $\pi_{\mathcal{H}}((\check{\nabla}_X \check{J})Y) = \pi_{\mathcal{H}}((\nabla_X J)Y)$ which vanishes by the property of Lemma 3.4.3 (ii) c). By an analogous argument we get from Eq. (3.36) the identity $\pi_{\mathcal{V}}((\check{\nabla}_X \check{J})V) = -\pi_{\mathcal{V}}((\nabla_X J)V)$. This vanishes by Lemma 3.4.3 (ii) b). From Eq. (3.37) we derive $-\pi_{\mathcal{H}}((\nabla_X J)V) \overset{n.K.}{=} \pi_{\mathcal{H}}((\nabla_V J)X) = \pi_{\mathcal{H}}((\check{\nabla}_V \check{J})X) + 2\pi_{\mathcal{H}}(JA_X V)$. The definition of $A_X V$ yields $\pi_{\mathcal{H}}((\check{\nabla}_V \check{J})X) = 0$. These are all the identities needed to apply Lemma 3.3.1. □

Proposition 3.4.6 *Let X, Y be vector fields in $\mathcal{H}$ and V_1, V_2, V_3 be vector fields in $\mathcal{V}$. Suppose that it holds $T(V, W) = 0$ for all $V, W \in \mathcal{V}$ then it is*

$$\bar{R}((\nabla_X J)JY, V_1, V_2, V_3) = g(JY, [\nabla_{V_1} J, [\nabla_{V_2} J, \nabla_{V_3} J]]X). \tag{3.42}$$

Moreover, one has $\bar{\nabla}_U \bar{R}(V_1, V_2, V_3, V_4) = 0$.

Proof For $V_1, V_2, V_3 \in \mathcal{V}$ and $X \in \mathcal{H}$ the second Bianchi identity gives

$$-\underset{XYV_1}{\sigma} \bar{\nabla}_X(\bar{R})(Y, V_1, V_2, V_3) = \underset{XYV_1}{\sigma} \bar{R}((\nabla_X J)JY, V_1, V_2, V_3).$$

As the decomposition $\mathcal{H} \oplus \mathcal{V}$ is $\bar{\nabla}$-parallel the terms on the left hand-side vanish due to the symmetries (3.24) of the curvature tensor $\bar{R}$. The right hand-side is determined with the help of Lemma 3.4.1 and Corollary 3.4.2. If we apply $\bar{\nabla}$ to the formula (3.42) we obtain by $\bar{\nabla}(\nabla J) = 0$ the identity $g\big(\bar{\nabla}_U(\bar{R})(V_1, V_2, V_3), (\nabla_X J)Z\big) = 0$ with $Z = JY$. This yields the proposition using Lemma 3.4.3 (ii) part d). □

3.4.3 Six-Dimensional Nearly Pseudo-Kähler Manifolds

Before analysing the general case we first focus on dimension 6.

Lemma 3.4.7 (Lemma 6.7 of [108]) *On a six-dimensional nearly pseudo-Kähler manifold (M^6, J, g) the integral manifolds of the foliation $\mathcal{V}$ have Gaussian curvature 4α and constant curvature $\kappa = 4\alpha$, where α is the type constant.*

Let us recall that the sign of α is completely determined by the signature of the metric g, cf. Remark 3.1.9.

Proposition 3.4.8 *The manifold (M, J, g) is the total space of a pseudo-Riemannian submersion $\pi : (M, g) \to (N, h)$ where (N, h) is an almost pseudo-Hermitian manifold and the fibres are totally geodesic Hermitian symmetric spaces. In particular, the fibres are simply connected.*

Proof The foliation which is induced by $\mathcal{V}$ is totally geodesic and each leaf is by Proposition 3.4.6 a locally Hermitian symmetric space of complex dimension 1.

It is shown in Lemma 3.4.7 that each leaf has constant curvature κ. In the case $\kappa > 0$ the leaves are compact and we can apply a result of Kobayashi, cf. [18, 11.26], to obtain that the leaves are simply connected. Since the leaves are also simply connected it follows, that the leaf holonomy is trivial and that the foliation comes from a (smooth) submersion (cf. p. 90 of [113]). In the case $\kappa < 0$ we observe, that $(M, J, -g)$ is a nearly pseudo-Kähler manifold of constant type $-\alpha$. The same argument shows that the fibres are simply connected. □

Lemma 3.4.9 (Lemma 6.9 of [108]) *Let (M^6, g, J) be a strict nearly pseudo-Kähler six-manifold of constant type α. For an arbitrary normalised*[5] *local vector field $V \in \mathcal{V}$, i.e. $\epsilon_V = g(V,V) \in \{\pm 1\}$, we consider the endomorphisms $\tilde{J}_1 := J_{|\mathcal{H}}$, $\tilde{J}_2 : \mathcal{H} \ni X \mapsto \sqrt{|\alpha|}^{-1}(\nabla_V J)X \in \mathcal{H}$ and $\tilde{J}_3 = \tilde{J}_1\tilde{J}_2$. Then the triple $(\tilde{J}_1, \tilde{J}_2, \tilde{J}_3)$ defines an ε-quaternionic triple on $\mathcal{H}$ with $\kappa_1 = -1$ and $\kappa_2 = \kappa_3 = sign(-\alpha\epsilon_V)$ and it is*

$$\pi^{\mathcal{H}}[(\nabla_\chi \tilde{J}_i)Y] = -\theta_k(\chi)\,\kappa_j \tilde{J}_j Y + \theta_j(\chi)\,\kappa_k \tilde{J}_k Y,$$

for a cyclic perm. of i,j,k and with $\theta_1(\chi) = sign(\alpha) g(JV, \bar{\nabla}_\chi V)$, $\theta_2(\chi) = -sign(\alpha)\sqrt{|\alpha|} g(V, J\chi)$ and $\theta_3(\chi) = sign(\alpha)\sqrt{|\alpha|} g(V,\chi)$. The sub-bundle of endomorphisms spanned by $(\tilde{J}_1, \tilde{J}_2, \tilde{J}_3)$ does not depend on the choice of V.

Lemma 3.4.10 *Let (M^6, g, J) be a strict nearly pseudo-Kähler six-manifold of constant type α. Let $s : U \subset N \to M$ be a (local) section*[6] *of π on some open set U. Define ϕ by*

$$\phi = s_* \circ \pi_* : \mathcal{H}_{s(n)} \overset{\pi_*}{\to} T_n N \overset{s_*}{\to} s_*(T_n N) \subset T_{s(n)} M, \text{ for } n \in N$$

and set $J_{i|n} := \pi_* \circ \tilde{J}_{i|s(n)} \circ (\pi_{*|\mathcal{H}})^{-1}$ for $i = 1, \dots, 3$, where $\tilde{J}_i$ are defined in Lemma 3.4.9. Then (J_1, J_2, J_3) defines a local ϵ-quaternionic basis preserved by the Levi-Civita connection ∇^N of N.

Proof We choose U such that the section s is a diffeomorphism onto $W = s(U)$ and a vector field V in $\mathcal{V}$ defined on a subset containing W. As π is a pseudo-Riemannian submersion we obtain from $\pi_* \circ s_* = Id$ that s is an isometry from U onto W. Therefore it holds $s_*(\nabla^N_X Y) = \pi^{s_* TN}[\nabla_{s_* X} s_* Y]$ which yields $\nabla^N_X Y = \pi_*(\nabla_{s_* X} s_* Y)$ and

$$(\pi_{*|\mathcal{H}})^{-1}(\nabla^N_X Y) = \pi^{\mathcal{H}}(\nabla_{s_* X} s_* Y). \tag{3.43}$$

For convenience let us identify U and W or in other words consider s as the inclusion $W \subset M$. Then the projection on $s_* TN$ is $\phi = s_*\pi_* = \pi_*|\mathcal{H}$. Moreover, we need the (tensorial) relation[7]

$$\nabla^N_X(\pi_* Z) - \pi_*\pi^{\mathcal{H}}(\nabla^M_X Z) = 0 \text{ or equivalently } \nabla^N_X \tilde{Z} - \pi_*\pi^{\mathcal{H}}(\nabla^M_X \phi^{-1}\tilde{Z}) = 0,$$

[5] Constant non-zero length suffices.

[6] Local sections exist, since π is locally trivial [18, 9.3].

[7] Here $\tilde{Z}$ is the horizontal lift of Z.

which can be directly checked for basic vector fields. Using this identity we get for $i = 1, \dots, 3$

$$\begin{aligned}\nabla^N_X(J_iY) &= \nabla^N_X(\phi\,\tilde{J}_i\phi^{-1}Y) = \phi\nabla^M_X(\tilde{J}_i\phi^{-1}Y) = \phi\,(\nabla^M_X\tilde{J}_i)\,\phi^{-1}Y + \phi\,\tilde{J}_i\,\nabla^M_X(\phi^{-1}Y)\\ &= \phi\,(\nabla^M_X\tilde{J}_i)\,\phi^{-1}Y + \phi\,\tilde{J}_i\,\phi^{-1}\nabla^N_XY = \phi\,(\nabla^M_X\tilde{J}_i)\,\phi^{-1}Y + J_i\nabla^N_XY,\end{aligned}$$

which reads $(\nabla^N_XJ_i)Y = \phi\,(\nabla^M_X\tilde{J}_i)\,\phi^{-1}Y$. This finishes the proof, since the right hand-side is completely determined by Lemma 3.4.9. Therefore we have checked the condition (3.40), i.e. the manifold N is endowed with a parallel skew-symmetric (para-)quaternionic structure, see also [18, 10.32 and 14.36]. □

3.4.4 General Dimension

In the last section we have seen that in dimension 6 the tensor ∇_VJ induces a (para-)complex structure on $\mathcal{H}$. This motivates the following definition.

Definition 3.4.11 The foliation induced by $TM = \mathcal{H} \oplus \mathcal{V}$ is called of twistorial type if for all $p \in M$ there exists a $V \in \mathcal{V}_p$ such that the endomorphism

$$\nabla_VJ : \mathcal{H}_p \to \mathcal{H}_p$$

is injective.

Obviously, if ∇_VJ defines a (para-)complex structure, then the foliation is of twistorial type.

Proposition 3.4.12

(a) *If the metric induced on $\mathcal{H}$ is definite, then the foliation is of twistorial type.*
(b) *If the foliation is of twistorial type, then for all $p \in M$ and all $0 \neq U \in \mathcal{V}_p$ the endomorphism*

$$\nabla_UJ : \mathcal{H}_p \to \mathcal{H}_p$$

is injective.
(c) *It holds with $A := \nabla_VJ$ for some vector field V in $\mathcal{V}$ of constant length and for vector fields $X \in \mathcal{H}$ and $\chi \in TM$*

$$\bar{\nabla}_\chi(A^2)X = 0. \tag{3.44}$$

Further it holds $[A^2, (\nabla_UJ)] = 0$ for all $U \in \mathcal{V}$ and

$$\nabla_U(A^2)X = 0 \tag{3.45}$$

for vector fields U in $\mathcal{V}$.

Proof Part (a) follows from $(\nabla_V J)X \in \mathcal{H}$ for $X \in \mathcal{H}$ and $V \in \mathcal{V}$, cf. Lemma 3.4.3 (i). For (b) we observe, that if $\nabla_V J$ is injective so is $\nabla_{JV} J = -J\nabla_V J$. As $\mathcal{V}$ is of dimension 2 $\{V, JV\}$ with $V \neq 0$ is an orthogonal basis. With $a, b \in \mathbb{R}$ it follows $g((a\nabla_V J + b\nabla_{JV} J)X, (a\nabla_V J + b\nabla_{JV} J)X) = (a^2 + b^2)\, g((\nabla_V J)X, (\nabla_V J)X)$, which yields, that $\nabla_{aV+bJV} J : \mathcal{H}_p \to \mathcal{H}_p$ is injective since $a \neq 0$ or $b \neq 0$. It remains to prove part (c). We first observe, that, since V has constant length and since $\bar{\nabla}$ is a metric connection and preserves $\mathcal{V}$, it follows $\bar{\nabla}_\chi V = \alpha(\chi)JV$ for some one-form α. From $\bar{\nabla}(\nabla J) = 0$ we obtain

$$(\bar{\nabla}_\chi A)X = (\bar{\nabla}_\chi(\nabla_V J))X = (\nabla_{\bar{\nabla}_\chi V} J)X = \alpha(\chi)(\nabla_{JV} J)X = -\alpha(\chi)JAX$$

and we compute using $\{A, J\} = 0$

$$\bar{\nabla}_\chi(A^2)X = A(\bar{\nabla}_\chi A)X + (\bar{\nabla}_\chi A)AX = -\alpha(\chi)[A(J(AX)) + JA^2X] = 0.$$

The equation $[A^2, (\nabla_U J)] = 0$ is tensorial in U and holds true for $U = V$. Therefore we only need to compute $[A^2, (\nabla_{JV} J)] = -[A^2, J(\nabla_V J)] = -J[A^2, (\nabla_V J)] = 0$, where we used that A^2 commutes with J. This implies

$$\nabla_U(A^2)X = \bar{\nabla}_U(A^2)X + \frac{1}{2}[J(\nabla_U J), A^2]X = -\frac{1}{2}[(\nabla_{JU} J), A^2]X = 0$$

and proves part (c). □

In the following V is a local vector field of constant length $\epsilon_V = g(V, V) \in \{\pm 1\}$.

We denote by Ω the curvature form of the connection induced by $\bar{\nabla}$ on the (complex) line bundle $\mathcal{V}$, which is given by

$$\bar{R}(X, Y)V = \Omega(X, Y)JV, \text{ for } X, Y \in TM, V \in \mathcal{V}.$$

Proposition 3.4.13 *If the foliation is of twistorial type,*

(i) *then the endomorphism* $A := \nabla_V J_{|\mathcal{H}}$ *satisfies* $A^2 = \kappa\epsilon_V Id_{\mathcal{H}}$ *for some real constant* $\kappa \neq 0$ *and*

$$\Omega = -2\kappa(2\omega^{\mathcal{V}} - \omega^{\mathcal{H}}),$$

where $\omega^{\mathcal{H}}(X, Y) = g(X, JY)$ *is the restriction of the fundamental two-form* ω *to* $\mathcal{H}$;

(ii) *for* X, Y *in* $\mathcal{H}$ *it is* $(\nabla_X J)Y \in \mathcal{V}$.

The proof of this proposition is divided in several steps.

Lemma 3.4.14

(i) *For* X, Y *in* $\mathcal{H}$ *and* V *in* $\mathcal{V}$ *it is* $\bar{R}(X, Y, V, JV) = -2g((\nabla_V J)^2X, JY)$.

(ii) *For a given* X *in* $\mathcal{H}$ *and* V *in* $\mathcal{V}$ *it follows* $\bar{R}(X, V, V, JV) = 0$.

Proof

(i) Since $\mathcal{H}$ is $\bar{\nabla}$-parallel we obtain, that $\underset{XYV}{\sigma}\bar{R}(X,Y,V,JV) = \bar{R}(X,Y,V,JV)$. This is the left hand-side of the first Bianchi identity (3.26) . The right hand-side reads

$$-\underset{XYV}{\sigma} g((\nabla_X J)Y, (\nabla_V J)JV) = -g((\nabla_V J)X, (\nabla_Y J)JV) - g((\nabla_Y J)V, (\nabla_X J)JV)$$
$$= -2g((\nabla_V J)^2 X, JY).$$

(ii) From the symmetries (3.24) of the curvature tensor $\bar{R}$ it follows $\bar{R}(X,V,V,JV) = \bar{R}(V,JV,X,V)$. This expression vanishes since $\mathcal{H}$ is $\bar{\nabla}$-parallel.

□

From the last lemma we derive the more explicit expression of the curvature form $\Omega(\cdot,\cdot)$:

$$\Omega(\cdot,\cdot) = f\omega^{\mathcal{V}}(\cdot,\cdot) + \epsilon_V \alpha(\cdot,\cdot), \tag{3.46}$$

where f is a smooth function, $\omega^{\mathcal{V}}$ is the restriction of the fundamental two-form $\omega = g(\cdot, J\cdot)$ to $\mathcal{V}$ and $\alpha(X,Y) = -2g(A^2X, JY)$.

Lemma 3.4.15 (Lemma 6.15 of [108]) *It holds with $U \in \mathcal{V}$ and $X, Y \in \mathcal{H}$:*

$$d\omega^{\mathcal{V}}(X,U,JU) = 0, \tag{3.47}$$
$$d\alpha(X,U,JU) = 0, \tag{3.48}$$
$$d\omega^{\mathcal{V}}(U,X,Y) = -g((\nabla_U J)X, Y), \tag{3.49}$$
$$d\alpha(U,X,Y) = 4g(A^2(\nabla_U J)X, Y). \tag{3.50}$$

Proof of the Proposition 3.4.13

(i) Let X, Y be vector fields in $\mathcal{H}$ and V be a local vector field in $\mathcal{V}$ of constant length. Since Ω as a curvature form of a (Hermitian) line bundle is closed, we obtain from Eq. (3.46) $-\epsilon_V d\alpha(\cdot,\cdot,\cdot) = f d\omega^{\mathcal{V}}(\cdot,\cdot,\cdot) + df \wedge \omega^{\mathcal{V}}(\cdot,\cdot,\cdot)$. Equations (3.47) and (3.48) imply $df_{|\mathcal{H}} = 0$. This implies $[X,Y]f = 0$ and using that $\mathcal{H}$ is $\bar{\nabla}$-parallel we obtain $(\bar{\nabla}_X Y)f = 0 = (\bar{\nabla}_Y X)f$ which yields finally $0 = T(X,Y)(f) = -[J(\nabla_X J)Y](f)$. By Lemma 3.4.3 (ii) d) the last equation shows $df_{|\mathcal{V}} = 0$. Since M is connected, it follows $f \equiv -\kappa$ for a constant κ.

Again using $d\Omega(V,X,Y) = 0$ Eqs. (3.49) and (3.50) yield for arbitrary X, Y

$$\kappa g((\nabla_V J)X, Y) + 4\epsilon_V g(A^2(\nabla_V J)X, Y) = 0.$$

This implies $(\nabla_V J)(\kappa Id_{\mathcal{H}} + 4\epsilon_V A^2) = 0$. It follows

$$A^2 = -\epsilon_V \frac{\kappa}{4} Id_{\mathcal{H}},$$

since the foliation is of twistorial type. If we set $4\alpha = \kappa$ in analogue to dimension 6[8] one gets $A^2 = -\epsilon_V \alpha Id_{\mathcal{H}}$.

(ii) Since Ω is closed, it follows from part (i) and $d\omega^{\mathcal{V}}(X, Y, Z) = 0$ for $X, Y, Z \in \mathcal{H}$ that it is $d\omega^{\mathcal{H}}(X, Y, Z) = 0$. Using $d\omega^{\mathcal{H}}(X, Y, Z) = 3g((\nabla_X J)Y, Z)$ yields part (ii).

□

Proposition 3.4.16 *Let (M^{4k+2}, g, J) be a strict nearly pseudo-Kähler manifold of twistorial type. Let $s : U \subset N \to M$ be a (local) section of π on some open set U. Define ϕ by*

$$\phi = s_* \circ \pi_* : \mathcal{H}_{s(n)} \xrightarrow{\pi_*} T_n N \xrightarrow{s_*} s_*(T_n N) \subset T_{s(n)} M, \text{ for } n \in N$$

and set $J_{i|n} := \pi_* \circ \tilde{J}_{i|s(n)} \circ (\pi_{*|\mathcal{H}})^{-1}$ for $i = 1, \ldots, 3$, where $\tilde{J}_i$ are defined in Lemma 3.4.9. Then (J_1, J_2, J_3) defines a local ϵ-quaternionic basis preserved by the Levi-Civita connection ∇^N of N.

Proof The proof of Lemma 3.4.9 only uses $A^2 = -\epsilon_V \frac{\kappa}{4} Id$ and $(\nabla_X J)Y \in \mathcal{V}$ for $X, Y \in \mathcal{H}$. Therefore we can generalise it by means of Proposition 3.4.13 to strict nearly pseudo-Kähler manifolds of twistorial type. □

Corollary 3.4.17

(i) *The tensor r has exactly two eigenvalues. More precisely, it has the eigenvalue κ on $\mathcal{H}$ and the eigenvalue $\epsilon_V \frac{\kappa}{2}(n-1)$ on $\mathcal{V}$ with $\kappa = 4\alpha$.*

(ii) *The Ricci-tensor has exactly two eigenvalues. More precisely, it has the eigenvalue $\frac{\kappa}{4}(\epsilon_V(n-1)+3)$ on $\mathcal{H}$ and the eigenvalue $\kappa\left(\epsilon_V \frac{(n-1)}{8} + 1\right)$ on $\mathcal{V}$. The base manifold (N, h) is an Einstein manifold with Einstein constant $\frac{\kappa}{4}\epsilon_V(n-1)$.*

Proof By definition we have

$$g(rX, Y) = \sum_{i=1}^{2n} \epsilon_i \, g((\nabla_X J)e_i, (\nabla_Y J)e_i) = -\sum_{i=1}^{2n} \epsilon_i \, g((\nabla_Y J)(\nabla_X J)e_i, e_i)$$

for some pseudo-orthogonal basis with $e_1, \ldots, e_{2n-2} \in \mathcal{H}$ and $e_{2n-1}, e_{2n} \in \mathcal{V}$. For $V \in \mathcal{V}$ with $g(V, V) = \epsilon_V$ we get

$$\begin{aligned} g(rV, V) &= -\sum_{i=1}^{2n} \epsilon_i \, g((\nabla_V J)(\nabla_V J)e_i, e_i) = -\sum_{i=1}^{2n-2} \epsilon_i \, g(A^2 e_i, e_i) \\ &= \epsilon_V \frac{\kappa}{4} \sum_{i=1}^{2n-2} \epsilon_i \, g(e_i, e_i) = \epsilon_V \frac{\kappa}{2}(n-1). \end{aligned}$$

[8] Without risk of confusion we use the same latter for the constant α as for the two-form $\alpha(\cdot, \cdot)$.

Let us now consider $X \in \mathcal{H}$ and V as before and compute

$$g(rX, X) = 2\epsilon_V g\left((\nabla_V J)X, (\nabla_V J)X\right) + \sum_{i=1}^{2n-2} \epsilon_i\, g((\nabla_X J)e_i, (\nabla_X J)e_i).$$

Since it is $(\nabla_X J)e_i \in \mathcal{V}$, we get

$$\epsilon_i\, g((\nabla_X J)e_i, (\nabla_X J)e_i) = \epsilon_i\, \epsilon_V (g((\nabla_X J)e_i, V)^2 + g((\nabla_X J)e_i, JV)^2)$$

and for the sum this gives

$$\begin{aligned} g(r^{\mathcal{H}}X, X) &= \sum_{i=1}^{2n-2} \epsilon_i\, g((\nabla_X J)e_i, (\nabla_X J)e_i) \\ &= \epsilon_V \sum_{i=1}^{2n-2} \epsilon_i\, \left(g((\nabla_X J)V, e_i)^2 + g((\nabla_X J)JV, e_i)^2\right) \\ &= 2\epsilon_V\, g\left((\nabla_V J)X, (\nabla_V J)X\right) = \frac{\kappa}{2} g(X, X). \end{aligned}$$

Summarizing it follows $g(rX, X) = 4\epsilon_V\, g\left((\nabla_V J)X, (\nabla_V J)X\right) = \kappa$. This shows part (i).

The statement (ii) follows from (i) using Lemma 3.2.7. Namely, for $X, Y \in \mathcal{H}$ it is

$$g(Ric(X), Y) = \frac{\kappa}{4} g(X, Y) + \epsilon_V \frac{1}{\kappa}\frac{\kappa}{2}(n-1)\, \underbrace{g(r^{\mathcal{V}}X, Y)}_{\frac{\kappa}{2}g(X,Y)} + g(r^{\mathcal{H}}X, Y) = \frac{\kappa}{4}(\epsilon_V(n-1)+3) g(X, Y),$$

since it is using $A^2 = -\epsilon_V \frac{\kappa}{4} Id_{\mathcal{H}}$

$$\begin{aligned} g(r^{\mathcal{V}}X, Y) &= -\mathrm{tr}_{\mathcal{V}}\left((\nabla_X J) \circ (\nabla_Y J)\right) = -2\epsilon_V\, g((\nabla_X J)(\nabla_Y J)V, V) \\ &= 2\epsilon_V\, g((\nabla_Y J)V, (\nabla_X J)V) = 2\epsilon_V\, g((\nabla_V J)Y, (\nabla_V J)X) \\ &= 2\epsilon_V\, g(AY, AX) = \frac{\kappa}{2} g(X, Y). \end{aligned}$$

Further, for $U, V \in \mathcal{V}$ it is

$$\begin{aligned} g(Ric(U), V) &= \epsilon_V \frac{\kappa(n-1)}{8} g(U, V) + \epsilon_V \frac{2}{\kappa(n-1)} \kappa \underbrace{g(r^{\mathcal{H}}U, V)}_{\epsilon_V \frac{\kappa}{2}(n-1) g(U,V)} \\ &= \kappa \left(\epsilon_V \frac{(n-1)}{8} + 1\right) g(U, V), \end{aligned}$$

where

$$g(r^{\mathcal{H}}V, V) = -\mathrm{tr}_{\mathcal{H}}\left((\nabla_V J) \circ (\nabla_V J)\right) = -\sum_{i=1}^{2n-2} \epsilon_i\, g((\nabla_V J)(\nabla_V J)e_i, e_i)$$
$$= 2(n-1)\frac{\kappa}{4}\epsilon_V = \epsilon_V \frac{\kappa}{2}(n-1).$$

The last statement follows from O'Neill's formula with the information, that the O'Neill tensor is $A_X Y = \frac{1}{2} J(\nabla_X J)Y$, c.f. Lemma 3.3.3. □

3.4.5 The Twistor Structure

In this subsection we finally characterise the nearly pseudo-Kähler structures, which are related to the canonical nearly Kähler structure of twistor spaces.

Theorem 3.4.18 *Suppose, that (M^{2n}, J, g) is a complex reducible nearly pseudo-Kähler manifold of twistorial type, then one has:*

(i) The manifold $(M, J = \check{J}, \check{g} = g_2)$ is a twistor space of a quaternionic pseudo-Kähler manifold, if it is $\epsilon_V \kappa > 0$.
(ii) The manifold $(M, J = \check{J}, \check{g} = g_2)$ is a twistor space of a para-quaternionic Kähler manifold, if it is $\epsilon_V \kappa < 0$.

Proof Denote by $\pi^{\mathcal{Z}} : \mathcal{Z} \to N$ the twistor space of the manifold N endowed with the parallel skew-symmetric (para-)quaternionic structure constructed from the foliation $\pi : M \to N$ of twistorial type, cf. Proposition 3.4.9 for dimension 6 and Proposition 3.4.16 for general dimension. We observe that the restriction of J to $\mathcal{H}$ yields a (smooth) map

$$\varphi : M \to \mathcal{Z}, \quad m \mapsto d\pi_m \circ J_{m\,|\mathcal{H}} \circ (d\pi_{m|\mathcal{H}})^{-1} =: j_{\pi(m)},$$

which by construction satisfies $\pi^{\mathcal{Z}} \circ \varphi = \pi$ and as a consequence $d\pi^{\mathcal{Z}} \circ d\varphi = d\pi$. Since π and $\pi^{\mathcal{Z}}$ are pseudo-Riemannian submersions, the last equation implies that $d\varphi$ induces an isometry of the according horizontal distributions and maps the vertical spaces into each other. Let us determine the differential of φ on $\mathcal{V}$.

Claim: For $V \in \mathcal{V}$ one has

$$d\varphi(V) = 2\, d\pi \circ (\nabla_V J) \circ (d\pi_{|\mathcal{H}})^{-1},$$
$$d\varphi(JV) = 2\, d\pi \circ (\nabla_{JV} J) \circ (d\pi_{|\mathcal{H}})^{-1} = -2\, d\pi \circ J(\nabla_V J) \circ (d\pi_{|\mathcal{H}})^{-1}.$$

To prove the claim we consider a (local) vector field $V \in \mathcal{V}$ and a (local) integral curve γ of V on some interval $I \ni 0$ with $\gamma(0) = m$. Let X be a vector field in N. Denote by $\tilde{X}$ the horizontal lift of X. The Lie transport of $\tilde{X}$ along the vertical curve γ projects to X, i.e. it holds $d\pi_{\gamma(t)}(\tilde{X}) = X$ for all $t \in I$ and in consequence $\left(d\pi_{\gamma(t)\,|\mathcal{H}}\right)^{-1} X = \tilde{X}$. In other words $d\pi$ commutes with this Lie transport, which implies

$$d\varphi(V)X = d\pi((\mathcal{L}_V J)\tilde{X}),$$

as one directly checks using basic vector fields. Therefore we need to determine the Lie-derivative $\mathcal{L}$ of J :

$$\begin{aligned}
\pi^{\mathcal{H}}((\mathcal{L}_V J)\tilde{X}) &= \pi^{\mathcal{H}}([V, J\tilde{X}] - J[V, \tilde{X}]) \\
&= \pi^{\mathcal{H}}\left(\nabla_V(J\tilde{X}) - \nabla_{J\tilde{X}}V - J\nabla_V\tilde{X} + J\nabla_{\tilde{X}}V\right) \\
&= \pi^{\mathcal{H}}\left((\nabla_V J)\tilde{X} - \frac{1}{2}J(\nabla_{J\tilde{X}}J)V + \frac{1}{2}J\left(J(\nabla_{\tilde{X}}J)\right)V\right) = 2(\nabla_V J)\tilde{X}.
\end{aligned}$$

This shows $d\varphi(V) = 2\, d\pi \circ (\nabla_V J) \circ (d\pi_{|\mathcal{H}})^{-1}$, which implies $d\varphi(JV) = 2\, d\pi \circ (\nabla_{JV} J) \circ (d\pi_{|\mathcal{H}})^{-1} = -2\, d\pi \circ J(\nabla_V J) \circ (d\pi_{|\mathcal{H}})^{-1}$. Given a local section $s : N \to M$ and the associated adapted frame of the (para-)quaternionic structure it follows that $\varphi \circ s$ is J_1, $d\varphi(V)$ is related to J_2 and $d\varphi(JV)$ to $-J_3$ which span the tangent space of the fibre $F_{\pi(m)} = S^2$ in $\varphi(m)$. The complex structure of $\mathcal{Z}$ maps J_2 to J_3. Hence $d\varphi$ is complex linear for the opposite complex structure $\check{J}$ on M. Further one sees in this local frame that φ maps horizontal part into horizontal part. Therefore φ is an isometry for the metric $\check{g} = g_2$, where the parameter in the canonical variation of the metric g is $t = 2$. This means that $(M, \check{J}, \check{g} = g_2)$ is isometrically biholomorph to $\mathcal{Z}$. □

Combining Theorems 3.2.10 and 3.4.18 we obtain the following result.

Theorem 3.4.19 *Let (M^{10}, J, g) be a nice decomposable nearly pseudo-Kähler manifold, then the universal cover of M is either the product of a pseudo-Kähler surface and a (strict) nearly pseudo-Kähler manifold M^6 or a twistor space of an eight-dimensional (para-)quaternionic Kähler manifold endowed with its canonical nearly pseudo-Kähler structure.*

3.5 A Class of Flat Pseudo-Riemannian Lie Groups

In this section we consider flat pseudo-Riemannian Lie groups which are closely related to nearly Kähler geometry (cf. Sect. 3.6). These geometric objects are also of independent interest [13]. Let $V = (\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ be the standard pseudo-Euclidian vector space of signature (k, l), $n = k + l$. Using the (pseudo-Euclidian) scalar product we shall identify $V \cong V^*$ and $\Lambda^2 V \cong \mathfrak{so}(V)$. These identifications provide the inclusion $\Lambda^3 V \subset V^* \otimes \mathfrak{so}(V)$. Using it we consider a three-vector $\eta \in \Lambda^3 V$ as an $\mathfrak{so}(V)$-valued one-form. Further we denote by $\eta_X \in \mathfrak{so}(V)$ the evaluation of this one-form on a vector $X \in V$. Let us recall, that the support of $\eta \in \Lambda^3 V$ is defined by

$$\Sigma_\eta := \mathrm{span}\{\eta_X Y \,|\, X, Y \in V\} \subset V. \tag{3.51}$$

Theorem 3.5.1 *Each*

$$\eta \in \mathcal{C}(V) := \{\eta \in \Lambda^3 V \,|\, \Sigma_\eta \text{ (totally) isotropic}\} = \bigcup_{L \subset V} \Lambda^3 L$$

defines a 2-step nilpotent simply transitive subgroup $\mathcal{L}(\eta) \subset Isom(V)$, where the union runs over all maximal isotropic subspaces. The subgroups $\mathcal{L}(\eta)$, $\mathcal{L}(\eta') \subset Isom(V)$ associated to $\eta, \eta' \in \mathcal{C}(V)$ are conjugated if and only if $\eta' = g \cdot \eta$ for some element of $g \in O(V)$.

Proof Using Lemma 2.3.2 of Chap. 2 any three-vector $\eta \in \Lambda^3 V$ satisfies $\eta \in \Lambda^3 \Sigma_\eta$. This implies the equation $\mathcal{C}(V) = \bigcup_{L \subset V} \Lambda^3 L$. Let an element $\eta \in \mathcal{C}(V)$ be given. By Lemma 2.3.4 of Chap. 2 its support Σ_η is isotropic if and only if the endomorphisms $\eta_X \in \mathfrak{so}(V)$ satisfy $\eta_X \circ \eta_Y = 0$ for all $X, Y \in V$. The 2-step nilpotent group

$$\mathcal{L}(\eta) := \left\{ \exp \begin{pmatrix} \eta_X & X \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} Id + \eta_X & X \\ 0 & 1 \end{pmatrix} \,\middle|\, X \in V \right\}$$

acts simply transitively on $V \cong V \times \{1\} \subset V \times \mathbb{R}$ by isometries:

$$\begin{pmatrix} Id + \eta_X & X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} X \\ 1 \end{pmatrix}.$$

Let us consider next $\eta, \eta' \in \mathcal{C}(V)$, $g \in O(V)$. The computation

$$g\mathcal{L}(\eta)g^{-1} = \left\{ \begin{pmatrix} Id + g\eta_X g^{-1} & gX \\ 0 & 1 \end{pmatrix} \,\middle|\, X \in V \right\} = \left\{ \begin{pmatrix} Id + g\eta_{g^{-1}Y} g^{-1} & Y \\ 0 & 1 \end{pmatrix} \,\middle|\, Y \in V \right\}$$

shows that $g\mathcal{L}(\eta)g^{-1} = \mathcal{L}(\eta')$ if and only if $\eta'_X = (g \cdot \eta)_X = g\, \eta_{g^{-1}X}\, g^{-1}$ for all $X \in V$. □

Let $\mathcal{L} \subset Isom(V)$ be a simply transitive group. Pulling back the scalar product on V by the orbit map $\mathcal{L} \ni g \mapsto g0 \in V$ yields a left-invariant flat pseudo-Riemannian metric h on $\mathcal{L}$. A pair $(\mathcal{L}, h)$ consisting of a Lie group $\mathcal{L}$ and a flat left-invariant pseudo-Riemannian metric h on $\mathcal{L}$ is called a flat pseudo-Riemannian Lie group.

Theorem 3.5.2

(i) *The class of flat pseudo-Riemannian Lie groups* $(\mathcal{L}(\eta), h)$ *defined in Theorem 3.5.1 exhausts all simply connected flat pseudo-Riemannian Lie groups with bi-invariant metric.*
(ii) *A Lie group with bi-invariant metric is flat if and only if it is 2-step nilpotent.*

Proof

(i) The group $\mathcal{L}(\eta)$ associated to a three-vector $\eta \in \mathcal{C}(V)$ is diffeomorphic to $\mathbb{R}^n$ by the exponential map. We have to show that the flat pseudo-Riemannian metric h on $\mathcal{L}(\eta)$ is bi-invariant. The Lie algebra of $\mathcal{L}(\eta)$ is identified with the vector space V endowed with the Lie bracket

$$[X, Y] := \eta_X Y - \eta_Y X = 2\eta_X Y, \quad X, Y \in V.$$

The left-invariant metric h on $\mathcal{L}(\eta)$ corresponds to the scalar product $\langle\cdot,\cdot\rangle$ on V. Since $\eta \in \Lambda^3 V$, the endomorphisms $\eta_X = \frac{1}{2} ad_X$ are skew-symmetric. This shows that h is bi-invariant.

Conversely, let $(V, [\cdot,\cdot])$ be the Lie algebra of a pseudo-Riemannian Lie group of dimension n with bi-invariant metric h. We can assume that the bi-invariant metric corresponds to the standard scalar product $\langle\cdot,\cdot\rangle$ of signature (k, l) on V. Let us denote by $\eta_X \in \mathfrak{so}(V)$, $X \in V$, the skew-symmetric endomorphism of V which corresponds to the Levi-Civita covariant derivative D_X acting on left-invariant vector fields. From the bi-invariance and the Koszul formula we obtain that $\eta_X = \frac{1}{2} ad_X$ and, hence, $R(X, Y) = -\frac{1}{4} ad_{[X,Y]}$ for the curvature. The last formula shows that h is flat if and only if the Lie group is 2-step nilpotent. This proves (ii). To finish the proof of (i) we have to show that, under this assumption, η is completely skew-symmetric and has isotropic support. The complete skew-symmetry follows from $\eta_X = \frac{1}{2} ad_X$ and the bi-invariance. Similarly, using the bi-invariance, we have

$$4\langle \eta_X Y, \eta_Z W\rangle = \langle [X, Y], [Z, W]\rangle = -\langle Y, [X, [Z, W]]\rangle = 0,$$

since the Lie algebra is 2-step nilpotent. This shows that Σ_η is isotropic.

□

Corollary 3.5.3 *With the above notations, let* $L \subset V$ *be a maximally isotropic subspace. The correspondence* $\eta \mapsto \mathcal{L}(\eta)$ *defines a bijection between the points of the orbit space* $\Lambda^3 L/GL(L)$ *and isomorphism classes of pairs* $(\mathcal{L}, h)$ *consisting of a simply connected Lie group* $\mathcal{L}$ *endowed with a flat bi-invariant pseudo-Riemannian metric* h *of signature* (k, l).

Corollary 3.5.4 *Any simply connected Lie group* $\mathcal{L}$ *with a flat bi-invariant metric* h *of signature* (k, l) *contains a normal subgroup of dimension* $\geq \max(k, l) \geq \frac{1}{2} \dim V$ *which acts by translations on the pseudo-Riemannian manifold* $(\mathcal{L}, h) \cong \mathbb{R}^{k,l}$.

Proof Let $\mathfrak{a} := ker(X \mapsto \eta_X) \subset V$ be the kernel of η. Then $\mathfrak{a} = \Sigma_\eta^\perp$ is co-isotropic and defines an Abelian ideal $\mathfrak{a} \subset \mathfrak{l} := Lie\,\mathcal{L} \cong V \cong \mathbb{R}^{k,l}$. The corresponding normal subgroup $A \subset \mathcal{L} = \mathcal{L}(\eta)$ is precisely the subgroup of translations. So we have shown that $\dim A \geq \max(k,l) \geq \frac{1}{2}\dim V$. □

Remarks 3.5.5

1) The number $\dim \Sigma_\eta$ is an isomorphism invariant of the groups $\mathcal{L} = \mathcal{L}(\eta)$, which is independent of the metric. We will denote it by $s(\mathcal{L})$. Let $L_3 \subset L_4 \subset \cdots \subset L$ be a filtration, where $\dim L_j = j$ runs from 3 to $\dim L$. The invariant $\dim \Sigma_\eta$ defines a decomposition of $\Lambda^3 L/GL(L)$ as a union
$$\{0\} \cup \bigcup_{j=3}^{\dim L} \Lambda^3_{reg} L_j/GL(L_j),$$
where $\Lambda^3_{reg}\mathbb{R}^j \subset \Lambda^3\mathbb{R}^j$ is the open subset of 3-vectors with j-dimensional support. The points of the stratum $\Lambda^3_{reg}L_j/GL(L_j) \cong \Lambda^3_{reg}\mathbb{R}^j/GL(j)$ correspond to isomorphism classes of pairs $(\mathcal{L}, h)$ with $s(\mathcal{L}) = j$.
2) Since in the above classification Σ_η is isotropic, it is clear that a flat (or 2-step nilpotent) bi-invariant metric on a Lie group is indefinite, unless $\eta = 0$ and the group is Abelian. It follows from Milnor's classification of Lie groups with a flat left-invariant Riemannian metric [95] that a 2-step nilpotent Lie group with a flat left-invariant Riemannian metric is necessarily Abelian.

Since a nilpotent Lie group with rational structure constants has a (co-compact) lattice [94], we obtain.

Corollary 3.5.6 *The groups $(\mathcal{L}(\eta), h)$ admit lattices $\Gamma \subset \mathcal{L}(\eta)$, provided that η has rational coefficients with respect to some basis. $M = M(\eta, \Gamma) := \Gamma \setminus \mathcal{L}(\eta)$ is a* flat compact homogeneous pseudo-Riemannian manifold. *The connected component of the identity in the isometry group of M is the image of the natural group homomorphism π from $\mathcal{L}(\eta)$ into the isometry group of M.*

Proof First we remark that the bi-invariant metric h induces an $\mathcal{L}(\eta)$-invariant metric on the homogeneous space $M = \Gamma\backslash\mathcal{L}(\eta)$. Let G be the connected component of the identity in the isometry group of $(\mathcal{L}(\eta), h) \cong \mathbb{R}^{k,l}$. The connected component of the identity in the isometry group of M is the image of the centraliser $Z_G(\Gamma)$ of Γ in G under the natural homomorphism $Z_G(\Gamma) \to Isom(M)$. Now the statement about the isometry group follows from the fact that the centraliser of the left-action of $\Gamma \subset \mathcal{L}(\eta)$ on $\mathcal{L}(\eta)$ is precisely the right-action of $\mathcal{L}(\eta)$ on $\mathcal{L}(\eta)$, since $\Gamma \subset \mathcal{L}(\eta)$ is Zariski-dense, see Theorem 2.1 of [101] . □

3.6 Classification Results for Flat Nearly ε-Kähler Manifolds

3.6.1 Classification Results for Flat Nearly Pseudo-Kähler Manifolds

In this section we denote by $\mathbb{C}^{k,l}$ the complex vector space $(\mathbb{C}^n, J_{can})$, $n = k + l$, endowed with the standard J_{can}-invariant pseudo-Euclidean scalar product g_{can} of signature $(2k, 2l)$.

Let (M, g, J) be a flat nearly pseudo-Kähler manifold. Then there exists for each point $p \in M$ an open set $U_p \subset M$ containing the point p, a connected open set U_0 of $\mathbb{C}^{k,l}$ containing the origin $0 \in \mathbb{C}^{k,l}$ and an isometry

$$\Phi \ : \ (U_p, g)\tilde{\to}(U_0, g_{can}),$$

such that at the point p we have:

$$\Phi_* J_p = J_{can}\Phi_*.$$

In other words, we can suppose, that locally M is a connected open subset of $\mathbb{C}^{k,l}$ containing the origin 0 and that $g = g_{can}$ and $J_0 = J_{can}$.

Proposition 3.6.1 *Let (M, g, J) be a flat nearly pseudo-Kähler manifold. Then*

1) $\eta_X \circ \eta_Y = 0$ *for all* X, Y,
2) $\nabla\eta = \bar{\nabla}\eta = 0$.

Proof From the curvature identity (3.15) we have for $X, Y, Z, W \in TM$

$$\begin{aligned} 0 = R(W, X, Y, Z) \ - \ R(W, X, JY, JZ) &= g((\nabla_X J)Y, (\nabla_Z J)W) = -g((\nabla_Z J)(\nabla_X J)Y, W) \\ &= -g(J(\nabla_Z J)J(\nabla_X J)Y, W) = -4g(\eta_Z\,\eta_X Y, W). \end{aligned}$$

This shows $\eta_X \circ \eta_Y = 0$ for all $X, Y \in TM$ and finishes the proof of part 1). The second part follows from 1) and $\bar{\nabla}\eta = 0$. In fact, one has

$$(\nabla_X\eta)_Y = (\bar{\nabla}_X\eta)_Y + \eta\eta_X Y + [\eta_X, \eta_Y] = 0, \ \text{for } X, Y \in TM,$$

which shows part 2). □

From Theorem 3.1.5 and Proposition 3.6.1 we obtain.

Corollary 3.6.2 *Let $M \subset \mathbb{C}^{k,l}$ be an open neighborhood of the origin endowed with a nearly pseudo-Kähler structure (g, J) such that $g = g_{can}$ and $J_0 = J_{can}$. Then the $(1, 2)$-tensor*

$$\eta := \frac{1}{2}J\nabla J$$

defines a constant three-form on $M \subset \mathbb{C}^{k,l} = \mathbb{R}^{2k,2l}$ *defined by*

$$\eta(X, Y, Z) := g(\eta_X Y, Z)$$

satisfying

(i) $\eta_X \eta_Y = 0, \quad \forall X, Y \in TM,$
(ii) $\{\eta_X, J_{can}\} = 0, \quad \forall X \in TM.$

Conversely, we have the next Lemma.

Lemma 3.6.3 (Lemma 5 of [41]) *Let* η *be a constant three-form on an open connected neighbourhood* $M \subset \mathbb{C}^{k,l}$ *of* 0 *satisfying (i) and (ii) of Corollary 3.6.2. Then there exists a unique almost complex structure* J *on* M *such that*

a) $J_0 = J_{can},$
b) $\{\eta_X, J\} = 0, \quad \forall X \in TM,$
c) $DJ = -2J\eta,$

where D *stands for the Levi-Civita connection of the pseudo-Euclidian vector space* $\mathbb{C}^{k,l}$*. With* $\bar{\nabla} := D - \eta$ *and assuming b), the last equation is equivalent to*

c') $\bar{\nabla} J = 0.$

More precisely, the almost complex structure is given by the formula

$$J = \exp\left(2\sum_{i=1}^{2n} x^i\, \eta_{\partial_i}\right) J_{can}$$
$$\overset{(i)}{=} \left(Id + 2\sum_{i=1}^{2n} x^i\, \eta_{\partial_i}\right) J_{can}, \tag{3.52}$$

where x^i are linear coordinates of $\mathbb{C}^{k,l} = \mathbb{R}^{2k,2l} = \mathbb{R}^{2n}$ and $\partial_i = \frac{\partial}{\partial x^i}$.

Theorem 3.6.4 *Let* η *be a constant three-form on a connected open set* $U \subset \mathbb{C}^{k,l}$ *containing* 0 *which satisfies (i) and (ii) of Corollary 3.6.2. Then there exists a unique almost complex structure given by Eq.* (3.52) *on* U *such that*

a) $J_0 = J_{can},$
b) $M(U, \eta) := (U, g = g_{can}, J)$ *is a flat nearly pseudo-Kähler manifold.*

Any flat nearly pseudo-Kähler manifold is locally isomorphic to a flat nearly pseudo-Kähler manifold of the form $M(U, \eta)$.

Now we discuss the general form of solutions of (i) and (ii) of Corollary 3.6.2. In the following we shall freely identify the real vector space $V := \mathbb{C}^{k,l} = \mathbb{R}^{2k,2l} = \mathbb{R}^{2n}$ with its dual V^* by means of the pseudo-Euclidian scalar product $g = g_{can}$.

Let us recall, that the support of $\eta \in \Lambda^3 V$ is defined by

$$\Sigma_\eta := \operatorname{span}\{\eta_X Y \mid X, Y \in V\} \subset V.$$

Proposition 3.6.5 *A three-form $\eta \in \Lambda^3 V^* \cong \Lambda^3 V$ satisfies (i) of Corollary 3.6.2 if and only if there exists an isotropic subspace $L \subset V$ such that $\eta \in \Lambda^3 L \subset \Lambda^3 V$. If η satisfies (i) and (ii) of Corollary 3.6.2 then there exists a J_{can}-invariant isotropic subspace $L \subset V$ with $\eta \in \Lambda^3 L$.*

Proof The proposition follows from Lemmata 2.3.2 and 2.3.4 of Chap. 2 by taking $L = \Sigma_\eta$. □

Remark 3.6.6 From the Proposition 3.6.5 we conclude that there are no strict flat nearly pseudo-Kähler manifolds of dimension less than 8. We shall see later that the dimension cannot be smaller than 12, see Corollary 3.6.8.
In the following we set $\Lambda^- W := [\![\Lambda^{3,0} W + \Lambda^{0,3} W]\!]$, where W is a complex vector space.

Theorem 3.6.7 *A three-form $\eta \in \Lambda^3 V^* \cong \Lambda^3 V$ satisfies (i) and (ii) of Corollary 3.6.2 if and only if there exists an isotropic J_{can}-invariant subspace $L \subset V$ such that $\eta \in \Lambda^- L \subset \Lambda^3 L \subset \Lambda^3 V$. (The smallest such subspace L is Σ_η.)*

Proof By Proposition 3.6.5, the conditions (i) and (ii) of Corollary 3.6.2 imply the existence of an isotropic J_{can}-invariant subspace $L \subset V$ such that $\eta \in \Lambda^3 L$. By Lemma 2.3.3 of Chap. 2 the condition (ii) is equivalent to $\eta \in \Lambda^- V$. Therefore $\eta \in \Lambda^3 L \cap \Lambda^- V = \Lambda^- L$. The converse statement follows from the same argument. □

Corollary 3.6.8 *There are no strict flat nearly pseudo-Kähler manifolds of dimension less than 12.*

Proof By Theorems 3.6.4 and 3.6.7 any flat nearly pseudo-Kähler manifold M is locally of the form $M(U, \eta)$, where $\eta \in \Lambda^- L$ for an isotropic J_{can}-invariant subspace $L \subset V$ and $U \subset V$ is an open subset. $M(U, \eta)$ is strict if and only if $\eta \neq 0$, which is possible only for $\dim_{\mathbb{C}} L \geq 3$, i.e. for $\dim M \geq 12$. □

Theorem 3.6.9 *Any strict flat nearly pseudo-Kähler manifold is locally a pseudo-Riemannian product $M = M_0 \times M(U, \eta)$ of a flat pseudo-Kähler factor M_0 of maximal dimension and a strict flat nearly pseudo-Kähler manifold $M(U, \eta)$ of (real) signature $(2m, 2m)$, $4m = \dim M(U, \eta) \geq 12$. The J_{can}-invariant isotropic support Σ_η has complex dimension m.*

Proof By Theorems 3.6.4 and 3.6.7, M is locally isomorphic to an open subset of a manifold of the form $M(V, \eta)$, where $\eta \in \Lambda^3 V$ has a J_{can}-invariant and isotropic support $L = \Sigma_\eta$. We choose a J_{can}-invariant isotropic subspace $L' \subset V$ such that $V' := L + L'$ is nondegenerate and $L \cap L' = 0$ and put $V_0 = (L + L')^\perp$. Then $\eta \in \Lambda^3 V' \subset \Lambda^3 V$ and $M(V, \eta) = M(V_0, 0) \times M(V', \eta)$. Notice that $M(V_0, 0)$ is simply the flat pseudo-Kähler manifold V_0 and that $M(V', \eta)$ is strict and of split signature $(2m, 2m)$, where $m = \dim_{\mathbb{C}} L \geq 3$. □

Corollary 3.6.10 *Let (M, g, J) be a flat nearly Kähler manifold with a (positive or negative) definite metric g then $\eta = 0$, $\bar{\nabla} = D$ and $DJ = 0$, i.e. (M, g, J) is a Kähler manifold.*

For the rest of this section we consider the case $V \cong \mathbb{C}^{m,m}$ and denote a maximal J_{can}-invariant isotropic subspace by L. We will say that a complex three-form $\zeta \in \Lambda^3(\mathbb{C}^m)^*$ has *maximal support* if $\mathrm{span}\{\zeta(Z, W, \cdot)|Z, W \in \mathbb{C}^m\} = (\mathbb{C}^m)^*$.

Corollary 3.6.11 *Any non-zero complex three-form $\zeta \in \Lambda^{3,0}L \cong \Lambda^3(\mathbb{C}^m)^*$ defines a complete flat simply connected strict nearly pseudo-Kähler manifold $M(\eta) := M(V, \eta)$, $\eta = \zeta + \bar{\zeta} \in \Lambda^3 L \subset \Lambda^3 V$, of split signature. $M(\eta)$ has no pseudo-Kähler de Rham factor if and only if ζ has maximal support.*

Conversely, any complete flat simply connected nearly pseudo-Kähler manifold without pseudo-Kähler de Rham factor is of this form.

Proof This follows from the previous results observing that the support of η is maximally isotropic if and only if ζ has maximal support. □

Corollary 3.6.12 *The map $\zeta \mapsto M(\zeta + \bar{\zeta})$ induces a bijective correspondence between $GL_m(\mathbb{C})$-orbits on the open subset $\Lambda^3_{reg}(\mathbb{C}^m)^* \subset \Lambda^3(\mathbb{C}^m)^*$ of three-forms ζ with maximal support and isomorphism classes of complete flat simply connected nearly pseudo-Kähler manifolds $M(\zeta + \bar{\zeta})$ of real dimension $4m \geq 12$ and without pseudo-Kähler de Rham factor.*

Example 3.6.13

1) The case $m \leq 5$.

   For $m = 3, 4, 5$ the group $GL_m(\mathbb{C})$ acts transitively on $\Lambda^3_{reg}(\mathbb{C}^m)^* = \Lambda^3(\mathbb{C}^m)^* \setminus \{0\}$. Therefore there exists precisely one complete flat simply connected strict nearly pseudo-Kähler manifold of dimension 12, 16 and 20 respectively.

2) The case $m = 6$.

   $GL_6(\mathbb{C})$ has precisely one open orbit in $\Lambda^3_{reg}(\mathbb{C}^6)^*$. This orbit consists of the stable three-forms $\Lambda^3_{stab}(\mathbb{C}^6)^*$ in the sense of Hitchin [75], cf. Sect. 2.1 in Chap. 2. We may recall, that a three-form ζ on $\mathbb{C}^6$ is stable if and only if $\zeta = e_1^* \wedge e_2^* \wedge e_3^* + e_4^* \wedge e_5^* \wedge e_6^*$ for some basis $(e_1, e_2, \ldots, e_6)$ of $\mathbb{C}^6$. $\Lambda^3(\mathbb{C}^6)^* \setminus \Lambda^3_{stab}(\mathbb{C}^6)^*$ is precisely the zero-set of the unique homogeneous quartic $SL_6(\mathbb{C})$-invariant and we have the following strict inclusions:

$$\Lambda^3_{stab}(\mathbb{C}^6)^* \subset \Lambda^3_{reg}(\mathbb{C}^6)^* \subset \Lambda^3(\mathbb{C}^6)^* \setminus \{0\}.$$

   An example of an instable regular form is

$$e_1^* \wedge e_2^* \wedge e_3^* + e_1^* \wedge e_4^* \wedge e_5^* + e_2^* \wedge e_4^* \wedge e_6^*.$$

3.6.2 Classification of Flat Nearly Para-Kähler Manifolds

In this subsection we consider (C^n, τ_{can}) endowed with the standard τ_{can}-anti-invariant pseudo-Euclidian scalar product g_{can} of signature (n, n).

Let (M, g, τ) be a flat nearly para-Kähler manifold. Then there exists for each point $p \in M$ an open set $U_p \subset M$ containing the point p, a connected open set U_0 of C^n containing the origin $0 \in C^n$ and an isometry $\Phi : (U_p, g) \tilde{\to} (U_0, g_{can})$, such that in $p \in M$ we have $\Phi_* \tau_p = \tau_{can} \Phi_*$. In other words, we can suppose, that locally M is a connected open subset of C^n containing the origin 0 and that $g = g_{can}$ and $\tau_0 = \tau_{can}$.

Proposition 3.6.14 *Let (M, g, τ) be a flat nearly para-Kähler manifold. Then*

1) $\eta_X \circ \eta_Y = 0$ *for all* $X, Y \in TM$,
2) $\nabla \eta = \bar{\nabla} \eta = 0$.

Summarising Theorem 3.1.5 and Proposition 3.6.14 we obtain the next Corollary.

Corollary 3.6.15 *Let $M \subset C^n$ be an open neighbourhood of the origin endowed with a nearly para-Kähler structure (g, τ) such that $g = g_{can}$ and $\tau_0 = \tau_{can}$. The $(1, 2)$-tensor*

$$\eta := -\frac{1}{2} \tau D \tau$$

defines a constant three-form on $M \subset C^n = \mathbb{R}^{n,n}$ given by $\eta(X, Y, Z) = g(\eta_X Y, Z)$ and satisfying

(i) $\eta \in \mathcal{C}(V)$, *i.e.* $\eta_X \eta_Y = 0, \quad \forall X, Y \in TM$,
(ii) $\{\eta_X, \tau_{can}\} = 0, \quad \forall X \in TM$.

The rest of this subsection is devoted to the local classification result. In Sect. 3.6.2 we study the structure of the subset of $\mathcal{C}(V)$ given by the condition (ii) in more detail and give global classification results. The converse statement of Corollary 3.6.15 is given in the next lemma.

Lemma 3.6.16 (Lemma 2.10 of [43]) *Let η be a constant three-form on an open connected neighbourhood $M \subset C^n$ of the origin 0 satisfying (i) and (ii) of Corollary 3.6.15. Then there exists a unique para-complex structure τ on M such that*

a) $\tau_0 = \tau_{can}$,
b) $\{\eta_X, \tau\} = 0, \quad \forall X \in TM$,
c) $D\tau = -2\tau\eta$,

where D is the Levi-Civita connection of the pseudo-Euclidian vector space C^n.
Let $\bar{\nabla} := D - \eta$ and assume b) then c) is equivalent to

c)' $\bar{\nabla} \tau = 0$.

Furthermore, this para-complex structure τ is skew-symmetric with respect to g_{can}.
In fact, one shows, that the para-complex structure τ is given by the following

formula

$$\tau = \exp\left(2\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right)\tau_{can} \overset{(i)}{=} \left(Id + 2\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right)\tau_{can}, \tag{3.53}$$

where x^i are linear coordinates of $C^n = \mathbb{R}^{n,n} = \mathbb{R}^{2n}$ and $\partial_i = \frac{\partial}{\partial x^i}$.

Theorem 3.6.17 *Let η be a constant three-form on a connected open set $U \subset C^n$ containing the origin 0 which satisfies (i) and (ii) of Corollary 3.6.15. Then there exists a unique almost para-complex structure*

$$\tau = \exp\left(2\sum_{i=1}^{2n} x^i\,\eta_{\partial_i}\right)\tau_{can} \tag{3.54}$$

on U such that a) $\tau_0 = \tau_{can}$, and b) $M(U,\eta) := (U, g = g_{can}, \tau)$ is a flat nearly para-Kähler manifold. Any flat nearly para-Kähler manifold is locally isomorphic to a flat nearly para-Kähler manifold of the form $M(U,\eta)$.

The Variety $\mathcal{C}_\tau(V)$

Now we discuss the solution of (i) and (ii) of Corollary 3.6.15. In the following we shall freely identify the real vector space $V := C^n = \mathbb{R}^{n,n} = \mathbb{R}^{2n}$ with its dual V^* by means of the pseudo-Euclidian scalar product $g = g_{can}$. The geometric interpretation is given in terms of an affine variety $\mathcal{C}_\tau(V) \subset \Lambda^3 V$.

Proposition 3.6.18 *A three-form $\eta \in \Lambda^3 V^* \cong \Lambda^3 V$ satisfies (i) of Corollary 3.6.15, i.e. $\eta_X \circ \eta_Y = 0, X, Y \in V$, if and only if there exists an isotropic subspace $L \subset V$ such that $\eta \in \Lambda^3 L \subset \Lambda^3 V$. If η satisfies (i) and (ii) of Corollary 3.6.15 then there exists a τ_{can}-invariant isotropic subspace $L \subset V$ with $\eta \in \Lambda^3 L$.*

Proof The proposition follows from Lemmata 2.3.2 and 2.3.4 of Chap. 2 by taking $L = \Sigma_\eta$. □

A three-form η on a para-complex vector space (W, τ_{can}) decomposes with respect to the grading induced by the decomposition $W^{1,0} \oplus W^{0,1}$ into $\eta = \eta^+ + \eta^-$. In the remainder of this subsection we set for convenience $\eta^+ \in \Lambda^+ W := \Lambda^{2,1} W + \Lambda^{1,2} W$ and $\eta^- \in \Lambda^- W := \Lambda^{3,0} W + \Lambda^{0,3} W$.

Theorem 3.6.19 *A three-form $\eta \in \Lambda^3 V^* \cong \Lambda^3 V$ satisfies (i) and (ii) of Corollary 3.6.15 if and only if there exists an isotropic τ_{can}-invariant subspace L such that $\eta \in \Lambda^- L = \Lambda^{3,0} L + \Lambda^{0,3} L \subset \Lambda^3 L \subset \Lambda^3 V$ (The smallest such subspace L is Σ_η.).*

Proof By Proposition 3.6.18, the conditions (i) and (ii) of Corollary 3.6.15 imply the existence of an isotropic τ_{can}-invariant subspace $L \subset V$ such that $\eta \in \Lambda^3 L$.

By Lemma 2.3.3 of Chap. 2 the condition (ii) is equivalent to $\eta \in \Lambda^- V$. Therefore $\eta \in \Lambda^3 L \cap \Lambda^- V = \Lambda^- L$. The converse statement follows from the same argument. □

Corollary 3.6.20

(i) *The conical affine variety*

$$\mathcal{C}_\tau(V) := \{\eta \mid \eta \text{ satisfies } (i) \text{ and } (ii)\} \subset \Lambda^3 V$$

has the following description

$$\mathcal{C}_\tau(V) = \bigcup_{L \subset V} \Lambda^- L = \bigcup_{L \subset V} (\Lambda^3 L^+ + \Lambda^3 L^-),$$

where the union is over all τ-invariant maximal isotropic subspaces.

(ii) *If* $\dim V < 12$ *then it holds* $\mathcal{C}_\tau(V) = \Lambda^3 V^+ \cup \Lambda^3 V^-$.

(iii) *Any flat nearly para-Kähler manifold M is locally of the form $M(U, \eta)$, for some $\eta \in \mathcal{C}_\tau(V)$ and some open subset $U \subset V$.*

(iv) *There are no strict flat nearly para-Kähler manifolds of dimension less than 6.*

Proof

(i) follows from Theorem 3.6.19.

(ii) Let $L \subset V$ be a τ-invariant isotropic subspace. If $\dim V < 12$, then $\dim L < 6$ and, hence, either $\dim L^+ < 3$ or $\dim L^- < 3$. In the first case we have

$$\Lambda^- L = \Lambda^3 L^+ + \Lambda^3 L^- = \Lambda^3 L^- \subset \Lambda^3 V^-,$$

in the second case it is $\Lambda^- L = \Lambda^3 L^+ + \Lambda^3 L^- = \Lambda^3 L^+ \subset \Lambda^3 V^+$.

(iii) is a consequence of (i), Theorems 3.6.17 and 3.6.19.

(iv) By (iii) the strict flat nearly para-Kähler manifold M is locally of the form $M(U, \eta)$, which is strict if and only if $\eta \neq 0$. This is only possible for $\dim L \geq 3$, i.e. for $\dim M \geq 6$.

□

Example 3.6.21 We have the following example which shows that part (ii) of Corollary 3.6.20 fails in dimension ≥ 12:

Consider $(V, \tau) = (C^6, \mathrm{i}_\varepsilon) = \mathbb{R}^6 \oplus \mathrm{i}_\varepsilon \mathbb{R}^6$, for $\varepsilon = 1$, with a basis given by $(e_1^+, \ldots, e_6^+, e_1^-, \ldots, e_6^-)$, such that $e_i^\pm$ form a basis of $V^\pm$ with $g(e_i^+, e_j^-) = \delta_{ij}$. Then the form $\eta := e_1^+ \wedge e_2^+ \wedge e_3^+ + e_4^- \wedge e_5^- \wedge e_6^-$ lies in the variety $\mathcal{C}_\tau(V)$.

Theorem 3.6.22 *Any strict flat nearly para-Kähler manifold is locally a pseudo-Riemannian product $M = M_0 \times M(U, \eta)$ of a flat para-Kähler factor M_0 of maximal dimension and a flat nearly para-Kähler manifold $M(U, \eta)$, $\eta \in C_\tau(V)$, of signature (m, m), $2m = \dim M(U, \eta) \geq 6$ such that Σ_η has dimension m.*

Proof By Theorems 3.6.17 and 3.6.19, M is locally isomorphic to an open subset of a manifold of the form $M(V, \eta)$, where $\eta \in \Lambda^3 V$ has a τ_{can}-invariant and isotropic support $L = \Sigma_\eta$. We choose a τ_{can}-invariant isotropic subspace $L' \subset V$ such that $V' := L + L'$ is nondegenerate and $L \cap L' = 0$ and put $V_0 = (L + L')^\perp$. Then $\eta \in \Lambda^3 V' \subset \Lambda^3 V$ and $M(V, \eta) = M(V_0, 0) \times M(V', \eta)$. Notice that $M(V_0, 0)$ is simply the flat para-Kähler manifold V_0 and that $M(V', \eta)$ is strict of split signature (m, m), where $m = \dim L \geq 3$. □

Corollary 3.6.23 *Any simply connected nearly para-Kähler manifold with a (geodesically) complete flat metric is a pseudo-Riemannian product $M = M_0 \times M(\eta)$ of a flat para-Kähler factor $M_0 = \mathbb{R}^{l,l}$ of maximal dimension and a flat nearly para-Kähler manifold $M(\eta) := M(V, \eta)$, $\eta \in C_\tau(V)$, of signature (m, m) such that Σ_η has dimension $m = 0, 3, 4, \ldots$.*

Next we wish to describe the moduli space of (complete simply connected) flat nearly para-Kähler manifolds M of dimension $2n$ up to isomorphism. Without restriction of generality we will assume that $M = M(\eta)$ has no para-Kähler de Rham factor, which means that $\eta \in C_\tau(V)$ has maximal support Σ_η, i.e. $\dim \Sigma_\eta = n$. We denote by $C_\tau^{reg}(V) \subset C_\tau(V)$ the open subset consisting of elements with maximal support. The group

$$G := \mathrm{Aut}(V, g_{can}, \tau_{can}) \cong GL(n, \mathbb{R})$$

acts on $C_\tau(V)$ and preserves $C_\tau^{reg}(V)$. Two nearly para-Kähler manifolds $M(\eta)$ and $M(\eta')$ are isomorphic if and only if η and η' are related by an element of the group G.

For $\eta \in C_\tau(V)$ we denote by p, q the dimensions of the eigenspaces of τ on Σ_η for the eigenvalues $1, -1$, respectively. We call the pair $(p, q) \in \mathbb{N}_0 \times \mathbb{N}_0$ the type of η. We will also say that the corresponding flat nearly para-Kähler manifold $M(\eta)$ has type (p, q). We denote by $C_\tau^{p,q}(V)$ the subset of $C_\tau(V)$ consisting of elements of type (p, q). Notice that $p + q \leq n$ with equality if and only if $\eta \in C_\tau^{reg}(V)$. We have the following decomposition

$$C_\tau^{reg}(V) = \bigcup_{(p,q)\in\Pi} C_\tau^{p,q}(V),$$

where $\Pi := \{(p, q) | p, q \in \mathbb{N}_0 \setminus \{1, 2\}, p + q = n\}$. The group $G = GL(n, \mathbb{R})$ acts on the subsets $C_\tau^{p,q}(V)$ and we are interested in the orbit space $C_\tau^{p,q}(V)/G$.

Fix a τ-invariant maximally isotropic subspace $L \subset V$ of type (p, q) and put $\Lambda^-_{reg} L := \Lambda^- L \cap C_\tau^{reg}(V) \subset C_\tau^{p,q}(V)$. The stabiliser $G_L \cong GL(L^+) \times GL(L^-) \cong GL(p, \mathbb{R}) \times GL(q, \mathbb{R})$ of $L = L^+ + L^-$ in G acts on $\Lambda^-_{reg} L$.

Theorem 3.6.24 *There is a natural one-to-one correspondence between complete simply connected flat nearly para-Kähler manifolds of type (p, q), $p + q = n$, and the points of the following orbit space:*

$$C_\tau^{p,q}(V)/G \cong \Lambda^-_{reg} L/G_L \subset \Lambda^- L/G_L = \Lambda^3 L^+/GL(L^+) \times \Lambda^3 L^-/GL(L^-).$$

Proof Consider two complete simply connected flat nearly para-Kähler manifolds M, M'. By the previous results we can assume that $M = M(\eta)$, $M' = M(\eta')$ are associated with $\eta, \eta' \in C_\tau^{p,q}(V)$. It is clear that M and M' are isomorphic if η and η' are related by an element of G. To prove the converse we assume that $\varphi : M \to M'$ is an isomorphism of nearly para-Kähler manifolds. By the results of Sect. 3.5 η defines a simply transitive group of isometries. This group preserves also the para-complex structure τ, which is ∇-parallel and hence left-invariant. This shows that M and M' admit a transitive group of automorphisms. Therefore, we can assume that φ maps the origin in $M = V$ to the origin in $M' = V$. Now φ is an isometry of pseudo-Euclidian vector spaces preserving the origin. Thus φ is an element of $O(V)$ preserving also the para-complex structure τ and hence $\varphi \in G$.

The identification of orbit spaces can be easily checked using Lemma 2.3.2 and the fact that any τ-invariant isotropic subspace $\Sigma = \Sigma^+ + \Sigma^-$ can be mapped onto L by an element of G. □

3.7 Conical Ricci-Flat Nearly Para-Kähler Manifolds

Definition 3.7.1 A conical semi-Riemannian manifold (M, g, ξ) is a semi-Riemannian manifold (M, g) endowed with a vector field ξ such that

$$\nabla \xi = \mathrm{Id}\,, \tag{3.55}$$

where ∇ is the Levi-Civita connection of g. It is called regular, if the function $k := g(\xi, \xi)$ has no zeros.

A conical nearly para-Kähler manifold (M, τ, g, ξ) is a nearly para-Kähler manifold (M, τ, g) such that (M, g, ξ) is conical and a conical para-Kähler manifold (M, P, g, ξ) is a para-Kähler manifold (M, P, g) such that (M, g, ξ) is conical.
For a proof of the following Proposition we refer to Proposition 6 of [39].

Proposition 3.7.2 *Let (M, g, ξ) be a regular conical semi-Riemannian manifold. Then the level sets $M_c := \{k = c\}, c \in \mathbb{R}$, are smooth hypersurfaces perpendicular to ξ or empty. If $M_c \neq \emptyset$, then g induces a semi-Riemannian metric g_c on M_c.*

Theorem 3.7.3 *Let (M, τ, g, ξ) be a Ricci-flat conical (strict) nearly para-Kähler manifold and define an endomorphism field P by*

$$P := \left(\mathrm{Id} + \frac{1}{4} N_\xi\right) \circ \tau. \tag{3.56}$$

(i) If $N(X, Y, Z)$ has isotropic support, then (M, P, g) is a para-Kähler manifold.
(ii) If the real dimension of M is 6, then (M, P, g) is a para-Kähler manifold.

We remark that the tuple (g, ξ) remains the same and in consequence (M, g, ξ) is conical and Ricci-flat. Hence (M, τ, g, ξ) is a Ricci-flat conical para-Kähler manifold.

Proof It suffices to show (i), since by Corollary 3.1.11 the Nijenhuis tensor $N(X, Y, Z)$ has isotropic support in dimension 6. Using $\{N_\xi, \tau\} = 0$, where N_ξ was defined as the endomorphism field given by $N_\xi Y = N(\xi, Y)$, we compute

$$\begin{aligned} P^2 &= \left(\left(\mathrm{Id} + \frac{1}{4}N_\xi\right) \circ \tau\right)^2 = \left(\mathrm{Id} + \frac{1}{4}N_\xi\right) \circ \left(\mathrm{Id} - \frac{1}{4}N_\xi\right) \circ \tau^2 \\ &= \mathrm{Id} - \frac{1}{16}N_\xi \circ N_\xi = \mathrm{Id}. \end{aligned}$$

Moreover, the condition

$$g(PX, Y) = -g(X, PY)$$

follows, since $N_\xi \circ \tau$ is skew-symmetric. In fact, we have

$$g(N_\xi \tau X, Y) = g(N(\xi, \tau X), Y) = 4g(\tau(\nabla_\xi \tau)\tau X, Y) = -4g((\nabla_\xi \tau)X, Y),$$

which is skew-symmetric in X, Y. Hence (M, P, g) defines an almost para-Hermitian structure. To show that it is para-Kähler we determine

$$\begin{aligned} (\nabla_X P)Y &= \nabla_X \left(\left(\mathrm{Id} + \frac{1}{4}N_\xi\right) \circ \tau\right) Y = (\nabla_X \tau)Y + \frac{1}{4}\nabla_X(N_\xi \circ \tau)Y \\ &= (\nabla_X \tau)Y + \frac{1}{4}\left[(\nabla_X N_\xi)\tau Y + N_\xi(\nabla_X \tau)Y\right] \\ &= (\nabla_X \tau)Y + \frac{1}{4}N_{\nabla_X \xi}(\tau Y) = (\nabla_X \tau)Y - \frac{1}{4}\tau N_X Y = 0. \end{aligned}$$

In this computation we used that $N(\cdot, \cdot, \cdot)$ has isotropic support and that $N(X, Y)$ is ∇-parallel (by Lemma 3.1.13). Namely, it is

$$(\nabla_X N_\xi)W = \nabla_X(N(\xi, W)) - N(\xi, \nabla_X W) = (\nabla_X N)(\xi, W) + N(\nabla_X \xi, W) = N_{\nabla_X \xi} W.$$

The statement that $N(X, Y, Z)$ is ∇-parallel is also shown in Lemma 3.1.13 and it does not vanish if (M, τ, g) is strict nearly para-Kähler. □

Remark 3.7.4 As the attentive reader observes, the ansatz $P = \left(\mathrm{Id} + \frac{1}{4}N_\xi\right) \circ \tau$ yields an almost para-complex structure, if N is of type $(3, 0) + (0, 3)$ and has isotropic support. This structure is para-Kähler if and only if it holds $N_{\nabla_X \xi} Y = 4\tau(\nabla_X \tau)Y$, i.e. $N_X Y = N_{\nabla_X \xi} Y$. If M is strict nearly para-Kähler, this implies $\nabla_X \xi = X$. This means ξ needs to be conical.

Theorem 3.7.5 *Let* (M, P, g, ξ) *be a Ricci-flat conical para-Kähler manifold of (real) dimension* $2m$ *carrying a (non-vanishing) parallel 3-form* $\varphi(X, Y, Z)$ *of type* $(3, 0) + (0, 3)$ *with isotropic support and define an endomorphism field* τ *by*

$$\tau = \left(\mathrm{Id} - \frac{1}{4}\varphi_\xi\right) \circ P. \tag{3.57}$$

Then (M, τ, g) *is a (strict) nearly para-Kähler manifold.*
As the pair (g, ξ) is not changed, (M, g, ξ) is conical and Ricci-flat. In consequence (M, τ, g, ξ) is a (strict) Ricci-flat conical nearly para-Kähler manifold.

Proof Here the endomorphism field φ_X is given by $\varphi_X Y := g^{-1} \circ \varphi(X, Y, \cdot)$.[9] Since $\varphi(X, Y, Z)$ has type $(3, 0) + (0, 3)$ one has $\{\varphi_\xi, P\} = 0$ (this follows from Eq. (2.35)) and we compute as before

$$\tau^2 = \left(\left(\mathrm{Id} - \frac{1}{4}\varphi_\xi\right) \circ P\right)^2 = \left(\mathrm{Id} - \frac{1}{16}\varphi_\xi \circ \varphi_\xi\right) \circ P^2 = \mathrm{Id}.$$

The last step follows, since $\varphi(X, Y, Z)$ has isotropic support (cf. the proof of Corollary 3.1.11 (b)). By the type condition it is $\varphi(\xi, PX, Y) = \varphi(P\xi, X, Y) = -\varphi(P\xi, Y, X)$ which means that $\varphi_\xi \circ P$ is skew-symmetric. From this it follows $g(\tau X, Y) = -g(X, \tau Y)$. It is left to check the nearly para-Kähler condition

$$\begin{aligned}(\nabla_X \tau)Y &= \nabla_X \left(\left(\mathrm{Id} - \frac{1}{4}\varphi_\xi\right) \circ P\right) Y = (\nabla_X P)Y - \frac{1}{4}\nabla_X(\varphi_\xi \circ P)Y \\ &= -\frac{1}{4}\left[(\nabla_X \varphi_\xi) \circ PY + \varphi_\xi(\nabla_X P)Y\right] = -\frac{1}{4}\varphi_{\nabla_X \xi}(PY) = \frac{1}{4}P(\varphi_X Y),\end{aligned}$$

which is skew-symmetric, since $\varphi(X, Y, Z)$ is a 3-form. Hence (M, τ, g) is a nearly para-Kähler manifold. If $\varphi(X, Y, Z)$ is non-vanishing, then (M, τ, g) is strict nearly para-Kähler. □

Remark 3.7.6 One may choose $\lambda\varphi(\cdot, \cdot, \cdot)$, $0 \neq \lambda \in \mathbb{R}$, in place of $\varphi(\cdot, \cdot, \cdot)$. Geometrically this corresponds to rescaling the conical vector field ξ by the factor λ.

Remark 3.7.7 Let us make an observation concerning Theorems 3.7.3 and 3.7.5. The Ansatz for τ only gives a para-complex structure, if it is $\varphi_\xi \circ \varphi_\xi = 0$. This implies, that $\varphi(X, Y, Z)$ has isotropic support and a para-Hermitian metric has automatically split signature. Therefore we only can have these examples for indefinite metrics (compare also Remark 3.7.11 for more comments).
In the following, we suppose that ξ is space-like, i.e. it is $g(\xi, \xi) > 0$. We can always achieve this by replacing the metric g by $-g$. Since $\pm g$ have the same Levi-Civita connection, this operation is compatible with the nearly para-Kähler condition (3.1)

[9] Here g^{-1} is the inverse of the map $g : TM \to T^*M,\ X \mapsto g(X, \cdot)$.

and the conical condition (3.55). One observes that one always can assume $M_1 = \{g(\xi,\xi) = 1\} \neq \emptyset$ after rescaling g by a positive constant without violating neither the nearly para-Kähler nor the conical condition.

Proposition 3.7.8 *Let (M, P, g, ξ) be a regular conical para-Kähler manifold with $M_1 \neq \emptyset$, then M_1 with the induced metric g_1 and the Reeb vector field $T = P\xi_{|M_1}$ is a para-Sasaki manifold.*

The manifold M is Ricci-flat if and only if M_1 is an Einstein manifold with scalar curvature $2m(2m+1)$.

Proof The conical vector field ξ is regular and Proposition 3.7.2 implies that (M_1, g_1) is a semi-Riemannian manifold. Denote by ∇^1 the Levi-Civita connection of g_1. By construction T is time-like, i.e. $g_1(T,T) = -1$ and tangential (see Proposition 3.7.2). Moreover, T is a Killing vector field, since one has for vector fields X, Y on M_1

$$\begin{aligned}\mathcal{L}_T g_1(X,Y) &= g_1(\nabla^1_X T, Y) + g_1(X, \nabla^1_Y T) = g(\nabla_X P\xi, Y) + g(X, \nabla_Y P\xi)\\ &= g(P\nabla_X\xi, Y) + g(X, P\nabla_Y\xi) = g(PX,Y) + g(X,PY) = 0.\end{aligned}$$

Additionally T is geodesic, since for $X \in TM_1$ it is

$$g_1(\nabla^1_T T, X) = g(\nabla_T T, X) = g(\nabla_T P\xi, X) = g(PT, X) = 0.$$

Since T is a Killing vector field $\Phi := \nabla^1 T$ is skew-symmetric and we have

$$\Phi X = \nabla^1_X T = (\nabla_X T)^{tan} = (PX)^{tan} = PX - g(\xi, PX)\xi = PX + g_1(T,X)\xi,$$

where $\cdot^{tan}$ is the projection on TM_1. This means $\Phi T = 0$ and $\Phi X = PX$ for $X \in TM_1$ perpendicular to T. It follows

$$\Phi^2(X) = P\Phi X + g_1(T, \Phi X)\xi = P\Phi X = X + g_1(T,X)T.$$

We compute $(\nabla^1_X\Phi)Y$ for $Y = T$

$$(\nabla^1_X\Phi)T = \nabla^1_X(\Phi T) - \Phi(\nabla^1_X T) = -\Phi^2(X) = -X - g_1(T,X)T$$

and for Y perpendicular to T

$$\begin{aligned}(\nabla^1_X\Phi)Y &= \nabla^1_X(\Phi Y) - \Phi(\nabla^1_X Y)\\ &\overset{(*)}{=} P(\nabla^1_X Y) - g(\xi, P\nabla^1_X Y)\xi - \Phi(\nabla^1_X Y) - g_1(X,Y)T\\ &\overset{(**)}{=} -g_1(X,Y)T.\end{aligned}$$

For $(*)$ we used

$$\begin{aligned}\nabla^1_X(\Phi Y) &= \nabla^1_X(PY) = (\nabla_X(PY))^{tan} = (P\nabla_X Y)^{tan} = (P(\nabla_X Y)^{tan})^{tan} + g(\nabla_X Y, \xi)T \\ &= (P(\nabla^1_X Y))^{tan} - g(Y, \nabla_X \xi)T = P(\nabla^1_X Y) - g(\xi, P\nabla^1_X Y)\xi - g_1(X, Y)T\end{aligned}$$

and $(**)$ follows from

$$P(\nabla^1_X Y) - g(\xi, P\nabla^1_X Y)\xi - \Phi(\nabla^1_X Y) = g_1(P\xi, \nabla^1_X Y)\xi - g_1(T, \nabla^1_X Y)\xi = 0.$$

Summarizing it holds

$$(\nabla^1_U \Phi)V = -g_1(U, V)T + g_1(V, T)U.$$

Hence we have checked all the conditions of Definition 2.6.1 and conclude that M_1 is a para-Sasaki manifold. In consequence the cone $\widehat{M_1}$ is a para-Kähler manifold, which is Einstein if and only if (M_1, g_1) is Einstein and the scalar curvature of g_1 equals $2m(2m+1)$ (cf. Remark 2.6.2 (i) and (ii)). □

Theorem 3.7.9 *Let (N^5, g, T) be a para-Sasaki Einstein manifold of dimension 5 and denote by $(M^6 = \widehat{N}, \widehat{g}, P, \xi)$ the associated conical Ricci-flat para-Kähler manifold on the cone $M = \widehat{N}$ over N, then the cone M can be endowed with the structure of a conical Ricci-flat strict nearly para-Kähler six-manifold $(M, \tau, \widehat{g}, \xi)$. Moreover, M is flat if and only if N has constant curvature.*

Proof By Remark 2.6.2 (ii) the cone $\widehat{N}$ is a conical Ricci-flat para-Kähler six-manifold $(\widehat{N}, \hat{g}, P, \xi)$ and hence admits a non-vanishing parallel three-form φ with isotropic support. From Theorem 3.7.5 we obtain a strict nearly para-Kähler structure τ on $\widehat{N}$ such that $(\widehat{N}, \tau, \hat{g}, \xi)$ is a conical Ricci-flat nearly para-Kähler six-manifold. The last statement follows from the fact that $\widehat{N}$ is flat if and only if N has constant curvature. □

In the following we call a nearly para-Kähler manifold M, which is the space-like metric cone $M = \widehat{N}$ over some semi-Riemannian manifold N a nearly para-Kähler cone. Summarising we have shown the following result.

Theorem 3.7.10 *There is a one to one correspondence between Ricci-flat strict nearly para-Kähler cones with isotropic Nijenhuis tensor and space-like cones over para-Sasaki Einstein manifolds endowed with a parallel 3-form having isotropic support.*

Remarks 3.7.11

(a) An analogous Ansatz can be made in almost complex geometry.

   (i) In this setting one still needs a form with isotropic support. Since non-trivial three-forms with isotropic support do not exist for Riemannian metrics, the Ansatz does only give something new, i.e. non-Kähler examples, for pseudo-Riemannian metrics and real dimension $\dim M \geq 12$, cf. Theorem 3.6.7 and Corollary 3.6.8.

(ii) Further one would get a cone over a Sasaki Einstein manifold with indefinite metric. Such manifolds can for instance be obtained as T-duals of homogeneous Sasaki manifolds of real dimension at least 11. These manifolds are only classified in dimensions ≤ 7 and the classification is possibly extended to dimension 9 and 11 using [19] (see Section 11.1.1 of [20]).

Details shall be postponed to future work.

(b) When we are not insisting on irreducible examples, one has the following construction in the almost pseudo-Hermitian world: Denote by (M^m, g_M, J_M) and (N^n, g_N, J_N) two nearly Kähler Einstein-manifolds with the same Einstein constant, then the pseudo-Riemannian product $(M \times N, g_M \oplus (-g_N), J_M \oplus J_N)$ is a nearly pseudo-Kähler manifold with vanishing Ricci curvature.

3.8 Evolution of Hypo Structures to Nearly Pseudo-Kähler Six-Manifolds

3.8.1 *Linear Algebra of Five-Dimensional Reductions of* SU(1, 2)*-Structures*

In this short section we prepare the linear algebra of dimensional reductions.

Lemma 3.8.1 *Let V be a six-dimensional real vector space and $(\omega, \rho) \in \Lambda^2 V^* \times \Lambda^3 V^*$ a compatible normalised pair of stable forms. Denote by $h = h_{(\omega,\rho)}$ the induced metric, let $N \in V$ be a unit vector with $h(N, N) = -\varepsilon \in \{\pm 1\}$ and denote by $W = N^{\perp}$ the orthogonal complement of $\mathbb{R}\cdot N$. Then the quadruple $(\eta, \omega_1, \omega_2, \omega_3)$ defined by*

$$\eta = \beta\,(N \lrcorner \omega), \quad \omega_1 = \alpha\,\omega|_W, \quad \omega_2 = N \lrcorner J_\rho^* \rho, \quad \omega_3 = -N \lrcorner \rho \tag{3.58}$$

with $\alpha, \beta \in \{\pm 1\}$ defines an $\mathrm{SU}^{\varepsilon}(p, q)$-structure with $p + q = 2$ on W.

Moreover, one has

$$\begin{aligned} \omega &= \alpha\,\omega_1 + \beta\, n \wedge \eta, \\ \rho &= -\varepsilon\beta\, \eta \wedge \omega_2 - n \wedge \omega_3, \\ J_\rho^* \rho &= -\varepsilon\beta\, \eta \wedge \omega_3 + n \wedge \omega_2, \end{aligned}$$

where $n \in V^*$ is the dual of N and

$$\eta \wedge \omega_2 = -\varepsilon\beta\, \rho_{|W} \text{ and } \eta \wedge \omega_3 = -\varepsilon\beta\, J_\rho^* \rho_{|W}.$$

Proof As above (see Eqs. (2.7) and (2.20) of Chap. 2) we may choose a basis $\{e_1, \dots, e_6\}$ of V, such that the stable forms ω, ρ and $J_\rho^* \rho$ are given in the normal forms

$$\omega = -e^{12} - e^{34} + e^{56},$$
$$\rho = e^{135} - (e^{146} + e^{236} + e^{245})$$

with $\lambda(\rho) = -4\nu^{\otimes 2}$ for $\nu = e^{123456} > 0$. Furthermore, it holds $J_\rho e_i = -e_{i+1}$, $J_\rho e_{i+1} = e_i$ for $i \in \{1, 3, 5\}$ and

$$J_\rho^* \rho = e^{246} - (e^{235} + e^{145} + e^{136}).$$

1) In the case $\varepsilon = 1$ we can suppose $N = e_1$ and obtain

$$\eta = -\beta\, e^2, \quad \alpha\, \omega_1 = -e^{34} + e^{56}, \quad \omega_2 = -e^{36} - e^{45}, \quad \omega_3 = -e^{35} + e^{46}.$$

One easily sees

$$\omega_1^2 = -\omega_2^2 = -\omega_3^2 = -2e^{3456} \text{ and } \eta \wedge \omega_j^2 = 2\beta\, e^{23456} \neq 0$$

and $\omega_j \wedge \omega_k = 0$ for $1 \leq j < k \leq 3$.
Moreover, one gets

$$\omega = \alpha\, \omega_1 + \beta\, n \wedge \eta,$$
$$\rho = -\beta\, \eta \wedge \omega_2 - n \wedge \omega_3 \text{ and}$$
$$J_\rho^* \rho = -\beta\, \eta \wedge \omega_3 + n \wedge \omega_2,$$

where n is the dual of N. Further, one has

$$\eta \wedge \omega_2 = -\beta\, \rho_{|W} \text{ and } \eta \wedge \omega_3 = -\beta\, J_\rho^* \rho_{|W}.$$

2) For $\varepsilon = -1$ we can choose $N = e_5$ and get

$$\eta = \beta\, e^6, \quad \alpha\, \omega_1 = -e^{12} - e^{34}, \quad \omega_2 = -e^{14} - e^{23}, \quad \omega_3 = -e^{13} + e^{24}.$$

One easily sees

$$\omega_1^2 = \omega_2^2 = \omega_3^2 = 2e^{1234} \text{ and } \eta \wedge \omega_j^2 = 2\beta\, e^{12346} \neq 0$$

and $\omega_j \wedge \omega_k = 0$ for $1 \leq j < k \leq 3$.

Moreover, one gets

$$\omega = \alpha\,\omega_1 + \beta\, n \wedge \eta,$$
$$\rho = \beta\,\eta \wedge \omega_2 - n \wedge \omega_3 \text{ and}$$
$$J_\rho^* \rho = \beta\,\eta \wedge \omega_3 + n \wedge \omega_2,$$

where n is the dual of N. Finally, one has

$$\eta \wedge \omega_2 = \beta\,\rho_{|W} \text{ and } \eta \wedge \omega_3 = \beta\, J_\rho^* \rho_{|W}.$$

□

3.8.2 Evolution of Hypo Structures

A five-manifold N^5 carries an $\mathrm{SU}^\varepsilon(p,q)$-structure with $p+q = 2$ provided, that its frame bundle admits a reduction to $\mathrm{SU}^\varepsilon(p,q)$. For the group SU(2) it is shown in [36], that such a structure is determined by a quadruple of differential forms $(\omega_1, \omega_2, \omega_3, \eta)$. We shortly derive the analogous statement for our setting. Let $f\colon N^5 \to M^6$ be an oriented hypersurface in a six-manifold M^6 endowed with an $\mathrm{SU}(1,2)$-structure given by a triple (ω, ψ^+, ψ^-) of compatible stable forms (cf. Sect. 2.1).

This $\mathrm{SU}(1,2)$-structure induces an $\mathrm{SU}^\varepsilon(p,q)$-structure with $p+q=2$ on N^5 via the definitions

$$\omega_1 = \alpha f^*\omega, \quad b^-\,\omega_2 = \nu\lrcorner\psi^-, \quad b^+\,\omega_3 = \nu\lrcorner\psi^+, \quad \eta = \beta\,\nu\lrcorner\omega, \tag{3.59}$$

where $\alpha, \beta, b^+, b^- \in \{\pm 1\}$ are real constants and ν denotes the unit normal vector field of N^5 of length $\varepsilon = -g(\nu,\nu)$.

In case, that the holonomy of M^6 is contained in $SU(1,2)$ or in other words the $SU(1,2)$-structure is integrable, which is equivalent to the equations

$$d\omega = 0, \quad d\psi^+ = 0 \quad \text{and} \quad d\psi^- = 0 \tag{3.60}$$

we obtain a hypo structure on N in the sense of the next Definition.

Definition 3.8.2 An $\mathrm{SU}^\varepsilon(p,q)$-structure with $p+q = 2$ determined by $(\eta, \omega_1, \omega_2, \omega_3)$ is called hypo provided, that it satisfies

$$d\omega_1 = 0, \quad d(\eta\wedge\omega_2) = 0 \quad \text{and} \quad d(\eta\wedge\omega_3) = 0.$$

For the Riemannian case the next lemma is shown in [36].

Lemma 3.8.3 *Let $f: N^5 \to M^6$ be an oriented hypersurface in a six-manifold M^6 endowed with an integrable* $\mathrm{SU}(1,2)$*-structure, then the induced* $\mathrm{SU}^\varepsilon(p,q)$*-structure, given by* (3.59) *and with $p+q=2$, on N^5 is a hypo-structure.*

Proof From $\alpha f^*\omega = \omega_1$ one has $d\omega_1 = \alpha\, d(f^*\omega) = \alpha f^*(d\omega) = 0$. Moreover, with help of the Lemma 3.8.1 one has

$$b^+ f^*\psi^+ = -\varepsilon\beta\, \eta \wedge \omega_2 \text{ and } b^- f^*\psi^- = -\varepsilon\beta\, \eta \wedge \omega_3,$$

which implies using (3.60)

$$\begin{aligned} -\varepsilon\beta\, d(\eta \wedge \omega_3) &= b^-\, d(f^*\psi^-) = b^- f^* d\psi^- = 0 \\ &= b^+\, d(f^*\psi^+) = b^+ f^* d\psi^+ = -\varepsilon\beta\, d(\eta \wedge \omega_2). \end{aligned}$$

This shows, that the induced $\mathrm{SU}^\varepsilon(p,q)$-structure is a hypo structure on N^5. □

Starting with an $\mathrm{SU}^\varepsilon(p,q)$-structure with $p+q=2$ on N^5 determined by $(\eta, \omega_1, \omega_2, \omega_3)$ we define a two-form

$$\omega = \alpha\omega_1 + \varepsilon\beta\, dt \wedge \eta \tag{3.61}$$

and three-forms $\psi^\pm$ on $N^5 \times \mathbb{R}$ by

$$\psi^+ = a^+\eta \wedge \omega_2 + b^+ dt \wedge \omega_3, \quad \psi^- = a^-\eta \wedge \omega_3 + b^- dt \wedge \omega_2, \tag{3.62}$$

where t is the coordinate on $\mathbb{R}$ and $\alpha, \beta, a^\pm, b^\pm \in \{\pm 1\}$ are non-zero real constants. Note, that the $a^\pm$ are determined from α, β and $b^\pm$ by Lemma 3.8.1. Then a partial converse of the result of the last Lemma is given in the next Proposition.

Proposition 3.8.4 *One can lift a hypo* $\mathrm{SU}^\varepsilon(p,q)$*-structure with $p+q=2$ to an integrable* $\mathrm{SU}(1,2)$*-structure on $N \times \mathbb{R}$ if it belongs to a one-parameter family of* $\mathrm{SU}^\varepsilon(p,q)$*-structures* $(\eta(t), \omega_1(t), \omega_2(t), \omega_3(t))$*, where t is the coordinate on $\mathbb{R}$, satisfying the Conti-Salamon type evolution equations*

$$\partial_t\omega_1 = \varepsilon\beta\alpha\, d^5\eta, \tag{3.63}$$

$$\partial_t(\eta \wedge \omega_3) = a^-\, b^-\, d\omega_2, \tag{3.64}$$

$$\partial_t(\eta \wedge \omega_2) = a^+\, b^+\, d\omega_3. \tag{3.65}$$

Proof In this proof we write d^5 and d^6 for the exterior differentials on N^5 and $M^6 = N \times \mathbb{R}$. With (3.61) it follows, that

$$0 = d^6\omega = \alpha d^5\omega_1 + (\alpha\partial_t\omega_1 - \varepsilon\beta\, d^5\eta) \wedge dt.$$

This is equivalent to $d^5\omega_1 = 0$ and $\partial_t\omega_1 = \varepsilon\beta\alpha\, d^5\eta$, i.e. Eq. (3.63). From the definition of ψ^+ one gets, that

$$0 = d^6\psi^+ = a^+d^5(\omega_2 \wedge \eta) + dt \wedge \left(a^+\partial_t(\omega_2 \wedge \eta) - b^+d^5\omega_3\right)$$

is equivalent to $d^5(\omega_2 \wedge \eta) = 0$ and $\partial_t(\omega_2 \wedge \eta) = a^+b^+\, d^5\omega_3$, i.e. Eq. (3.65). For ψ^- as given in (3.62) we obtain, that

$$0 = d^6\psi^- = a^-d^5(\omega_3 \wedge \eta) + dt \wedge (a^-\partial_t(\omega_3 \wedge \eta) - b^-d^5\omega_2)$$

itself is equivalent to $d^5(\omega_3 \wedge \eta) = 0$ and Eq. (3.64). □

Examples of this type are given by the pseudo-Riemannian cousins of Sasaki-Einstein manifolds, namely para-Sasaki-Einstein and Lorentzian-Sasaki-Einstein manifolds. These can be characterised by the fact, that the space-like/time-like cone is a Ricci-flat Kähler-Einstein manifold or equivalently this cone has an integrable $\mathrm{SU}(1,2)$-structure. Here one considers the special solution of the above evolution equations on $N^5 \times \mathbb{R}$ given by

$$\omega = t^2\alpha\omega_1 + t\varepsilon\beta dt \wedge \eta, \tag{3.66}$$

$$\psi^+ = a^+t^3\eta \wedge \omega_2 + t^2b^+dt \wedge \omega_3, \tag{3.67}$$

$$\psi^- = a^-t^3\eta \wedge \omega_3 + t^2b^-dt \wedge \omega_2. \tag{3.68}$$

The integrability conditions read

$$\begin{aligned}
0 &= d\omega = d(t^2\alpha\omega_1 + t\varepsilon\beta dt \wedge \eta) = tdt \wedge (2\alpha\omega_1 - \varepsilon\beta d\eta),\\
0 &= d\psi^+ = d(a^+t^3\eta \wedge \omega_2 + t^2b^+dt \wedge \omega_3)\\
&= t^2dt \wedge (3a^+\eta \wedge \omega_2 - b^+d^5\omega_3) + t^3a^+d^5(\eta \wedge \omega_2),\\
0 &= d\psi^- = d(a^-t^3\eta \wedge \omega_3 + t^2b^-dt \wedge \omega_2)\\
&= t^2dt \wedge (3a^-\eta \wedge \omega_3 - b^-d^5\omega_2) + t^3a^-d^5(\eta \wedge \omega_3).
\end{aligned}$$

This is equivalent to the para-Sasaki-Einstein or Lorentz-Sasaki-Einstein equations

$$d\eta = 2\varepsilon\alpha\,\beta\,\omega_1, \quad d\omega_2 = 3b^-a^-\omega_3 \wedge \eta, \quad d\omega_3 = 3b^+a^+\omega_2 \wedge \eta. \tag{3.69}$$

Obviously, Eq. (3.69) imply the hypo equations.

The next result has been discovered in the Riemannian case in [53].

Proposition 3.8.5 *Let $f\colon N^5 \to M^6$ be a totally geodesic oriented hypersurface in a nearly pseudo-Kähler manifold M^6 with unit normal vector field ν, then the induced $\mathrm{SU}^\varepsilon(p,q)$-structure with $p+q=2$ and $\varepsilon = -g(\nu,\nu)$ satisfies the hypo equations.*

Proof From the first nearly Kähler equation one has

$$d\omega_1 = \alpha d(f^*\omega) = \alpha f^* d\omega = 3\alpha f^*\psi^+ = 3(\alpha a^+)\eta \wedge \omega_2. \tag{3.70}$$

Moreover, we compute

$$\beta^{-1}\, d\eta = d(\nu \lrcorner \omega) = (\mathcal{L}_\nu \omega) - \nu \lrcorner d\omega = (\mathcal{L}_\nu \omega) - 3\nu \lrcorner \psi^+ = (\mathcal{L}_\nu \omega) - 3b^+\omega_3,$$

where $\mathcal{L}_\nu$ is the Lie-derivative. Since N^5 is totally geodesic, it is $(\mathcal{L}_\nu \omega) = \nabla_\nu \omega$. Using $\bar{\nabla}\omega = 0$ we get with $\bar{\nabla} = \nabla + \frac{1}{2}T$

$$\begin{aligned} \nabla_\nu \omega(X,Y) &= \bar{\nabla}_\nu \omega(X,Y) + \frac{1}{2}\left[\omega(T(\nu,X),Y) + \omega(X,T(\nu,Y))\right] \\ &= \omega(T(\nu,X),Y) = b^+\omega_3(X,Y), \end{aligned}$$

as with $\omega(\cdot,\cdot) = g(\cdot, J\cdot)$ it is

$$\omega(T(\nu,X),Y) = -g(J(\nabla_\nu J)X, JY) = g(X,(\nabla_\nu J)Y) = \psi^+(\nu,X,Y) = b^+\omega_3(X,Y),$$

which shows

$$d\eta = -2\beta b^+\,\omega_3.$$

Finally, we compute with help of $d\psi^- = -2\omega \wedge \omega$, i.e. the second nearly Kähler equation

$$\begin{aligned} b^- d\omega_2 &= d(\nu \lrcorner \psi^-) = \mathcal{L}_\nu \psi^- - \nu \lrcorner d\psi^- = \nabla_\nu \psi^- + 2\nu \lrcorner (\omega \wedge \omega) \\ &= \bar{\nabla}_\nu \psi^- - \frac{1}{4}\nu \lrcorner \sum_k \sigma_k (e_k \lrcorner \psi^-) \wedge (e_k \lrcorner \psi^-) + 2\nu \lrcorner (\omega \wedge \omega) \\ &\overset{(*)}{=} -\frac{1}{4}\nu \lrcorner (2\omega \wedge \omega) + 2\nu \lrcorner (\omega \wedge \omega) = \frac{3}{2}\nu \lrcorner (\omega \wedge \omega) = 3(\alpha\beta)\eta \wedge \omega_1, \end{aligned}$$

where $\{e_1, \ldots, e_6 = \nu\}$ is some adapted basis and where in $(*)$ we used

$$\sum_{k=1}^{6} \sigma_k (e_k \lrcorner \psi^-) \wedge (e_k \lrcorner \psi^-) = 2\omega \wedge \omega,$$

which holds for a nearly pseudo-Kähler six-manifold. □

Theorem 3.8.6 *Any totally geodesic hypersurface N^5 of a nearly pseudo-Kähler six-manifold M^6 carries a hypo structure given by the quadruplet $(\tilde{\eta}, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3) = (\eta, -\varepsilon\omega_3, \omega_2, \omega_1)$, and in consequence the Conti-Salamon type evolution equations can be solved on $N \times \mathbb{R}$.*

Proof By the last Lemma we obtain a hypo $SU^{\varepsilon}(p, q)$-structure $(\tilde{\eta}, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)$, which is a solution of the first and the third para-Sasaki-Einstein or Lorentz-Sasaki-Einstein equations (3.69). Using $b^+ = -1$ and $a^+ = -\beta$ (by Lemma 3.8.1) and setting $\alpha = -1$ yields

$$d\tilde{\eta} = d\eta = -2(\beta b^+)\,\omega_3 = 2\beta\,\omega_3 \overset{!!}{=} 2\,\varepsilon(\beta\alpha)\,\tilde{\omega}_1,$$
$$d\tilde{\omega}_3 = d\omega_1 \overset{(3.70)}{=} 3(\alpha a^+)\,\eta \wedge \omega_2 = -3(\alpha\beta)\,\tilde{\eta} \wedge \tilde{\omega}_2 \overset{!!}{=} 3(b^+ a^+)\tilde{\eta} \wedge \tilde{\omega}_2,$$

since again by Lemma 3.8.1,[10] it is $b^+ a^+ = \beta = -a^- b^-$. The remaining para-Sasaki-Einstein or Lorentz-Sasaki-Einstein equation

$$d\tilde{\omega}_2 = d\omega_2 = 3(b^-\alpha\beta)\eta \wedge \omega_1 = 3(\alpha\beta)\tilde{\eta} \wedge \tilde{\omega}_3 \overset{!!}{=} 3(b^- a^-)\tilde{\eta} \wedge \tilde{\omega}_3$$

holds true using $b^- = 1$ (by Lemma 3.8.1). □

3.8.3 Evolution of Nearly Hypo Structures

In this subsection we generalise results of [53] to construct examples of nearly pseudo-Kähler manifolds via the nearly hypo evolution equations.

Definition 3.8.7 An $SU^{\varepsilon}(p, q)$-structure with $p + q = 2$ determined by $(\eta, \omega_1, \omega_2, \omega_3)$ is called nearly hypo provided, that it satisfies the conditions

$$d\omega_1 = 3\alpha a^+ \eta \wedge \omega_2, \qquad d(\eta \wedge \omega_3) = -2a^- \omega_1 \wedge \omega_1. \tag{3.71}$$

Proposition 3.8.8 *An $SU^{\varepsilon}(p, q)$-structure $(\eta, \omega_1, \omega_2, \omega_3)$ with $p + q = 2$ can be lifted to a nearly pseudo-Kähler structure $(\omega(t), \psi^+(t), \psi^-(t))$ on $N^5 \times \mathbb{R}$ defined in* (3.61) *and* (3.62) *if and only if it is a nearly hypo structure which generates a* 1*-parameter family of $SU^{\varepsilon}(p, q)$-structures $(\eta(t), \omega_k(t))$ satisfying the following nearly hypo evolution equations*

$$\begin{aligned} \partial_t \omega_1 &= 3b^+\alpha\,\omega_3 + \varepsilon\beta\alpha\,d\eta, \\ \partial_t(\eta \wedge \omega_3) &= a^- b^- d\omega_2 - 4\varepsilon a^- \alpha\beta\omega_1 \wedge \eta, \\ \partial_t(\eta \wedge \omega_2) &= a^+ b^+\,d\omega_3. \end{aligned} \tag{3.72}$$

[10] Observe, that there is a relative factor ε between dual and the metric dual of ν.

Remark 3.8.9 Setting $\beta = 1$ and with $b^- = 1 = -b^+$ and $b^+a^+ = -b^-a^- = \beta = 1$ (by Lemma 3.8.1) we get

$$\begin{aligned} \partial_t(\alpha\omega_1) &= -3\,\omega_3 + \varepsilon\, d\eta, \\ \partial_t(\eta \wedge \omega_3) &= -d\omega_2 + 4\varepsilon\eta \wedge (\alpha\omega_1), \\ \partial_t(\eta \wedge \omega_2) &= d\omega_3. \end{aligned} \tag{3.73}$$

For $(\tilde{\eta}, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3) = (-\eta, \alpha\omega_1, \omega_2, \omega_3)$, this yields

$$\begin{aligned} \partial_t\tilde{\omega}_1 &= -3\,\tilde{\omega}_3 - \varepsilon\, d\tilde{\eta}, \\ \partial_t(\tilde{\eta} \wedge \tilde{\omega}_3) &= d\tilde{\omega}_2 - 4\varepsilon\tilde{\eta} \wedge \tilde{\omega}_1, \\ \partial_t(\tilde{\eta} \wedge \tilde{\omega}_2) &= -d\tilde{\omega}_3. \end{aligned} \tag{3.74}$$

Hence the exterior differential system on the modified data $(\tilde{\eta}, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)$ looks formally the same as the system found in the Riemannian case [53].

Proof From the definitions of ω_1, ω_2 and η we get, that the first nearly Kähler equation is equivalent to

$$\begin{aligned} d\omega &= d(\alpha\omega_1 + \varepsilon\beta dt \wedge \eta) = \alpha d\omega_1 + dt \wedge (\alpha\partial_t\omega_1 - \varepsilon\beta d\eta) \\ &= 3\psi^+ = 3a^+\eta \wedge \omega_2 + 3b^+ dt \wedge \omega_3, \end{aligned}$$

i.e. $d\omega_1 = 3\alpha a^+\eta \wedge \omega_2$ and $\partial_t\omega_1 = 3b^+\alpha\,\omega_3 + \varepsilon\beta\alpha\, d\eta$. For the second nearly Kähler equation we have

$$\begin{aligned} d\psi^- &= d(a^-\eta \wedge \omega_3 + b^- dt \wedge \omega_2) = a^- d(\eta \wedge \omega_3) + dt \wedge (a^-\partial_t(\eta \wedge \omega_3) - b^- d\omega_2) \\ &= -2(\alpha\omega_1 + \varepsilon\beta dt \wedge \eta)^2 = -2\,\omega_1 \wedge \omega_1 - 4\varepsilon\alpha\beta\, dt \wedge \omega_1 \wedge \eta, \end{aligned}$$

which is equivalent to

$$d(\eta \wedge \omega_3) = -2a^-\omega_1 \wedge \omega_1 \quad \text{and} \quad \partial_t(\eta \wedge \omega_3) = a^-b^-d\omega_2 - 4\varepsilon a^-\alpha\beta\omega_1 \wedge \eta.$$

These are the first two evolution equations and the nearly hypo equations. The third equation is needed to show, that the nearly hypo property is conserved along the evolution. Firstly, one has

$$\begin{aligned} \partial_t(d\omega_1 - 3\alpha a^+\eta \wedge \omega_2) &= d(\partial_t\omega_1) - 3\alpha a^+\partial_t(\eta \wedge \omega_2) \\ &= d(3b^+\alpha\,\omega_3 + \varepsilon\beta\alpha\, d\eta) - 3\alpha a^+\partial_t(\eta \wedge \omega_2) \\ &= 3b^+\alpha\, d\omega_3 - 3\alpha a^+\partial_t(\eta \wedge \omega_2), \end{aligned}$$

which vanishes by the third evolution equation $\partial_t(\eta \wedge \omega_2) = a^+b^+ d\omega_3$. For the other nearly hypo equation we compute

$$\begin{aligned}\partial_t[d(\eta \wedge \omega_3) + 2a^-\omega_1 \wedge \omega_1] &= d[\partial_t(\eta \wedge \omega_3)] + 2a^-\partial_t(\omega_1 \wedge \omega_1)\\ &= d[a^-b^-d\omega_2 - 4\varepsilon a^-\alpha\beta\omega_1 \wedge \eta]\\ &\quad +4a^-\partial_t(\omega_1) \wedge \omega_1\\ &= -4\varepsilon a^-\alpha\beta d(\omega_1 \wedge \eta)\\ &\quad +4a^-\omega_1 \wedge (3b^+\alpha\, \omega_3 + \varepsilon\beta\alpha\, d\eta)\\ &= -4\varepsilon a^-\alpha\beta(d(\omega_1 \wedge \eta) - \omega_1 \wedge d\eta)\\ &= -4\varepsilon a^-\alpha\beta\, d\omega_1 \wedge \eta = 0,\end{aligned}$$

where we used, that by the already proven first hypo equation $d\omega_1$ is (along the flow) a multiple of $\eta \wedge \omega_2$. Hence the nearly hypo condition is preserved along a solution of the system (3.72). □

Proposition 3.8.10 *Any* $\mathrm{SU}^\varepsilon(p,q)$*-structure with* $p+q=2$ *satisfying the para- or pseudo-Sasaki equations* (3.69) *defines a nearly hypo structure* $(\tilde\eta, \tilde\omega_1, \tilde\omega_2, \tilde\omega_3) = (\eta, \omega_3, \omega_2, \omega_1)$.

Proof From (3.69) one has after setting $\alpha = b^+ = -1$ (by Lemma 3.8.1)

$$d\tilde\omega_1 = d\omega_3 = 3b^+a^+\,\omega_2 \wedge \eta = 3\alpha a^+\,\tilde\eta \wedge \tilde\omega_2$$

and

$$d(\tilde\eta\wedge\tilde\omega_3) = d(\eta\wedge\omega_1) = d\eta\wedge\omega_1 = 2\varepsilon\alpha\beta\,\omega_1\wedge\omega_1 = -2\alpha\beta\,\omega_3\wedge\omega_3 \overset{!!}{=} -2a^-\tilde\omega_1\wedge\tilde\omega_1,$$

where we used $\omega_1 \wedge \omega_1 = -\varepsilon\,\omega_3 \wedge \omega_3$. This yields the claim, since one has $a^- = -\beta = \alpha\beta$ (by Lemma 3.8.1). □

Proposition 3.8.11 *Let* $f\colon N^5 \to M^6$ *be an immersion of an oriented 5-manifold into a 6-dimensional nearly pseudo-Kähler manifold, then the induced* $\mathrm{SU}^\varepsilon(p,q)$*-structure* $(\tilde\eta, \tilde\omega_1, \tilde\omega_2, \tilde\omega_3) = (\eta, \omega_1, \omega_2, \omega_3)$ *with* $p+q=2$ *is a nearly hypo structure.*

Proof Let us first observe, that one has

$$a^+\,\eta \wedge \omega_2 = f^*\psi^+ \text{ and } a^-\,\eta \wedge \omega_3 = f^*\psi^-,$$

which implies

$$\begin{aligned}d\tilde\omega_1 &= d\omega_1 = \alpha\, d(f^*\omega) = \alpha f^*(d\omega)\\ &= 3\,\alpha f^*\psi^+ = 3\,\alpha\, a^+\,\eta \wedge \omega_2 = 3\alpha a^+\,\tilde\eta \wedge \tilde\omega_2,\end{aligned}$$

$$
\begin{aligned}
a^- d(\tilde{\eta} \wedge \tilde{\omega}_3) = \; & a^- d(\eta \wedge \omega_3) = f^* d\psi^- \\
& = -2 f^*(\omega \wedge \omega) = -2\,\omega_1 \wedge \omega_1 = -2\,\tilde{\omega}_1 \wedge \tilde{\omega}_1 .
\end{aligned}
$$

This proves the nearly hypo equations. □

Theorem 3.8.12 *Let* $(N^5, \eta, \omega_1^0, \omega_2^0, \omega_3^0)$ *be an* ε*-Sasaki-Einstein* $SU^\varepsilon(p,q)$*-structure, then*

$$\omega_1 = f_\varepsilon^2 \left(f_\varepsilon \omega_1^0 + f_\varepsilon' \omega_3^0 \right), \tag{3.75}$$

$$\omega_2 = f_\varepsilon^2 \omega_2^0, \tag{3.76}$$

$$\omega_3 = -f_\varepsilon^2 \left(f_\varepsilon' \omega_1^0 + \varepsilon f_\varepsilon \omega_3^0 \right), \tag{3.77}$$

$$\eta = f_\varepsilon \eta^0 \tag{3.78}$$

with

$$f_\varepsilon(t) := \sin_\varepsilon(t) = \begin{cases} \sinh(t), & \textit{for } t \in \mathbb{R} \textit{ and } \varepsilon = 1, \\ \sin(t), & \textit{for } t \in [0,\pi] \textit{ and } \varepsilon = -1 \end{cases} \qquad \textit{and } I_\varepsilon := \begin{cases} \mathbb{R}^* \textit{ for } \varepsilon = 1, \\ (0,\pi) \textit{ for } \varepsilon = -1 \end{cases}$$

is a solution of the nearly hypo evolution equations and yields a nearly pseudo-Kähler structure on $N \times I_\varepsilon$ *Metrik angeben, anpassen with metric* $g = dt^2 + f_\varepsilon^2 g^N$ *with conical singularities in* $\{0\}$ *and* $\{0, \pi\}$ *respectively.*

Proof Recall, that we have to solve (where we omit the $\tilde{\cdot}$) the following system

$$
\begin{aligned}
\partial_t \omega_1 &= -3\,\omega_3 - \varepsilon\, d\eta, \\
\partial_t (\eta \wedge \omega_3) &= d\omega_2 - 4\varepsilon \eta \wedge \omega_1, \\
\partial_t (\eta \wedge \omega_2) &= -d\omega_3.
\end{aligned} \tag{3.79}
$$

As Ansatz we consider the following family of $\mathrm{SU}^\varepsilon(p,q)$-structures with $p + q = 2$

$$
\begin{aligned}
\omega_1 &= f^2 \left(f \omega_1^0 \pm f' \omega_3^0 \right), \\
\omega_2 &= \sigma f^2 \omega_2^0, \\
\omega_3 &= -f^2 \left(f' \omega_1^0 \pm \varepsilon f \omega_3^0 \right), \\
\eta &= \sigma f \eta^0,
\end{aligned}
$$

where we set $f(t) = sin_\varepsilon(t)$, which yields $f'' = \varepsilon f$ and $f'^2 - \varepsilon f^2 = 1$. First we compute $\alpha\beta = 1$

$$\begin{aligned}
\partial_t\omega_1 &= 3f^2f'\omega_1^0 \pm \left(2ff'^2 + f^2f''\right)\omega_3^0 \\
&\overset{(*)}{=} 3\left(f^2f'\omega_1^0 \pm \varepsilon f^3\omega_3^0\right) \pm 2f\omega_3^0 \\
&= -3\,\omega_3 \pm \varepsilon f d\eta^0 = -3\,\omega_3 \pm \sigma\varepsilon d\eta \overset{!!}{=} -3\,\omega_3 - \varepsilon d\eta,
\end{aligned}$$

where in $(*)$ we used

$$2f(f')^2 + \varepsilon f^3 = f(2f'^2 + \varepsilon f^2) = 2f(\varepsilon f^2 + 1) + \varepsilon f^3 = 3\,\varepsilon f^3 + 2f$$

and $d\eta^0 = 2\varepsilon\omega_3^0$, since N^5 is an ε-Sasaki manifold. Hence we need to fix $\sigma = \mp 1$.

Next we calculate

$$\partial_t\,(\eta\wedge\omega_2) = \partial_t\left(f^3\eta^0\wedge\omega_2^0\right) = 3f^2f'\,\eta^0\wedge\omega_2^0$$

and using $d\omega_3^0 = 0$ and $d\omega_1^0 = 3a^+b^+\eta^0\wedge\omega_2^0$ yields

$$-b^+a^+\,d\omega_3 = b^+a^+\;d^5\left(f^2\left(f'\omega_1^0 \pm \varepsilon f\omega_3^0\right)\right) = b^+a^+f^2f'd^5\omega_1^0 = 3f^2f'\,\eta^0\wedge\omega_2^0,$$

which shows $\partial_t\,(\eta\wedge\omega_2) = -b^+a^+\,d\omega_3 = -d\omega_3$. It remains to determine the evolution of $\eta\wedge\omega_3$

$$\begin{aligned}
-\partial_t\,(\eta\wedge\omega_3) &= \sigma\partial_t\left(\pm\varepsilon f^4\eta^0\wedge\omega_3^0 + f^3f'\eta^0\wedge\omega_1^0\right) \\
&= \sigma\left(\pm 4\varepsilon f^3f'\eta^0\wedge\omega_3^0 + \left(3f^2(f')^2 + f^3f''\right)\eta^0\wedge\omega_1^0\right) \\
&\overset{(*)}{=} 4\varepsilon\sigma\left(\pm f^3f'\eta^0\wedge\omega_3^0 + f^4\eta^0\wedge\omega_1^0\right) + 3\sigma f^2\eta^0\wedge\omega_1^0 \\
&= 4\varepsilon\,\eta\wedge\omega_1 + \sigma b^-a^-f^2d\omega_2^0 \overset{b^-a^-=-1}{=} 4\varepsilon\,\eta\wedge\omega_1 - \sigma f^2d\omega_2^0 \\
&= 4\varepsilon\,\eta\wedge\omega_1 - d\omega_2,
\end{aligned}$$

where in $(*)$ we used

$$3f^2f'^2 + \varepsilon f^4 = 3f^2(\varepsilon f^2 + 1) + \varepsilon f^4 = 4\,\varepsilon f^4 + 3f^2.$$

This yields

$$\partial_t\,(\eta\wedge\omega_3) = -4\varepsilon\,\eta\wedge\omega_1 + d\omega_2$$

and finishes the proof of the Theorem. □

3.9 Results in the Homogeneous Case

3.9.1 Consequences for Automorphism Groups

An automorphism of an $SU^{\varepsilon}(p,q)$-structure on a six-manifold M is an automorphism of principal fibre bundles or equivalently, a diffeomorphism of M preserving all tensors defining the $SU^{\varepsilon}(p,q)$-structure. By our discussion on stable forms in Sect. 2.1 of Chap. 2, an $SU^{\varepsilon}(p,q)$-structure is characterised by a pair of compatible stable forms $(\omega,\rho) \in \Omega^2 M \times \Omega^3 M$. Since the construction of the remaining tensors J, ψ^- and g is invariant, a diffeomorphism preserving the two stable forms is already an automorphism of the $SU^{\varepsilon}(p,q)$-structure and in particular an isometry.

This easy observation has the following consequences when combined with the exterior systems of the previous section and the naturality of the exterior derivative.

Proposition 3.9.1 *Let (ω,ψ^+) be an $SU^{\varepsilon}(p,q)$-structure on a six-manifold M.*

(i) If the exterior differential equation

$$d\omega = \mu\, \psi^+$$

is satisfied for a constant $\mu \neq 0$, then a diffeomorphism Φ of M preserving ω is an automorphism of the $SU^{\varepsilon}(p,q)$-structure and in particular an isometry.

(ii) If the exterior differential equation

$$d\psi^- = \nu\, \omega \wedge \omega$$

is satisfied for a constant $\nu \neq 0$, then a diffeomorphism Φ of M preserving

(a) the real volume form and ψ^+,
(b) or the real volume form and ψ^-,
(c) or the ε-complex volume form $\Psi = \psi^+ + i_\varepsilon \psi^-$,

is an automorphism of the $SU^{\varepsilon}(p,q)$-structure and in particular an isometry.

We like to emphasise that both parts of the Proposition apply to strict nearly ε-Kähler structures of non-zero type. The same holds true for the following Proposition.

Proposition 3.9.2 *Let $(M^6, g, J^{\varepsilon}, \omega)$ be an almost ε-Hermitian six-manifold with totally skew-symmetric Nijenhuis tensor and Φ be a diffeomorphism of M preserving the almost ε-complex structure J^{ε}. Suppose, that the structure J^{ε} is quasi-integrable,*

(i) then Φ is a conformal map.

(ii) and additionally, assume, that one has $d\omega^2 = 0$, then Φ is a homothety on connected components of M. If moreover, Φ preserves the volume, then it is an isometry.

Proof As Φ preserves the ε-complex structure, it also preserves the Nijenhuis tensor. From Corollary 3.1.15 it follows $g(X, Y) = f\,\mathrm{tr}(N_X \circ N_Y)$ for some function f on M. This yields in $p \in M$

$$\begin{aligned} g_{\Phi(p)}(\Phi_* X, \Phi_* Y) &= f(\Phi(p))\mathrm{tr}(N_{\Phi_* X} \circ N_{\Phi_* Y}) \\ &= f(\Phi(p))\,\mathrm{tr}\left((\Phi^* N)_X \circ (\Phi^* N)_Y\right) = f(\Phi(p)) f(p)\, g_p(X, Y), \end{aligned}$$

i.e. the conformal factor is $c := f \cdot \Phi^* f$. Further let us assume, that one has $d\omega^2 = 0$. From above we know $\Phi^*(\omega \wedge \omega) = c^2(\omega \wedge \omega)$, which yields

$$0 = d(c^2 \omega \wedge \omega)) = d(c^2) \wedge \omega^2 + c^2 d\omega^2 = d(c^2) \wedge \omega^2.$$

Using, that the map $\eta \in \Lambda^1 T^* M^6 \to \eta \wedge \omega^2 \in \Lambda^5 T^* M^6$ is an isomorphism, we obtain $d(c^2) = 0$ and hence the function c is constant on connected components of M. Recall, that the metric volume is a multiple of ω^3. This implies, that one has $c = 1$. □

Corollary 3.9.3 *Let* $(M, J^\varepsilon, g, \omega)$ *be a nearly* ε*-Kähler six-manifold with* $\|\nabla J^\varepsilon\|^2 \neq 0$, *then a diffeomorphism* Φ *of* M *preserving* J^ε *is an automorphism of the* $\mathrm{SU}^\varepsilon(p, q)$*-structure and in particular an isometry.*

Proof The second nearly ε-Kähler equation implies $d\omega^2 = 0$. Hence we obtain from Proposition 3.9.2, that one has $\Phi^*(\omega) = c\,\omega$, for some constant c (on each connected component) and by the first nearly pseudo-Kähler equation

$$\Phi^*(\psi^+) = \frac{d(\Phi^* \omega)}{3} = \frac{c}{3} d\omega = c\psi^+.$$

As J^ε is preserved, this yields $\Phi^*(\psi^-) = c\psi^-$ and another time using the second nearly pseudo-Kähler equation

$$c d\psi^- = \Phi^*(d\psi^-) = \nu \Phi^*(\omega^2) = \nu c^2 \omega^2,$$

forces $c = 1$. □

Conversely, it is known for *complete* Riemannian nearly Kähler manifolds, that orientation-preserving isometries are automorphism of the almost Hermitian structure except for the round sphere S^6, see for instance [26, Proposition 4.1]. However, this is not true if the metric is incomplete. In [53, Theorem 3.6], a nearly Kähler structure is constructed on the incomplete sine-cone over a Sasaki-Einstein five-manifold $(N^5, \eta, \omega_1, \omega_2, \omega_3)$. In fact, the Reeb vector field dual to the one-form η is a Killing vector field which does not preserve ω_2 and ω_3. Thus, by the formulae given in [53], its lift to the nearly Kähler six-manifold is a Killing field for the sine-cone metric which does neither preserve Ψ nor ω nor J.

3.9.2 Left-Invariant Nearly ε-Kähler Structures on $\mathbf{SL(2,\mathbb{R})\times SL(2,\mathbb{R})}$

The following lemma is the key to proving the forthcoming structure result, since it considerably reduces the number of algebraic equations on the nearly ε-Kähler candidates.

Lemma 3.9.4 (Lemma 4.1 of [110]) *Denote by* $(\mathbb{R}^{1,2}, \langle\cdot,\cdot\rangle)$ *the vector space* $\mathbb{R}^3$ *endowed with its standard Minkowskian scalar-product and denote by* $\mathrm{SO}_0(1,2)$ *the connected component of the identity of its group of isometries. Consider the action of* $\mathrm{SO}_0(1,2)\times\mathrm{SO}_0(1,2)$ *on the space of real* 3×3 *matrices* $\mathrm{Mat}(3,\mathbb{R})$ *given by*

$$\begin{aligned}\Phi\,:\,\mathrm{SO}_0(1,2)\times\mathrm{Mat}(3,\mathbb{R})\times\mathrm{SO}_0(1,2)&\to\mathrm{Mat}(3,\mathbb{R})\\ (A,C,B)&\mapsto A^tCB.\end{aligned}$$

Then any invertible element $C\in\mathrm{Mat}(3,\mathbb{R})$ *lies in the orbit of an element of the form*

$$\begin{pmatrix}\alpha & x & y\\ 0 & \beta & z\\ 0 & 0 & \gamma\end{pmatrix}\quad or\quad\begin{pmatrix}0 & \beta & z\\ \alpha & x & y\\ 0 & 0 & \gamma\end{pmatrix}$$

with $\alpha,\beta,\gamma,x,y,z\in\mathbb{R}$ *and* $\alpha\beta\gamma\neq 0$.

Finally, we prove our main result of this subsection which is the following theorem. By a homothety, we define the rescaling of the metric by a real number which we do not demand to be positive since we are working with all possible signatures.

Theorem 3.9.5 *Let G be a Lie group with Lie algebra* $\mathfrak{sl}(2,\mathbb{R})$. *Up to homothety, there is a unique left-invariant nearly ε-Kähler structure with* $\|\nabla J^{\varepsilon}\|^2\neq 0$ *on* $G\times G$. *This is the nearly pseudo-Kähler structure of signature (4,2) constructed as 3-symmetric space in Sect. 3.9.4. In particular, there is no left-invariant nearly para-Kähler structure.*

Remark 3.9.6 The proof also shows that there is a left-invariant nearly ε-Kähler structure of non-zero type on $G\times H$ with $Lie(G)=Lie(H)=\mathfrak{sl}(2,\mathbb{R})$ if $G\neq H$ which is unique up to homothety *and* exchanging the orientation.

Proof More precisely, we will prove uniqueness up to equivalence of left-invariant almost ε-Hermitian structures and homothety. We will consider the algebraic exterior system

$$d\omega = 3\,\psi^+, \tag{3.80}$$

$$d\psi^- = 2\,\omega\wedge\omega \tag{3.81}$$

on the Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$. By Theorem 3.1.19, solutions of this system are in one-to-one correspondence to left-invariant nearly ε-Kähler structure on $G \times G$ with $\|\nabla J^\varepsilon\|^2 = 4$. This normalisation can always be achieved by applying a homothety. Furthermore, two solutions which are isomorphic under an inner Lie algebra automorphism from

$$\mathrm{Inn}(\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})) = \mathrm{SO}_0(1,2) \times \mathrm{SO}_0(1,2)$$

are equivalent under the corresponding Lie group isomorphism. Since both factors are equal, we can also lift the outer Lie algebra automorphism exchanging the two summands to the group level. In summary, it suffices to show the existence of a solution of the algebraic exterior system (3.80), (3.81) on the Lie algebra which is unique up to inner Lie algebra automorphisms and exchanging the summands.

A further significant simplification is the observation that all tensors defining a nearly ε-Kähler structure of non-zero type can be constructed out of the fundamental two-form ω with the help of the first nearly Kähler equation (3.80) and the stable form formalism described in Sect. 2.4 of Chap. 2. We break the main part of the proof into three lemmas, step by step simplifying ω under Lie algebra automorphisms in a fixed Lie bracket.

We call $\{e_1, e_2, e_3\}$ a *standard basis* of $\mathfrak{so}(1,2)$ if the Lie bracket satisfies

$$de^1 = -e^{23}\,, \quad de^2 = e^{31}\,, \quad de^3 = e^{12}.$$

In this basis, an inner automorphism in $\mathrm{SO}_0(1,2)$ acts by usual matrix multiplication on $\mathfrak{so}(1,2)$.

Lemma 3.9.7 *Let $\mathfrak{g} = \mathfrak{h} = \mathfrak{so}(1,2)$ and let ω be a non-degenerate two-form in*

$$\Lambda^2(\mathfrak{g} \oplus \mathfrak{h})^* = \Lambda^2\mathfrak{g}^* \oplus (\mathfrak{g} \otimes \mathfrak{h}) \oplus \Lambda^2\mathfrak{h}^*.$$

Then we have

$$d\omega^2 = 0 \quad \Leftrightarrow \quad \omega \in \mathfrak{g} \otimes \mathfrak{h}. \tag{3.82}$$

Proof By inspecting the standard basis, we observe that all two-forms on $\mathfrak{so}(1,2)$ are closed whereas no non-trivial 1-form is closed. Thus, when separately taking the exterior derivative of the components of ω^2 in $\Lambda^4 = (\Lambda^3\mathfrak{g}^* \otimes \mathfrak{h}^*) \oplus (\Lambda^2\mathfrak{g}^* \otimes \Lambda^2\mathfrak{h}^*) \oplus (\mathfrak{g}^* \otimes \Lambda^3\mathfrak{h}^*)$, the equivalence is easily deduced. □

Lemma 3.9.8 *Let $\mathfrak{g} = \mathfrak{h} = \mathfrak{so}(1,2)$ and let $\{e^1, e^2, e^3\}$ be a basis of $\mathfrak{g}^*$ and $\{e^4, e^5, e^6\}$ a basis of $\mathfrak{h}^*$ such that the Lie brackets are given by*

$$de^1 = -e^{23}, \quad de^2 = e^{31}, \quad de^3 = \tau e^{12} \quad \textit{and} \quad de^4 = -e^{56}, \quad de^5 = e^{64}, \quad de^6 = e^{45} \tag{3.83}$$

for some $\tau \in \{\pm 1\}$. *Then, every non-degenerate two-form* ω *on* $\mathfrak{g} \oplus \mathfrak{h}$ *satisfying* $d\omega^2 = 0$ *can be written*

$$\omega = \alpha\, e^{14} + \beta\, e^{25} + \gamma\, e^{36} + x\, e^{15} + y\, e^{16} + z\, e^{26} \tag{3.84}$$

for $\alpha, \beta, \gamma \in \mathbb{R} - \{0\}$ *and* $x, y, z \in \mathbb{R}$ *modulo an automorphism in* $\mathrm{SO}_0(1,2) \times \mathrm{SO}_0(1,2)$.

Proof We choose standard bases $\{e^1, e^2, e^3\}$ for $\mathfrak{g}$ and $\{e^4, e^5, e^6\}$ for $\mathfrak{h}$. Using the previous lemma and the assumption $d\omega^2 = 0$, we may write $\omega = \sum_{i,j=1}^{3} c_{ij} e^{i(j+3)}$ for an invertible matrix $C = \left(c_{ij}\right) \in \mathrm{Mat}(3, \mathbb{R})$. When a pair $(A, B) \in \mathrm{SO}_0(1,2) \times \mathrm{SO}_0(1,2)$ acts on the two-form ω, the matrix C is transformed to $A^t CB$. Applying Lemma 3.9.4, we can achieve by an inner automorphism that C is in one of the normal forms given in that lemma. However, an exchange of the base vectors e_1 and e_2 corresponds exactly to exchanging the first and the second row of C. Therefore, we can always write ω in the claimed normal form by adding the sign τ in the Lie bracket of the first summand $\mathfrak{g}$. □

Lemma 3.9.9 (Lemma 4.6 of [110]) *Let* $\{e^1, \ldots, e^6\}$ *be a basis of* $\mathfrak{so}(1,2) \times \mathfrak{so}(1,2)$ *such that*

$$de^1 = -e^{23}, \quad de^2 = e^{31}, \quad de^3 = e^{12} \quad \text{and} \quad de^4 = -e^{56}, \quad de^5 = e^{64}, \quad de^6 = e^{45}. \tag{3.85}$$

Then the only $\mathrm{SU}^{\varepsilon}(p,q)$*-structure* (ω, ψ^+) *modulo inner automorphisms and modulo exchanging the summands, which solves the two nearly* ε*-Kähler equations* (3.80) *and* (3.81), *is determined by*

$$\omega = \frac{\sqrt{3}}{18}(e^{14} + e^{25} + e^{36}). \tag{3.86}$$

In fact, the uniqueness, existence and non-existence statements claimed in the theorem follow directly from this lemma and formula obtained for the quartic invariant which implies $\lambda\left(\frac{1}{3}d\omega\right) < 0$.

As explained in Sect. 3.9.4, we know that there is a left-invariant nearly pseudo-Kähler structure of indefinite signature on all the groups in question. After applying a homothety, we can achieve $\|\nabla J^{\varepsilon}\|^2 = 4$ and this structure has to coincide with the unique structure we just constructed. Therefore, the indefinite metric has to be of signature (4,2) by our sign conventions.

We summarise the data of the unique nearly pseudo-Kähler structure in the basis (3.85) and can easily double-check the signature of the metric explicitly:

$$\omega = \frac{1}{18}\sqrt{3}\,(e^{14} + e^{25} + e^{36})$$
$$\psi^+ = \frac{1}{54}\sqrt{3}\,(e^{126} - e^{135} + e^{156} - e^{234} + e^{246} - e^{345})$$

$$
\begin{aligned}
\psi^- &= -\frac{1}{54}(2\,e^{123} + e^{126} - e^{135} - e^{156} - e^{234} - e^{246} + e^{345} + 2\,e^{456})\\
J(e_1) &= -\frac{1}{3}\sqrt{3}\,e_1 - \frac{2}{3}\sqrt{3}\,e_4\,, \quad J(e_4) = \frac{2}{3}\sqrt{3}\,e_1 + \frac{1}{3}\sqrt{3}\,e_4\\
J(e_2) &= -\frac{1}{3}\sqrt{3}\,e_2 + \frac{2}{3}\sqrt{3}\,e_5\,, \quad J(e_5) = -\frac{2}{3}\sqrt{3}\,e_2 + \frac{1}{3}\sqrt{3}\,e_5\\
J(e_3) &= -\frac{1}{3}\sqrt{3}\,e_3 + \frac{2}{3}\sqrt{3}\,e_6\,, \quad J(e_6) = -\frac{2}{3}\sqrt{3}\,e_3 + \frac{1}{3}\sqrt{3}\,e_6\\
g &= \frac{1}{9}\,(\,(e^1)^2 - (e^2)^2 - (e^3)^2 + (e^4)^2 - (e^5)^2\\
&\qquad - (e^6)^2 - e^1\cdot e^4 - e^2\cdot e^5 - e^3\cdot e^6).
\end{aligned}
$$

□

Observing that in [25] very similar arguments have been applied to the Lie group $S^3 \times S^3$, we find the following non-existence result.

Proposition 3.9.10 *On the Lie groups $G \times H$ with $Lie(G) = Lie(H) = \mathfrak{so}(3)$, there is neither a left-invariant nearly para-Kähler structure of non-zero type nor a left-invariant nearly pseudo-Kähler structure with an indefinite metric.*

Proof The unicity of the left-invariant nearly Kähler structure $S^3 \times S^3$ is proved in [25, Section 3], with a strategy analogous to the proof of Theorem 3.9.5. In the following, we will refer to the English version [26]. There, it is shown in the proof of Proposition 2.5, that for any solution of the exterior system

$$
\begin{aligned}
d\omega &= 3\psi^+\\
d\psi^+ &= -2\mu\omega^2
\end{aligned}
$$

there is a basis of the Lie algebra of $S^3 \times S^3$ and a real constant α such that

$$
de^1 = e^{23}\,, \quad de^2 = e^{31}\,, \quad de^3 = e^{12} \quad \text{and} \quad de^4 = e^{56}\,, \quad de^5 = e^{64}\,, \quad de^6 = e^{45},
$$
$$
\omega = \alpha(e^{14} + e^{25} + e^{36}).
$$

In this basis, a direct computation or formula (18) in [26] show that the quartic invariant that we denote by λ is

$$
\lambda = -\frac{1}{27}\alpha^4
$$

with respect to the volume form e^{123456}. Therefore, a nearly para-Kähler structure cannot exist on all the Lie groups with the same Lie algebra as $S^3 \times S^3$ by Theorem 3.1.19. A nearly pseudo-Kähler structure with an indefinite metric cannot exist either, since the induced metric is always definite as computed in the second part of Lemma 2.3 in [26]. □

3.9.3 Real Reducible Holonomy

Nearly pseudo-Kähler manifolds admitting a J-invariant and $\bar{\nabla}$-parallel decomposition of the tangent bundle TM are related to twistor spaces [108] (cf. [16, 98] for Riemannian metrics) and Sects. 3.3 and 3.4 of this chapter. The next Proposition considers the complementary situation, i.e. the case where TM decomposes into two sub-bundles and J interchanges these sub-bundles and generalises a result of [99] to pseudo-Riemannian metrics.

Proposition 3.9.11 *Let (M, g, J) be a complete, strict, simply connected nearly pseudo-Kähler manifold. Suppose, that TM admits an orthogonal, $\bar{\nabla}$-parallel decomposition $TM = \mathcal{V} \oplus \mathcal{V}'$ with $\mathcal{V}' = J\mathcal{V}$, then (M, g) is a homogeneous space.*

Proof For a vector field $X = JV_1$ in $\mathcal{V}' = J\mathcal{V}$, a vector field $Y = JV_2$ in TM and vector fields V_3, V_4 in $\mathcal{V}$ by the same argument as in Lemma 3.4.1 it is

$$\begin{aligned}\bar{R}(JV_1, JV_2, V_3, V_4) &= g\left([\nabla_{V_3}J, \nabla_{V_4}J]JV_1, JV_2\right) - g\left((\nabla_{JV_1}J)JV_2, (\nabla_{V_3}J)V_4\right)\\ &= g\left([\nabla_{V_3}J, \nabla_{V_4}J]V_1, V_2\right) + g\left((\nabla_{V_1}J)V_2, (\nabla_{V_3}J)V_4\right) \quad (3.87)\end{aligned}$$

By the symmetries (3.24) and (3.25) of the curvature tensor $\bar{R}$ the last equation determines $\bar{R}$. By Proposition 3.1.7 the torsion T and ∇J are $\bar{\nabla}$-parallel. In particular, we have

$$\bar{\nabla}_U((\nabla_X J)Y) = (\nabla_X J)\bar{\nabla}_U Y + (\nabla_{\bar{\nabla}_U X} J)Y.$$

Deriving (3.87) this implies $\bar{\nabla}\bar{R} = 0$. Hence $\bar{\nabla}$ is an Ambrose-Singer connection and as M is simply connected and complete it follows, that (M, J, g) is a homogeneous space (see [116]). □

3.9.4 3-Symmetric Spaces

The idea of a three-symmetric space is to replace the symmetry of order two as in the case of a symmetric space by a symmetry of order three. Nearly Kähler geometry on such spaces was first studied in [66, 69].

Like symmetric spaces three-symmetric spaces have a homogenous model, which we shortly resume: Let G be a connected Lie group and s an automorphism of order 3 and let $G_0^s \subset H \subset G^s$ be a subgroup contained in the fix-point set G^s of s. The differential s_* decomposes

$$\mathfrak{g} \otimes \mathbb{C} = \mathfrak{h} \otimes \mathbb{C} \oplus \mathfrak{m}^+ \oplus \mathfrak{m}^-$$

into the eigenspaces of s_* with eigenvalues 1 and $\frac{1}{2}(-1 \pm \sqrt{-3})$. With the definition $\mathfrak{m} := (\mathfrak{m}^+ \oplus \mathfrak{m}^-) \cap \mathfrak{g}$ the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \tag{3.88}$$

is reductive and G/H is a reductive homogenous space. The characteristic complex structure is then defined by

$$s_{*|\mathfrak{m}} = -\frac{1}{2}Id + \frac{\sqrt{3}}{2}J. \tag{3.89}$$

The choice of an $Ad(H)$-invariant and s_*-invariant (pseudo-)Euclidean scalar-product B on $\mathfrak{m}$ endows G/H with the structure of a (pseudo-)Riemannian three-symmetric space, such that B is almost Hermitian with respect to J. The next Theorem locally relates homogeneous spaces to three-symmetric spaces.

Theorem 3.9.12 *An almost pseudo-Hermitian manifold (M, J, g) is a locally three-symmetric space if and only if it is a quasi-Kähler manifold and the torsion T and the curvature tensor $\bar{R}$ of the characteristic Hermitian connection $\bar{\nabla}$ are parallel, i.e. $\bar{\nabla}T = 0$ and $\bar{\nabla}\bar{R} = 0$.*

The proof of Theorem 3.9.12 is based on the following description of locally three-symmetric spaces given in [66].

Theorem 3.9.13 *Let (M, J, g) be an almost pseudo-Hermitian manifold. Then there exists a family of local cubic diffeomorphisms $(s_x)_{x\in M}$ such that J is the induced complex structure and such that M is a three-symmetric space if and only if*

(i) *M is quasi-Kähler, i.e. one has $(\nabla_X J)Y + (\nabla_{JX} J)JY = 0$,*
(ii) $\sigma = s_*$ preserves $\nabla^2 J$,
(iii) *for $X, Y, Z, T \in \Gamma(TM)$ one has*

$$\begin{aligned} R(X, Y, Z, T) = {} & R(JX, JY, Z, T) + R(JX, Y, JZ, T) \\ & + R(JX, Y, Z, JT), \end{aligned} \tag{3.90}$$

(iv) *for $X, Y, Z, T \in \Gamma(TM)$ one has*

$$(\nabla_W R)(X, Y, Z, T) + (\nabla_W R)(JX, JY, JZ, JT) = 0.$$

Proof of Theorem 3.9.12 We claim that the conditions (i)-(iv) of Theorem 3.9.13 are equivalent to the following system of equations:

$$\eta_X Y + \eta_{JX} JY = 0, \tag{3.91}$$

$$\bar{\nabla}\eta = 0, \tag{3.92}$$

$$\bar{\nabla}\bar{R} = 0, \tag{3.93}$$

Let us recall the definition $\eta := \frac{1}{2}J(\nabla J)$ which yields that the condition (3.91) is equivalent to Theorem 3.9.13 part (i). From $\nabla_X J = -2J\eta_X$ it follows

$$(\nabla^2_{X,Y}J)Z = 2J(2\eta_X\eta_Y Z - (\nabla_X\eta)_Y Z) = 2J\left(2\eta_X\eta_Y Z - (\bar{\nabla}_X\eta)_Y Z - \eta_{\eta_X Y}Z + [\eta_Y, \eta_X]Z\right).$$

By Eq. (3.92) this expression only depends on η and J, which are both preserved by σ.

Conversely, we suppose that σ preserves ∇J and $\nabla^2 J$. From Proposition 3.3 of Gray [66] one obtains

$$\begin{aligned}\bar{\nabla}\eta(X, Y, Z, T) &= \bar{\nabla}\eta(JX, JY, Z, T) + \bar{\nabla}\eta(JX, Y, JZ, T)\\ &\quad + \bar{\nabla}\eta(JX, Y, Z, JT) = 3\bar{\nabla}\eta(JX, JY, Z, T),\end{aligned}$$

where the last equality follows from $\bar{\nabla}J = 0$ and Eq. (3.91). This implies replacing X by JX and Y by JY

$$3\bar{\nabla}\eta(JX, JY, Z, T) = \bar{\nabla}\eta(X, Y, Z, T) = \frac{1}{3}\bar{\nabla}\eta(JX, JY, Z, T)$$

and finally we obtain $\bar{\nabla}\eta = 0$.

Moreover, the condition (3.92) implies Theorem 3.9.13 part (iii). In fact, we claim that Theorem 3.9.13 part (iii) can be re-written as $R \in \mathcal{L}_2$, where $\mathcal{L}_2$ is one of the irreducible components of the space of curvature tensors considered as a $\mathrm{GL}(n, \mathbb{C})$ representation [52]. More precisely, in our case we identify $\mathfrak{u}(p, q)$ with $[\lambda^{1,1}]$ (instead of $\mathfrak{u}(n)$) and $\mathfrak{u}(p, q)^{\perp}$ (rather than $\mathfrak{u}(n)^{\perp}$) with $[\![\lambda^{2,0}]\!]$ and then apply the results of [52] to obtain an analogous decomposition. In particular, it follows in the quasi-Kähler case, that the complement of $\mathcal{L}_2$ only depends on $\bar{\nabla}\eta$ and we conclude that Eq. (3.92) implies Theorem 3.9.13 part (iii).

It remains to relate Theorem 3.9.13 part (iv) and Eq. (3.93). From $\nabla = \bar{\nabla} + \eta$ it follows

$$\bar{R}(X, Y, Z, W) = R(X, Y, Z, W) + g([\eta_X, \eta_Y]Z, W) - g(\eta_{\eta_X Y - \eta_Y X}Z, W),$$

where we use the condition (3.92). Using a second time the condition (3.92) and $\bar{\nabla}g = 0$ we get

$$\bar{\nabla}\bar{R} = \bar{\nabla}R = \nabla R - \eta \cdot R \tag{3.94}$$

with

$$\eta \cdot R = R(\eta\cdot, \cdot, \cdot, \cdot) + R(\cdot, \eta\cdot, \cdot, \cdot) + R(\cdot, \cdot, \eta\cdot, \cdot) + R(\cdot, \cdot, \cdot, \eta\cdot).$$

Now Eq. (3.90) implies $R(JX, JY, JZ, JW) = R(X, Y, Z, W)$, see for instance Corollary 3.4 of [66]. As η and J anti-commute, it follows from (3.90)

$$(\eta \cdot R)(JX, JY, JZ, JW) = -(\eta \cdot R)(X, Y, Z, W) \tag{3.95}$$

and as $\bar{\nabla}$ is Hermitian we have

$$\bar{\nabla} R(X, Y, Z, W) = \bar{\nabla} R(JX, JY, JZ, JW). \tag{3.96}$$

Equation (3.94) and (3.96) yield

$$\begin{aligned} 2(\bar{\nabla}_W \bar{R})(X, Y, Z, T) &= (\bar{\nabla}_W \bar{R})(X, Y, Z, T) + (\bar{\nabla}_W \bar{R})(JX, JY, JZ, JT) \\ &\overset{Eqs.\,(3.94),(3.95)}{=} (\nabla_W R)(X, Y, Z, T) + (\nabla_W R)(JX, JY, JZ, JT), \end{aligned}$$

which shows the equivalence of Theorem 3.9.13 part (iv) and Eq. (3.93). □

The following proposition relates the information coming from the Hermitian structure to the data of the homogeneous space.

One may suppose G to be simply connected, since otherwise one considers its universal cover $\pi : \tilde{G} \to G$ and the isomorphic homogeneous space $\tilde{G}/\tilde{H}$ with $\tilde{H} = \pi^{-1}(H)$.

Proposition 3.9.14 *Let $(M = G/H, J, g)$ be a (simply connected) reductive homogeneous almost pseudo-Hermitian manifold, then $M = G/H$ is three-symmetric if and only if it is quasi-Kähler and the connection $\bar{\nabla}$ coincides with the normal connection ∇^{nor} of the reductive homogeneous space G/H.*

Proof Let $(M = G/H, J, g)$ be a reductive homogeneous space with adapted reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. The invariant almost complex structure J induces a complex structure on $\mathfrak{m}$ and an invariant decomposition

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^{1,0} \oplus \mathfrak{m}^{0,1}. \tag{3.97}$$

The invariance of $\mathfrak{m}$ and J implies

$$[\mathfrak{h}, \mathfrak{m}^{1,0}] \subset \mathfrak{m}^{1,0} \text{ and } [\mathfrak{h}, \mathfrak{m}^{0,1}] \subset \mathfrak{m}^{0,1}. \tag{3.98}$$

The three-symmetry s is now obtained by the integration of the map σ

$$\sigma_{|\mathfrak{h}} = Id_{|\mathfrak{h}},\ \sigma_{|\mathfrak{m}^{1,0}} = jId_{|\mathfrak{m}^{1,0}}, \sigma_{|\mathfrak{m}^{0,1}} = j^2 Id_{|\mathfrak{m}^{0,1}},$$

where $j = -\frac{1}{2}Id + \frac{\sqrt{3}}{2}i$. The map σ integrates (since G is supposed to be simply connected) to s if and only if it is an automorphism of the Lie algebra $\mathfrak{g}$. By Eq. (3.98) and the definition of σ this is the case if and only if one has

$$[\mathfrak{m}^{1,0}, \mathfrak{m}^{1,0}] \subset \mathfrak{m}^{0,1}, [\mathfrak{m}^{0,1}, \mathfrak{m}^{0,1}] \subset \mathfrak{m}^{1,0} \text{ and } [\mathfrak{m}^{1,0}, \mathfrak{m}^{0,1}] \subset \mathfrak{h}. \tag{3.99}$$

Recall, that the torsion of the normal connection (see [87, Chapter X]) is given by the invariant tensor

$$T^{nor}(u,v) = -[u,v]^{\mathfrak{m}}.$$

In terms of the torsion T^{nor} the integrability conditions (3.99) are

$$\begin{aligned} T^{nor}(u,v) &= -[u,v]^{\mathfrak{m}^{0,1}}, && \text{for } u,v \in \mathfrak{m}^{1,0},\\ T^{nor}(u,v) &= -[u,v]^{\mathfrak{m}^{1,0}}, && \text{for } u,v \in \mathfrak{m}^{0,1},\\ T^{nor}(u,v) &= \qquad 0, && \text{for } u \in \mathfrak{m}^{1,0} \text{ and } v \in \mathfrak{m}^{0,1}. \end{aligned}$$

In other words T^{nor} is a multiple of the Nijenhuis tensor. Contraction with the metric yields a tensor $g(T^{nor}(\cdot,\cdot),\cdot)$ which is of type $\otimes^3\left(\mathfrak{m}^{1,0}\right)^* \oplus \otimes^3\left(\mathfrak{m}^{0,1}\right)^*$ w.r.t. the complex structure induced by $\mathfrak{m}^{1,0} \oplus \mathfrak{m}^{0,1}$ and skew-symmetric in the first two entries. These symmetries exclude contributions of the pseudo-Riemannian version of class $\mathcal{W}_3$ and $\mathcal{W}_4$ in the Gray-Hervella list [68] and hence (M,J,g) is of type $\mathcal{W}_1 \oplus \mathcal{W}_2$, i.e. (M,J,g) is a quasi-Kähler manifold. Moreover, we have $T^{nor} \in [\![\lambda^{2,0} \otimes \lambda^{1,0}]\!]$ and we obtain $\nabla^{nor} - \nabla^g \in \mathcal{W}_1 \oplus \mathcal{W}_2 \subset T^*M \otimes \mathfrak{u}^{\perp}_{p,q}$. This means that ∇^{nor} is the intrinsic connection which equals the characteristic Hermitian connection. Summarizing (3.97) is the decomposition into eigenspaces of an automorphism of order three if and only if $(M = G/H, J, g)$ is quasi-Kähler and the normal connection coincides with the intrinsic connection. □

As a consequence the torsion and the curvature of the connection $\bar{\nabla}$ are given by

$$T(u,v) = -[u,v]^{\mathfrak{m}} \text{ and } \bar{R}(u,v) = [u,v]^{\mathfrak{h}} \text{ with } u,v \in \mathfrak{m}. \tag{3.100}$$

reductive, if it holds

$$B([X,Y]^{\mathfrak{m}}, Z) = B(X, [Y,Z]^{\mathfrak{m}}) \text{ for } X,Y,Z \in \mathfrak{m}.$$

The next result was already shown in [66] Proposition 5.6 for pseudo-Riemannian metrics. It is a consequence of $\nabla^{nor} = \bar{\nabla}$ for three-symmetric spaces.

Proposition 3.9.15 *A three-symmetric space is a nearly pseudo-Kähler manifold if and only if it is a naturally reductive homogeneous space.*

In the sequel, we consider two homogeneous spaces G/H and G'/H' which are T-dual to each other in the sense of the construction given in Sect. 2.6.1 of Chap. 2 and we are going to show that this construction is compatible with 3-symmetry.

As a preparation we recall the construction of the related complex structures. Let us suppose, that $\mathfrak{g}$ is a compact Lie algebra with a subalgebra $\mathfrak{h}$ and that $(M = G/H, g, J)$ is a Riemannian 3-symmetric space with a nearly Kähler structure of above discussed type. Moreover, denote by

$$\mathfrak{g}' = \mathfrak{g}_+ \oplus i\mathfrak{g}_-, \;\; \mathfrak{m}' = \mathfrak{m}_+ \oplus i\mathfrak{m}_- \text{ and } \mathfrak{h}' = \mathfrak{h}_+ \oplus i\mathfrak{h}_-$$

the associated decompositions of a fixed T-dual space $M' = G'/H'$.

In this situation, there exists a natural almost complex structure J' on M' which we shortly recall next, cf. Section 3.4 of [82]. Firstly, one decomposes $\mathfrak{gl}(\mathfrak{m})$ into

$$\mathfrak{gl}(\mathfrak{m})_+ := \{A \in \mathfrak{gl}(\mathfrak{m}) \,|\, A(\mathfrak{m}_+) \subset \mathfrak{m}_+, \; A(\mathfrak{m}_-) \subset \mathfrak{m}_-\},$$
$$\mathfrak{gl}(\mathfrak{m})_- := \{B \in \mathfrak{gl}(\mathfrak{m}) \,|\, B(\mathfrak{m}_+) \subset \mathfrak{m}_-, \; B(\mathfrak{m}_-) \subset \mathfrak{m}_+\}.$$

The Lie algebra $\mathfrak{gl}(\mathfrak{m})_+ \oplus i\mathfrak{gl}(\mathfrak{m})_- \subset \mathfrak{gl}(\mathfrak{m})^{\mathbb{C}}$ is isomorphic to $\mathfrak{gl}(\mathfrak{m}')$ via the extension of the following definition

$$A(iX) := iA(X), \; A(Y) := A(Y), \; (iB)(iX) := -B(X), \; (iB)(Y) := iB(Y),$$

where $X \in \mathfrak{m}_-$ and $Y \in \mathfrak{m}_+$ and $A \in \mathfrak{gl}(\mathfrak{m})_+$ and $B \in \mathfrak{gl}(\mathfrak{m})_-$. Further denote by $j \in \mathfrak{gl}(\mathfrak{m})$ the linear map associated to the invariant complex structure J on G/H.

Assume on the one hand, that one even has $j \in \mathfrak{gl}(\mathfrak{m})_+$, then (as shown in Proposition 3.5 of [82]) using the above identification of $\mathfrak{gl}(\mathfrak{m})_+ \oplus i\mathfrak{gl}(\mathfrak{m})_-$ and $\mathfrak{gl}(\mathfrak{m}')$ the map $j \in \mathfrak{gl}(\mathfrak{m})_+ \subset \mathfrak{gl}(\mathfrak{m}')$ induces an invariant almost complex structure J' on G'/H', such that if g is Hermitian for J then J' is pseudo-Hermitian for g'. If on the other hand one has $j \in \mathfrak{gl}(\mathfrak{m})_-$, then $ij \in \mathfrak{gl}(\mathfrak{m})_- \subset \mathfrak{gl}(\mathfrak{m}')$ is a para-complex structure on G'/H'.

For the 3-symmetric case we recover the 3-symmetry using Eq. (3.89), i.e. for $j \in \mathfrak{gl}(\mathfrak{m})_+$ one has

$$\sigma_{|\mathfrak{m}} = s_{*|\mathfrak{m}} = -\frac{1}{2}Id + \frac{\sqrt{3}}{2}j \in \mathfrak{gl}(\mathfrak{m})_+,$$

which induces as before an endomorphism of $\mathfrak{m}'$

$$\sigma_{|\mathfrak{m}'} = -\frac{1}{2}Id + \frac{\sqrt{3}}{2}j \in \mathfrak{gl}(\mathfrak{m})_+ \subset \mathfrak{gl}(\mathfrak{m}'),$$

i.e. after extending σ by the identity on $\mathfrak{h}'$ this yields a local 3-symmetry for G'/H' and assuming G' to be simply connected one may integrates σ to a 3-symmetry of G'.

By construction it follows, that if $(\mathfrak{g}, \mathfrak{m}, \mathfrak{h}, \langle\cdot,\cdot\rangle)$ is naturally reductive, the T-dual $(\mathfrak{g}', \mathfrak{m}', \mathfrak{h}', \langle\cdot,\cdot\rangle')$ is naturally reductive, too. Using Proposition 3.9.15 it follows, that the T-dual of a nearly Kähler 3-symmetric space is nearly pseudo-Kähler. Summarising our discussion we have shown.

Theorem 3.9.16 *Let* $(G/H, J, g)$ *be a nearly Kähler 3-symmetric space (with compact* G*) associated to* $(\mathfrak{g}, \mathfrak{m}, \mathfrak{h}, \langle\cdot,\cdot\rangle)$ *with the above described nearly Kähler* *structure and* G'/H' *be a T-dual of* G/H *with data* $(\mathfrak{g}', \mathfrak{m}', \mathfrak{h}', \langle\cdot,\cdot\rangle')$ *such that the map* j *associated to the invariant complex structure* J *lies in* $\mathfrak{gl}(\mathfrak{m})_+$*, then* $(G'/H', J', g')$ *is a nearly pseudo-Kähler 3-symmetric space.*

A natural question is starting with some homogeneous nearly Kähler manifold G/H as above to give a classification of all T-dual spaces. Even though there is no general answer to this question the cases of interest for the sequel are discussed in [82]. Let us recall, that by [25] the list of homogenous strict nearly Kähler six-manifolds is

$$\begin{aligned} S^6 &= G_2/SU(3), \\ \mathbb{C}P^3 &= Sp(2)/(SU(2) \times U(1)), \\ \mathbb{F}(1,2) &= SU(3)/(U(1) \times U(1)) \text{ and} \\ S^3 \times S^3 &= (SU(2) \times SU(2) \times SU(2))/\Delta(SU(2)). \end{aligned}$$

For these spaces all possible T-duals (given in [82]) are the following **pruefen**

$$\begin{aligned} S^{2,4} &= G_2^*/SU(1,2), \text{cf. Chap. 4 or [82]}, \\ \mathcal{Z}(S^{2,2}) &= SO^+(2,3)/U(1,1), \text{ c.f. Example 3.1 of [82]}, \\ \mathcal{Z}(S^{4,0}/\mathbb{Z}_2) &= SO^+(4,1)/U(2), \text{ c.f. Example 3.1 of [82]}, \\ \mathcal{Z}(\mathbb{C}P^{2,0}) &= SU(2,1)/(U(1) \times U(1)), \text{ c.f. Example 3.2 of [82]}, \\ \mathcal{Z}((SL(3,\mathbb{R})/GL^+(2,\mathbb{R})) &= SL^+(3,\mathbb{R})/\mathbb{R}^* \cdot SO(2), \text{ c.f. Example 3.2 of [82]}, \\ SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) &= (SU(1,1) \times SU(1,1) \times SU(1,1))/\Delta(SU(1,1)). \end{aligned}$$

Let us recall, that the twistor spaces already appeared in Sects. 3.3 and 3.4 of the present chapter. Moreover, one may wonders, if one obtains all nearly pseudo-Kähler structures as T-duals of some homogeneous space G/H with a compact Lie group G. The answer can be found in a recent preprint [11], where six-dimensional homogeneous almost complex structures with semi-simple isotropy have been classified. In this list an example of a left invariant nearly pseudo-Kähler structure on a solvable Lie group is given (cf. Remark 3 of [11]), which does not appear in the above list of T-dual spaces.

3.10 Lagrangian Submanifolds in Nearly Pseudo-Kähler Manifolds

This section is based on results with Smoczyk and Schäfer [111] which are extended to pseudo-Riemannian signature in Sects. 3.10.3 and 3.10.4.

3.10.1 Definitions and Geometric Identities

For the rest of this section let us assume that $L \subset M$ is a Lagrangian submanifold[11] of a nearly pseudo-Kähler manifold (M^{2n}, J, g) in the sense of the next definition.

Definition 3.10.1 Let (M^{2n}, J, g, ω) be a nearly pseudo-Kähler manifold. A submanifold $\iota : L^n \to M^{2n}$ is called Lagrangian submanifold, provided that one has $\omega_{|TL} = \iota^*\omega = 0$, the dimension n of L is half the dimension $2n$ of M and that $\iota^* g$ is non-degenerate.
Since $n = \dim(L) = \frac{1}{2}\dim(M)$ we have, that for a Lagrangian submanifold

$$g(JX, Y) = 0\,, \quad \forall\, X,\, Y \in TL \quad \Leftrightarrow \quad J : TL \to T^{\perp}L \quad \text{is an isomorphism.}$$

We observe that the (symmetric) signature of the metric g restricted to L is (p, q) if the signature of g is $(2p, 2q)$. From $\omega_{|TL} = 0$ we deduce $d\omega_{|TL} = 0$. On the other hand (3.1) implies

$$d\omega(X, Y, Z) = 3g((\nabla_X J)Y, Z)\,.$$

From this and the symmetries of ∇J the following Lemma easily follows (see also [77]).

Lemma 3.10.2 *Suppose $L \subset M$ is a Lagrangian submanifold in a nearly Kähler manifold (M, J, g) with (possibly) indefinite metric. Then*

$$(\nabla_X J)Y \in T^{\perp}L\,, \quad \forall\, X,\, Y \in TL\,, \tag{3.101}$$

$$(\nabla_X J)Y \in T^{\perp}L\,, \quad \forall\, X,\, Y \in T^{\perp}L\,, \tag{3.102}$$

$$(\nabla_X J)Y \in TL\,,\ \text{if } X \in TL, Y \in T^{\perp}L \text{ or if } X \in T^{\perp}L, Y \in TL\,. \tag{3.103}$$

Denote by II the second fundamental form of the Lagrangian immersion $L \subset M^{2n}$ into a nearly Kähler manifold M.

Proposition 3.10.3 *For a Lagrangian submanifold $L \subset M^{2n}$ in a nearly Kähler manifold (with possibly indefinite metric) we have the following information.*

(i) *The second fundamental form is given by $\langle II(X, Y), U\rangle = \langle \bar{\nabla}_X Y, U\rangle$ for $X, Y \in \Gamma(TL)$ and $U \in \Gamma(T^{\perp}L)$.*
(ii) *The tensor $C(X, Y, Z) := \langle II(X, Y), JZ\rangle = \omega(II(X, Y), Z)$, $\forall\, X, Y, Z \in TL$ is totally symmetric.*

[11] In order to compute expressions like for example $\nabla_X Y$ one needs to extend the vector fields on L to vector fields on M. It is common to use the same symbols for the extended vector fields, since the induced objects do not depend on the choice of extension.

Proof From Lemma 3.10.2 we compute for $X, Y \in \Gamma(TL)$ and $U \in \Gamma(T^{\perp}L)$ the second fundamental form II

$$\langle II(X,Y), U\rangle = \langle \nabla_X Y, U\rangle = \langle \bar{\nabla}_X Y - \frac{1}{2}J(\nabla_X J)Y, U\rangle = \langle \bar{\nabla}_X Y, U\rangle\,.$$

This yields part (i). Next we prove (ii): First we observe for $X, Y, Z \in \Gamma(TL)$

$$\begin{aligned} C(X,Y,Z) &= \langle II(X,Y), JZ\rangle = \langle \bar{\nabla}_X Y, JZ\rangle = -\langle Y, \bar{\nabla}_X(JZ)\rangle \\ &= -\langle Y, J\bar{\nabla}_X Z\rangle = \langle \bar{\nabla}_X Z, JY\rangle = C(X,Z,Y). \end{aligned}$$

Since the second fundamental form is symmetric, it follows that C is totally symmetric. □

Next we generalise an identity of [51] to nearly Kähler manifolds of arbitrary dimension and signature of the metric. This and the next lemma will be crucial to prove that Lagrangian submanifolds in strict nearly (pseudo-)Kähler six-manifolds and in twistor spaces Z^{4n+2} over quaternionic Kähler manifolds with their canonical nearly Kähler structure are minimal. A six-dimensional version of the Lemma was also proved in [71], see also Remark 3.10.7.

Lemma 3.10.4 *The second fundamental form II of a Lagrangian immersion $L \subset M^{2n}$ into a nearly (pseudo-)Kähler manifold and the tensor ∇J satisfy the following identity*

$$\langle II(X, J(\nabla_Y J)Z), U\rangle = \langle J(\nabla_{II(X,Y)}J)Z, U\rangle + \langle J(\nabla_Y J)II(X,Z), U\rangle \tag{3.104}$$

with $X, Y \in TL$ and $U \in T^{\perp}L$.

Proof The proof of this identity uses $\bar{\nabla}J = 0$, $\bar{\nabla}(\nabla J) = 0$ and Lemma 3.10.2. With $X, Y, Z \in \Gamma(TL)$ and $U \in \Gamma(T^{\perp}L)$ we obtain

$$\begin{aligned} \langle II(X, J(\nabla_Y J)Z), U\rangle &= \langle \bar{\nabla}_X(J(\nabla_Y J)Z), U\rangle \overset{\bar{\nabla}J=0}{=} \langle J\bar{\nabla}_X(\nabla_Y J)Z), U\rangle \\ &\overset{\bar{\nabla}((\nabla J)=0}{=} \langle J\left[(\nabla_{\bar{\nabla}_X Y}J)Z + (\nabla_Y J)\bar{\nabla}_X Z\right], U\rangle \\ &= \langle J[-(\nabla_Z J)\bar{\nabla}_X Y + (\nabla_Y J)\bar{\nabla}_X Z], U\rangle \\ &= -\langle \bar{\nabla}_X Y, (\nabla_Z J)JU\rangle + \langle \bar{\nabla}_X Z, (\nabla_Y J)JU\rangle \\ &= -\langle II(X,Y), (\nabla_Z J)JU\rangle + \langle II(X,Z), (\nabla_Y J)JU\rangle \\ &= -\langle J(\nabla_Z J)II(X,Y), U\rangle + \langle J(\nabla_Y J)II(X,Z), U\rangle \\ &= \langle J(\nabla_{II(X,Y)}J)Z, U\rangle + \langle J(\nabla_Y J)II(X,Z), U\rangle. \end{aligned}$$

This is exactly the claim of the Lemma. □

Given the tensor $C(X, Y, T(Z, V))$ we define the following traces

$$\alpha(X, Y) := \sum_{i=1}^{n} \sigma_i\, C(e_i, X, T(e_i, Y))\,,$$
$$\beta(X, Y, Z) := \sum_{i=1}^{n} \sigma_i\, C(T(e_i, X), Y, T(e_i, Z))\,,$$

where $\{e_1, \ldots, e_n\}$ is a local (pseudo-)orthonormal frame of TL and where we set $\sigma_i = g(e_i, e_i)$.

Lemma 3.10.5 *For a Lagrangian immersion in a nearly (pseudo-)Kähler manifold and any $X, Y, Z, V \in TL$ holds:*

$$C(X, Y, T(Z, V)) + C(X, Z, T(V, Y)) + C(X, V, T(Y, Z)) = 0\,, \tag{3.105}$$
$$\alpha(X, Y) - \alpha(Y, X) = \langle \overrightarrow{H}, JT(X, Y)\rangle\,, \tag{3.106}$$
$$\beta(X, Y, Z) = \beta(Z, Y, X) = \beta(Y, X, Z) + \alpha(T(Y, X), Z)\,, \tag{3.107}$$
$$\alpha(T(X, Y), Z) + \alpha(T(Y, Z), X) + \alpha(T(Z, X), Y) = 0\,. \tag{3.108}$$

Here $\overrightarrow{H}$ denotes the mean curvature vector of L.

Proof Let us first rewrite the identity in Lemma 3.10.4 in terms of the tensor C and the torsion $T(X, Y) = -J(\nabla_X J)Y$ of $\bar{\nabla}$. Let $X, Y, Z, V \in \Gamma(TL)$ be arbitrary. Then Lemma 3.10.4 gives

$$\begin{aligned}
C(X, T(Z, Y), V) &= \langle II(X, J(\nabla_Y J)Z), JV\rangle \\
&= \langle J(\nabla_{II(X,Y)} J)Z, JV\rangle + \langle J(\nabla_Y J)II(X, Z), JV\rangle \\
&= \langle J(\nabla_Z J)(JV), II(X, Y)\rangle - \langle J(\nabla_Y J)(JV), II(X, Z)\rangle \\
&= -\langle J^2(\nabla_Z J)V, II(X, Y)\rangle + \langle J^2(\nabla_Y J)V, II(X, Z)\rangle \\
&= \langle JT(Z, V), II(X, Y)\rangle - \langle JT(Y, V), II(X, Z)\rangle \\
&= C(X, Y, T(Z, V)) - C(X, Z, T(Y, V))\,.
\end{aligned}$$

This is (3.105). Taking a trace gives

$$\begin{aligned}
\alpha(X, Y) &= \sum_{i=1}^{n} \sigma_i\, C(e_i, X, T(e_i, Y)) \\
&\overset{(3.105)}{=} \sum_{i=1}^{n} \sigma_i\, C(e_i, e_i, T(X, Y)) + \sum_{i=1}^{n} \sigma_i\, C(e_i, Y, T(e_i, X)) \\
&= \langle \overrightarrow{H}, JT(X, Y)\rangle + \alpha(Y, X)\,,
\end{aligned}$$

which is (3.106). The first identity in (3.107) is clear since C is fully symmetric. If we apply (3.105) to $\beta(X, Y, Z)$, then we get

$$\begin{aligned}
\beta(X, Y, Z) &= \sum_{i=1}^{n} \sigma_i\, C(T(e_i, X), Y, T(e_i, Z)) \\
&= -\sum_{i=1}^{n} \sigma_i\, C(T(X, Y), e_i, T(e_i, Z)) - \sum_{i=1}^{n} \sigma_i\, C(T(Y, e_i), X, T(e_i, Z)) \\
&= \sum_{i=1}^{n} \sigma_i\, C(e_i, T(Y, X), T(e_i, Z)) + \sum_{i=1}^{n} \sigma_i\, C(T(e_i, Y), X, T(e_i, Z)) \\
&= \alpha(T(Y, X), Z) + \beta(Y, X, Z)\,.
\end{aligned}$$

This is the second identity in (3.107). In view of this we also get

$$\begin{aligned}
&\alpha(T(X, Y), Z) + \alpha(T(Y, Z), X) + \alpha(T(Z, X), Y) \\
&= \beta(Y, X, Z) - \beta(X, Y, Z) + \beta(Z, Y, X) \\
&\quad -\beta(Y, Z, X) + \beta(X, Z, Y) - \beta(Z, X, Y) \\
&= 0
\end{aligned}$$

and this is (3.108). □

3.10.2 Lagrangian Submanifolds in Nearly Kähler Six-Manifolds

By a well known theorem of Ejiri [51] Lagrangian submanifolds of S^6 are minimal. In this section we will see that this is a special case of a much more general theorem which is a consequence of Lemma 3.10.4 and was shown independently for Riemannian metrics in Theorem 7 of [71].

Theorem 3.10.6 *Let L^3 be a Lagrangian immersion in a strict nearly (pseudo-)Kähler six-manifold M^6. Then we have*

$$\alpha = 0\,, \tag{3.109}$$

$$\overrightarrow{H} = 0\,. \tag{3.110}$$

In particular, any Lagrangian immersion in a strict nearly (pseudo-)Kähler six-manifold is orientable and minimal.

Proof Let $\{e_1, e_2, e_3\}$ be an orthonormal basis of T_pL for a fixed point $p \in L$. From the skew-symmetry of $\langle T(X, Y), Z\rangle$ we see that there exists a (nonzero) constant a such that $T(e_1, e_2) = a\sigma_3 e_3$. Then we also have $T(e_2, e_3) = a\sigma_1 e_1$, $T(e_3, e_1) = a\sigma_2 e_2$. The symmetry of C implies

$$\begin{aligned}\alpha(e_1, e_1) &= \sigma_1 C(e_1, e_1, T(e_1, e_1)) + \sigma_2 C(e_2, e_1, T(e_2, e_1)) + \sigma_3 C(e_3, e_1, T(e_3, e_1))\\ &= 0 - a\sigma_2\sigma_3 C(e_2, e_1, e_3) + a\sigma_3\sigma_2 C(e_3, e_1, e_2) = 0\end{aligned}$$

and

$$\begin{aligned}\alpha(e_1, e_2) &= \sigma_1 C(e_1, e_1, T(e_1, e_2)) + \sigma_2 C(e_2, e_1, T(e_2, e_2)) + \sigma_3 C(e_3, e_1, T(e_3, e_2))\\ &= a\sigma_1\sigma_3 C(e_1, e_1, e_3) + 0 - a\sigma_3\sigma_1 C(e_3, e_1, e_1) = 0\,.\end{aligned}$$

Similarly we prove that $\alpha(e_i, e_j) = 0$ for all $i, j = 1, \ldots, 3$. This shows $\alpha = 0$. But then (3.106) also implies $\overrightarrow{H} = 0$. The observation, that the frame $\{e_1, e_2, e_3\}$ defines an orientation on L, finishes the proof. The fact that a is a constant was not used in the proof. □

Remark 3.10.7 The constant a in the formula $T(e_1, e_2) = ae_3$ from above is related to the type constant α of the nearly Kähler manifold M, cf. Sect. 3.1.1 of this chapter, by the formula

$$a^2 = \alpha\,.$$

A six-dimensional strict nearly Kähler manifold is of constant type and a nearly Kähler manifold of constant type has dimension 6 [67]. The authors of [71] used the Eq. (3.5) to obtain a six-dimensional version of Lemma 3.10.4 for arbitrary nearly Kähler six-manifolds.

In the pseudo-Riemannian case we only have the relation $a^2 = |\alpha|$. The sign of the type constant depends on the signature $(2p, 2q)$ of g by $sign(p - q)$, see for example [82], see also Sect. 3.1.1 of this chapter.
The connection induced on L by $\bar{\nabla}$ is intrinsic in the following sense.

Proposition 3.10.8 *Let L be a Lagrangian submanifold in a strict nearly (pseudo-)Kähler six-manifold. Then the connection $\bar{\nabla}^L$ on L induced by the connection $\bar{\nabla}$ is completely determined by the restriction of g to L.*

Proof We observe that the torsion of $\bar{\nabla}^L$ considered as a three-form is a constant multiple of the volume form $T^L = c\,\mathrm{vol}^L_g$. A metric connection D with prescribed torsion T^D is known to be unique. If the torsion is totally skew-symmetric we can recover it from the formula

$$g(D_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{2}T(X, Y, Z).$$

This finishes the proof, since $T^L = c\mathrm{vol}^L_g$ is determined by g. □

3.10.3 The Splitting Theorem

The following example shows that Theorem 3.10.6 does not extend to eight dimensions:

Example 3.10.9 Let $L' \subset M^6_{SNK}$ be a (minimal) Lagrangian submanifold in a strict nearly Kähler manifold M^6_{SNK} and suppose $\gamma \subset \Sigma$ is a curve on a Riemann surface Σ. Then the Lagrangian submanifold $L := \gamma \times L' \subset M$ in the nearly Kähler manifold $M := \Sigma \times M_{SNK}$ is minimal, if and only if γ is a geodesic in Σ.
In this section we will see that this is basically the only counterexample to Theorem 3.10.6 that occurs in dimension 8. Nearly Kähler manifolds (M, J, g) split locally into a Kähler factor and a strict nearly Kähler factor and under the assumption, that M is complete and simply connected this splitting is global [98], cf. Theorem 3.2.1 of this chapter for the pseudo-Riemannian case. The natural question answered in the following theorem is in which way Lagrangian submanifolds lie in this decomposition.

These facts motivate the next Theorem.

Theorem 3.10.10 *Let M be a nearly Kähler manifold and L be a Lagrangian submanifold. Then M and L decompose locally into products $M = M_K \times M_{SNK}$, $L = L_K \times L_{SNK}$, where M_K is Kähler, M_{SNK} is strict nearly Kähler and $L_K \subset M_K$, $L_{SNK} \subset M_{SNK}$ are both Lagrangian. The dimension of L_K is given by*

$$\dim L_K = \frac{1}{2} \dim \ker(r)$$

Moreover, if the splitting of M is global and L is simply connected, then L decomposes globally as well.

Proof

i) We define

$$K_p := \{X \in T_pM : rX = 0\}, \quad K_p^{\perp} := \{\, Y \in T_pM : \langle X, Y\rangle = 0\,, \forall\, X \in K_p\}\,.$$

Because of $\bar{\nabla} r = 0$, $\bar{\nabla} g = 0$ this defines two orthogonal smooth distributions

$$\mathscr{D}_K := \bigcup_{p \in M} K_p\,, \quad \mathscr{D}_{SNK} := \bigcup_{p \in M} K_p^{\perp}$$

on M.

ii) The splitting theorem of de Rham can be applied, see Sect. 3.2.1 of this chapter, to the distributions $\mathscr{D}_K$ and $\mathscr{D}_{SNK}$ and the nearly Kähler manifold (M, J, g) splits (locally) into a Riemannian product

$$(M, J, g) = (M_K, J_K, g_K) \times (M_{SNK}, J_{SNK}, g_{SNK})\,,$$

where $TM_K = \mathscr{D}_K$, $TM_{SNK} = \mathscr{D}_{SNK}$. Here (M_K, J_K, g_K) is Kähler and $(M_{SNK}, J_{SNK}, g_{SNK})$ is strict nearly Kähler.

iii) Now let $L \subset M = M_K \times M_{SNK}$ be Lagrangian. We prove that r leaves tangent and normal spaces of L invariant. To see this, we fix an adapted local orthonormal frame field $\{e_1, \ldots, e_{2n}\}$ of M such that $e_1, \ldots, e_n$ are tangent to L and $e_{n+1} = Je_1, \ldots, e_{2n} = Je_n$ are normal to L. Since for any three vectors X, Y, Z we have

$$\langle(\nabla_X J)JZ, (\nabla_Y J)JZ\rangle = \langle J(\nabla_X J)Z, J(\nabla_Y J)Z\rangle = \langle(\nabla_X J)Z, (\nabla_Y J)Z\rangle\,,$$

we obtain

$$\begin{aligned}\langle rX, Y\rangle &= \sum_{i=1}^{2n}\langle(\nabla_X J)e_i, (\nabla_Y J)e_i\rangle \\ &= \sum_{i=1}^{n}\langle(\nabla_X J)e_i, (\nabla_Y J)e_i\rangle + \sum_{i=1}^{n}\langle(\nabla_X J)Je_i, (\nabla_Y J)Je_i\rangle \\ &= 2\sum_{i=1}^{n}\langle(\nabla_X J)e_i, (\nabla_Y J)e_i\rangle.\end{aligned}$$

Now, if $X \in TL$, $Y \in T^\perp L$, then by Lemma 3.10.2 we have

$$(\nabla_X J)e_i \in T^\perp L\,, \quad (\nabla_Y J)e_i \in TL\,,$$

so that

$$\langle(\nabla_X J)e_i, (\nabla_Y J)e_i\rangle = 0\,, \quad \forall\, i = 1, \ldots, n\,.$$

Further it follows

$$\langle rX, Y\rangle = 2\sum_{i=1}^{n}\langle(\nabla_X J)e_i, (\nabla_Y J)e_i\rangle = 0\,.$$

Since this works for any $X \in TL, Y \in T^\perp L$ and since r is selfadjoint we conclude

$$r(TL) \subset TL\,, \quad r(T^\perp L) \subset T^\perp L\,.$$

At a given point $p \in L$ we may now choose an orthonormal basis $\{f_1, \ldots, f_n\}$ of T_pL that consists of eigenvectors of $r_{|TL}$ considered as an endomorphism of TL. Since $[r, J] = 0$ and L is Lagrangian, the set $\{f_1, \ldots, f_n, Jf_1, \ldots, Jf_n\}$ then also determines an orthonormal eigenbasis of $r \in \mathrm{End}(TM)$. In particular, since J leaves the eigenspaces invariant, $K_p = \ker(r(p))$ and T_pL intersect in a subspace

K_p^L of dimension $\frac{1}{2}\dim(K_p) = \frac{1}{2}\dim(M_K)$. For the same reason $K_p^\perp \cap T_pL$ gives an $\frac{1}{2}\dim(M_{SNK})$-dimensional subspace. The corresponding distributions, denoted by $\mathscr{D}_K^L$ and $\mathscr{D}_{SNK}^L$ are orthogonal and both integrable, since in view of

$$\mathscr{D}_K^L = \mathscr{D}_K \cap TL\,, \quad \mathscr{D}_{SNK}^L = \mathscr{D}_{SNK} \cap TL$$

they are given by intersections of integrable distributions. We may now apply again the splitting theorem of de Rham to the Lagrangian submanifold. This completes the proof.

□

Remark 3.10.11 A more detailed analysis of the proof of the last theorem shows, that the result can be shown in the pseudo-Riemannian setting:

Let (M, J, g) be a nearly pseudo-Kähler manifold. Suppose, that the distribution $\mathcal{K}$ has constant dimension and admits an orthogonal complement, and the kernel of $r_{|L}$ admits an orthogonal complement in TL, then M and L decompose locally into products $M = M_K \times M_{SNK}$, $L = L_K \times L_{SNK}$, where M_K is Kähler, M_{SNK} is strict nearly pseudo-Kähler and $L_K \subset M_K$, $L_{SNK} \subset M_{SNK}$ are both Lagrangian. Moreover, if the splitting of M is global and L is simply connected, then L decomposes globally as well.

Corollary 3.10.12 *If $L \subset M$ is Lagrangian and $p \in L$ a fixed point, then to each eigenvalue λ of the operator r at p there exists a basis $e_1, \dots, e_k, f_1, \dots, f_k$ of eigenvectors of* $\mathrm{Eig}(\lambda)$ *such that $e_1, \dots, e_k \in T_pL$, $f_1, \dots, f_k \in T_p^\perp L$. Here, $2k$ denotes the multiplicity of λ.*

Proposition 3.10.13 *Let (M, J, g) be a nearly Kähler manifold and $L \subset M$ be a Lagrangian submanifold. Then TL and $T^\perp L$ are invariant by the Ricci tensor. In particular the spectrum of* Ric *is compatible with $TL \oplus T^\perp L$.*

Proof Let us recall (cf. [98]) that the Ricci-tensor satisfies $\langle \mathrm{Ric}\, X, Y\rangle = 0$ if X and Y are vector fields in eigenbundles $Eig(\lambda_X)$ and $Eig(\lambda_Y)$ of the tensor r with different eigenvalue $\lambda_X \neq \lambda_Y$. If X, Y belong to the same eigenvalue λ then $\langle \mathrm{Ric}\, X, Y\rangle$ is given by the following formula

$$\langle \mathrm{Ric}\, X, Y\rangle = \frac{\lambda}{4}\langle X, Y\rangle + \frac{1}{\lambda}\sum_{i=1}^{\mu} \lambda_i \langle r^{Eig(\lambda_i)} X, Y\rangle, \tag{3.111}$$

where μ is the number of different eigenvalues of r and $r^{Eig(\lambda_i)}$ is defined by

$$\langle r^{Eig(\lambda_i)} X, Y\rangle = -\operatorname{tr}_{Eig(\lambda_i)}[(\nabla_X J) \circ (\nabla_Y J)].$$

Like in the proof of Theorem 3.10.10 we obtain using Corollary 3.10.12

$$r^{Eig(\lambda_i)}(TL) \subset TL, \quad r^{Eig(\lambda_i)}(T^\perp L) \subset T^\perp L.$$

Equation (3.111) implies that

$$\mathrm{Ric}(TL) \subset TL, \quad \mathrm{Ric}(T^{\perp}L) \subset T^{\perp}L.$$

This further implies that the spectrum of Ric is compatible with the decomposition $TL \oplus T^{\perp}L$ and finishes the proof. □

Remark 3.10.14 The last proposition gives in the minimal case (only) a partial information on the Ricci curvature of $L \subset M$. Recall the Gauss equation

$$\langle R^L(V,W)X,Y\rangle = \langle R(V,W)X,Y\rangle + \langle II(V,X), II(W,Y))\rangle - \langle II(V,Y), II(W,X)\rangle$$

which implies using minimality

$$\sum_{i=1}^{\dim L}\langle R^L(e_i,W)e_i,Y\rangle = \sum_{i=1}^{\dim L}\langle R(e_i,W)e_i,Y\rangle - \sum_{i=1}^{\dim L}\langle II(e_i,Y), II(e_i,W)\rangle.$$

It is straight-forward to show, that the second term on the right hand-side vanishes if and only if II is zero, i.e. L is a totally geodesic manifold. In that case Proposition 3.10.13 yields the Ricci tensor of L.

Let us recall the situation in dimension 8 and 10 [67, 98].

Proposition 3.10.15

(i) *Let M^8 be a simply connected complete nearly Kähler manifold of dimension 8. Then M^8 is a Riemannian product $M^8 = \Sigma \times M^6_{SNK}$ of a Riemannian surface Σ and a six-dimensional strict nearly Kähler manifold M^6_{SNK}.*

(ii) *Let M^{10} be a simply connected complete nearly Kähler manifold of dimension 10. Then M^{10} is either the product $M^4_K \times M^6_{SNK}$ of a Kähler surface M^4_K and a six-dimensional strict nearly Kähler manifold M^6_{SNK} or M is a twistor space over a positive, eight dimensional quaternionic Kähler manifold.*

Note, that any complete, simply connected eight dimensional positive quaternionic Kähler manifold equals one of the following three spaces: $\mathbb{HP}^2$, $\mathbb{G}r_2(\mathbb{C}^2)$, $G_2/SO(4)$.

In the next theorem, part (i) and (ii) collect the information on Lagrangian submanifolds in nearly Kähler manifolds of dimension 8 and 10.

Theorem 3.10.16

(i) Let L be a Lagrangian submanifold in a simply connected nearly Kähler manifold M^8. Then $M^8 = \Sigma \times M^6_{SNK}$, where Σ is a Riemann surface, M^6_{SNK} is strict nearly Kähler and $L = \gamma \times L'$ is a product of a (real) curve $\gamma \subset \Sigma$ and a minimal Lagrangian submanifold $L' \subset M^6_{SNK}$.

(ii) Let L be a Lagrangian submanifold in a simply connected complete nearly Kähler manifold M^{10}, then either

(a) *$M^{10} = M^4_K \times M^6_{SNK}$ and the manifold $L = S \times L'$ is a product of a Lagrangian (real) surface $S \subset M^4_K$ and a minimal Lagrangian submanifold $L' \subset M^6_{SNK}$ or*
(b) *the manifold L is a Lagrangian submanifold in a twistor space over a positive, eight dimensional quaternionic Kähler manifold.*

(iii) *Let M_1, M_2 be two nearly Kähler manifolds. Denote the operator r on M_i by r_i, $i = 1, 2$. If $\mathrm{Spec}(r_1) \cap \mathrm{Spec}(r_2) = \emptyset$ and $L \subset M_1 \times M_2$ is Lagrangian, then L splits (locally) into $L = L_1 \times L_2$, where $L_i \subset M_i$, $i = 1, 2$ are Lagrangian. If L is simply connected, then the decomposition is global.*

Proof This is a combination of the results of Theorem 3.10.10, Corollary 3.10.12 and Proposition 3.10.15. □

Remark 3.10.17 As in Remark 3.10.11 we may note, that using our results one can generalise Theorem 3.10.16 to the case of indefinite metrics, where we omit part (iii):

Theorem 3.10.18

(i) *Let L be a Lagrangian submanifold in a simply connected nice nearly pseudo-Kähler manifold M^8. Then $M^8 = \Sigma \times M^6_{SNK}$, where Σ is a Riemann surface,*[12] *M^6_{SNK} is strict nearly Kähler and $L = \gamma \times L'$ is a product of a (real) curve $\gamma \subset \Sigma$ and a minimal Lagrangian submanifold $L' \subset M^6_{SNK}$.*
(ii) *Let L be a Lagrangian submanifold in a simply connected complete nice decomposable nearly pseudo-Kähler manifold M^{10}, such that the kernel*[13] *of $r_{|TL}$ admits an orthogonal complement, then either*
   (a) *$M^{10} = M^4_K \times M^6_{SNK}$ and the manifold $L = S \times L'$ is a product of a Lagrangian (real) surface $S \subset M^4_K$ and a minimal Lagrangian submanifold $L' \subset M^6_{SNK}$ or*
   (b) *the manifold L is a Lagrangian submanifold in a twistor space over a negative, eight dimensional quaternionic Kähler manifold or a para-quaternionic Kähler manifold.*

Theorems 3.10.16 (3.10.16) and 3.10.18 (3.10.18), parts (b) motivate the discussion of Lagrangian submanifolds in twistor spaces in the subsequent section. Indeed, the results derived in the next section imply that Lagrangian submanifolds in twistor spaces are, regardless their dimension, always minimal.

[12]Remark, that the restriction of g to Σ is always definite.

[13]Let us recall, that in the case (b) r has trivial kernel.

3.10.4 Lagrangian Submanifolds in Twistor Spaces

An important class of examples for nearly pseudo-Kähler manifolds is given by twistor spaces Z^{4n+2} over quaternionic Kähler or para-quaternionic Kähler manifolds N^{4n}, as we have seen in Sects. 3.3 and 3.4 of this chapter.

For the readers convenience, let us shortly recall that the twistor space is the bundle of almost complex structures in the quaternionic bundle Q over the (para-)quaternionic Kähler manifold N. It can be endowed with a Kähler structure (Z, J^Z, g^Z), such that the projection $\pi : Z \to N$ is a Riemannian submersion with totally geodesic fibres S^2. Denote by $\mathcal{H}$ and $\mathcal{V}$ the horizontal and the vertical distributions of the submersion π. Then the direct sum decomposition

$$TZ = \mathcal{H} \oplus \mathcal{V} \tag{3.112}$$

is orthogonal and compatible with the complex structure J^Z. Let us consider now a second almost Hermitian structure (J, g) on Z which is defined by

$$g := \begin{cases} g^Z(X, Y), \text{ for } X, Y \in \mathcal{H}, \\ \frac{1}{2} g^Z(V, W), \text{ for } V, W \in \mathcal{V}, \\ g^Z(V, X) = 0, \text{ for } V \in \mathcal{V}, X \in \mathcal{H} \end{cases}$$

and

$$J := \begin{cases} J^Z \text{ on } \mathcal{H}, \\ -J^Z \text{ on } \mathcal{V}. \end{cases}$$

Note, that in view of (3.112), the decomposition $TZ = \mathcal{H} \oplus \mathcal{V}$ is also compatible w.r.t. J and orthogonal w.r.t. g.

The manifold (Z, J, g) is a strict nearly pseudo-Kähler manifold and the distributions $\mathcal{V}$ and $\mathcal{H}$ are parallel w.r.t. the connection $\bar{\nabla}$. The projection π is also a Riemannian submersion with totally geodesic fibres for the metric g.

Let us summarise some information which will be useful later in this section.

Lemma 3.10.19 *In this situation we have the following information:*

(a) The torsion $T = -J\nabla J$ of the characteristic connection satisfies (see Lemma 3.4.3)

$$T(X, Y) \in \mathcal{V}, \quad \textit{for } X, Y \in \mathcal{H}, \tag{3.113}$$

$$T(X, V) \in \mathcal{H}, \quad \textit{for } X \in \mathcal{H}, V \in \mathcal{V}, \tag{3.114}$$

$$T(U, V) = 0, \quad \textit{for } U, V \in \mathcal{V}. \tag{3.115}$$

(b) *The association (see Lemma 3.4.3)*

$$\mathcal{H} \ni X \mapsto T(Y, X) \in \mathcal{V} \tag{3.116}$$

is surjective for $0 \neq Y \in \mathcal{H}$ *and the map*

$$\Phi^V : \mathcal{H} \ni X \mapsto T(V, X) \tag{3.117}$$

with $0 \neq V \in \mathcal{V}$ *is invertible and squares to* $\varepsilon k^2 Id_{\mathcal{H}}$ *for some* $k \in \mathbb{R}, \varepsilon \in \{\pm 1\}$, *cf. Lemma 3.4.9.*

(c) *The operator* r *has eigenvalues* $\lambda_{\mathcal{H}} = 4k^2$, $\lambda_{\mathcal{V}} = \frac{n-1}{2}\varepsilon\lambda_{\mathcal{H}}$. *If* $n > 1$, *then the eigenbundle of* $\lambda_{\mathcal{H}}$ *is* $\mathcal{H}$ *and* $\mathcal{V}$ *is the eigenbundle of* $\lambda_{\mathcal{V}}$, *cf. Corollary 3.4.17.*

In the rest of this section we consider a nearly pseudo-Kähler manifold $(M = Z, J, g)$ of twistor type and study Lagrangian submanifolds $L \subset M$.

Remark 3.10.20 As will be shown in the next theorem, for $n > 1$ we have

$$\pi^{\mathcal{H}}(TL) = \mathcal{H} \cap TL\,, \quad \pi^{\mathcal{V}}(TL) = \mathcal{V} \cap TL\,,$$

where $\pi^{\mathcal{H}}(TL), \pi^{\mathcal{H}}(T^{\perp}L)$ and $\pi^{\mathcal{V}}(TL), \pi^{\mathcal{V}}(T^{\perp}L)$ are the orthogonal projections of TL and $T^{\perp}L$ w.r.t. $\mathcal{H} \oplus \mathcal{V}$.

Lemma 3.10.21 *Let* $L^{2n+1} \subset M^{4n+2}$ *with* $n > 1$ *be a Lagrangian submanifold in a twistor space as described above. Then the second fundamental form* II *satisfies*

$$II(X, Y) \in \pi^{\mathcal{H}}(T^{\perp}L), \text{ for } X, Y \in \pi^{\mathcal{H}}(TL), \tag{3.118}$$

$$II(X, Y) \in \pi^{\mathcal{V}}(T^{\perp}L), \text{ for } X, Y \in \pi^{\mathcal{V}}(TL), \tag{3.119}$$

$$II(X, Y) = 0, \quad \text{for } X \in \pi^{\mathcal{H}}(TL), Y \in \pi^{\mathcal{V}}(TL). \tag{3.120}$$

Proof The second fundamental form is given by $C(X, Y, Z) = \langle \bar{\nabla}_X Y, JZ \rangle$ for $X, Y, Z \in \Gamma(TL)$. The lemma follows since the decomposition (3.112) is $\bar{\nabla}$-parallel, orthogonal and J-invariant and as the tensor C is completely symmetric. □

With these preparations we prove the next result.

Theorem 3.10.22 *Let* $L^{2n+1} \subset M^{4n+2}$ *be a Lagrangian submanifold in a nearly pseudo-Kähler manifold of the above type. Then L is minimal. If* $n > 1$, *then the tangent space of L splits into a one-dimensional vertical part and a* $2n$*-dimensional horizontal part. Moreover, the second fundamental form II of the vertical normal direction vanishes completely if* $n > 1$.

Proof

i) By Theorem 3.10.6 it suffices to consider the case $n > 1$.
ii) Let $L \subset M$ be a Lagrangian submanifold. Since $n > 1$, the two eigenvalues $\lambda_{\mathcal{H}}$, $\lambda_{\mathcal{V}}$ of r are distinct and the eigenspace $\mathcal{V}$ of $\lambda_{\mathcal{V}}$ is two-dimensional. By Corollary 3.10.12 this induces a one-dimensional vertical tangential distribution

$\mathcal{D}$ on L in the Riemannian case. In the pseudo-Riemannian case this follows, since the restriction of the metric to $\mathcal{V}$ is definite. In particular, $\mathcal{D}$ is not isotropic. Then, by the Lagrangian condition, we get $\mathcal{D} := \pi^{\mathcal{V}}(TL)$.

iii) Denote by $\mathcal{D}^{\perp}$ the orthogonal complement of $\mathcal{D}$ in TL. We claim, that the trace of the second fundamental form II of L restricted to $\mathcal{D}^{\perp}$ is zero.

Proof First we observe that by Lemma 3.10.21 we can restrict the second fundamental form II to $\mathcal{D}^{\perp} = \pi^{\mathcal{H}}(TL)$. We fix $U \in \mathcal{D}$ of unit length. Using Lemmas 3.10.2 and 3.10.19 we observe, that $\Phi(X) := \frac{1}{k}J(\nabla_U J)X$ defines an (almost) (para-)complex structure on $\mathcal{D}^{\perp}$ which is compatible with the metric. With $X \in \mathcal{D}^{\perp}$ and $\Phi^2 = \varepsilon Id$ we compute

$$\begin{aligned} II(X,X) &= \varepsilon II(X, \Phi(\Phi(X))) = \varepsilon \frac{1}{k} II(X, J(\nabla_U J)\,\Phi(X)) \\ &= \varepsilon \frac{1}{k} J\left[(\nabla_{II(X,U)} J)\Phi(X) + (\nabla_U J) II(X, \Phi(X))\right] \\ &= \varepsilon \frac{1}{k} J(\nabla_U J)\, II(X, \Phi(X)) = \varepsilon \Phi II(X, \Phi(X))\,. \end{aligned}$$

After polarising we obtain

$$II(\Phi X, \Phi Y) = \varepsilon II(X, Y)\,, \quad \forall\, X,\, Y \in \mathcal{D}^{\perp}\,. \tag{3.121}$$

In particular, taking a trace over (3.121) we get

$$\mathrm{tr}^{\mathcal{D}^{\perp}}\, II = 0\,,$$

where we keep in mind, that it holds $g(\Phi\cdot, \Phi\cdot) = -\varepsilon g(\cdot, \cdot)$.

iv) We have

$$\alpha(X, Y) = 0\,, \quad \forall\, X,\, Y \in \mathcal{D}^{\perp}\,.$$

Proof By (ii) we may choose a pseudo-orthonormal frame $\{e_1, \dots, e_{2n+1}\}$ of TL such that $e_1, \dots, e_{2n} \in \mathcal{D}^{\perp}$ and $e_{2n+1} \in \mathcal{D}$. Since

$$\alpha(X, Y) = \sum_{i=1}^{2n+1} \sigma_i\, C(e_i, X, T(e_i, Y))$$

and the tensor C is fully symmetric we see that by Lemma 3.10.21 all terms on the RHS vanish since either $e_i \in \mathcal{D} = \pi^{\mathcal{V}}(TL)$, $X \in \mathcal{D}^{\perp} = \pi^{\mathcal{H}}(TL)$ or $e_i, X \in \mathcal{D}^{\perp}$ and $T(e_i, Y) \in \mathcal{D}$ (cf. Lemma 3.10.19).

v) By Lemma 3.10.21 and (iii) the mean curvature vector $\overrightarrow{H}$ satisfies $J\overrightarrow{H} \in \mathcal{D}$. From (3.106) and (iv) we get

$$\langle J\overrightarrow{H}, T(X, Y)\rangle = 0\,, \quad \forall\, X,\, Y \in \mathcal{D}^{\perp}\,. \tag{3.122}$$

Since J maps $\mathcal{V}$ to itself, we also have $J\overrightarrow{H} \in \mathcal{D} \subset \mathcal{V}$. Now we choose $X \in \mathcal{D}^{\perp}$ and $\tilde{Y} \in \mathcal{H}$ with

$$T(X, \tilde{Y}) = J\overrightarrow{H}\,.$$

This is possible since the map $\mathcal{H} \ni \tilde{Y} \mapsto T(X, \tilde{Y}) \in \mathcal{V}$ is surjective by (3.116). Let $\tilde{Y} = Y + Y^{\perp}$ be the orthogonal decomposition of $\tilde{Y}$ into the tangent and normal parts of $\tilde{Y}$. Note that $Y, Y^{\perp}$ are both horizontal. We claim

$$T(X, Y^{\perp}) = 0.$$

This follows, since $T(X, \cdot)$ maps tangent to tangent and normal to normal vectors and one has

$$T(X, \tilde{Y}) = T(X, Y) + T(X, Y^{\perp}) = J\overrightarrow{H} \in TL\,.$$

Therefore there exist two tangent vectors $X, Y \in \mathcal{D}^{\perp}$ with

$$T(X, Y) = J\overrightarrow{H}\,.$$

This implies

$$|\overrightarrow{H}|^2 = \langle J\overrightarrow{H}, T(X, Y)\rangle \overset{(3.122)}{=} 0,$$

which proves that the mean curvature vector vanishes, as the metric restricted to $\mathcal{V}$ is definite (even in the pseudo-Riemannian case). From this, the fact that $\mathcal{D}$ is one-dimensional and from Lemma 3.10.21 it follows that $II(V, \cdot) = 0$ for any $V \in \mathcal{D}$.

□

Corollary 3.10.23 *Let $L \subset M$ be a Lagrangian submanifold in a twistor space M^{4n+2} as above with $n > 1$. Then the integral manifolds c of the distribution $\mathcal{D}$ are geodesics (hence locally great circles) in the totally geodesic fibres S^2.*

Proof The last theorem implies that the geodesic curvature vanishes and that in consequence an integral manifold c of $\mathcal{D}$ is totally geodesic in the fibres. □

Remark 3.10.24

(i) It is well-known, that the twistor space of $\mathbb{H}P^n$ is $\mathbb{C}P^{2n+1}$. Therefore the above result applies to $(\mathbb{C}P^{2n+1}, J, g)$ endowed with its canonical nearly Kähler structure.
(ii) Using Remark 3.10.14 (for Riemannian metrics) and Lemma 3.10.19 (c) we observe that totally geodesic Lagrangian submanifolds in twistor spaces have two different Ricci eigenvalues with multiplicities $2n$ and 1.

3.10.5 Deformations of Lagrangian Submanifolds in Nearly Kähler Manifolds

Our aim in this section is to study the space of deformations of a given Lagrangian (and hence minimal Lagrangian) submanifold L in a strict six-dimensional nearly (pseudo-)Kähler manifold M^6. In an article by Moroianu et al. [96] the deformation space of nearly Kähler structures on six-dimensional nearly Kähler manifolds has been related to the space of coclosed eigenforms of the Hodge-Laplacian. As we will show below, a similar statement holds for the deformation of Lagrangian submanifolds in strict nearly (pseudo-)Kähler six-manifolds. To this end we assume that

$$F : L \times (-\epsilon, \epsilon) \to M$$

is a smooth variation of Lagrangian immersions $F_t := F(\cdot, t) : L \to M$, $t \in (-\epsilon, \epsilon)$ into a nearly (pseudo-)Kähler manifold M. Let

$$V := \frac{d}{dt} F_t$$

denote the variation vector field. Since tangential deformations correspond to diffeomorphisms acting on L, we may assume w.l.o.g. that $V \in \Gamma(T^{\perp}L)$ is a normal vector field. The Cartan formula and $F_t^* \omega = 0$ for all t then imply that

$$0 = d(i_V \omega) + i_V d\omega$$

holds everywhere on L. By the nearly Kähler condition this is equivalent to

$$d(V \lrcorner \omega) + 3V \lrcorner \nabla\omega = 0 \tag{3.123}$$

on L. Let us define the variation 1-form $\theta \in \Omega^1(L)$ by

$$\theta := V \lrcorner \omega\,.$$

This Theorem has recently been used in [97]. In this paper the authors relate generalised Killing spinors on spheres to Lagrangian graphs in the nearly Kähler manifold $S^3 \times S^3$.

Theorem 3.10.25 *Let $F_t : L \to M$ be a variation of Lagrangian immersions in a six-dimensional nearly (pseudo-)Kähler manifold M. Then the variation 1-form θ is a coclosed eigenform of the Hodge-Laplacian, where the eigenvalue λ satisfies $\lambda = 9\alpha$ with the type constant α of M. If the metric is positive definite this space is finite dimensional.*

In the case of Riemannian metrics a similar result was also shown in Theorem 7 of [71]. For a more recent study of deformations of Lagrangian submanifolds we refer to [91].

Proof For $X, Y \in TL$ and $V \in T^{\perp}L$ we compute

$$\begin{aligned}(V \lrcorner \nabla\omega)(X, Y) &= \nabla\omega(V, X, Y)\\ &= \langle(\nabla_X J)Y, V\rangle\\ &= \langle J(\nabla_X J)Y, JV\rangle\\ &= -\langle JV, T(X, Y)\rangle\,.\end{aligned}$$

Since T induces an orientation on the Lagrangian submanifold by the three-form

$$\tau(X, Y, Z) := \langle T(X, Y), Z\rangle\,,$$

we obtain a naturally defined $*$-operator $* : \Omega^p(L) \to \Omega^{3-p}(L)$ which for 1-forms is given by

$$*\phi := \frac{\sigma}{\sqrt{|\alpha|}}\,\phi \circ T\,.$$

Here, α is the type constant of M (cf. Remark 3.10.7) and $\sigma \in \{\pm 1\}$ depends only on the signature. This implies that equation (3.123) can be rewritten in the form

$$d\theta = 3\sigma\sqrt{|\alpha|}\,*\theta\,. \tag{3.124}$$

Consequently, if the signature of the metric g restricted to L is (p, q), we obtain

$$\mathrm{sign}(p-q)\,\delta\theta = *d*\theta = 0$$

and

$$\begin{aligned}\mathrm{sign}(p-q)\,\delta d\theta &= 3\sigma\sqrt{|\alpha|}\,*d**\theta\\ &\overset{(*^2=\mathrm{Id})}{=} 3\sigma\sqrt{|\alpha|}\,*d\theta\\ &\overset{(3.124)}{=} 9|\alpha| * *\theta\\ &\overset{(*^2=\mathrm{Id})}{=} 9|\alpha|\theta\,.\end{aligned}$$

In total as the sign of α is also $\operatorname{sign}(p-q)$ we get

$$\Delta_{\mathrm{Hodge}}\theta = (\delta d + d\delta)\theta = 9\alpha\theta \, .$$

This proves the theorem. Since one has $Ric = 5\alpha g$ this is equivalent to

$$\Delta_{\mathrm{Hodge}}\theta = \frac{3}{10} scal\, \theta \, ,$$

where *scal* is the scalar curvature of M. □

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