Chapter 2
Linear Programming Algorithms

Abstract LPs can be formulated in various forms. An LP problem consists of the objective function, the constraints, and the decision variables. This chapter presents the theoretical background of LP. More specifically, the different formulations of the LP problem are presented. Detailed steps on how to formulate an LP problem are given. In addition, the geometry of the feasible region and the duality principle are also covered. Finally, a brief description of LP algorithms that will be used in this book is also presented.

2.1 Chapter Objectives

- Present the different formulations of linear programming problems.
- Formulate linear programming problems.
- Solve linear programming problems graphically.
- Introduce the duality principle.
- Present the linear programming algorithms that will be used in this book.

2.2 Introduction

The LP problem is an optimization problem with a linear objective function and constraints. We can write the LP problem in its general form as shown in Equation (2.1):

$$\begin{array}{llllllll} \min z = & c_1x_1 & + & c_2x_2 & + \cdots + & c_nx_n & & \\ \text{s.t.} & A_{11}x_1 & + & A_{12}x_2 & + \cdots + & A_{1n}x_n & \bigoplus & b_1 \\ & A_{21}x_1 & + & A_{22}x_2 & + \cdots + & A_{2n}x_n & \bigoplus & b_2 \\ & \vdots & \vdots & \vdots & \vdots \ldots \vdots & \vdots & \vdots & \vdots \\ & A_{m1}x_1 & + & A_{m2}x_2 & + \cdots + & A_{mn}x_n & \bigoplus & b_m \end{array} \tag{2.1}$$

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Coefficients A_{ij}, c_j, and b_i, where $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$, are known real numbers and are the input data of the problem. Symbol $\bigoplus$ defines the type of the constraints and can take the form of:

- $=$: equality constraint
- $\geq$: greater than or equal to constraint
- $\leq$: less than or equal to constraint

The word *min* shows that the problem is to minimize the objective function and *s.t.* is the shorthand for *subject to* that is followed by the constraints. Variables $x_1, x_2, \cdots, x_n$ are called decision variables and they are the unknown variables of the problem. When solving an LP problem, we want to find the values of the decision variables while minimizing the value of the objective function subject to the constraints.

Writing Equation (2.1) using matrix notation, the following format is extracted:

$$\begin{aligned} \min\ & z = c^T x \\ \text{s.t.}\ & Ax \bigoplus b \end{aligned} \tag{2.2}$$

where $c, x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $A \in \mathbb{R}^{m \times n}$. x is the vector of the decision variables (size $n \times 1$), A is the matrix of coefficients of the constraints (size $m \times n$), c is the vector of coefficients of the objective function (size $n \times 1$), b is the vector of the right-hand side of the constraints (size $m \times 1$), and $\bigoplus$ (or *Eqin*) is the vector of the type of the constraints (size $m \times 1$) where:

- 0: defines an equality constraint ($=$)
- -1: defines a less than or equal to constraint ($\leq$)
- 1: defines greater than or equal to constraint ($\geq$)

Most LPs are applied problems and the decision variables are usually nonnegative values. Hence, nonnegativity or natural constraints must be added to Equation (2.2):

$$\begin{aligned} \min\ & z = c^T x \\ \text{s.t.}\ & Ax \bigoplus b \\ & x \geq 0 \end{aligned} \tag{2.3}$$

All other constraints are called technological constraints. Equations (2.2) and (2.3) are called general LPs.

Let's transform a general problem written in the format shown in Equation (2.1) in matrix notation. The LP problem that will be transformed is the following:

$$
\begin{aligned}
\min z = \; & 3x_1 - 4x_2 + 5x_3 - 2x_4 \\
\text{s.t.} \; & -2x_1 - x_2 - 4x_3 - 2x_4 \leq 4 \\
& 5x_1 + 3x_2 + x_3 + 2x_4 \leq 18 \\
& 5x_1 + 3x_2 + x_4 \geq -13 \\
& 4x_1 + 6x_2 + 2x_3 + 5x_4 \geq -10 \\
& x_j \geq 0, \quad (j = 1, 2, 3, 4)
\end{aligned}
$$

In matrix notation, the above LP problem is written as follows:

$$
A = \begin{bmatrix} -2 & -1 & -4 & -2 \\ 5 & 3 & 1 & 2 \\ 5 & 3 & 0 & 1 \\ 4 & 6 & 2 & 5 \end{bmatrix}, c = \begin{bmatrix} 3 \\ -4 \\ 5 \\ -2 \end{bmatrix}, b = \begin{bmatrix} 4 \\ 18 \\ -13 \\ -10 \end{bmatrix}, Eqin = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}
$$

This chapter presents the theoretical background of LP. More specifically, the different formulations of the LP problem are presented. Moreover, detailed steps on how to formulate an LP problem are given. In addition, the geometry of the feasible region and the duality principle are also covered. Finally, a brief description of LP algorithms that will be used in this book is also presented.

The structure of this chapter is as follows. In Section 2.3, the different formulations of the LP problem are presented. Section 2.4 presents the steps to formulate an LP problem. Section 2.5 presents the geometry of the feasible region, while Section 2.6 presents the duality principle. Section 2.7 provides a brief description of the LP algorithms that will be used in this book. Finally, conclusions and further discussion are presented in Section 2.8.

2.3 Linear Programming Problem

The general LP problem (Equation (2.2)) can contain both equality ($=$) and inequality ($\leq, \geq$) constraints, ranges in the constraints, and the following types of variable constraints (bounds):

- lower bound: $x_j \geq l_j$
- upper bound: $x_j \leq u_j$
- lower and upper bound: $l_j \leq x_j \leq u_j$
- fixed value (i.e., the lower and upper bounds are the same): $x_j = k, k \in \mathbb{R}$
- free variable: $-\infty \leq x_j \leq \infty$
- minus infinity: $x_j \geq -\infty$
- plus infinity: $x_j \leq \infty$

This format cannot be used easily to describe algorithms, because there are many different forms of constraints that can be included. There are two other formulations of LPs that are more appropriate: (i) canonical form, and (ii) standard form.

An LP problem in its canonical form has all its technological constraints in the form 'less than or equal to' ($\leq$) and all variables are nonnegative, as shown in Equation (2.4):

$$\begin{aligned}
\min z = \; & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{s.t.} \; & A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n \leq b_1 \\
& A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n \leq b_2 \\
& \vdots \quad \vdots \quad \vdots \quad \vdots \ldots \vdots \quad \vdots \quad \vdots \quad \vdots \\
& A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n \leq b_m \\
x_j \geq 0, \; & (j = 1, 2, \cdots, n)
\end{aligned} \tag{2.4}$$

Equation (2.4) can be written in matrix notation as shown in Equation (2.5):

$$\begin{aligned}
\min z &= c^T x \\
\text{s.t.} \; Ax &\leq b \\
x &\geq 0
\end{aligned} \tag{2.5}$$

An LP problem in its standard form has all its technological constraints as equalities ($=$) and all variables are nonnegative, as shown in Equation (2.6):

$$\begin{aligned}
\min z = \; & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{s.t.} \; & A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = b_1 \\
& A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n = b_2 \\
& \vdots \quad \vdots \quad \vdots \quad \vdots \ldots \vdots \quad \vdots \quad \vdots \quad \vdots \\
& A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n = b_m \\
x_j \geq 0, \; & (j = 1, 2, \cdots, n)
\end{aligned} \tag{2.6}$$

Equation (2.6) can be written in matrix notation as shown in Equation (2.7):

$$\begin{aligned}
\min z &= c^T x \\
\text{s.t.} \; Ax &= b \\
x &\geq 0
\end{aligned} \tag{2.7}$$

All LPs can be transformed to either the canonical form or the standard form. Below, we describe the steps that must be performed to transform:

- a general LP problem to its canonical form,
- a general LP to its standard form,
- an LP problem in its canonical form to its standard form, and
- an LP problem in its standard form to its canonical form.

Definition 2.1 Two LPs are equivalent, iff there is a one-to-one correspondence between their feasible points or solutions and their objective function values.

Or equivalently, if an LP problem is optimal, infeasible, or unbounded, then its equivalent LP problem is also optimal, infeasible, or unbounded, respectively.

2.3.1 General to Canonical Form

The steps that must be performed to transform a general LP problem to its canonical form are the following:

1. If the LP problem is a maximization problem, transform it to a minimization problem. Since maximizing a quantity is equivalent to minimizing its negative, any maximization objective function

$$\max z = c^T x = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \tag{2.8}$$

   can be transformed to

$$\min z = -c^T x = -c_1 x_1 - c_2 x_2 - \cdots - c_n x_n \tag{2.9}$$

   or in a more general form

$$\max \left\{ c^T x : x \in \top \right\} = -\min \left\{ -c^T x : x \in \top \right\} \tag{2.10}$$

   where $\top$ is a set of numbers
2. An inequality constraint in the form 'greater than or equal to' ($\geq$)

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \geq b \tag{2.11}$$

   can be transformed in the form 'less than or equal to' ($\leq$) if we multiply both the left-hand and the right-hand sides of that constraint by -1

$$-a_1 x_1 - a_2 x_2 - \cdots - a_n x_n \leq -b \tag{2.12}$$

3. An equality constraint

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b \tag{2.13}$$

   is equivalent with two inequality constraints

$$\begin{aligned} a_1 x_1 + a_2 x_2 + \cdots + a_n x_n &\leq b \\ a_1 x_1 + a_2 x_2 + \cdots + a_n x_n &\geq b \end{aligned} \tag{2.14}$$

   Following the same procedure as previously, we can transform the second constraint of Equation (2.14) in the form 'greater than or equal to' ($\geq$) to an inequality constraint in the form 'less than or equal to' ($\leq$).

$$\begin{aligned} a_1x_1 &+ a_2x_2 + \cdots + a_nx_n \le b \\ -a_1x_1 &- a_2x_2 - \cdots - a_nx_n \le -b \end{aligned} \tag{2.15}$$

However, this procedure increases the number of constraints (for each equality constraint we add two inequality constraints) and it is not efficient in practice. Another way to transform an equality to an inequality constraint is to solve the equation in terms of a variable a_jx_j with $a_j \neq 0$

$$x_1 = (b - a_2x_2 - \cdots - a_nx_n) \ / \ a_1, a_1 \neq 0 \tag{2.16}$$

and extract the inequality (since $x_1 \geq 0$)

$$(a_2/a_1)x_2 + \cdots + (a_n/a_1)x_n \leq b/a_1, a_1 \neq 0 \tag{2.17}$$

Finally, we remove variable x_1 from the objective function and the other constraints.

Let's transform a general problem to its canonical form. The LP problem that will be transformed is the following:

$$\begin{array}{lrrrrrr} \max z = & 3x_1 & - 4x_2 & + 5x_3 & - 2x_4 & & \\ \text{s.t.} & -2x_1 & - x_2 & - 4x_3 & - 2x_4 & \leq & 4 \\ & 5x_1 & + 3x_2 & + x_3 & + 2x_4 & \leq & 18 \\ & 5x_1 & + 3x_2 & & + x_4 & \geq & -13 \\ & 4x_1 & + 6x_2 & + 2x_3 & + 5x_4 & = & 10 \\ x_j \geq 0, & \multicolumn{6}{l}{(j = 1, 2, 3, 4)} \end{array}$$

Initially, we need to transform the objective function to minimization. According to the first step of the above procedure, we take its negative.

$$\begin{array}{lrrrrrrl} \min z = & -3x_1 & + 4x_2 & - 5x_3 & + 2x_4 & & & \\ \text{s.t.} & -2x_1 & - x_2 & - 4x_3 & - 2x_4 & \leq & 4 & (1) \\ & 5x_1 & + 3x_2 & + x_3 & + 2x_4 & \leq & 18 & (2) \\ & 5x_1 & + 3x_2 & & + x_4 & \geq & -13 & (3) \\ & 4x_1 & + 6x_2 & + 2x_3 & + 5x_4 & = & 10 & (4) \\ x_j \geq 0, & \multicolumn{7}{l}{(j = 1, 2, 3, 4)} \end{array}$$

Next, we transform all constraints in the form 'greater than or equal to' ($\geq$) to 'less than or equal to' ($\leq$). According to the second step of the aforementioned procedure, we multiply the third constraint by -1.

$$
\begin{array}{lrrrrll}
\min z = & -3x_1 & + 4x_2 & - 5x_3 & + 2x_4 & & \\
\text{s.t.} & -2x_1 & - x_2 & - 4x_3 & - 2x_4 & \leq 4 & (1) \\
 & 5x_1 & + 3x_2 & + x_3 & + 2x_4 & \leq 18 & (2) \\
 & -5x_1 & - 3x_2 & & - x_4 & \leq 13 & (3) \\
 & 4x_1 & + 6x_2 & + 2x_3 & + 5x_4 & = 10 & (4) \\
x_j \geq 0, & (j = 1, 2, 3, 4) & & & & &
\end{array}
$$

Finally, we transform all equalities (=) to inequality constraints in the form 'less than or equal to' ($\leq$). According to the third step of the aforementioned procedure, we divide each term of the fourth constraint by 4 (the coefficient of the variable x_1) and remove variable x_1 from the objective function and other constraints.

$$
\begin{array}{lrrrll}
\min z = & 17/2x_2 & - 7/2x_3 & + 23/4x_4 & - 15/2 & \\
\text{s.t.} & 2x_2 & - 3x_3 & + 1/2x_4 & \leq 9 & (1) \\
 & -9/2x_2 & - 3/2x_3 & - 17/4x_4 & \leq 11/2 & (2) \\
 & 9/2x_2 & + 5/2x_3 & + 21/4x_4 & \leq 51/2 & (3) \\
 & 3/2x_2 & + 1/2x_3 & + 5/4x_4 & \leq 5/2 & (4) \\
x_j \geq 0, & (j = 2, 3, 4) & & & &
\end{array}
$$

The following function presents the implementation in MATLAB of the transformation of a general LP problem to its canonical form (filename: `general2canonical.m`). Some necessary notations should be introduced before the presentation of the implementation for transforming a general LP problem to its canonical form. Let A be a $m \times n$ matrix with the coefficients of the constraints, c be a $n \times 1$ vector with the coefficients of the objective function, b be a $m \times 1$ vector of the right-hand side of the constraints, *Eqin* be a $m \times 1$ vector of the type of the constraints (-1 for inequality constraints in the form 'less than or equal to' ($\leq$), 0 for equalities, and 1 for inequality constraints in the form 'greater than or equal to' ($\geq$)), *MinMaxLP* be a variable denoting the type of optimization (-1 for minimization and 1 for maximization), and $c0$ be the constant term of the objective function.

The function for the transformation of a general LP problem to its canonical form takes as input the matrix of coefficients of the constraints (matrix A), the vector of coefficients of the objective function (vector c), the vector of the right-hand side of the constraints (vector b), the vector of the type of the constraints (vector *Eqin*), a variable denoting the type of optimization (variable *MinMaxLP*), and the constant term of the objective function (variable $c0$), and returns as output the transformed matrix of coefficients of the constraints (matrix A), the transformed vector of coefficients of the objective function (vector c), the transformed vector of the right-hand side of the constraints (vector b), the transformed vector of the type of the constraints (vector *Eqin*), a variable denoting the type of optimization (variable *MinMaxLP*), and the updated constant term of the objective function (variable $c0$).

In lines 39–42, if the LP problem is a maximization problem, we transform it to a minimization problem. Finally, we find all constraints that are not in the form ‘less than or equal to’ and transform them in that form (lines 45–68).

```
1.   function [A, c, b, Eqin, MinMaxLP, c0] = ...
2.       general2canonical(A, c, b, Eqin, MinMaxLP, c0)
3.   % Filename: general2canonical.m
4.   % Description: the function is an implementation of the
5.   % transformation of a general LP problem to its
6.   % canonical form
7.   % Authors: Ploskas, N., & Samaras, N.
8.   %
9.   % Syntax: [A, c, b, Eqin, MinMaxLP, c0] = ...
10.  %   general2canonical(A, c, b, Eqin, MinMaxLP, c0)
11.  %
12.  % Input:
13.  % -- A: matrix of coefficients of the constraints
14.  %    (size m x n)
15.  % -- c: vector of coefficients of the objective function
16.  %    (size n x 1)
17.  % -- b: vector of the right-hand side of the constraints
18.  %    (size m x 1)
19.  % -- Eqin: vector of the type of the constraints
20.  %    (size m x 1)
21.  % -- MinMaxLP: the type of optimization
22.  % -- c0: constant term of the objective function
23.  %
24.  % Output:
25.  % -- A: transformed matrix of coefficients of the
26.  %    constraints (size m x n)
27.  % -- c: transformed vector of coefficients of the objective
28.  %    function (size n x 1)
29.  % -- b: transformed vector of the right-hand side of the
30.  %    constraints (size m x 1)
31.  % -- Eqin: transformed vector of the type of the
32.  %    constraints (size m x 1)
33.  % -- MinMaxLP: the type of optimization
34.  % -- c0: updated constant term of the objective function
35.
36.  [m, ~] = size(A); % size of matrix A
37.  % if the LP problem is a maximization problem, transform it
38.  % to a minimization problem
39.  if MinMaxLP == 1
40.     MinMaxLP = -1;
41.     c = -c;
42.  end
43.  % find all constraints that are not in the form 'less than
44.  % or equal to'
45.  for i = 1:m
46.     % transform constraints in the form 'greater than or
47.     % equal to'
48.     if Eqin(i) == 1
49.         A(i, :) = -A(i, :);
50.         b(i) = -b(i);
```

```
       Eqin(i) = -1;
   elseif Eqin(i) == 0 % transform equality constraints
       f = find(A(i, :) ~= 0);
       f = f(1);
       b(i) = b(i) / A(i, f);
       A(i, :) = A(i, :) / A(i, f);
       b([1:i - 1, i + 1:m]) = b([1:i - 1, i + 1:m]) - ...
            A([1:i - 1, i + 1:m], f) .* b(i);
       A([1:i - 1, i + 1:m], :) = ...
            A([1:i - 1, i + 1:m], :) - ...
            A([1:i - 1, i + 1:m], f) * A(i, :);
       c0 = c0 + c(f) * b(i);
       c = c - c(f) * A(i, :);
       A(:, f) = [];
       c(f) = [];
       Eqin(i) = -1;
   end
end
end
```

2.3.2 General to Standard Form

The steps that must be performed to transform a general LP problem to its standard form are the following:

1. If the LP problem is a maximization problem, transform it to a minimization problem. Since maximizing a quantity is equivalent to minimizing its negative, any maximization objective function

$$\max z = c^T x = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \tag{2.18}$$

can be transformed to

$$\min z = -c^T x = -c_1 x_1 - c_2 x_2 - \cdots - c_n x_n \tag{2.19}$$

or in a more general form

$$\max \left\{ c^T x : x \in \top \right\} = -\min \left\{ -c^T x : x \in \top \right\} \tag{2.20}$$

where $\top$ is a set of numbers

2. An inequality constraint in the form 'less than or equal to' ($\leq$)

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \leq b \tag{2.21}$$

can be transformed to an equality ($=$) if we add a deficit slack variable to its left-hand side and add a corresponding nonnegativity constraint

$$\begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_nx_n + x_{n+1} &= b \\ \text{and} \quad x_{n+1} &\geq 0 \end{aligned} \tag{2.22}$$

3. An inequality constraint in the form 'greater than or equal to' ($\geq$)

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq b \tag{2.23}$$

can be transformed to an equality ($=$) if we subtract a surplus slack variable to its left-hand side and add a corresponding nonnegativity constraint

$$\begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_nx_n - x_{n+1} &= b \\ \text{and} \quad x_{n+1} &\geq 0 \end{aligned} \tag{2.24}$$

Let's transform a general problem to its standard form. The LP problem that will be transformed is the following:

$$\begin{aligned} \max z = \quad & 3x_1 - 4x_2 + 5x_3 - 2x_4 \\ \text{s.t.} \quad & -2x_1 - x_2 - 4x_3 - 2x_4 \leq 4 \\ & 5x_1 + 3x_2 + x_3 + 2x_4 \leq 18 \\ & 5x_1 + 3x_2 + x_4 \geq -13 \\ & 4x_1 + 6x_2 + 2x_3 + 5x_4 \geq -10 \\ & x_j \geq 0, \quad (j = 1, 2, 3, 4) \end{aligned}$$

Initially, we need to transform the objective function to minimization. According to the first step of the aforementioned procedure, we take its negative.

$$\begin{aligned} \min z = \quad & -3x_1 + 4x_2 - 5x_3 + 2x_4 \\ \text{s.t.} \quad & -2x_1 - x_2 - 4x_3 - 2x_4 \leq 4 \\ & 5x_1 + 3x_2 + x_3 + 2x_4 \leq 18 \\ & 5x_1 + 3x_2 + x_4 \geq -13 \\ & 4x_1 + 6x_2 + 2x_3 + 5x_4 \geq -10 \\ & x_j \geq 0, \quad (j = 1, 2, 3, 4) \end{aligned}$$

Next, we transform all constraints in the form 'less than or equal to' ($\leq$) to equalities ($=$). According to the second step of the above procedure, we add a deficit slack variable to the left-hand side and add a corresponding nonnegativity constraint.

$$
\begin{aligned}
\min z = & -3x_1 + 4x_2 - 5x_3 + 2x_4 \\
\text{s.t.} \quad & -2x_1 - x_2 - 4x_3 - 2x_4 + x_5 = 4 \\
& 5x_1 + 3x_2 + x_3 + 2x_4 + x_6 = 18 \\
& 5x_1 + 3x_2 + x_4 \geq -13 \\
& 4x_1 + 6x_2 + 2x_3 + 5x_4 \geq -10 \\
& x_j \geq 0, \quad (j = 1, 2, 3, 4, 5, 6)
\end{aligned}
$$

Finally, we transform all constraints in the form 'greater than or equal to' ($\geq$) to equalities ($=$). According to the third step of the previously presented procedure, we subtract a surplus slack variable to the left-hand side and add a corresponding nonnegativity constraint.

$$
\begin{aligned}
\min z = & -3x_1 + 4x_2 - 5x_3 + 2x_4 \\
\text{s.t.} \quad & -2x_1 - x_2 - 4x_3 - 2x_4 + x_5 = 4 \\
& 5x_1 + 3x_2 + x_3 + 2x_4 + x_6 = 18 \\
& 5x_1 + 3x_2 + x_4 - x_7 = -13 \\
& 4x_1 + 6x_2 + 2x_3 + 5x_4 - x_8 = -10 \\
& x_j \geq 0, \quad (j = 1, 2, 3, 4, 5, 6, 7, 8)
\end{aligned}
$$

The following function presents the implementation in MATLAB of the transformation of a general LP problem to its standard form (filename: `general2standard.m`). Some necessary notations should be introduced before the presentation of the implementation for transforming a general LP problem to its standard form. Let A be a $m \times n$ matrix with the coefficients of the constraints, c be a $n \times 1$ vector with the coefficients of the objective function, b be a $m \times 1$ vector of the right-hand side of the constraints, $Eqin$ be a $m \times 1$ vector of the type of the constraints (-1 for inequality constraints in the form 'less than or equal to' ($\leq$), 0 for equalities, and 1 for inequality constraints in the form 'greater than or equal to' ($\geq$)), and $MinMaxLP$ be a variable denoting the type of optimization (-1 for minimization and 1 for maximization).

The function for the transformation of a general LP problem to its standard form takes as input the matrix of coefficients of the constraints (matrix A), the vector of coefficients of the objective function (vector c), the vector of the right-hand side of the constraints (vector b), the vector of the type of the constraints (vector $Eqin$), and a variable denoting the type of optimization (variable $MinMaxLP$), and returns as output the transformed matrix of coefficients of the constraints (matrix A), the transformed vector of coefficients of the objective function (vector c), the transformed vector of the right-hand side of the constraints (vector b), the transformed vector of the type of the constraints (vector $Eqin$), and a variable denoting the type of optimization (variable $MinMaxLP$).

In order to improve the performance of memory bound code in MATLAB, in lines 35–36, we pre-allocate matrix *A* and vector *c* before accessing them within the for-loop. In lines 39–42, if the LP problem is a maximization problem, we transform it to a minimization problem. Finally, we find all constraints that are not equalities and transform them to equalities (lines 44–58).

```
1.   function [A, c, b, Eqin, MinMaxLP] = ...
2.       general2standard(A, c, b, Eqin, MinMaxLP)
3.   % Filename: general2standard.m
4.   % Description: the function is an implementation of the
5.   % transformation of a general LP problem to its
6.   % standard form
7.   % Authors: Ploskas, N., & Samaras, N.
8.   %
9.   % Syntax: [A, c, b, Eqin, MinMaxLP] = ...
10.  %   general2standard(A, c, b, Eqin, MinMaxLP)
11.  %
12.  % Input:
13.  % -- A: matrix of coefficients of the constraints
14.  %    (size m x n)
15.  % -- c: vector of coefficients of the objective function
16.  %    (size n x 1)
17.  % -- b: vector of the right-hand side of the constraints
18.  %    (size m x 1)
19.  % -- Eqin: vector of the type of the constraints
20.  %    (size m x 1)
21.  % -- MinMaxLP: the type of optimization
22.  %
23.  % Output:
24.  % -- A: transformed matrix of coefficients of the
25.  %    constraints (size m x n)
26.  % -- c: transformed vector of coefficients of the objective
27.  %    function (size n x 1)
28.  % -- b: transformed vector of the right-hand side of the
29.  %    constraints (size m x 1)
30.  % -- Eqin: transformed vector of the type of the
31.  %    constraints (size m x 1)
32.  % -- MinMaxLP: the type of optimization
33.
34.  [m, n] = size(A); % size of matrix A
35.  A = [A zeros(m, n)]; % preallocate matrix A
36.  c = [c; zeros(m, 1)]; % preallocate vector c
37.  % if the LP problem is a maximization problem, transform it
38.  % to a minimization problem
39.  if MinMaxLP == 1
40.     MinMaxLP = -1;
41.     c = -c;
42.  end
43.  % find all constraints that are not equalities
44.  for i = 1:m
45.     % transform constraints in the form 'less than or
46.     % equal to' to equality constraints
47.     if Eqin(i) == -1
```

```
            A(i, n + 1) = 1;
            Eqin(i) = 0;
            n = n + 1;
        % transform constraints in the form 'greater than or
        % equal to' to equality constraints
        elseif Eqin(i) == 1
            A(i, n + 1) = -1;
            Eqin(i) = 0;
            n = n + 1;
        end
    end
end
```

2.3.3 Canonical to Standard Form

The steps that must be performed to transform an LP problem in its canonical form to its standard form are the same steps described previously in this section to transform a general LP problem to its standard form except that we do not have to check if the problem is a maximization problem and transform it to a minimization problem because an LP problem in its canonical form is already a minimization problem. Moreover, an LP problem in its canonical form contains only inequality constraints in the form ‘less than or equal to’. The function with the implementation in MATLAB of the transformation of an LP problem in its canonical form to its standard form can be found in the function `canonical2standard` (filename: `canonical2standard.m`).

2.3.4 Standard to Canonical Form

The steps that must be performed to transform an LP problem in its standard form to its canonical form are the same steps described previously in this section to transform a general LP problem to its canonical form except that we do not have to check if the problem is a maximization problem and transform it to a minimization problem because an LP problem in its standard form is already a minimization problem. Moreover, an LP problem in its standard form contains only equality constraints. The function with the implementation in MATLAB of the transformation of an LP problem in its standard form to its canonical form can be found in the function `standard2canonical` (filename: `standard2canonical.m`).

2.3.5 Transformations of Constraint Ranges and Variable Constraints

As already mentioned in this section, a general LP problem can include ranges in the constraints and variable constraints (bounds).

If a constraint has ranges

$$l_b \leq a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq u_b \tag{2.25}$$

then this restriction can be substituted by two constraints (one in the form of 'less than or equal to' and one in the form of 'greater than or equal to')

$$\begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_nx_n &\geq l_b \\ a_1x_1 + a_2x_2 + \cdots + a_nx_n &\leq u_b \end{aligned}$$

Constraints (bounds) can also be defined for variables:

1. **lower bound** ($x_j \geq l_j$): We can add the bound to our constraints. Moreover, if we want to transform our LP problem to its standard form, then we subtract a slack variable to the left-hand side of this constraint ($x_j - y_j = l_j$) and transform it to an equality constraint. However, that would add an extra constraint and an extra variable to the LP problem. Instead, we can substitute the initial decision variable (x_j) with the slack variable plus the lower bound ($x_j = y_j + l_j$) in the constraints and the objective function. Using the latter procedure, the transformed LP problem has the same dimensions with the initial LP problem.
2. **upper bound** ($x_j \leq u_j$): We can add the bound to our constraints. Moreover, if we want to transform our LP problem to its standard form, then we add a slack variable to the left-hand side of this constraint ($x_j + y_j = u_j$) and transform it to an equality constraint. However, that would add an extra constraint and an extra variable to the LP problem. Instead, we can substitute the initial decision variable (x_j) with the upper bound minus the slack variable ($x_j = u_j - y_j$) in the constraints and the objective function. Using the latter procedure, the transformed LP problem has the same dimensions with the initial LP problem.
3. **lower and upper bounds** ($l_j \leq x_j \leq u_j$): We can add the bounds to our constraints as two separate constraints ($x_j \geq l_j$ and $x_j \leq u_j$). Moreover, if we want to transform our LP problem to its standard form, then we subtract a slack variable to the left-hand side of the first constraint ($x_j - y_j = l_j$) and add a slack variable to the left-hand side of the second constraint ($x_j + y_{j+1} = u_j$) and transform them to equality constraints. However, that would add two extra constraints and two extra variables to the LP problem. Instead, we can write the compound inequality as $0 \leq x_j - l_j \leq u_j - l_j$. If we set $y_j = x_j - l_j \geq 0$, then we can substitute the initial decision variable (x_j) with the slack variable plus the lower bound ($x_j = y_j + l_j$) in the constraints and the objective function. Finally, we add a constraint $y_j \leq u_j - l_j$ and add a slack variable to that constraint ($y_j + y_{j+1} = u_j - l_j$) if we want to transform the LP problem to its standard form. Using the latter procedure, the transformed LP problem has one more constraint and one more variable than the initial LP problem.
4. **fixed variable** ($x_j = k$): We can add the bound to our constraints. However, that would add an extra constraint to the LP problem. Instead, we can substitute the fixed bound (k) to the initial decision variable (x_j) in the constraints and the

objective function and eliminate that variable. Using the latter procedure, the transformed LP problem will have one variable less than the initial LP problem.

5. **free variable**: We can substitute a free variable x_j by $x_j^+ - x_j^-$, where x_j^+ and x_j^- are nonnegative, and eliminate that variable. However, that would add one more variable to the LP problem. Instead, we can find an equality constraint where we can express variable $A_{ij}x_j$, where $A_{ij} \neq 0$, as a function of the other variables and substitute x_j in the constraints and the objective function. Next, we can eliminate variable x_j from the constraints and the objective function and delete the equality constraint from which we expressed variable x_j as a function of the other variables. Using the latter procedure, the transformed LP problem has one less constraint and one less variable than the initial LP problem.

Let's transform a general problem to its standard form. The LP problem that will be transformed is the following:

$$\begin{array}{llllllll} \min z = & -2x_1 & - \ 3x_2 & + \ 4x_3 & + \ x_4 & + \ 5x_5 & & \\ \text{s.t.} & 2x_1 & - \ x_2 & + \ 3x_3 & + \ x_4 & + \ x_5 & = 5 & (1) \\ & x_1 & + \ 3x_2 & + \ 2x_3 & - \ 2x_4 & & \geq 4 & (2) \\ & -x_1 & - \ x_2 & + \ 3x_3 & + \ x_4 & & \leq 3 & (3) \\ \multicolumn{8}{l}{x_1 - free, x_2 \geq -1, x_3 \leq 1, -1 \leq x_4 \leq 1, x_5 = 1} \end{array}$$

Initially, we check if the problem is a maximization problem and we need to transform it to a minimization problem. This is not the case in that problem. Next, we transform all inequality constraints ($\leq$, $\geq$) to equalities ($=$). We subtract a surplus slack variable to the left-hand side of the second constraint and add a corresponding nonnegativity constraint. Moreover, we add a deficit slack variable to the left-hand side of the third constraint and add a corresponding nonnegativity constraint.

$$\begin{array}{llllllllll} \min z = & -2x_1 & - \ 3x_2 & + \ 4x_3 & + \ x_4 & + \ 5x_5 & & & & \\ \text{s.t.} & 2x_1 & - \ x_2 & + \ 3x_3 & + \ x_4 & + \ x_5 & & & = 5 & (1) \\ & x_1 & + \ 3x_2 & + \ 2x_3 & - \ 2x_4 & & - \ x_6 & & = 4 & (2) \\ & -x_1 & - \ x_2 & + \ 3x_3 & + \ x_4 & & & + \ x_7 & = 3 & (3) \\ \multicolumn{10}{l}{x_1 - free, x_2 \geq -1, x_3 \leq 1, -1 \leq x_4 \leq 1, x_5 = 1, x_6 \geq 0, x_7 \geq 0} \end{array}$$

Next, we handle the lower bound of variable x_2. According to the first case of the aforementioned transformations, we can substitute the decision variable (x_2) with the slack variable plus the lower bound ($x_2 = y_2 - 1$).

$$\begin{array}{llllllllll} \min z = & -2x_1 & - \ 3(y_2 - 1) & + \ 4x_3 & + \ x_4 & + \ 5x_5 & & & & \\ \text{s.t.} & 2x_1 & - \ (y_2 - 1) & + \ 3x_3 & + \ x_4 & + \ x_5 & & & = 5 & (1) \\ & x_1 & + \ 3(y_2 - 1) & + \ 2x_3 & - \ 2x_4 & & - \ x_6 & & = 4 & (2) \\ & -x_1 & - \ (y_2 - 1) & + \ 3x_3 & + \ x_4 & & & + \ x_7 & = 3 & (3) \\ \multicolumn{10}{l}{x_1 - free, y_2 \geq 0, x_3 \leq 1, -1 \leq x_4 \leq 1, x_5 = 1, x_6 \geq 0, x_7 \geq 0} \end{array}$$

Note, that the objective function also includes a constant term, $c_0 = 3$.

$$
\begin{array}{llr}
\min z = -2x_1 - 3y_2 + 4x_3 + x_4 + 5x_5 + 3 & & \\
\text{s.t.} \quad 2x_1 - y_2 + 3x_3 + x_4 + x_5 = 4 & & (1) \\
\quad x_1 + 3y_2 + 2x_3 - 2x_4 - x_6 = 7 & & (2) \\
\quad -x_1 - y_2 + 3x_3 + x_4 + x_7 = 2 & & (3) \\
x_1 - free, y_2 \geq 0, x_3 \leq 1, -1 \leq x_4 \leq 1, x_5 = 1, x_6 \geq 0, x_7 \geq 0 & &
\end{array}
$$

Next, we handle the upper bound of variable x_3. According to the second case of the aforementioned transformations, we can substitute the decision variable (x_3) with the upper bound minus the slack variable ($x_3 = 1 - y_3$).

$$
\begin{array}{llr}
\min z = -2x_1 - 3y_2 - 4y_3 + x_4 + 5x_5 + 7 & & \\
\text{s.t.} \quad 2x_1 - y_2 - 3y_3 + x_4 + x_5 = 1 & & (1) \\
\quad x_1 + 3y_2 - 2y_3 - 2x_4 - x_6 = 5 & & (2) \\
\quad -x_1 - y_2 - 3y_3 + x_4 + x_7 = -1 & & (3) \\
x_1 - free, y_2 \geq 0, y_3 \geq 0, -1 \leq x_4 \leq 1, x_5 = 1, x_6 \geq 0, x_7 \geq 0 & &
\end{array}
$$

Next, we handle the lower and upper bounds of variable x_4. According to the third case of the aforementioned transformations, we can substitute the initial decision variable (x_4) with the slack variable plus the lower bound ($x_4 = y_4 - 1$) in the constraints and the objective function. Finally, we add a new constraint $y_4 \leq 2$ and a slack variable y_5 to that constraint.

$$
\begin{array}{llr}
\min z = -2x_1 - 3y_2 - 4y_3 + y_4 + 5x_5 + 6 & & \\
\text{s.t.} \quad 2x_1 - y_2 - 3y_3 + y_4 + x_5 = 2 & & (1) \\
\quad x_1 + 3y_2 - 2y_3 - 2y_4 - x_6 = 3 & & (2) \\
\quad -x_1 - y_2 - 3x_3 + y_4 + x_7 = 0 & & (3) \\
\quad y_4 + y_5 = 2 & & (4) \\
x_1 - free, y_2 \geq 0, y_3 \geq 0, y_4 \geq 0, x_5 = 1, x_6 \geq 0, x_7 \geq 0, y_5 \geq 0 & &
\end{array}
$$

Next, we handle the fixed variable x_5. According to the fourth case of the aforementioned transformations, we can substitute the fixed bound (1) to the initial decision variable (x_5) and eliminate that variable.

$$
\begin{array}{llr}
\min z = -2x_1 - 3y_2 - 4y_3 + y_4 + 11 & & \\
\text{s.t.} \quad 2x_1 - y_2 - 3y_3 + y_4 = 1 & & (1) \\
\quad x_1 + 3y_2 - 2y_3 - 2y_4 - x_6 = 3 & & (2) \\
\quad -x_1 - y_2 - 3x_3 + y_4 + x_7 = 0 & & (3) \\
\quad y_4 + y_5 = 2 & & (4) \\
x_1 - free, y_2 \geq 0, y_3 \geq 0, y_4 \geq 0, x_6 \geq 0, x_7 \geq 0, y_5 \geq 0 & &
\end{array}
$$

Finally, we handle the free variable x_1. According to the fifth case of the aforementioned transformations, we can find an equality constraint where we can express variable x_1 as a function of the other variables. Let's choose the second constraint to express variable x_1 as a function of the other variables. Then, we substitute x_1 in the constraints and the objective function. Next, we eliminate variable x_1 from the constraints and the objective function and delete the second constraint. The final transformed LP problem is shown below:

$$
\begin{array}{llr}
\min z = & 3y_2 - 8y_3 - 3y_4 - 2x_6 + 5 & \\
\text{s.t.} & -7y_2 + y_3 + 5y_4 + 2x_6 = -5 & (1) \\
 & 2y_2 - 5x_3 - y_4 - x_6 + x_7 = 3 & (2) \\
 & y_4 + y_5 = 2 & (3) \\
\multicolumn{3}{l}{y_2 \geq 0, y_3 \geq 0, y_4 \geq 0, x_6 \geq 0, x_7 \geq 0, y_5 \geq 0}
\end{array}
$$

2.4 Modeling Linear Programming Problems

Given a description of the problem, you should be able to formulate it as an LP problem following the steps below:

- Identify the decision variables,
- Identify the objective function,
- Identify the constraints, and
- Write down the entire problem adding nonnegativity constraints if necessary.

Let's follow these steps to formulate the associated LP problem in the following three examples.

Example 1 A company produces two products, A and B. One unit of product A is sold for \$50, *while one unit of product B is sold for* \$35. *Each unit of product A requires* 3 *kg of raw material and* 5 *labor hours for processing, while each unit of product B requires* 4 *kg of raw material and* 4 *labor hours for processing. The company can buy* 200 *kg of raw material every week. Moreover, the company has currently* 4 *employees that work* 8*-hour shifts per day (Monday - Friday). The company wants to find the number of units of each product that should produce in order to maximize its revenue.*

Initially, we identify the decision variables of the problem. The number of units of each product that the company should produce are the decision variables of this problem. Let x_1 and x_2 be the number of units of product A and B per week, respectively.

Next, we define the objective function. The company wants to maximize its revenue and we already know that the price for product A is \$50 per unit and the price for product B is \$35 per unit. Hence, the objective function is:

$$\max z = 50x_1 + 35x_2$$

Then, we identify the technological constraints of the given problem. First of all, there is a constraint about the raw material that should be used to produce the two products. Each unit of product A requires 3 kg of raw material and each unit of product B requires 4 kg of raw material, while the raw material used every week cannot exceed 200 kg. Hence, the first constraint is given by

$$3x_1 + 4x_2 \leq 200$$

There is another technological constraint about the labor hours that should be used to produce the two products. Each unit of product A requires 5 labor hours for processing and each unit of product B requires 4 labor hours for processing, while the available labor hours every week cannot exceed 160 h (4 employees × 5 days per week × 8-h shifts per day). Hence, the second constraint is given by

$$5x_1 + 4x_2 \leq 160$$

Moreover, we also add the nonnegativity constraints for variables x_1 and x_2. Hence, the LP problem is the following:

$$\begin{array}{ll} \max z = & 50x_1 + 35x_2 \\ \text{s.t.} & 3x_1 + 4x_2 \leq 200 \\ & 5x_1 + 4x_2 \leq 160 \\ \multicolumn{2}{l}{x_1 \geq 0, x_2 \geq 0, \{x_1, x_2\} \in \mathbb{Z}} \end{array}$$

Example 2 A student is usually purchasing a snack every day from a small store close to his university. The store offer two choices of food: brownies and chocolate ice cream. One brownie costs \$2 *and one scoop of chocolate ice cream costs* \$1. *The student purchases his snack from the same small store each day for many years now, so the owner of the store allows him to purchase a fraction of a product if he wishes. The student knows that each brownie contains* 4 *ounces of chocolate and* 3 *ounces of sugar, while each scoop of chocolate ice cream contains* 3 *ounces of chocolate and* 2 *ounces of sugar. The student has also decided that he needs at least* 8 *ounces of chocolate and* 11 *ounces of sugar per snack. He wants to find the amount of each product that should buy to meet his requirements by minimizing the cost.*

Initially, we identify the decision variables of the problem. The amount of each product that the student should buy are the decision variables of this problem. Let x_1 and x_2 be the amount of brownies and the number of scoops of chocolate ice cream, respectively.

Next, we define the objective function. The student wants to minimize his cost and we already know that the cost for each brownie is \$2 and the cost for a scoop of chocolate ice cream is \$1. Hence, the objective function is:

$$\min z = 2x_1 + x_2$$

Then, we identify the technological constraints of the given problem. First of all, there is a constraint about the chocolate intake per snack. The student wants 8 ounces of chocolate per snack and each brownie contains 4 ounces of chocolate, while each scoop of chocolate ice cream contains 3 ounces of chocolate. Hence, the first constraint is given by

$$4x_1 + 3x_2 \geq 8$$

There is another technological constraint about the sugar intake per snack. The student wants 11 ounces of sugar per snack and each brownie contains 3 ounces of sugar, while each scoop of chocolate ice cream contains 2 ounces of sugar. Hence, the second constraint is given by

$$3x_1 + 2x_2 \geq 11$$

Moreover, we also add the nonnegativity constraints for variables x_1 and x_2. Hence, the LP problem is the following:

$$\begin{aligned} \min z = {} & 2x_1 + x_2 \\ \text{s.t.} \quad & 4x_1 + 3x_2 \geq 8 \\ & 3x_1 + 2x_2 \geq 11 \\ & x_1 \geq 0, x_2 \geq 0, \{x_1, x_2\} \in \mathbb{R} \end{aligned}$$

Example 3 A 24/7 store has decided its minimal requirements for employees as follows:

1. 6 *a.m.*–10 *a.m.*–2 *employees*
2. 10 *a.m.*–2 *p.m.*–5 *employees*
3. 2 *p.m.*–6 *p.m.*–4 *employees*
4. 6 *p.m.*–10 *p.m.*–3 *employees*
5. 10 *p.m.*–2 *a.m.*–2 *employees*
6. 2 *a.m.*–6 *a.m.*–1 *employee*

Employees start working at the beginning of one of the above periods and work for 8 *consecutive hours. The store wants to determine the minimum number of employees to be employed in order to have a sufficient number of employees available for each period.*

Initially, we identify the decision variables of the problem. The number of employees start working at the beginning of each period are the decision variables of this problem. Let x_1, x_2, x_3, x_4, x_5, and x_6 be the number of employees start working at the beginning of each of the aforementioned 6 periods.

Next, we define the objective function. The store wants to minimize the number of employees to be employed. Hence, the objective function is:

$$\min z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

Then, we identify the technological constraints of the given problem. We take into account that an employee that starts working in a specific period will also work in the next period (8-h shift). So, the number of employees start working in the first period and the number of employees start working in the second period must be at least 5 (the maximum of 2 employees needed in the first period and 5 employees needed in the second period). Hence, the first constraint is given by

$$x_1 + x_2 \geq 5$$

The number of employees start working in the second period and the number of employees start working in the third period must be at least 5 (the maximum of 5 employees needed in the second period and 4 employees needed in the third period). Hence, the second constraint is given by

$$x_2 + x_3 \geq 5$$

The number of employees start working in the third period and the number of employees start working in the fourth period must be at least 4 (the maximum of 4 employees needed in the third period and 3 employees needed in the fourth period). Hence, the third constraint is given by

$$x_3 + x_4 \geq 4$$

The number of employees start working in the fourth period and the number of employees start working in the fifth period must be at least 3 (the maximum of 3 employees needed in the fourth period and 2 employees needed in the fifth period). Hence, the fourth constraint is given by

$$x_4 + x_5 \geq 3$$

The number of employees start working in the fifth period and the number of employees start working in the sixth period must be at least 2 (the maximum of 2 employees needed in the fifth period and 1 employee needed in the sixth period). Hence, the fifth constraint is given by

$$x_5 + x_6 \geq 2$$

The number of employees start working in the sixth period and the number of employees start working in the first period must be at least 2 (the maximum of 1 employee needed in the sixth period and 2 employees needed in the first period). Hence, the sixth constraint is given by

$$x_1 + x_6 \geq 2$$

Moreover, we also add the nonnegativity constraints for variables x_1, x_2, x_3, x_4, x_5, and x_6. Hence, the LP problem is the following:

$$\begin{aligned}
\min z = {} & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \\
\text{s.t.} \quad & x_1 + x_2 \geq 5 \\
& x_2 + x_3 \geq 5 \\
& x_3 + x_4 \geq 4 \\
& x_4 + x_5 \geq 3 \\
& x_5 + x_6 \geq 2 \\
& x_1 + x_6 \geq 2 \\
& x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0, \{x_1, x_2, x_3, x_4, x_5, x_6\} \in \mathbb{Z}
\end{aligned}$$

2.5 Geometry of Linear Programming Problems

One way to solve LPs that involve two (or three) variables is the graphical solution of the LP problem. The steps that we must follow to solve an LP problem with this technique are the following:

- Draw the constraints and find the half-planes that represent them,
- Identify the feasible region that satisfies all constraints simultaneously, and
- Locate the corner points, evaluate the objective function at each point, and identify the optimum value of the objective function.

2.5.1 Drawing Systems of Linear Equalities

Definition 2.2 A set of points $x \in \mathbb{R}^n$ whose coordinates satisfy a linear equation of the form

$$A_1x_1 + A_2x_2 + \cdots + A_nx_n = b$$

where at least an $A_j \neq 0, j = 1, 2, \cdots, n$, is called an $(n-1)$ dimensional hyperplane.

Definition 2.3 Let $(x, y, z) \in \mathbb{R}^N$ and $t \geq 0$. A half line is the set of points $\{x | x = y + tz\}$, $t \in [0, \infty)$.

In order to graph linear inequalities with two variables, we need to:

- Draw the graph of the equation obtained for the given inequality by replacing the inequality sign with an equal sign, and
- Pick a point lying in one of the half-planes determined by the line sketched in the previous step and substitute the values of x and y into the given inequality. If the inequality is satisfied, the solution is the half-plane containing the point. Otherwise, the solution is the half-plane that does not contain the point.

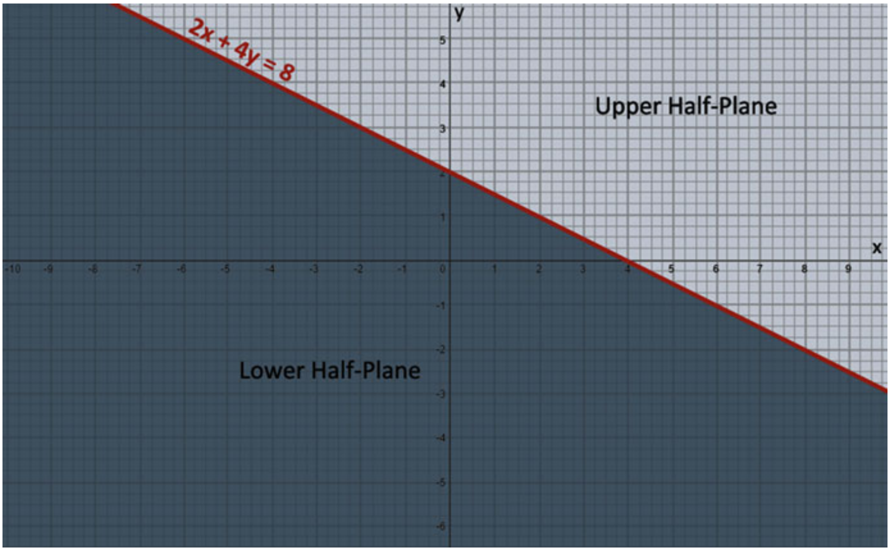


Fig. 2.1 Graph of the equation $2x + 4y = 8$

Let's consider the graph of the equation $2x + 4y \leq 8$. Initially, we draw the line $2x + 4y = 8$ (Figure 2.1).

Next, we determine which half-plane satisfies the inequality $2x + 4y \leq 8$. Pick any point lying in one of the half-planes and substitute the values of x and y into the inequality. Let's pick the origin $(0, 0)$, which lies in the lower half-plane. Substituting $x = 0$ and $y = 0$ into the given inequality, we find

$$2(0) + 4(0) \leq 8$$

or $0 \leq 8$, which is true. This tells us that the required half-plane is the one containing the point $(0, 0)$, i.e., the lower half-plane (Figure 2.2).

2.5.2 Identify the Feasible Region

Definition 2.4 Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^n$, and let the polyhedron $P = \{x \in \mathbb{R}^n | Ax \leq b\}$. Then, polyhedron P is convex.

The above definition implies that the feasible region of any LP problem in canonical form is convex, or equivalently, the set of points corresponding to feasible or optimal solutions of the LP problem is a convex set.

A set of values of the variables of an LP problem that satisfies the constraints and the nonnegative restrictions is called a feasible solution. The feasible region is a convex polytope with polygonal faces. A feasible solution of an LP problem

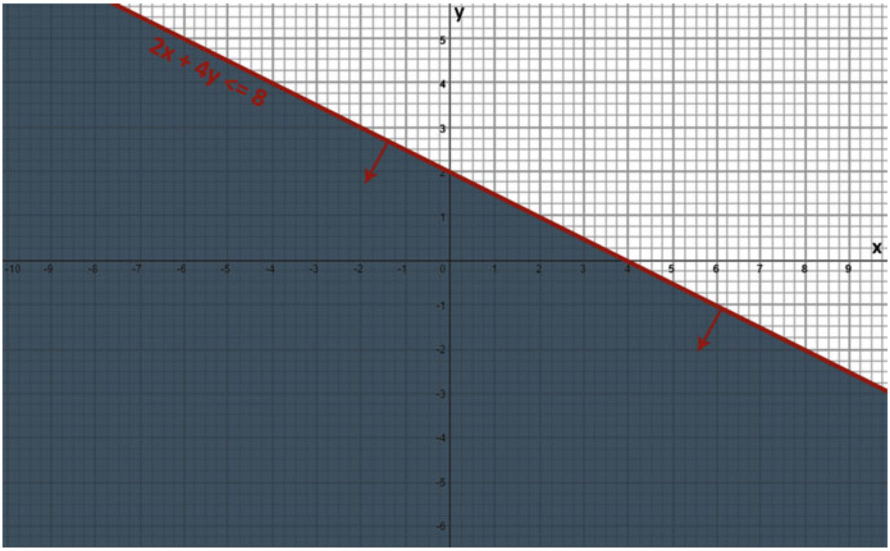


Fig. 2.2 Half-plane of the equation $2x + 4y \leq 8$

that optimizes its objective function is called the optimal solution. In order to solve geometrically an LP problem, we locate the half-planes determined by each constraint following the procedure that was described in the previous step (taking also into account the nonnegativity constraints of the variables). The solution set of a system of linear equalities is the feasible region. The feasible region can be bounded (Figure 2.3) or unbounded (Figure 2.4). In the latter case, if the optimal solution is also unbounded, then the LP problem is unbounded. Otherwise, the region can be unbounded but the LP can have an optimal solution. If the feasible region is empty, then the LP problem is infeasible (Figure 2.5).

2.5.3 Evaluate the Objective Function at Each Corner Point and Identify the Optimum Value of the Objective Function

Definition 2.5 A point $x \in \mathbb{R}^n$ in a convex set is said to be an extreme (corner) point if there are no two points x and y such that $z = tx + (1-t)\,y$ for some $t \in (0, 1)$.

Definition 2.6 If the convex set corresponding to $\{Ax = b, x \geq 0\}$ is nonempty, then it has at least one extreme (corner) point.

Definition 2.7 The constraint set corresponding to $\{Ax = b, x \geq 0\}$ has a finite number of extreme (corner) points.

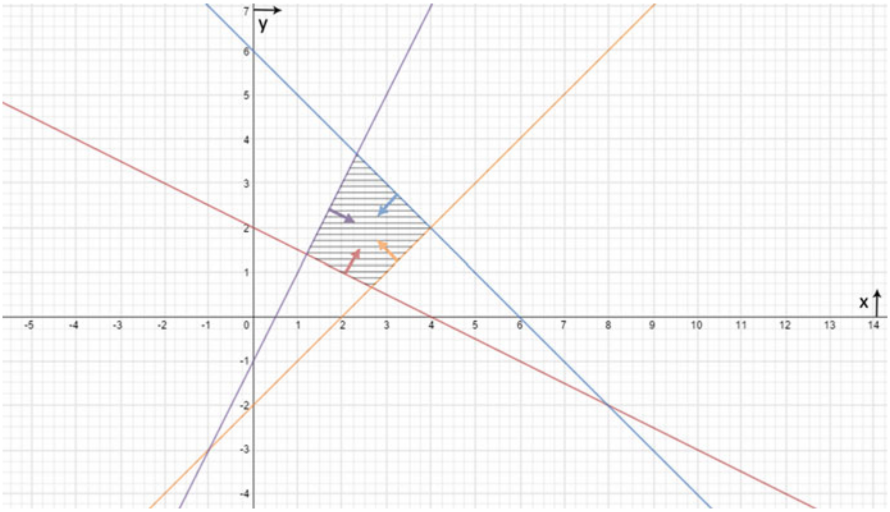


Fig. 2.3 Bounded feasible region

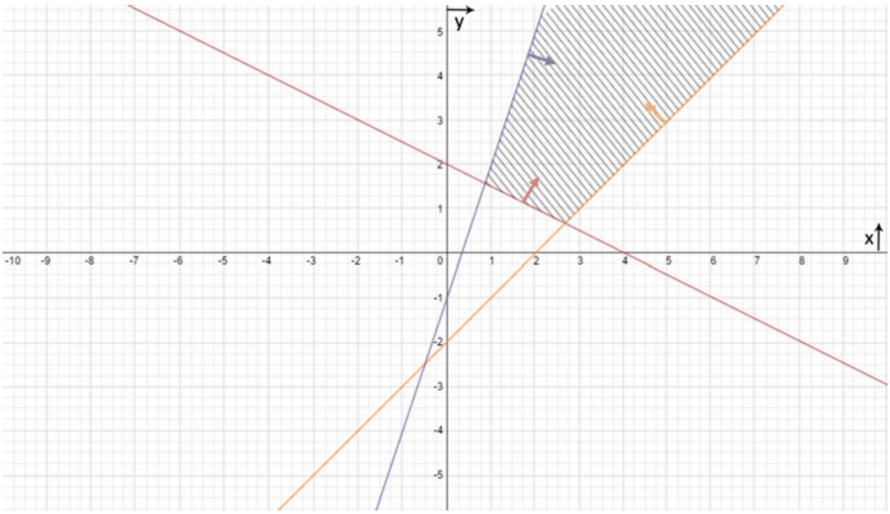


Fig. 2.4 Unbounded feasible region

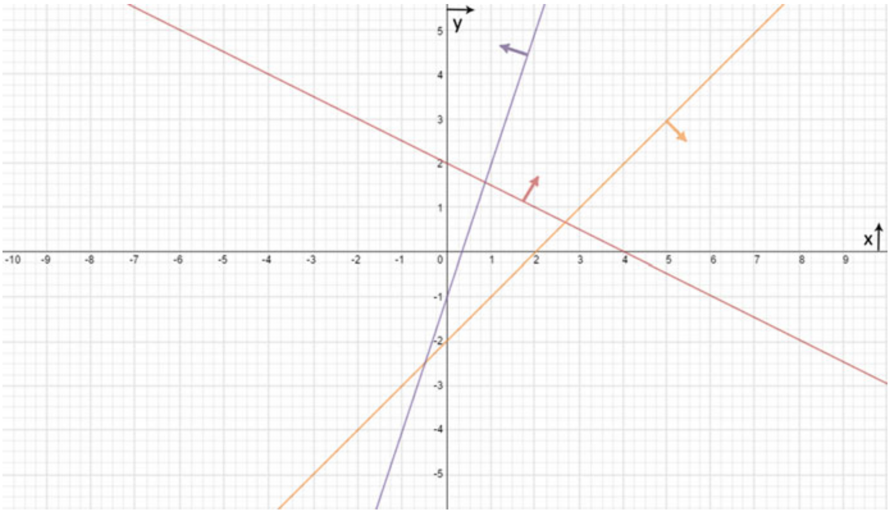


Fig. 2.5 Infeasible LP problem

Definition 2.8 A path in a polyhedral set is a sequence of adjacent extreme (corner) points.

If an LP problem has a solution, then it is located at a vertex (extreme point) of the feasible region. If an LP problem has multiple solutions, then at least one of them is located at a vertex of the set of feasible solutions. In order to find the optimum value of the objective function, we find all the points of intersection of the straight lines bounding the region, evaluate the objective function at each point, and identify the optimum value of the objective function.

Let's consider the bounded feasible region of the following LP problem (Figure 2.6).

We evaluate the objective function at each point of intersection of the straight lines bounding the region. Points 0, F, G, H, I, J, K, L, M, N, O, P, Q, and R are in the intersection of the straight lines of two constraints, but they are not in the feasible region. Hence, we evaluate the objective function only at points A, B, C, D, and E. Substituting x and y values of these points to the objective function, we find the value of the objective function at each point. The minimum or maximum value (depending if the problem is a minimization or a maximization problem) is the optimum value of the objective function and x and y values of the specific point are the optimum values for the LP problem.

The constraints in which the intersection lies the optimal point are called active because these constraints are satisfied as equalities. All other constraints are called inactive because these constraints are satisfied as inequalities.

Another way to find the optimal value of the objective function is to look at the contours of the objective function. The contours of the function $f(x_1, x_2) =$

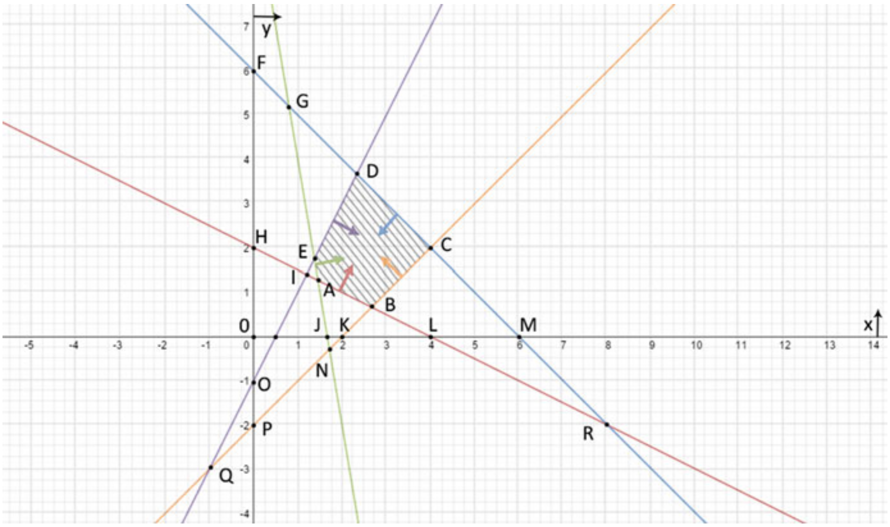


Fig. 2.6 Bounded feasible region of an LP problem

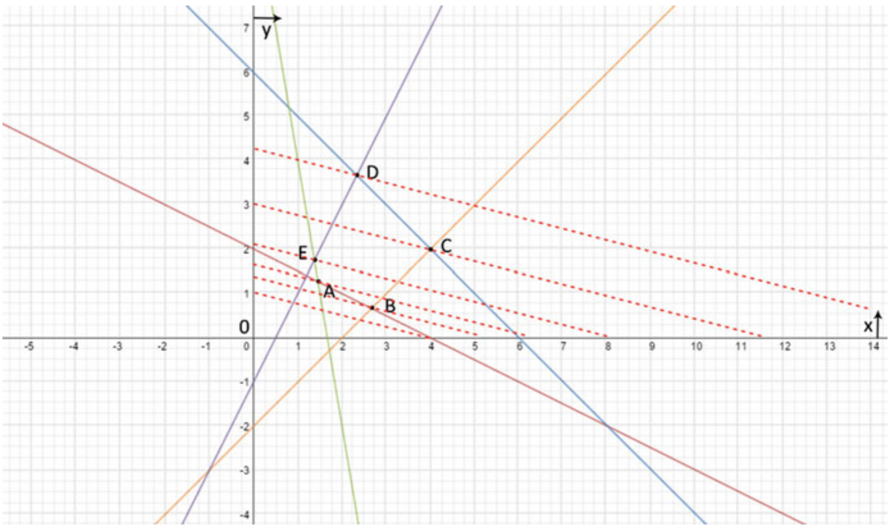


Fig. 2.7 Contours of the objective function

$a_1x_1 + a_2x_2$ are defined by $a_1x_1 + a_2x_2 = c$ for different values of c. The contours are all straight lines and for different values of c are all parallel. Let max $z = x + 4y$ be the objective function of the LP problem shown in Figure 2.6. Figure 2.7 includes the contours of the objective function for different values of c (dashed lines). We sketch a contour of the objective function and move perpendicular to this line in the direction in which the objective function increases until the boundary of the feasible region is reached. Hence, the optimum value of the objective function is at point D, because if we move further the contour then it will be outside of the feasible region boundary.

Example 1 Let's consider the following LP problem:

$$
\begin{array}{lrl}
\max z = & 2x + 3y & \\
\text{s.t.} & 2x + 4y \geq 8 & (1) \\
& 2x + 5y \leq 18 & (2) \\
& 3x + y \geq 5 & (3) \\
& x - 2y \leq 2 & (4) \\
& x, y \geq 0 &
\end{array}
$$

Initially, we draw the constraints and find the half-planes that represent them. Graphing the first constraint

$$2x + 4y \geq 8$$

and taking also into account the nonnegativity constraints, we get the shaded region shown in Figure 2.8.

Adding the second constraint

$$2x + 5y \leq 18$$

into the previous graph, we now get the shaded region shown in Figure 2.9.

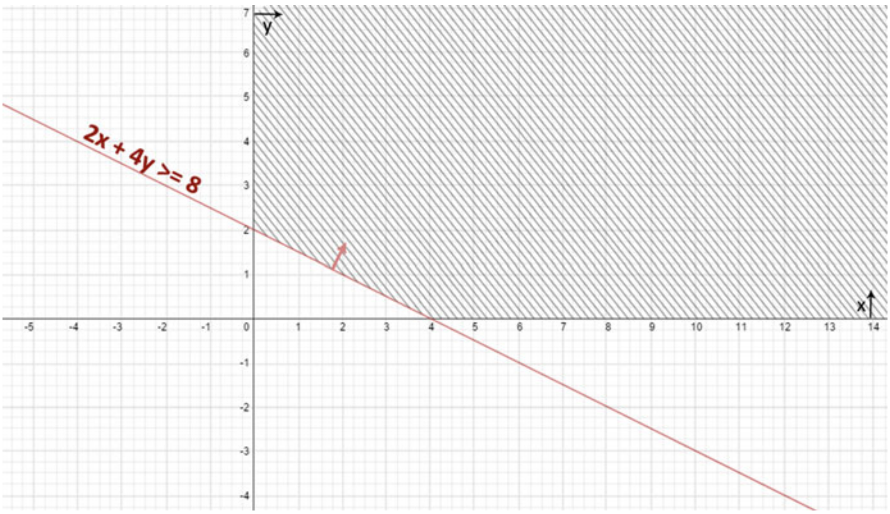


Fig. 2.8 Drawing the nonnegativity constraints and the first constraint

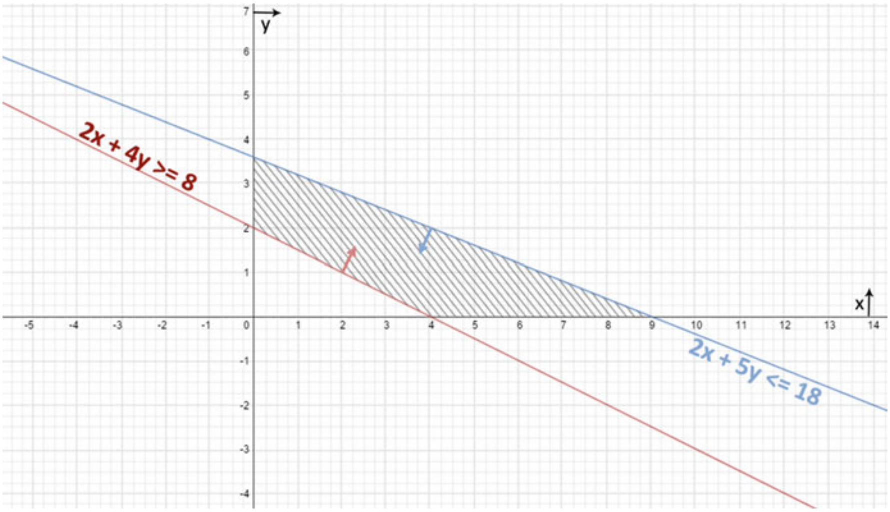


Fig. 2.9 Drawing the second constraint

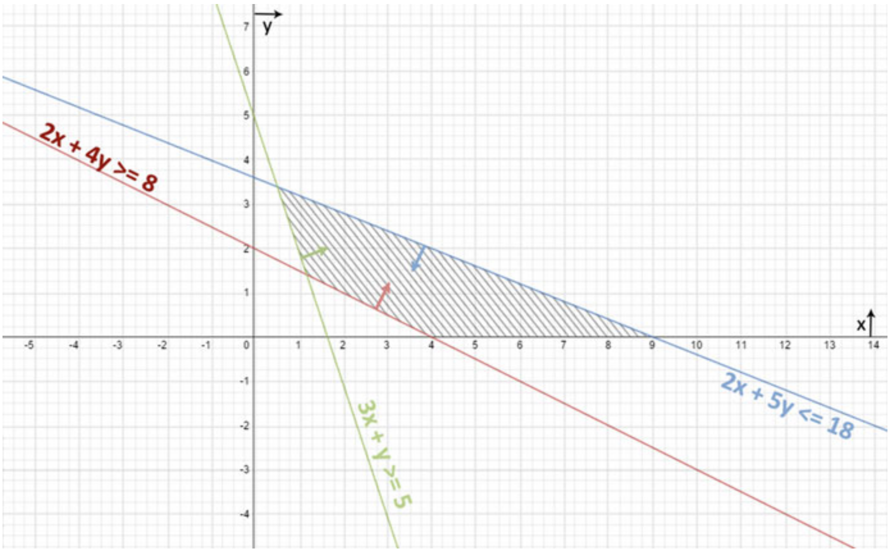


Fig. 2.10 Drawing the third constraint

Adding the third constraint

$$3x + y \geq 5$$

into the previous graph, we now get the shaded region shown in Figure 2.10.

Adding the fourth constraint

$$x - 2y \leq 2$$

into the previous graph, we now get the shaded region shown in Figure 2.11.

We have identified the feasible region (shaded region in Figure 2.11) which satisfies all the constraints simultaneously. Now, we locate the corner points, evaluate the objective function at each point, and identify the optimum value of the objective function. Figure 2.12 shows the points of intersection of the straight

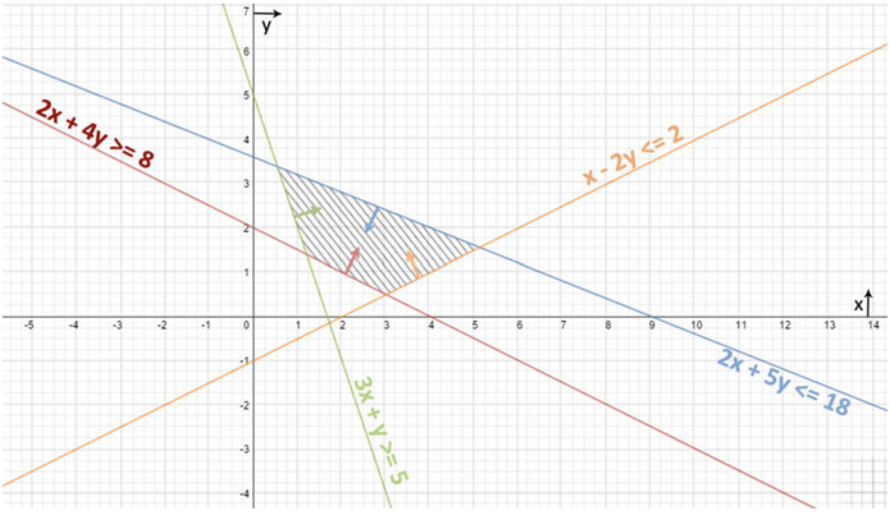


Fig. 2.11 Drawing the fourth constraint

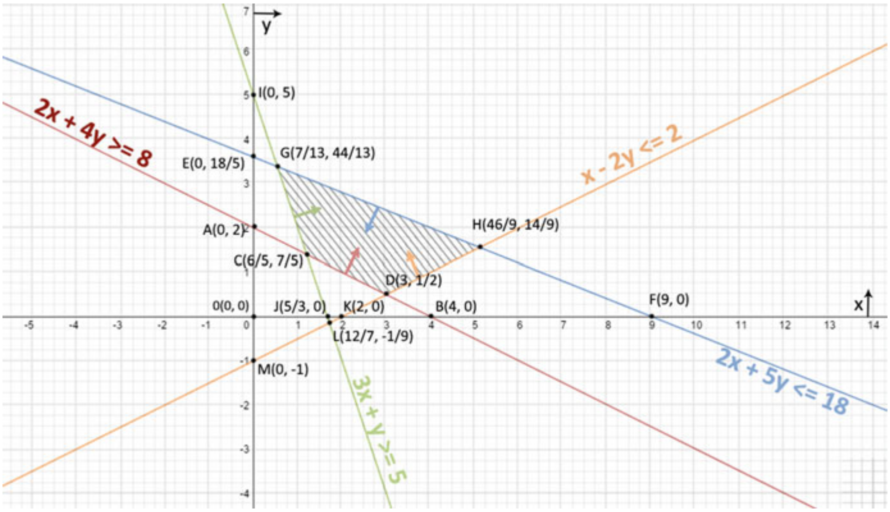


Fig. 2.12 Feasible region of the LP problem

Table 2.1 Evaluation of the objective function for the feasible boundary points

Point	x	y	z
C	6/5	7/5	33/5
D	3	1/2	15/2
G	7/13	44/13	146/13
H	46/9	14/9	134/9

lines of two constraints. Points $O, A, B, E, F, I, J, K, L$, and M are in the intersection of the straight lines of two constraints, but they are not in the feasible region. Hence, we evaluate the objective function only at points C, D, G, and H. Substituting the x and y values of these points to the objective function, we find the value of the objective function at each point (Table 2.1). The maximum value (the LP problem is a maximization problem) is $z = 134/9$ and the optimum values of the decision variables for the LP problem are $x = 46/9$ and $y = 14/9$.

Example 2 Let's consider the following LP problem:

$$
\begin{array}{llll}
\max z = & x + 2y & & \\
\text{s.t.} & x + 4y \geq 15 & & (1) \\
 & 5x + y \geq 10 & & (2) \\
 & x - y \leq 3 & & (3) \\
x, y \geq 0 & & &
\end{array}
$$

Initially, we draw the constraints and find the half-planes that represent them. Graphing the first constraint

$$x + 4y \geq 15$$

and taking also into account the nonnegativity constraints, we get the shaded region shown in Figure 2.13.

Adding the second constraint

$$5x + y \geq 10$$

into the previous graph, we now get the shaded region shown in Figure 2.14.

Adding the third constraint

$$x - y \leq 3$$

into the previous graph, we now get the shaded region shown in Figure 2.15.

We have identified the feasible region (shaded region in Figure 2.15) which satisfies all the constraints simultaneously. Now, we locate the corner points, evaluate the objective function at each point, and identify the optimum value of the objective function. Figure 2.16 shows the points of intersection of the straight

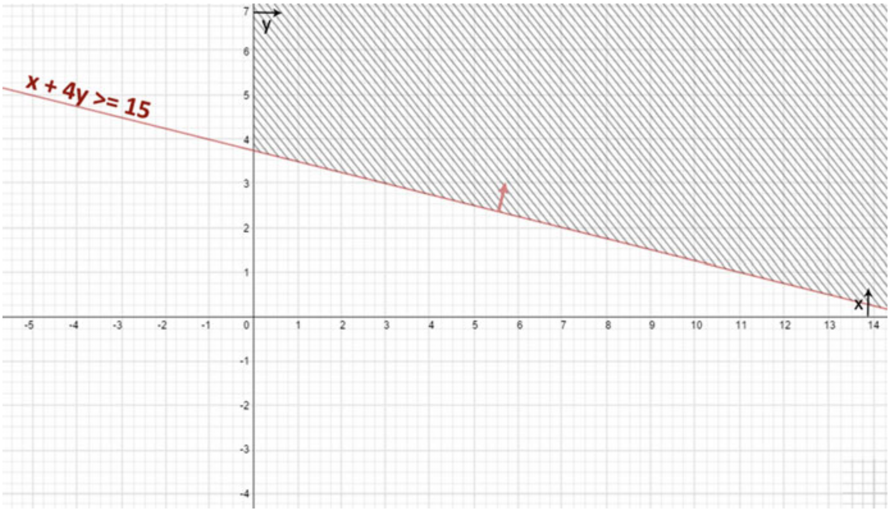


Fig. 2.13 Drawing the nonnegativity constraints and the first constraint

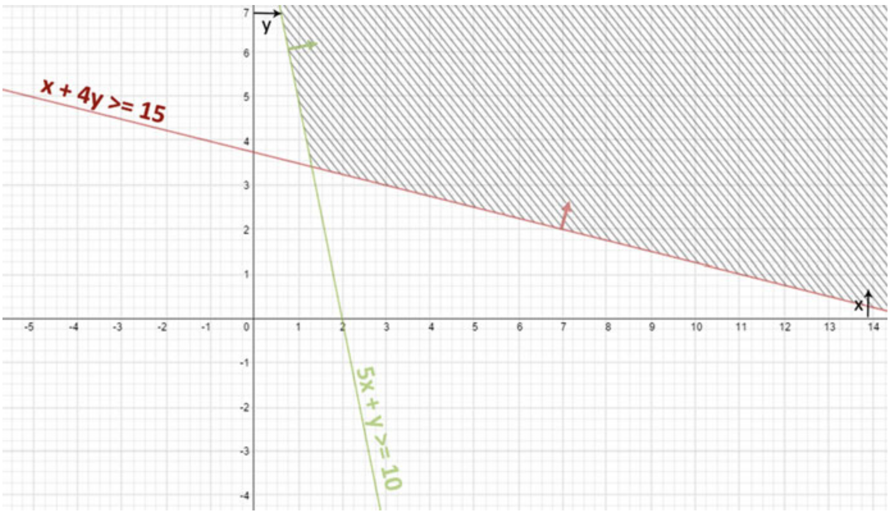


Fig. 2.14 Drawing the second constraint

lines of two constraints. Points O, A, B, F, G, H, and I are in the intersection of the straight lines of two constraints, but they are not in the feasible region. Hence, we evaluate the objective function only at points C, D, and E. Substituting the x and y values of these points to the objective function, we find the value of the objective function at each point (Table 2.2). However, the feasible region is unbounded and

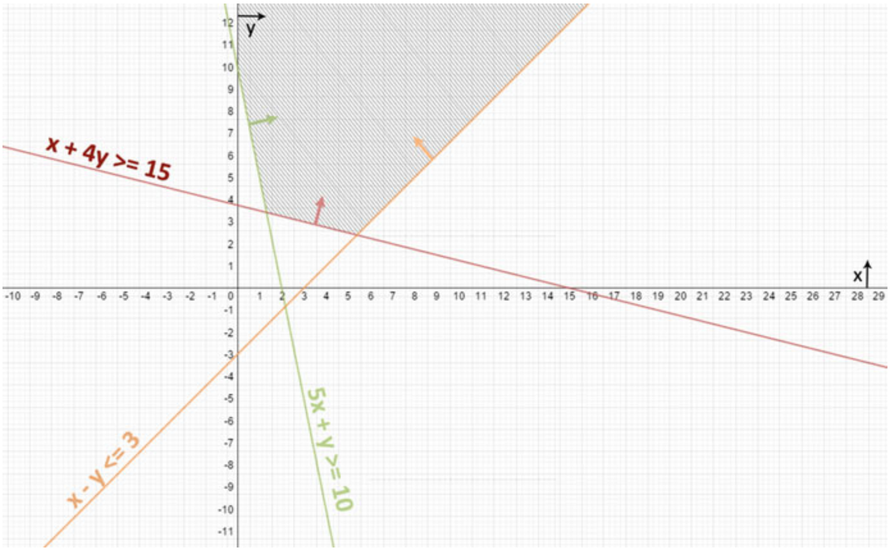


Fig. 2.15 Drawing the third constraint

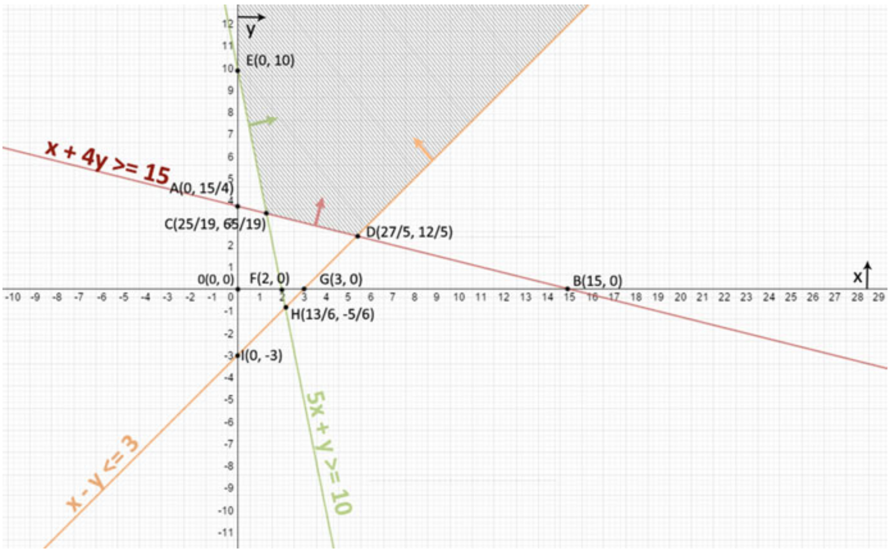


Fig. 2.16 Feasible region of the LP problem

the LP problem is a maximization problem, so we can find other points that have greater value than $z = 20$ (the value of the objective function if we substitute the x and y values of point E). Hence, the specific LP problem is unbounded.

Example 3 Let's consider the following LP problem:

Table 2.2 Evaluation of the objective function for the feasible boundary points

Point	x	y	z
C	25/19	65/19	155/19
D	27/5	12/5	51/5
E	0	10	20

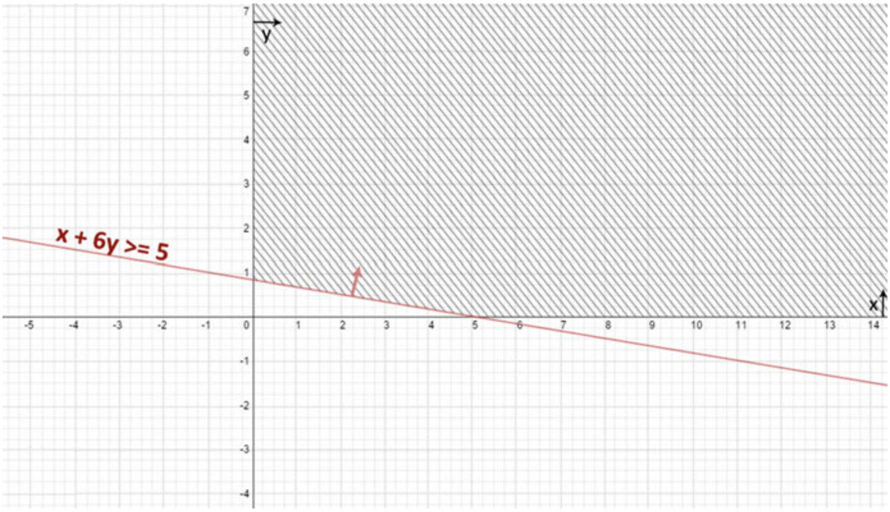


Fig. 2.17 Drawing the nonnegativity constraints and the first constraint

$$
\begin{array}{llll}
\min z = & 3x + y & & \\
\text{s.t.} & x + 6y \geq 5 & & (1) \\
 & 3x + 3y \geq 5 & & (2) \\
 & x - 2y \leq 1 & & (3) \\
x, y \geq 0 & & &
\end{array}
$$

Initially, we draw the constraints and find the half-planes that represent them. Graphing the first constraint

$$x + 6y \geq 5$$

and taking also into account the nonnegativity constraints, we get the shaded region shown in Figure 2.17.

Adding the second constraint

$$3x + 3y \geq 5$$

into the previous graph, we now get the shaded region shown in Figure 2.18.

Adding the third constraint

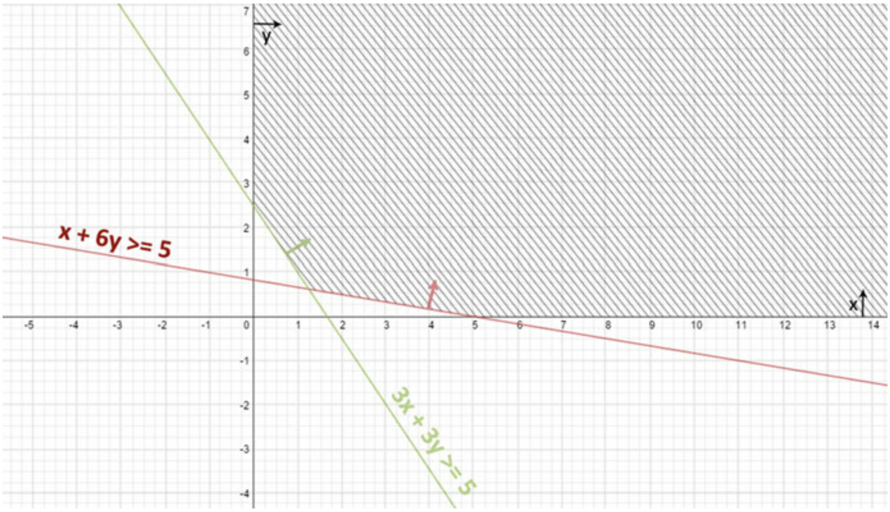


Fig. 2.18 Drawing the second constraint

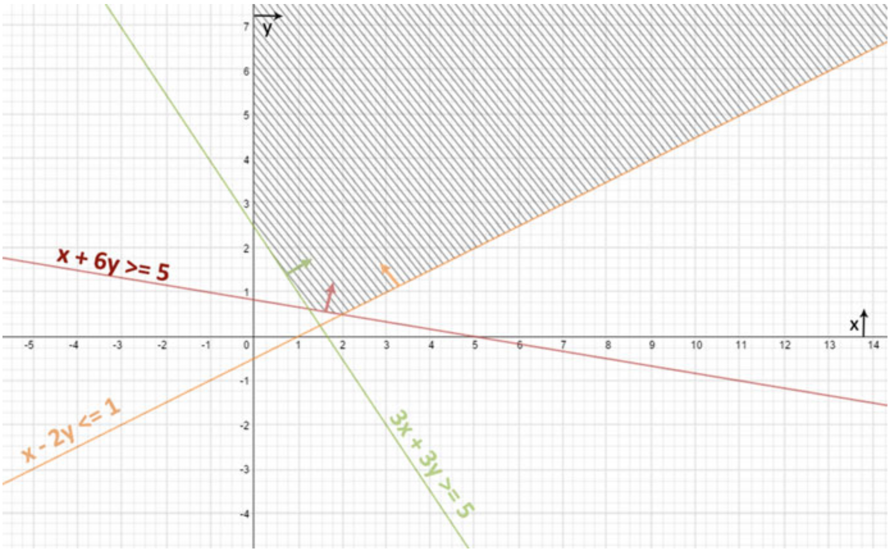


Fig. 2.19 Drawing the third constraint

$$x - 2y \leq 1$$

into the previous graph, we now get the shaded region shown in Figure 2.19.

We have identified the feasible region (shaded region in Figure 2.19) which satisfies all the constraints simultaneously. Now, we locate the corner points,

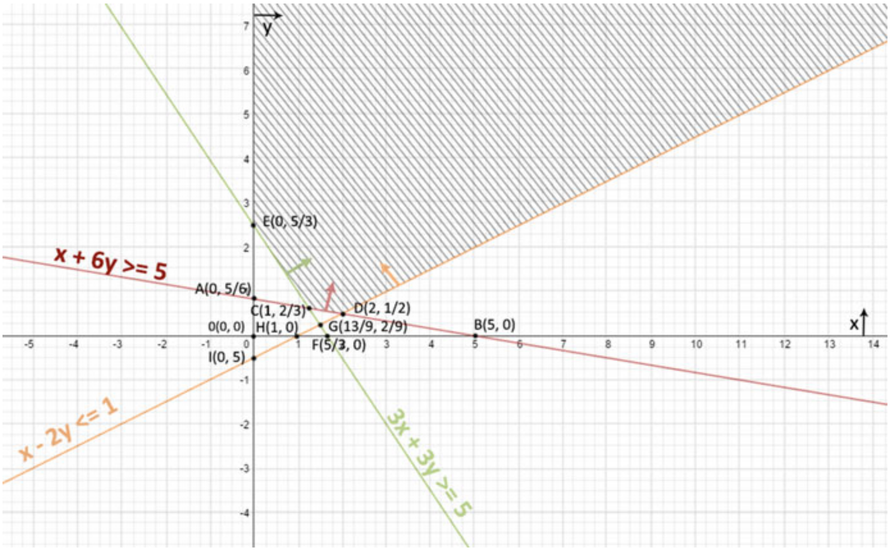


Fig. 2.20 Feasible region of the LP problem

Table 2.3 Evaluation of the objective function for the feasible boundary points

Point	x	y	z
C	1	2/3	11/3
D	2	1/2	13/2
E	0	5/3	5/3

evaluate the objective function at each point, and identify the optimum value of the objective function. Figure 2.20 shows the points of intersection of the straight lines of two constraints. Points O, A, B, F, G, H, and I are in the intersection of the straight lines of two constraints, but they are not in the feasible region. Hence, we evaluate the objective function only at points C, D, and E. Substituting the x and y values of these points to the objective function, we find the value of the objective function at each point (Table 2.3). The feasible region is unbounded, but the LP problem is a minimization problem. Hence, the minimum value is $z = 5/3$ and the optimum values of the decision variables for the LP problem are $x = 0$ and $y = 5/3$.

Example 4 Let's consider the following LP problem:

$$
\begin{array}{lrcrll}
\min z = & 2x & - & y & & \\
\text{s.t.} & x & + & 6y & \leq 5 & (1) \\
& -3x & + & y & \geq 4 & (2) \\
& 2x & - & y & \leq 3 & (3) \\
x, y \geq 0 & & & & &
\end{array}
$$

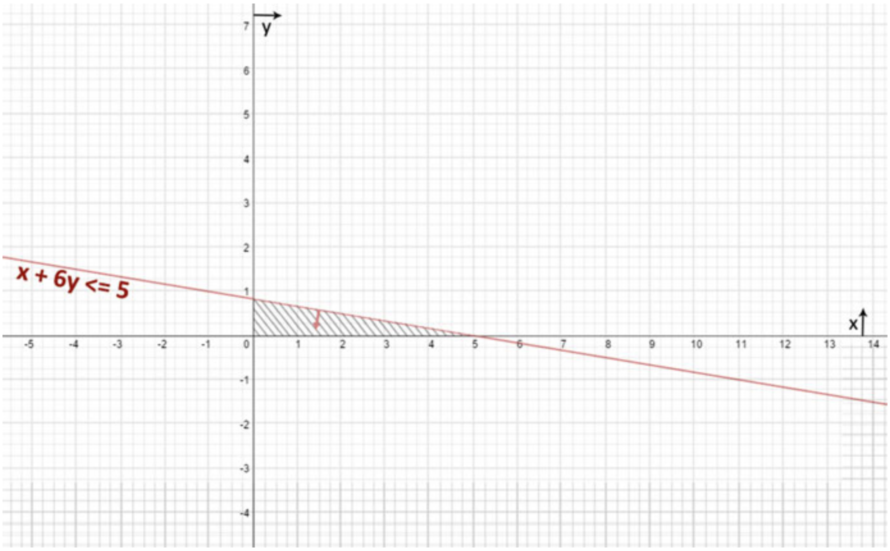


Fig. 2.21 Drawing the nonnegativity constraints and the first constraint

Initially, we draw the constraints and find the half-planes that represent them. Graphing the first constraint

$$x + 6y \leq 5$$

and taking also into account the nonnegativity constraints, we get the shaded region shown in Figure 2.21.

Adding the second constraint

$$-3x + y \geq 4$$

into the previous graph, we now see that there is not any feasible region, so the LP problem is infeasible (Figure 2.22).

For the sake of completeness, adding the third constraint

$$2x - y \leq 3$$

into the previous graph, we get the graph shown in Figure 2.23.

The LP problem is infeasible and there is no solution.

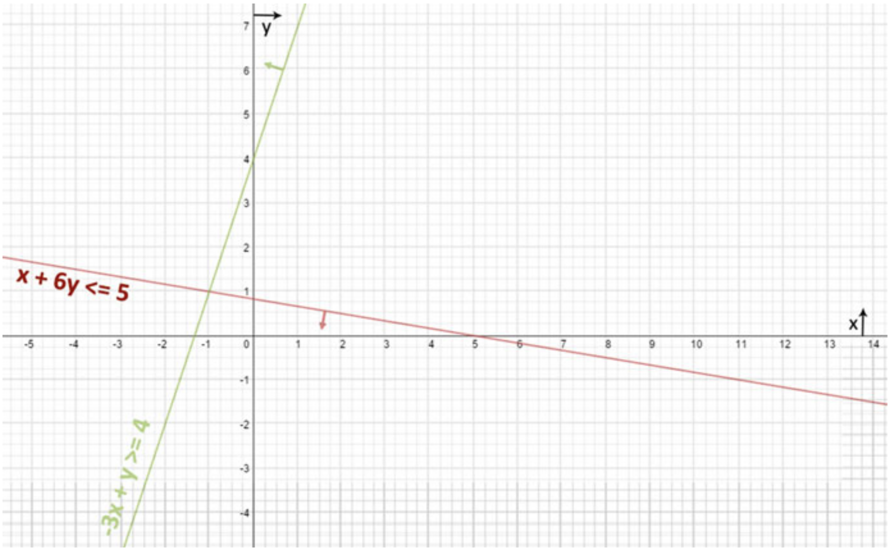


Fig. 2.22 Drawing the second constraint

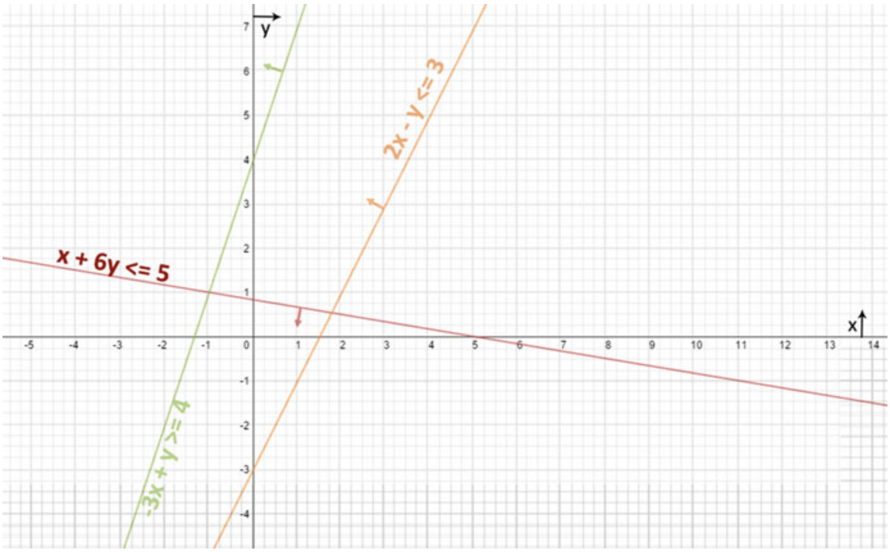


Fig. 2.23 Drawing the third constraint

2.5.4 Solve an LP Problem Graphically Using MATLAB

MATLAB's Symbolic Math Toolbox provides a powerful symbolic engine, named MuPAD. MuPAD is an optimized language for symbolic math expressions and provides an extensive set of mathematical functions and libraries. Among others, MuPAD provides the linopt library. The linopt library consists of algorithms for linear and integer programming. We use the linopt algorithm to solve LPs graphically in this subsection. However, we will use MATLAB's linprog algorithms to solve LPs algebraically in Appendix A because linprog consists of sophisticated LP algorithms. We will discuss linprog algorithms in Appendix A.

The linopt library includes the function 'linopt::plot_data' that returns a graphical representation of the feasible region of an LP problem and the contour of the objective function through the corner with the optimal objective function value found. In order to use that function, the LP problem must be a maximization problem. If the problem is a minimization problem, then we transform it to a maximization problem, as shown earlier in this chapter.

The following list provides the different syntax that can be used when calling the 'linopt::plot_data' function [3]:

- $linopt :: plot\_data([constr, obj, < NonNegative >, < seti >], vars)$
- $linopt :: plot\_data([constr, obj, < NonNegative >, < All >], vars)$
- $linopt :: plot\_data([constr, obj, < setn >, < seti >], vars)$
- $linopt :: plot\_data([constr, obj, < setn >, < All >], vars)$

The 'linopt::plot_data' function returns a graphical description of the feasible region of the LP problem $[constr, obj]$, and the contour of the objective function through the corner with the maximal objective function value found. $[constr, obj]$ is an LP problem with exactly two variables. The expression *obj* is the linear objective function to be maximized subject to the linear constraints *constr*. The second parameter *vars* specifies which variable belongs to the horizontal and vertical axis. Parameters *seti* and *setn* are sets which contain identifiers interpreted as indeterminates. Option 'All' defines that all variables are constrained to be integer and option 'NonNegative' defines that all variables are constrained to be nonnegative.

Let's solve the following LP problem using the 'linopt::plot_data' function:

$$
\begin{array}{lrrl}
\max z = & x & +\ 2y & \\
\text{s.t.} & 2x & +\ 2y \geq & 8 \\
 & 2x & +\ 5y \leq & 18 \\
 & -3x & -\ 4y \geq & -17 \\
x, y \geq 0 & & &
\end{array}
$$

To open the MuPAD Notebook, type *mupad* in the Command Window or start it from the MATLAB Apps tab (Figure 2.24).

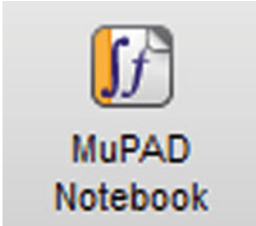


Fig. 2.24 MuPAD notebook in the MATLAB Apps tab

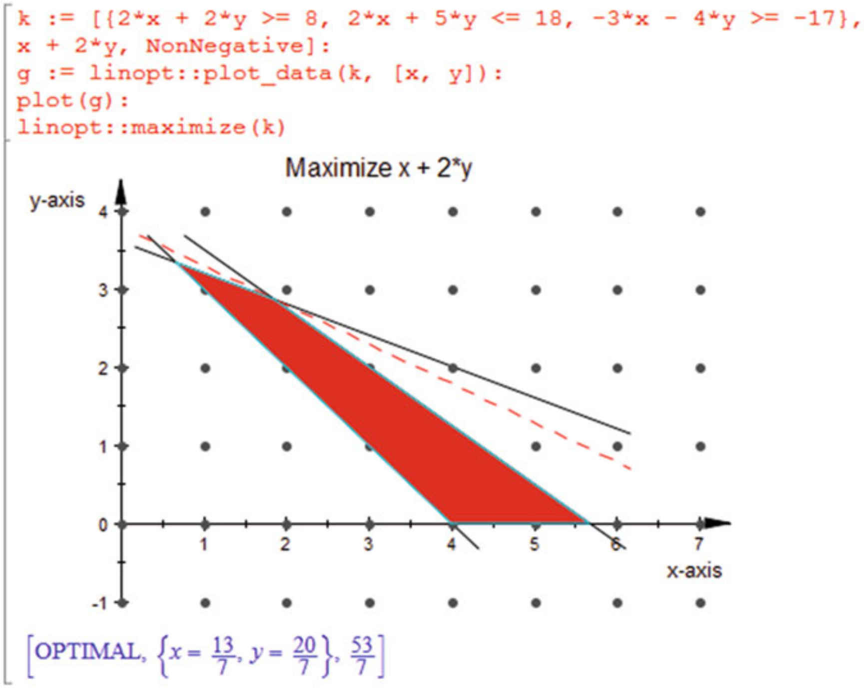


Fig. 2.25 Solve an LP problem graphically using MATLAB

Then, write the LP problem in the following form and plot it using the functions 'linopt::plot_data' and 'plot' (filename: `example1.mn`). Moreover, call 'linopt:maximize' function to obtain the optimal values of the decision variables and the value of the objective function (Figure 2.25). In this case, the LP problem is optimal with an objective value $z = \frac{53}{7}$.

```
k := [{2*x + 2*y >= 8, 2*x + 5*y <= 18, -3*x - 4*y >= -17},
      x + 2*y, NonNegative]:
g := linopt::plot_data(k, [x, y]):
plot(g):
linopt::maximize(k)
```

Let's also solve the following LP problem using the 'linopt::plot_data' function:

$$\begin{array}{llcl}
\max z = & x + 2y & & \\
\text{s.t.} & x + 4y & \geq & 15 \\
 & 5x + y & \geq & 10 \\
 & x - y & \leq & 3 \\
x, y \geq 0 & & &
\end{array}$$

Write the LP problem in the following form and plot it using the functions 'linopt::plot_data' and 'plot' (filename: `example2.mn`). Moreover, call 'linopt:maximize' function to obtain the optimal values of the decision variables and the value of the objective function (Figure 2.26). In this case, the LP problem is unbounded.

```
k := [{x + 4*y >= 15, 5*x + y >= 10, x - y <= 3},
      x + 2*y, NonNegative]:
g := linopt::plot_data(k, [x, y]):
plot(g):
linopt::maximize(k)
```

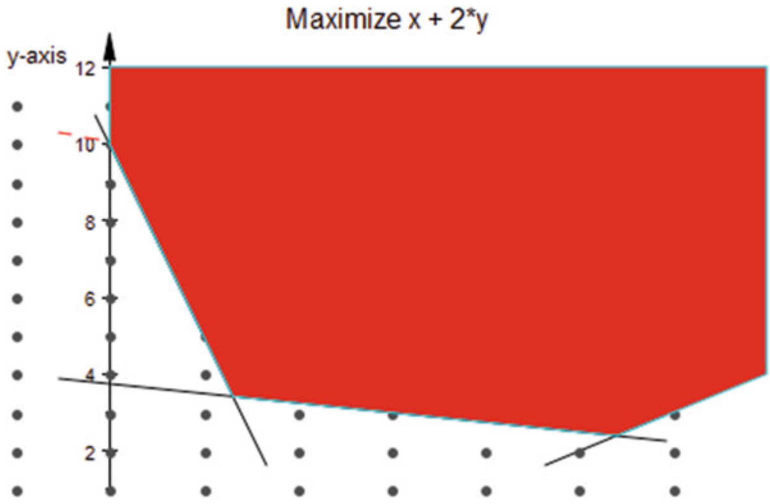
Finally, let's also solve the following LP problem using the 'linopt::plot_data' function:

$$\begin{array}{llcl}
\max z = & 2x - y & & \\
\text{s.t.} & x + 6y & \leq & 5 \\
 & -3x + y & \geq & 4 \\
 & 2x - y & \leq & 3 \\
x, y \geq 0 & & &
\end{array}$$

Write the LP problem in the following form and plot it using the functions 'linopt::plot_data' and 'plot' (filename: `example3.mn`). Moreover, call 'linopt:maximize' function to solve the LP problem (Figure 2.27). In this case, the LP problem is infeasible.

```
k := [{x + 4*y >= 15, 5*x + y >= 10, x - y <= 3},
      x + 2*y, NonNegative]:
g := linopt::plot_data(k, [x, y]):
plot(g):
linopt::maximize(k)
```

```
k := [{x + 4*y >= 15, 5*x + y >= 10, x - y <= 3},
x + 2*y, NonNegative]:
g := linopt::plot_data(k, [x, y]):
plot(g):
linopt::maximize(k)
```



[UNBOUNDED, $\{y = \text{PHI3} + 10, x = 0\}$, 2 PHI3 + 20]

Fig. 2.26 Solve an LP problem graphically using MATLAB

```
k := [{x + 6*y <= 5, -3*x + y >= 4, 2*x - y <= 3},
2*x - y, NonNegative]:
g := linopt::plot_data(k, [x, y]):
plot(g):
linopt::maximize(k)
Error: No feasible corners found. The feasible area is empty. [linopt::plot_data]
```

Fig. 2.27 Solve an LP problem graphically using MATLAB

2.6 Duality

Let's consider again the second example presented in Section 2.4 about the student's diet. The owner of the store informs his supplier that he needs at least 8 ounces of chocolate and 11 ounces of sugar to meet his client minimum requirements. He also provides the recipes that he uses to produce a brownie (4 ounces of chocolate and 3 ounces of sugar) and a scoop of chocolate ice cream (3 ounces of chocolate and 2 ounces of sugar). Moreover, he also informs the supplier that he sells each brownie for \$2 and each scoop of chocolate ice cream for \$1. The supplier now wants to determine the prices per ounce of chocolate and sugar so that he will maximize his revenue and the owner of that store will keep buying these products from him.

Let's formulate the problem that the supplier has to solve. Initially, we identify the decision variables of the problem. The price per ounce of chocolate and the price per ounce of sugar that the supplier should sell are the decision variables of this problem. Let w_1 and w_2 be the price per ounce of chocolate and sugar, respectively.

Next, we define the objective function. The supplier wants to maximize his revenue and we already know that the owner of the store will buy 8 ounces of chocolate and 11 ounces of sugar to meet his client minimum requirements. Hence, the objective function is:

$$\max z = 8w_1 + 11w_2$$

Then, we identify the technological constraints of the given problem. The cost to create a brownie should be below \$2, otherwise the owner of the shop will not buy from the supplier, because he runs the risk of making a loss if the student decides to buy brownies. Hence, the first constraint is given by

$$4w_1 + 3w_2 \leq 2$$

Similarly, the cost to create a chocolate ice cream should be below \$1, otherwise the owner of the shop will not buy the raw ingredients from the supplier, because he runs the risk of making a loss if the student decides to buy chocolate ice cream. Hence, the second constraint is given by

$$3w_1 + 2w_2 \leq 1$$

Moreover, we also add the nonnegativity constraints for variables w_1 and w_2. Hence, the LP problem is the following:

$$\begin{array}{ll}\max z = & 8w_1 + 11w_2 \\ \text{s.t.} & 4w_1 + \ 3w_2 \ \leq 2 \\ & 3w_1 + \ 2w_2 \ \leq 1 \\ \multicolumn{2}{c}{w_1 \geq 0, w_2 \geq 0, \{w_1, w_2\} \in \mathbb{R}}\end{array}$$

Let's take a closer look in the LP problem we formulated in Section 2.4 and the LP problem we formulated in this section.

$$\begin{array}{llll}\min z = & 2x_1 + \ x_2 & \max z = & 8w_1 + 11w_2 \\ \text{s.t.} & 4x_1 + 3x_2 \geq \ 8 & \text{s.t.} & 4w_1 + \ 3w_2 \ \leq 2 \\ & 3x_1 + 2x_2 \geq 11 & & 3w_1 + \ 2w_2 \ \leq 1 \\ \multicolumn{2}{l}{x_1 \geq 0, x_2 \geq 0, \{x_1, x_2\} \in \mathbb{R}} & \multicolumn{2}{l}{w_1 \geq 0, w_2 \geq 0, \{w_1, w_2\} \in \mathbb{R}}\end{array}$$

The first problem (student's problem) is a minimization problem, while the second problem (supplier's problem) is a maximization problem. The coefficients of the objective function in the first problem are the same with the right-hand side in the second problem. The coefficients of the objective function in the second problem are the same with the right-hand side in the first problem. Moreover, the matrix of coefficients in the second problem is the transpose of the matrix of coefficients of the first problem. All these findings show that these two problems have a relation. Indeed, the second problem is the dual of the first problem. The first problem is called primal.

It turns out that LPs come in pairs. We can automatically derive two LPs from the same data. The relation of these problems is close. If the primal LP problem refers to the employer, then the dual LP problem refers to the employees. If the primal LP problem refers to the clients of a store (student's problem), then the dual LP problem refers to the suppliers of a store (supplier's problem).

Generalizing the previous findings, for a specific LP problem

$$\begin{aligned} \min\ z &= c^T x \\ \text{s.t.}\ \ Ax &\geq b \\ x &\geq 0 \end{aligned} \tag{2.26}$$

We can find its dual problem in its canonical form

$$\begin{aligned} \max\ z &= b^T w \\ \text{s.t.}\ \ A^T w &\leq c \\ w &\geq 0 \end{aligned} \tag{2.27}$$

Transforming an LP problem in the form shown in Equation (2.26), we can find its dual LP problem. For example, let's consider the following LP problem:

$$\begin{array}{lrcrcrcr} \min z = & 3x_1 & - & 2x_2 & + & 2x_3 & & \\ \text{s.t.} & -2x_1 & + & 3x_2 & + & 5x_3 & \geq & 10 \\ & -x_1 & - & 2x_2 & + & 2x_3 & \geq & -5 \\ \multicolumn{8}{l}{x_1 \geq 0, x_2 \geq 0, x_3 \geq 0} \end{array}$$

The dual of the previous LP problem is the following:

$$\begin{array}{lrcrcr} \max z = & 10w_1 & - & 5w_2 & & \\ \text{s.t.} & -2w_1 & - & w_2 & \leq & 3 \\ & 3w_1 & - & 2w_2 & \leq & -2 \\ & 5w_1 & + & 2w_2 & \leq & 2 \\ \multicolumn{6}{l}{w_1 \geq 0, w_2 \geq 0} \end{array}$$

The aforementioned procedure to transform a primal LP problem to its dual has a major drawback. The primal LP problem must be in the form shown in

Equation (2.26) or otherwise we convert it to that form. Let's describe a more general procedure to find the dual LP problem without the need to perform any transformations to the primal problem. The following steps will help us to find the dual LP problem:

- **Optimization type**: if the primal LP problem is a minimization problem, then the dual LP problem is a maximization problem. Similarly, if the primal LP problem is a maximization problem, then the dual LP problem is a minimization problem.
- **Decision variables**: the number of the decision variables of the dual LP problem is the same with the number of the constraints of the primal LP problem. The ith variable of the dual LP problem is related to the ith constraint of the primal LP problem. Hence:
  - if the ith constraint of the primal LP problem is an equality, then the ith variable of the dual LP problem is a free variable.
  - if the ith constraint of the primal LP problem is in the form 'greater than or equal to' ($\geq$), then the ith variable of the dual LP problem is greater than or equal to zero (nonnegativity constraint).
  - if the ith constraint of the primal LP problem is in the form 'less than or equal to' ($\leq$), then the ith variable of the dual LP problem is less than or equal to zero (upper bound).
- **Coefficients of the objective function**: The coefficients of the objective function of the dual LP problem are the right-hand side values of the primal LP problem.
- **Constraints**: the number of the constraints of the dual LP problem is the same with the number of the decision variables of the primal LP problem. The ith constraint of the dual LP problem is related to the ith variable of the primal LP problem. Hence:
  - if the ith variable (x_1) of the primal LP problem is a free variable, then the ith constraint of the dual LP problem is an equality.
  - if the ith variable of the primal LP problem is greater than or equal to zero (nonnegativity constraint), then the ith constraint of the dual LP problem is in the form of 'less than or equal to' ($\leq$).
  - if the ith variable of the primal LP problem is less than or equal to zero, then the ith constraint of the dual LP problem is in the form of 'greater than or equal to' ($\geq$).
- **Coefficient matrix**: The coefficient matrix of the dual LP problem is the transpose of the coefficient matrix of the primal LP problem.
- **Right-hand side values**: The right-hand side values of the dual LP problem are the coefficients of the objective function of the primal LP problem.

Table 2.4 summarizes the above procedure.

So, let's follow the aforementioned steps to find the dual LP problem of the following LP problem:

Table 2.4 Primal to dual transformations

min		$\leftrightarrow$	max	
Constraint	$=$	$\leftrightarrow$	Variable	Free
Constraint	$\geq$	$\leftrightarrow$	Variable	≥ 0
Constraint	$\leq$	$\leftrightarrow$	Variable	≤ 0
Variable	free	$\leftrightarrow$	Constraint	$=$
Variable	≥ 0	$\leftrightarrow$	Constraint	$\leq$
variable	≤ 0	$\leftrightarrow$	Constraint	$\geq$

$$\begin{array}{llr}
\max z = & 2x_1 + 4x_2 & \\
\text{s.t.} & -3x_1 + 4x_2 - 3x_3 = -3 & (1) \\
& x_1 - 2x_2 + 4x_3 \geq 4 & (2) \\
& 2x_1 - x_2 - 2x_3 \leq 6 & (3) \\
x_2 \geq 0, x_3 \leq 0 & &
\end{array}$$

Since the primal LP problem is a maximization problem, the dual LP problem will be a minimization problem. The primal LP problem has 3 constraints, so the dual LP problem will have 3 decision variables. The first constraint of the primal LP problem is an equality, so the first variable of the dual LP problem will be a free variable. The second constraint of the primal LP is in the form of ‘greater than or equal to’ ($\geq$), so the second variable of the dual LP problem will be greater than or equal to zero. The third constraint of the primal LP problem is in the form of ‘less than or equal to’ ($\leq$), so the third variable of the dual LP problem will be less than or equal to zero. Moreover, the coefficients of the objective function of the dual LP problem will be the same as the right-hand side values of the primal LP problem. Hence, the objective function of the dual LP problem is:

$$\min z = -3w_1 + 4w_2 + 6w_3$$

The primal LP problem has 3 variables, so the dual LP problem will have 3 constraints. The first variable (x_1) of the primal LP problem is a free variable, so the first constraint of the dual LP problem will be an equality. The second variable (x_2) of the primal LP problem is greater than or equal to zero, so the second constraint of the dual LP problem will be in the form of ‘less than or equal to’ ($\leq$). The third variable (x_3) of the primal LP problem is less than or equal to zero, so the third constraint of the dual LP problem will be in the form of ‘greater than or equal to’ ($\geq$). The coefficient matrix of the dual LP problem will be the transpose of the coefficient matrix of the primal LP problem. The right-hand side values of the dual LP problem are the coefficients of the objective function of the primal LP problem. Hence, the constraints of the dual LP problem are:

$$\begin{aligned} -3w_1 + w_2 + 2w_3 &= 2 \\ 4w_1 - 2w_2 - w_3 &\le 4 \\ -3w_1 + 4w_2 - 2w_3 &\ge 0 \end{aligned}$$

So, the dual LP problem is the following:

$$\begin{aligned} \min z = -3w_1 + 4w_2 + 6w_3 & \\ -3w_1 + w_2 + 2w_3 &= 2 \\ 4w_1 - 2w_2 - w_3 &\le 4 \\ -3w_1 + 4w_2 - 2w_3 &\ge 0 \\ w_2 \ge 0, w_3 \le 0 & \end{aligned}$$

A code that transforms a primal LP problem to its dual is presented (filename: `primal2dual.m`). Some necessary notations should be introduced before the presentation of this code. Let A be a $m \times n$ matrix with the coefficients of the constraints of the primal LP problem, c be a $n \times 1$ vector with the coefficients of the objective function of the primal LP problem, b be a $m \times 1$ vector with the right-hand side of the constraints of the primal LP problem, *Eqin* be a $m \times 1$ vector with the type of the constraints of the primal LP problem, *MinMaxLP* be a variable denoting the type of optimization of the primal LP problem (-1 for minimization and 1 for maximization), *VarConst* be a $n \times 1$ vector of the variables' constraints of the primal LP problem (0—free variable, 1—variable ≤ 0, 2—variable ≥ 0), *DA* be a $n \times m$ matrix with the coefficients of the constraints of the dual LP problem, *Dc* be a $m \times 1$ vector with the coefficients of the objective function of the dual LP problem, *Db* be a $n \times 1$ vector with the right-hand side of the constraints of the dual LP problem, *DEqin* be a $n \times 1$ vector with the type of the constraints of the dual LP problem, *DMinMaxLP* be a variable denoting the type of optimization of the dual LP problem (-1 for minimization and 1 for maximization), and *DVarConst* be a $m \times 1$ vector of the variables' constraints of the dual LP problem (0—free variable, 1—variable ≤ 0, 2—variable ≥ 0).

The function takes as input the matrix of coefficients of the constraints of the primal LP problem (matrix A), the vector of the coefficients of the objective function of the primal LP problem (vector c), the vector of the right-hand side of the constraints of the primal LP problem (vector b), the vector of the type of the constraints of the primal LP problem (vector *Eqin*), a variable denoting the type of optimization of the primal LP problem (variable *MinMaxLP*), and the vector of the variables' constraints of the primal LP problem (vector *VarConst*), and returns as output the matrix of coefficients of the constraints of the dual LP problem (matrix *DA*), the vector of the coefficients of the objective function of the dual LP problem (vector *Dc*), the vector of the right-hand side of the constraints of the dual LP problem (vector *Db*), the vector of the type of the constraints of the dual LP problem (vector *DEqin*), a variable denoting the type of optimization of the dual LP problem (variable *DMinMaxLP*), and the vector of the variables' constraints of the dual LP problem (vector *DVarConst*).

If the primal LP problem is a minimization problem, then the dual LP problem is a maximization one and vice versa (line 43). The coefficients of the objective function of the dual LP problem are the right-hand side values of the primal LP problem (line 47). The right-hand side values of the dual LP problem are the coefficients of the objective function of the primal LP problem (line 51). The coefficient matrix of the dual LP problem is the transpose of the coefficient matrix of the primal LP problem (line 55). Then, we calculate the vector of the type of the constraints (lines 58–72). Finally, we calculate the vector of the variables' constraints (lines 74–88).

```
1.    function [DA, Dc, Db, DEqin, DMinMaxLP, DVarConst] = ...
2.        primal2dual(A, c, b, Eqin, MinMaxLP, VarConst)
3.    % Filename: primal2dual.m
4.    % Description: the function is an implementation of the
5.    % transformation of a primal LP problem to its dual
6.    % Authors: Ploskas, N., & Samaras, N.
7.    %
8.    % Syntax: [DA, Dc, Db, DEqin, DMinMaxLP, DVarConst] = ...
9.    %   primal2dual(A, c, b, Eqin, MinMaxLP, VarConst)
10.   %
11.   % Input:
12.   % -- A: matrix of coefficients of the constraints of the
13.   %    primal LP problem (size m x n)
14.   % -- c: vector of coefficients of the objective function
15.   %    of the primal LP problem (size n x 1)
16.   % -- b: vector of the right-hand side of the constraints
17.   %    of the primal LP problem (size m x 1)
18.   % -- Eqin: vector of the type of the constraints of the
19.   %    primal LP problem (size m x 1)
20.   % -- MinMaxLP: the type of optimization of the primal
21.   %    LP problem
22.   % -- VarConst: vector of the variables' constraints of the
23.   %    primal LP problem (0 - free variable, 1 - variable
24.   %    <= 0, 2 - variable >= 0) (size n x 1)
25.   %
26.   % Output:
27.   % -- DA: matrix of coefficients of the constraints of the
28.   %    dual LP problem (size n x m)
29.   % -- Dc: vector of coefficients of the objective function
30.   %    of the dual LP problem (size m x 1)
31.   % -- Db: vector of the right-hand side of the constraints
32.   %    of the dual LP problem (size n x 1)
33.   % -- DEqin: vector of the type of the constraints of the
34.   %    dual LP problem (size n x 1)
35.   % -- DMinMaxLP: the type of optimization of the dual
36.   %    LP problem
37.   % -- DVarConst: vector of the variables' constraints of the
38.   %    dual LP problem (0 - free variable, 1 - variable
39.   %    <= 0, 2 - variable >= 0) (size m x 1)
40.
41.   % if the primal is a minimization problem, then the
42.   % dual is a maximization one and vice versa
```

```
DMinMaxLP = -MinMaxLP;
% the coefficients of the objective function of the
% dual LP problem are the right-hand side values of
% the primal LP problem
Dc = b;
% the right-hand side values of the dual LP problem
% are the coefficients of the objective function of the
% primal LP problem
Db = c;
% the coefficient matrix of the dual LP problem is the
% transpose of the coefficient matrix of the primal LP
% problem
DA = A';
[m, n] = size(A); % size of matrix A
% find the vector of the type of the constraints
DEqin = zeros(n, 1);
for i = 1:n
    % free variable -> equality constraint
    if VarConst(i) == 0
        DEqin(i) = 0;
    % variable <= 0 -> 'greater than or equal to'
    % constraint
    elseif VarConst(i) == 1
        DEqin(i) = 1;
    % variable >= 0 -> 'less than or equal to'
    % constraint
    elseif VarConst(i) == 2
        DEqin(i) = -1;
    end
end
% find the vector of variables' constraints
DVarConst = zeros(m, 1);
for j = 1:m
    % equality constraint -> free variable
    if Eqin(j) == 0
        DVarConst(j) = 0;
    % 'less than or equal to' constraint ->
    % variable <= 0
    elseif Eqin(j) == -1
        DVarConst(j) = 1;
    % 'greater than or equal to' constraint ->
    % variable >= 0
    elseif Eqin(j) == 1
        DVarConst(j) = 2;
    end
end
end
```

As we already know, an LP problem can be infeasible, unbounded, or have a finite optimal solution or infinite optimal solutions. Suppose that x is a feasible solution of the primal and w is a feasible solution of the dual. Then, $Ax \leq b$, $w^T A \geq c^T$, $x \geq 0$, and $w \geq 0$. It follows that $w^T Ax \geq c^T x$ and $w^T Ax \leq w^T b$. Hence, $c^T x \leq b^T w$. This theorem is known as the weak duality theorem and we can conclude that if x

Table 2.5 Relationship between the primal and dual LPs

Primal \ Dual	Finite optimum	Unbounded	Infeasible
Finite optimum	Yes; values equal	Impossible	Impossible
Unbounded	Impossible	Impossible	Possible
Infeasible	Impossible	Possible	Possible

and w are feasible solutions of the primal and dual problems and $c^T x = b^T w$, then x and w must be optimal solutions to the primal and dual LP problems, respectively. However, this does not imply that there are feasible solutions x and w such that $c^T x = b^T w$.

The strong duality theorem guarantees that the dual has an optimal solution if and only if the primal does. If x and w are optimal solutions to the primal and dual LP problems, then $c^T x = b^T w$. As a result, the objective values of the primal and dual LP problems are the same. Moreover, the primal and dual LP problems are related. We can conclude to the following relationship between the primal and dual LP problems (Table 2.5):

- If the primal LP problem has a finite solution, then the dual LP problem has also a solution. The values of the objective function of the primal and the dual LPs are equal.
- If the primal LP problem is unbounded, then the dual LP problem is infeasible.
- If the primal LP problem is infeasible, then the dual LP problem may be either unbounded or infeasible.

Similarly:

- If the dual LP problem has a finite solution, then the primal LP problem has also a solution. The values of the objective function of the primal and the dual LPs are equal.
- If the dual LP problem is unbounded, then the primal LP problem is infeasible.
- If the dual LP problem is infeasible, then the primal LP problem may be either unbounded or infeasible.

The duality theorem implies a relationship between the primal and dual LPs that is known as complementary slackness. As you already know, the number of the variables in the dual LP problem is equal to the number of the constraints in the primal LP problem. Moreover, the number of the constraints in the dual LP problem is equal to the number of the variables in the primal LP problem. This relationship suggests that the variables in one problem are complementary to the constraints in the other. A constraint is binding if changing it also changes the optimal solution. A constraint has slack if it is not binding. An inequality constraint has slack if the slack variable is positive.

Assume that the primal LP problem has a solution x and the dual LP problem has a solution w. The complementary slackness theorem states that:

- If $x_j > 0$, then the jth constraint in the dual LP problem is binding.
- If the jth constraint in the dual LP problem is not binding, then $x_j = 0$.
- If $w_i > 0$, then the ith constraint in the primal LP problem is binding.
- If the ith constraint in the primal LP problem is not binding, then $w_i = 0$.

The complementary slackness theorem is useful, because it helps us to interpret the dual variables. If we already know the solution to the primal LP problem, then we can easily find the solution to the dual LP problem.

2.7 Linear Programming Algorithms

As already discussed in Chapter 1, there are two type of LP algorithms: (i) simplex-type algorithms, and (ii) interior point algorithms. The most widely used simplex-type algorithm is the revised simplex algorithm proposed by Dantzig [1]. The revised simplex algorithm begins with a feasible basis and uses pricing operations until an optimum solution is computed. The revised simplex algorithm moves from extreme point to extreme point on the boundary of the feasible region. In addition, the revised dual simplex algorithm, which was initially proposed by Lemke [2], is another alternative for solving LPs. The revised dual simplex algorithm begins with a dual feasible basis and uses pricing operations until an optimum solution is computed. Another efficient simplex-type algorithm is the exterior point simplex algorithm that was proposed by Paparrizos [5, 6]. The main idea of the exterior point simplex algorithm is that it moves in the exterior of the feasible region and constructs basic infeasible solutions instead of feasible solutions calculated by the simplex algorithm. This book describes in detail these three algorithms. Moreover, we also present another family of LP algorithms, the interior point algorithms. Interior point algorithms traverse across the interior of the feasible region.

We use four LP algorithms in the following chapters of this book:

- Revised primal simplex algorithm (presented in detail in Chapter 8).
- Revised dual simplex algorithm (presented in detail in Chapter 9).
- Exterior point simplex algorithm (presented in detail in Chapter 10).
- Mehrotra's predictor-corrector method (presented in detail in Chapter 11).

A brief description of these algorithms follows.

2.7.1 Revised Primal Simplex Algorithm

A formal description of the revised primal simplex algorithm is given in Table 2.6 and a flow diagram of its major steps in Figure 2.28. Initially, the LP problem is presolved. Presolve methods are important in solving LPs, as they reduce LPs' size and discover whether the problem is unbounded or infeasible. Presolve methods are

Table 2.6 Revised primal simplex algorithm

Step 0. *(Initialization).*
Presolve the LP problem.
Scale the LP problem.
Select an initial basic solution (B, N).
if the initial basic solution is feasible then proceed to step 2.
Step 1. *(Phase I).*
Construct an auxiliary problem by adding an artificial variable x_{n+1} with a coefficient vector equal to $-A_B e$, where $e = (1, 1, \cdots, 1)^T \in \mathbb{R}^m$.
Apply the revised simplex algorithm. If the final basic solution (B, N) is feasible, then proceed to step 2 in order to solve the initial problem. The auxiliary LP problem can be either optimal or infeasible.
Step 2. *(Phase II).*
Step 2.0. *(Initialization).*
Compute $(A_B)^{-1}$ and vectors x_B, w, and s_N.
Step 2.1. *(Test of Optimality).*
if $s_N \geq 0$ then STOP. The LP problem is optimal.
else
  Choose the index l of the entering variable using a pivoting rule.
  Variable x_l enters the basis.
Step 2.2. *(Pivoting).*
Compute the pivot column $h_l = (A_B)^{-1} A_l$.
if $h_l \leq 0$ then STOP. The LP problem is unbounded.
else
Choose the leaving variable $x_{B[r]} = x_k$ using the following relation:

$$x_k = x_{B[r]} = \frac{x_{B[r]}}{h_{il}} = \min\left\{\frac{x_{B[i]}}{h_{il}} : h_{il} > 0\right\}$$

Step 2.3. *(Update).*
Swap indices k and l. Update the new basis inverse $\left(A_{\overline{B}}\right)^{-1}$, using a basis update scheme. Update vectors x_B, w, and s_N.
Go to Step 2.1.

used prior to the application of an LP algorithm in order to: (i) eliminate redundant constraints, (ii) fix variables, (iii) transform bounds of single structural variables, and (iv) reduce the number of variables and constraints by eliminations. A detailed analysis of presolve methods is presented in Chapter 4.

Next, the LP problem is scaled. Scaling is used prior to the application of an LP algorithm in order to: (i) produce a compact representation of the variable bounds, (ii) reduce the condition number of the constraint matrix, (iii) improve the numerical behavior of the algorithms, (iv) reduce the number of iterations required to solve LPs, and (v) simplify the setup of the tolerances. A detailed analysis of scaling techniques is presented in Chapter 5.

Then, we calculate an initial basis. If the basis is feasible then we proceed to apply the revised primal simplex algorithm to the original problem (Phase II). If the

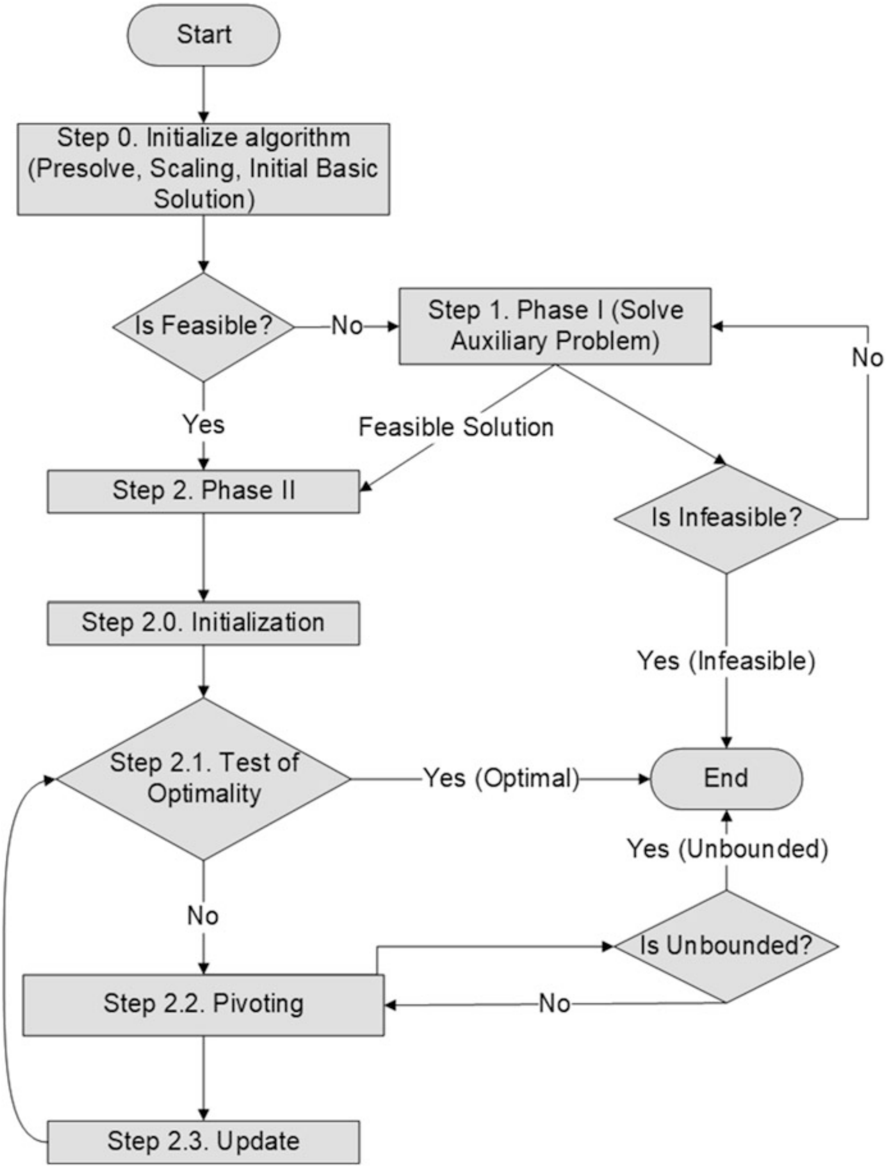


Fig. 2.28 Revised primal simplex algorithm

initial basis is not feasible, then we construct an auxiliary problem to find an initial basis (Phase I). In Phase I, it is possible to find out that the LP problem is infeasible. More details about this step will be given in Chapter 8.

Next, we perform the test of optimality to identify if the current basis is optimal and terminate the algorithm. If not, then we apply a pivoting rule to select the entering variable. The choice of the pivot element at each iteration is one of the most critical steps in simplex-type algorithms. A detailed analysis of pivoting rules is presented in Chapter 6.

Then, we perform the pivoting step where we select the leaving variable. At this step, it is possible to identify that the LP problem is unbounded. More details about this step will be given in Chapter 8.

Next, we update the appropriate variables. At this step, we need to update the basis inverse using a basis update scheme. A detailed analysis of basis update schemes is presented in Chapter 7. Finally, we continue with the next iteration of the algorithm.

Initially, we will present the different methods available for each step of the algorithm in Chapters 4–7 (presolve methods, scaling techniques, pivoting rules, and basis update schemes) and then present the whole algorithm in Chapter 8.

2.7.2 Revised Dual Simplex Algorithm

A formal description of the revised dual simplex algorithm is given in Table 2.7 and a flow diagram of its major steps in Figure 2.29. Initially, the LP problem is presolved. Next, the LP problem is scaled. Then, we calculate an initial basis. If the basis is dual feasible then we proceed to apply the revised dual simplex algorithm to the original problem. If the initial basis is not dual feasible, then we construct an auxiliary problem using the big-M method and solve this problem with the revised dual simplex algorithm. More details about this step will be given in Chapter 9.

At each iteration of the revised dual simplex algorithm, we perform the test of optimality to identify if the current basis is optimal and terminate the algorithm. If not, then we select the leaving variable. Then, we perform the pivoting step where we select the entering variable using the minimum ratio test. At this step, it is possible to identify that the LP problem is infeasible. More details about this step will be given in Chapter 9.

Next, we update the appropriate variables. At this step, we need to update the basis inverse using a basis update scheme. Finally, we continue with the next iteration of the algorithm.

Initially, we will present the different methods available for each step of the algorithm in Chapters 4, 5, and 7 (presolve methods, scaling techniques, and basis update schemes) and then present the whole algorithm in Chapter 9.

2.7.3 Exterior Point Simplex Algorithm

A formal description of the exterior point simplex algorithm is given in Table 2.8 and a flow diagram of its major steps in Figure 2.30. Similar to the revised simplex algorithm, the LP problem is presolved and scaled.

Table 2.7 Revised dual simplex algorithm

Step 0. *(Initialization).*
Presolve the LP problem.
Scale the LP problem.
Select an initial basic solution (B, N).
if the initial basic solution is dual feasible then proceed to step 2.
Step 1. *(Dual Algorithm with big-M Method).*
Add an artificial constraint $e^T x_N + x_{n+1} = M$, $x_{n+1} \geq 0$. Construct the vector of the coefficients of M in the right-hand side, $\bar{\bar{b}}$.
Set $s_p = \min\left\{s_j : j \in N\right\}$, $\bar{B} = B \cup p$, and $\bar{N} = N \cup \{n+1\}$.
Find $\bar{x}_B$, $\bar{\bar{x}}_B$, w, and s_N. Now, apply the algorithm of Step 2
to the modified big-M problem. The original LP problem can be either
optimal or infeasible or unbounded.
Step 2. *(Dual Simplex Algorithm).*
Step 2.0. *(Initialization).*
Compute $(A_B)^{-1}$ and vectors x_B, w, and s_N.
Step 2.1. *(Test of Optimality).*
if $x_B \geq 0$ then STOP. The primal LP problem is optimal.
else
  Choose the leaving variable k such that $x_{B[r]} = x_k = \min\left\{x_{B[i]} : x_{B[i]} \leq 0\right\}$.
  Variable x_k leaves the basis.
Step 2.2. *(Pivoting).*
Compute vector $H_{rN} = \left(A_B^{-1}\right)_{r.} A_N$.
if $H_{rN} \geq 0$ then STOP. The primal LP problem is infeasible.
else
  Choose the entering variable $x_{N[t]} = x_l$ using the minimum ratio test:
$$x_l = x_{N[t]} = \frac{-s_{N[t]}}{H_{rN}} = \min\left\{\frac{-s_{N[i]}}{H_{iN}} : H_{iN} < 0\right\}$$
Step 2.3. *(Update).*
Swap indices k and l. Update the new basis inverse $\left(A_{\overline{B}}\right)^{-1}$, using a basis
update scheme. Update vectors x_B, w, and s_N.
Go to Step 2.1.

Then, we calculate an initial basis. If $P \neq \emptyset$ and the improving direction crosses the feasible region then we proceed to apply the exterior point simplex algorithm to the original problem (Phase II). Otherwise, we construct an auxiliary problem and apply the revised simplex algorithm to find an initial basis (Phase I). In Phase I, it is possible to find out that the LP problem is infeasible. More details about this step will be given in Chapter 10.

Next, we perform the test of optimality to identify if the current basis is optimal and terminate the algorithm. If not, then we choose the leaving variable. At this step, it is possible to identify that the LP problem is unbounded. Then, we perform the pivoting step to find the entering variable. More details about these steps will be given in Chapter 10.

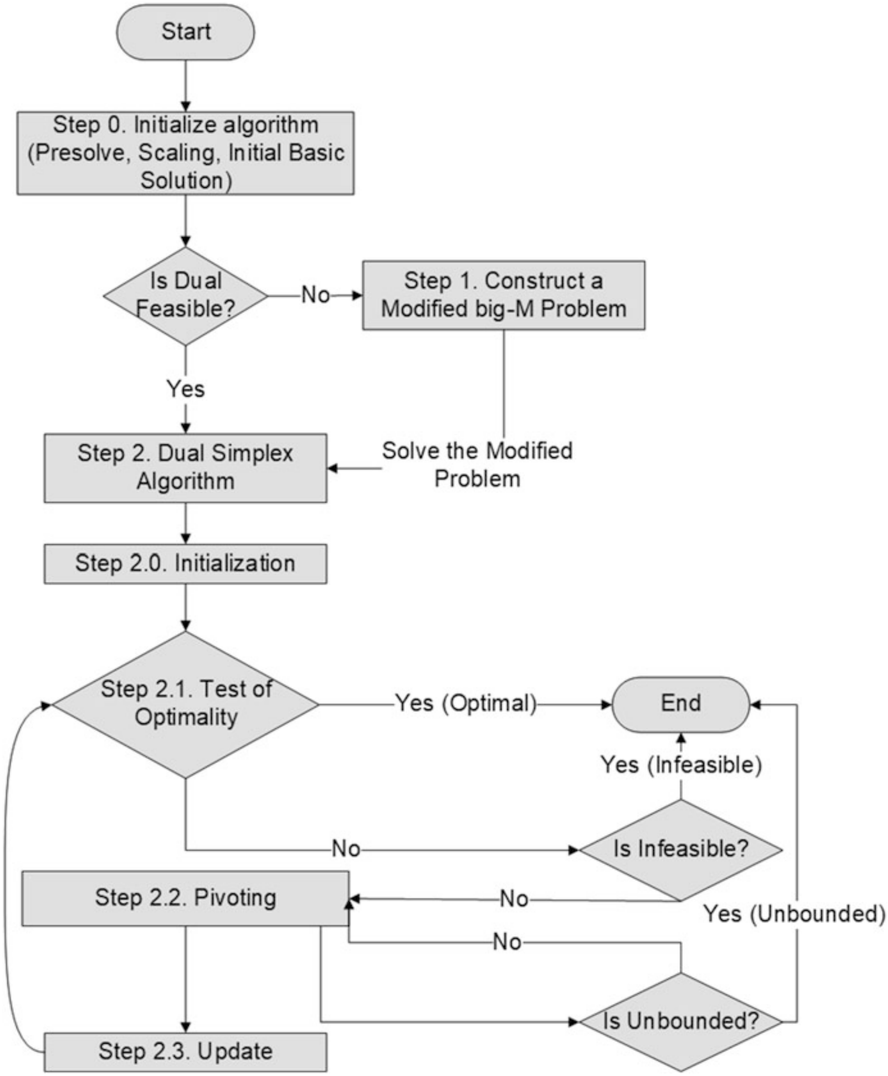


Fig. 2.29 Revised dual simplex algorithm

Next, we update the appropriate variables. At this step, we need to update the basis inverse using a basis update scheme. Finally, we continue with the next iteration of the algorithm.

Initially, we will present the different methods available for each step of the algorithm in Chapters 4, 5, and 7 (presolve methods, scaling techniques and basis update schemes) and then present the whole algorithm in Chapter 10.

Table 2.8 Exterior point simplex algorithm

Step 0. *(Initialization).*
Presolve the LP problem.
Scale the LP problem.
Select an initial basic solution (B, N).
Find the set of indices $P = \{j \in N : s_j < 0\}$.
If $P \neq \emptyset$ and the improving direction crosses the feasible region then proceed to step 2.
Step 1. *(Phase I).*
Construct an auxiliary problem by adding an artificial variable x_{n+1} with a coefficient vector equal to $-A_B e$, where $e = (1, 1, \cdots, 1)^T \in \mathbb{R}^m$.
Apply the steps of the revised simplex algorithm in Phase I. If the final basic solution (B, N) is feasible, then proceed to step 2 in order to solve the initial problem. The auxiliary LP problem can be either optimal or infeasible.
Step 2. *(Phase II).*
Step 2.0. *(Initialization).*
Compute $(A_B)^{-1}$ and vectors x_B, w, and s_N.
Find the sets of indices $P = \{j \in N : s_j < 0\}$ and $Q = \{j \in N : s_j \geq 0\}$.
Define an arbitrary vector $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{|P|}) > 0$ and compute s_0 as follows:
$s_0 = \sum_{j \in P} \lambda_j s_j$
and the direction
$d_B = -\sum_{j \in P} \lambda_j h_j$, where $h_j = A_B^{-1} A_{.j}$.
Step 2.1. *(Test of Optimality).*
if $P = \emptyset$ then STOP. The LP problem is optimal.
else
if $d_B \geq 0$ then
if $s_0 = 0$ then STOP. The LP problem is optimal.
else
choose the leaving variable $x_{B[r]} = x_k$ using the following relation:
$a = \frac{x_{B[r]}}{-d_{B[r]}} = \min \left\{ \frac{x_{B[i]}}{-d_{B[i]}} : d_{B[i]} < 0 \right\}, i = 1, 2, \cdots, m$
if $a = \infty$, the LP problem is unbounded.
Step 2.2. *(Pivoting).*
Compute the row vectors: $H_{rP} = \left(A_B^{-1}\right)_{r.} A_P$ and $H_{rQ} = \left(A_B^{-1}\right)_{r.} A_Q$.
Compute the ratios θ_1 and θ_2 using the following relations:
$\theta_1 = \frac{-s_P}{H_{rP}} = \min \left\{ \frac{-s_j}{H_{rj}} : H_{rj} > 0 \wedge j \in P \right\}$ and
$\theta_2 = \frac{-s_Q}{H_{rQ}} = \min \left\{ \frac{-s_j}{H_{rj}} : H_{rj} < 0 \wedge j \in Q \right\}$
Determine the indices t_1 and t_2 such that $P[t_1] = p$ and $Q[t_2] = q$.
if $\theta_1 \leq \theta_2$ then set $l = p$
else set $l = q$.
Step 2.3. *(Update).*
Swap indices k and l. Update the new basis inverse $\left(A_{\overline{B}}\right)^{-1}$, using a basis update scheme. Set $B(r) = l$. If $\theta_1 \leq \theta_2$, set $P = P \setminus \{l\}$ and $Q = Q \cup \{k\}$. Otherwise, set $Q(t_2) = k$. Update vectors x_B, w, and s_N.
Update the new improving direction $\overline{d}_B$ using the relation
$d_{\overline{B}} = E^{-1} d_B$. If $l \in P$, set $d_{\overline{B}[r]} = d_{\overline{B}[r]} + \lambda_l$.
Go to Step 2.1.

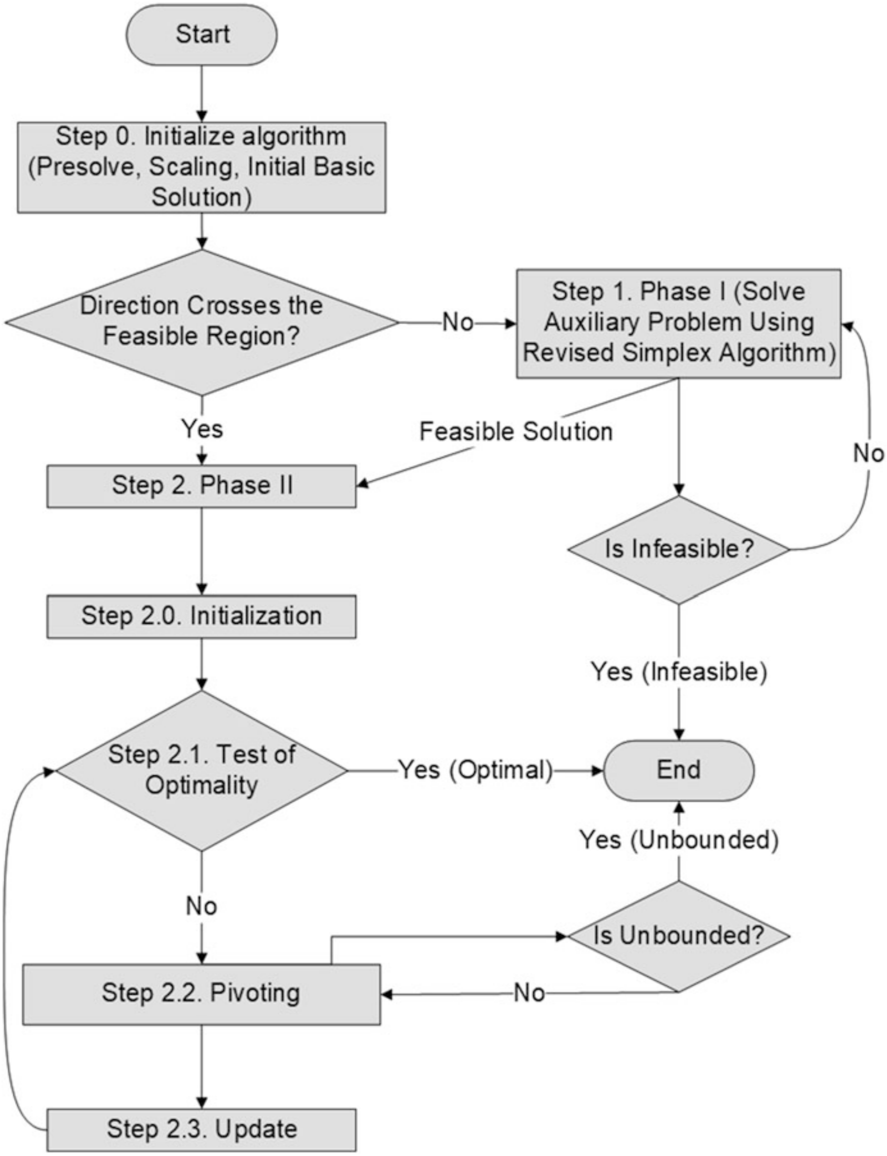


Fig. 2.30 Exterior point simplex algorithm

2.7.4 Mehrotra's Predictor-Corrector Method

A formal description of Mehrotra's Predictor-Corrector (MPC) method [4] is given in Table 2.9 and a flow diagram of its major steps in Figure 2.31. Similar to the revised simplex algorithm, the LP problem is presolved and scaled.

Table 2.9 Mehrotra's predictor-corrector method

Step 0. *(Initialization).*
Presolve the LP problem.
Scale the LP problem.
Find a starting point (x^0, w^0, s^0).
Step 1. *(Test of Optimality).*
Calculate the primal (r_p), dual (r_d), and complementarity (r_c) residuals.
Calculate the duality measure (μ).
if $\max(\mu, \lvert\lvert r_p \rvert\rvert, \lvert\lvert r_d \rvert\rvert) <= tol$ then STOP. The LP problem is optimal.
Step 2. *(Predictor Step).*
Solve a system to calculate $(\Delta x^p, \Delta w^p, \Delta s^p)$.
Calculate the largest possible step lengths α_p^p, α_d^p.
Step 3. *(Centering Parameter Step).*
Compute the centering parameter σ.
Step 4. *(Corrector Step).*
Solve a system to calculate (11.25) for $(\Delta x, \Delta w, \Delta s)$.
Calculate the primal and dual step lengths α_p, α_d.
Step 5. *(Update Step).*
Update the solution (x, w, s).
Go to Step 1.

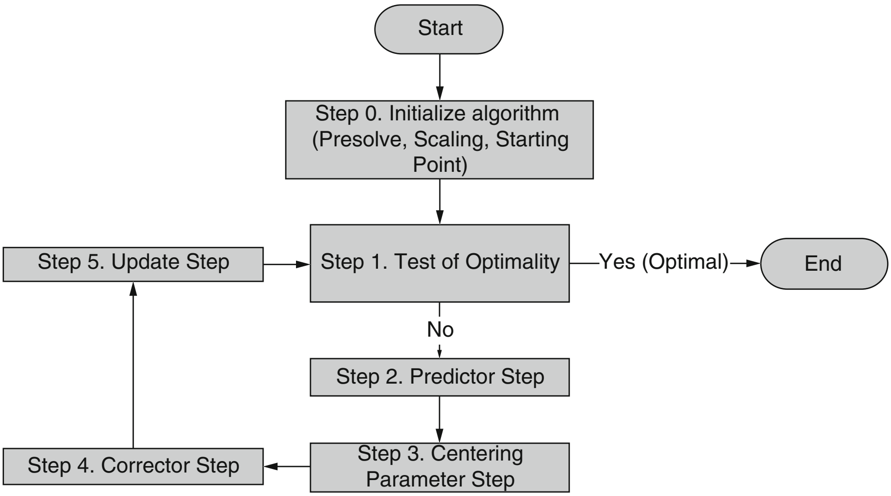


Fig. 2.31 Mehrotra's predictor-corrector method

Then, we find a starting point. Next, we perform the test of optimality to identify if the current solution is optimal and terminate the algorithm. In the predictor step, we solve a system to calculate $(\Delta x^p, \Delta w^p, \Delta s^p)$ and obtain the largest possible step lengths α_p^p and α_d^p. Next, we compute the centering parameter σ. In the corrector step, we solve a system to calculate $(\Delta x, \Delta w, \Delta s)$ and obtain the primal and dual step lengths α_p and α_d. Finally, we update the solution (x, w, s) and continue to apply these steps until the solution is optimal.

This algorithm will be presented in detail in Chapter 11, because it requires the introduction of some basic principles of the interior point methods.

2.8 Chapter Review

In this chapter, we presented the theoretical background of LP. More specifically, the different formulations of the LP problem were presented. Moreover, detailed steps on how to model an LP problem were given. In addition, the geometry of the feasible region and the duality principle were also covered. Finally, a brief description of LP algorithms that will be used in this book was also given.

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