Chapter 15
Completions

One can sometimes establish a certain property for a class of relation algebras by applying the following strategy. First, prove that all complete and atomic relation algebras in the class possess the property, and then show that the property is inherited by subalgebras. If the class in question is closed under the formation of canonical extensions, that is to say, if the canonical extension of every algebra in the class also belongs to the class, then every algebra in the class will possess the desired property. Canonical extensions, however, do have a serious disadvantage from this perspective, because all properly infinite sums and products in the original algebra are changed in the passage to the canonical extension, and (in the infinite case) new atoms are introduced even when the original algebra is atomic. Consequently, if the property in question involves properly infinite sums or products, or atoms, in the original algebra, then the above strategy may fail. What one would like is an extension that is complete, with the same atoms as in the original algebra, and in which all existing infinite sums and products of the original algebra are preserved. Fortunately, such an extension exists, and in fact it satisfies a very nice minimality condition.

The construction actually goes through in the context of arbitrary Boolean algebras with quasi-complete operators. As in the case of canonical extensions, in order to simplify notation we shall assume that the similarity type of the algebras under discussion is the same as the similarity type of relation algebras. The extension of the results to Boolean algebras with quasi-complete operators of arbitrary ranks is straightforward.

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15.1 Complete Boolean ideals

The construction of canonical extensions involves (explicitly or implicitly) the notion of an ultrafilter—a maximal Boolean filter. In rough analogy, the construction of a completion involves the notion of a complete Boolean ideal. We therefore begin with the study of such ideals. *In this and the next three sections, all ideals under consideration are assumed to be Boolean ideals*, so we will just refer to them as ideals. Recall that Boolean algebras are denoted with italic letters. An ideal in a Boolean algebra A is defined to be a subset X of A with following three properties: (1) 0 is in X; (2) X is closed under addition in the sense that if r and s are in X, then so is $r + s$; (3) X is closed under multiplication by elements from A in the sense that if r is in X and s in A, then $r \cdot s$ is in X. Condition (3) is equivalent to the condition that X be *downward closed* in the sense that if r is in X and $s \leq r$, then s is in X. In the presence of condition (3), condition (1) is equivalent to the condition that X be non-empty (see Lemma 8.8).

An ideal X in a Boolean algebra A is said to be *complete* if it is *closed under* (the formation of) *suprema*. This means that if a subset of X has a supremum in A, then that supremum must also belong to X. A set that is closed under formation of suprema is certainly closed under addition, so a subset X of A is a complete ideal just in case X is non-empty, downward closed, and closed under suprema. Every principal ideal is an example of a complete ideal. Indeed, if Y is a subset of the principal ideal generated by an element r in A, and if Y has a supremum s in A, then s is below r (because r is an upper bound of Y, and s is the least upper bound of Y), and therefore s is in the ideal generated by r. We shall see other examples of complete ideals in the next section.

The intersection of an arbitrary system of complete ideals in a Boolean algebra A is again a complete ideal in A. Indeed, the intersection of such a system is an ideal, by (the Boolean version of) the remarks at the beginning of Section 8.7. If Y is a subset of this intersection, and if Y has a supremum r in A, then r belongs to each of the ideals in the system, by the assumption that these ideals are complete, and therefore r belongs to the intersection of the system. Consequently, the intersection is a complete ideal.

It follows that if Y is an arbitrary subset of A, then the intersection of the set of complete ideals in A that include Y is itself a complete ideal. (There is always at least one complete ideal that includes Y, namely the improper ideal A, so the set of complete ideals that include Y is never empty.) That intersection—call it X—is the smallest complete ideal that includes Y, In other words, X is a complete ideal in A that includes Y, and every complete ideal in A that includes Y also includes X, by the very definition of X. The set X is called the complete ideal *generated* by Y.

The definition just given is non-constructive and gives no idea which elements in A actually belong to the complete ideal generated by Y. There is another description that is quite useful. Let Y^d be the set of elements in A that are below some element in Y, so that

$$Y^d = \{s \in A : s \leq r \text{ for some } r \in Y\}.$$

The set Y^d is called the *downward closure* of Y (in A).

Lemma 15.1. *An element r in a Boolean algebra belongs to the complete ideal generated by a set Y if and only if r is the supremum of some subset of Y^d.*

Proof. Let X be the complete ideal generated by a set Y in a Boolean algebra A, and let Z be the set of elements in A that are suprema of subsets of Y^d. It is to be shown that X and Z are equal.

The first step is to verify that Z is indeed a complete ideal that includes Y. Each element r in Y is the supremum of a subset of Y^d, namely the subset $\{r\}$, so Y is certainly included in Z. The element 0 is the supremum of the empty subset, so 0 belongs to Z. To verify the closure of Z under suprema, consider a subset W of Z that has a supremum r in A. Each element s in W is the supremum of some subset Y_s of Y^d, by the definition of Z. The union $\bigcup_{s \in W} Y_s$ is a subset of Y^d, and r is the supremum of this union, since

$$r = \sum\{s : s \in W\} = \sum\{\sum Y_s : s \in W\} = \sum(\bigcup_{s \in W} Y_s),$$

by the general associative law for addition. Therefore, r belongs to Z, by the definition of Z.

It remains to check that Z is downward closed. Observe first that the set Y^d is obviously downward closed. Let r be an element in Z, and assume that $s \leq r$. The definition of Z implies that r is the supremum of some subset Y_r of Y^d. The set

$$\{t \cdot s : t \in Y_r\}$$

is also included in Y^d, by the downward closure of Y^d, and

$$s = r \cdot s = (\sum Y_r) \cdot s = \sum\{t \cdot s : t \in Y_r\},$$

so s is the supremum of a subset of Y^d and therefore belongs to Z.

So far, it has been shown that Z is a complete ideal and includes Y. The set X is the smallest complete ideal that includes Y, so X must be included in Z. On the other hand, every element in Y^d certainly belongs to X, because Y is included in X, and X is downward closed. The completeness of the ideal X therefore implies that whenever a subset of Y^d has a supremum in A, that supremum must belong to X. Thus, every element in Z belongs to X, by the definition of Z. Conclusion: $X = Z$, as claimed. □

There is another description of the complete ideal generated by a set Y in a Boolean algebra A that is worth mentioning, because it gives a different perspective on these ideals. Recall that the set U of upper bounds of Y is the set of elements in A that are above every element in Y. The set of lower bounds of U is the set L of elements in A that are below every element in U. It turns out that L is the complete ideal generated by Y.

Lemma 15.2. *The complete ideal generated by a set Y in a Boolean algebra is the set of lower bounds of the set of upper bounds of Y.*

Proof. Let X be the complete ideal generated by a set Y in a Boolean algebra A, let U be the set of upper bounds of Y, and let L be the set of lower bounds of U. It is to be shown that X and L coincide.

The first step is to verify that L is a complete ideal that includes Y. Each element in Y is a lower bound of U, by the definition of U, so Y is certainly included in the set L, by the definition of L. It is equally obvious that 0 is in L, since 0 is below every element in U. To verify that L is closed under suprema, consider an arbitrary set W of elements in L, and suppose that W has a supremum r in A. The elements in U are upper bounds of the set L, by the definition of L, so they are upper bounds of the set W, and consequently they are all above the least upper bound r of W. Thus, r is a lower bound of U, so it belongs to L, by the definition of L. The downward closure of L is equally easy to check. Any element r in L is, by definition, a lower bound of U, and therefore so is every element that is below r. It follows that all such elements are in L, by the definition of L.

The set X is, by assumption, the smallest complete ideal that includes Y. We have seen that L is also a complete ideal that includes Y, so X must be included in L.

To establish the reverse inclusion, consider an arbitrary element r in L, and let W be the set of elements in Y^d that are below r. The goal is to show that r is the supremum of the set W. Lemma 15.1 then implies that r is in X, so the set L must be included in X.

The element r is certainly an upper bound of W, by the definition of W. Consider any other upper bound of W, say s, with the aim of showing that $r \leq s$. The first step is to prove that $s + -r$ is an upper bound of Y. For each element t in Y, the product $t \cdot r$ is below r and also below t. In particular, this product must belong to the set Y^d, by the definition of Y^d and the assumption that t is in Y. Combine these observations with the definition of W to see that $t \cdot r$ must be in W. Every element in W is below s, by assumption, so $t \cdot r \leq s$. A straightforward computation yields

$$t = t \cdot 1 = t \cdot (r + -r\,) = (t \cdot r) + (t \cdot -r\,) \leq s + -r.$$

It has been shown that the sum $s + -r$ is an upper bound of Y, so this sum belongs to the set U of upper bounds of Y. The element r belongs to the set L of lower bounds of U, by assumption, so $r \leq s+-r$ and consequently

$$r = r \cdot (s + -r\,) = (r \cdot s) + (r \cdot -r\,) = (r \cdot s) + 0 = r \cdot s.$$

Thus, $r \leq s$, as was to be shown. □

There is a close connection between complete ideals and the "cuts" that play a crucial role in Dedekind's classical construction of the real numbers from the rational numbers. A *Dedekind cut* in the set of rational numbers is a pair (P, Q) of non-empty sets that partition the rational numbers and have the property that every number in P is less than every number in Q. The set P has the characteristic property that it is downward closed in the sense that if p is in P and if q is a rational number less than p, then q is in P. Similarly, the set Q is upward closed in the sense that if p is in Q and if q is a rational number greater than p, then q is in Q. The set Q can be reconstructed from P (it is just the complement of P in the set of rational numbers), so one could define a Dedekind cut to be simply a non-empty, downward closed set of rational numbers that does not coincide with the set of all rational numbers.

Consider now a subset Y of a Boolean algebra A. The set U of upper bounds of Y is upward closed, and the set L of lower bounds of U is downward closed, and the two sets L and U have at most one element in common. The pair (L, U) is therefore a kind of Dedekind cut in the partial ordering of the algebra A. (The fact that L and U may have one element in common—namely the supremum of Y, if this supremum exists—is of no real significance.) The set U can of course be reconstructed from the set L—it is just the set of upper bounds of L—so one can consider L itself to be a Dedekind cut in A, In view of Lemma 15.2, we may conclude that complete ideals are the analogues for Boolean algebras of Dedekind cuts of rational numbers.

The complete ideals in a Boolean algebra A form a complete lattice. The infimum of any system of complete ideals is the intersection of the system, and the supremum of the system is the complete ideal generated by the union of the system, or in different words, it is the intersection of the complete ideals in A that include the union of the system. Warning: the lattice of complete ideals in A is not a sublattice of the lattice of all ideals in A. The binary operation of meet is the same in both lattices, but the operation of join is not the same. In the lattice of all ideals, the join of two complete ideals X an Y is the intersection of all ideals that include $X \cup Y$; in the lattice of complete ideals, it is the intersection of all complete ideals that include $X \cup Y$.

The lattice of complete ideals in A is not only complete, it is also distributive. For the proof, consider three complete ideals X, Y, and Z in A. We shall show that meet distributes over join in the sense that

$$X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z).$$

The ideals $X \wedge Y$ and $X \wedge Z$ are obviously included in the ideal on the left side of the equation, since Y and Z are each included in $Y \vee Z$. Consequently, the ideal on the right side of the equation—the smallest complete ideal that includes the ideals $X \wedge Y$ and $X \wedge Z$—is included in the ideal on the left side. To establish the reverse inclusion, consider an arbitrary element r in the ideal on the left side. Since r belongs to the complete ideal generated by $Y \vee Z$, and since the ideals Y and Z coincide with their downward closures, there must be a subset Y_0 of Y and a subset Z_0 of Z such that r is the supremum of the set $Y_0 \cup Z_0$, by Lemma 15.1. The element r also belongs to X, by assumption, so the set $Y_0 \cup Z_0$ must be included in X, by the downward closure of the ideal X. Consequently, the set Y_0 is a subset of $X \wedge Y$, and the set Z_0 is a subset of $X \wedge Z$, so the union $Y_0 \cup Z_0$ of these two sets is included in the ideal on the right side of the equation. It follows that the supremum r of this union also belongs to the ideal on the right, because the ideal on the right is complete.

The dual distributive law for join over meet,

$$X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z),$$

can be derived in a dual fashion. Alternatively, this law is lattice-theoretically derivable from the distributive law for meet over join; the proof is left as an exercise.

There is another rather surprising difference between the lattice of all ideals and the lattice of all complete ideals in a Boolean algebra A. In the former, ideals may fail to have a complement. In the latter, this never happens; every complete ideal has a complement. For the proof, consider a complete ideal X in A, and define the *annihilator* of X to be the set Y defined by

$$Y = \{r \in A : r \cdot t = 0 \text{ for all } t \in X\}.$$

It is easy to check that Y is a complete ideal in A. The element 0 belongs to Y, because $0 \cdot t = 0$ for all t in X. The set Y is downward closed, because if r is in Y and if $s \leq r$, then

$$s \cdot t \leq r \cdot t = 0$$

for all t in X, and therefore s is in Y. To see that Y is closed under suprema, let Y_0 be a subset of Y that has a supremum r in A. For each element t in X,

$$r \cdot t = (\textstyle\sum Y_0) \cdot t = \textstyle\sum\{s \cdot t : s \in Y_0\} = 0,$$

by the complete distributivity of multiplication and the assumption that Y_0 is a subset of the annihilator of X. Conclusion: r belongs to Y.

We proceed to verify that Y is the complement of X in the sense that

$$X \wedge Y = \{0\} \qquad \text{and} \qquad X \vee Y = A$$

in the lattice of complete ideals in A. The first equation is almost immediate: if r is in X and also in the annihilator Y, then $r \cdot r = 0$, by the definition of the annihilator, and therefore $r = 0$. The second equation is equivalent to the assertion that the unit 1 belongs to the complete ideal generated by the set $X \cup Y$. In fact, 1 is the supremum of this set. For the proof, observe that 1 is obviously an upper bound of the set. If r is any upper bound of the set, then every element s in X is below r, and therefore $-r \cdot s = 0$, by Boolean algebra. It follows that $-r$ belongs to the annihilator of X, which is Y. Since r is also assumed to be an upper bound of Y, we must have $-r \leq r$, which can only happen if $r = 1$. Thus, 1 is the least upper bound of the set $X \cup Y$, so 1 belongs to the complete ideal generated by this set.

A distributive, complemented lattice with zero and one is automatically a Boolean algebra, so the following theorem has been proved.

Theorem 15.3. *The set of all complete ideals in a Boolean algebra A is a complete Boolean algebra under the following operations*: *the product of two complete ideals X and Y is their intersection*, *the sum of X and Y is the complete ideal generated by their union*, *and the complement of X is the annihilator of X.*

15.2 Completions of Boolean algebras

We now turn to the study of completions, and we begin with the case of Boolean algebras without any additional operations.

Definition 15.4. A *completion* of a Boolean algebra A is a Boolean Boolean algebra B with the following properties.

(i) The algebra B is complete, and A is a subalgebra of B.
(ii) Every non-zero element in B is above a non-zero element in A.

□

A subset X of a Boolean algebra B is said to be *dense* (in B) if every non-zero element in B is above a non-zero element in X. Condition (ii) in the preceding definition requires that the (universe of the) subalgebra A be dense in B. For that reason, the condition is often called the *density property.* Density can be characterized in several different ways.

Lemma 15.5. *The following conditions on a subset X of a Boolean algebra B are equivalent.*

(i) *X is dense in B.*
(ii) *Every element in B is the supremum of some subset of X.*
(iii) *Every element p in B is the supremum of the set of all elements in X that are below p.*

Proof. Condition (iii) obviously implies condition (ii). If condition (ii) holds, then every non-zero element p in B must be above a non-zero element in X; for if the set of elements in X that are below an element p is either empty or contains only zero, then the supremum of this set—which is p, by condition (ii)—must be zero. Thus, condition (ii) implies condition (i).

Assume now that condition (i) holds, with the goal of deriving condition (iii). To this end, consider an arbitrary element p in B, and let Y be the set of elements in X that are below p. It is to be shown that p is the supremum of Y. Clearly, p is an upper bound of Y, by the definition of Y. Let q be any other upper bound of Y in B, and assume, for contradiction, that the difference $p-q$ is not zero. There is then a non-zero element r in X that is below this difference, by condition (i), so r is below both p and $-q$. Because r is in X and below p, it must belong to the set Y, by the definition of Y, and therefore it must be below q, which is assumed to be an upper bound of Y. Because r is also below $-q$, it must be below the product of q and $-q$, which is zero. This forces r to be zero, in contradiction to the assumption that r is non-zero. Conclusion: the difference $p - q$ is zero, so $p \leq q$, and therefore p is the least upper bound of Y. □

The density property is the requirement that forces all infinite sums existing in A to be left intact in the passage to B, so that A is a regular (Boolean) subalgebra of B (see Section 6.5).

Lemma 15.6. *A dense subalgebra of a Boolean algebra B is always a regular subalgebra of B.*

Proof. Let A be a dense subalgebra of B, and consider a subset Y of A with a supremum p in A. It is to be shown that p remains the supremum of Y in B. The argument is analogous to the final argument in the preceding proof.

Obviously, p is an upper bound of Y in B. Let q be any upper bound of Y in B, and suppose, for contradiction, that the difference $p-q$ is not zero. There must then be a non-zero element r in A that is below $p-q$, by the assumed density of A in B. The difference $p - r$ belongs to A, because p and r are both in A, and this difference is strictly below p, because r is non-zero and below p. We shall show that $p - r$ is also an upper bound of Y in A. This contradicts the assumption that p is the least upper bound of Y in A, so $p - q$ must in fact be zero. It follows that $p \leq q$ and therefore p is the least upper bound of Y, as claimed.

To see that $p - r$ is an upper bound of Y, fix an element s in Y. Certainly, s is below q, because q is assumed to be an upper bound of Y. Therefore, s must be disjoint from $-q$, and as a result, s must also be disjoint from the difference $p - q$. The element r is below this difference, so s must be disjoint from r, and therefore s must be below the complement $-r$. Of course, s is also below p, since p is an upper bound of Y, so s is below the difference $p - r$. Thus, $p - r$ is an upper bound of Y, as claimed. □

Lemma 15.6 implies that completions of a Boolean algebra A *preserve existing sums and products* in A in the sense that if a subset Y of A has a supremum (or an infimum) r in A, then r remains the supremum (or infimum) of Y in each completion of A.

Corollary 15.7. *A completion of a Boolean algebra A preserves all sums and products that exist in A.*

Proof. Suppose B is a completion of A. Condition (ii) in Definition 15.4 implies that A is a dense subalgebra of B, and therefore A is a regular subalgebra of B, by Lemma 15.6. By definition, this means that if a subset Y of A has a supremum r in A, then r remains the supremum of Y in B. Similarly, if Y has an infimum s in A, then s remains the infimum of Y in B. In more detail, since s is the infimum of Y in A, its complement $-s$ must be the supremum in A of the complementary set $\{-t : t \in Y\}$, because

$$-s = -\prod Y = \sum\{-t : t \in Y\}.$$

in A. Consequently, $-s$ must be the supremum of the complementary set in B, because A is a regular subalgebra of B. The computation

$$s = -(-s) = -\sum\{-t : t \in Y\} = \prod\{-(-t) : t \in Y\} = \prod Y$$

in B implies that s is the infimum of Y in B. □

The Boolean algebra B in Lemmas 15.5 and 15.6 is not required to be complete. In the presence of the density property, the completeness of B is equivalent to the weaker requirement that every subset of A has a supremum in B.

Lemma 15.8. *If X is a dense subset of a Boolean algebra B, then B is complete if and only if every subset of X has a supremum in B.*

Proof. If B is complete, then obviously every subset of X has a supremum in B. To establish the reverse implication, assume that every subset of X has a supremum in B, and consider an arbitrary subset Y of B. It is to be shown that Y has a supremum in B.

Each element q in Y is the supremum of the set Z_q of elements in X that are below q, by Lemma 15.5 and the assumed density of X. The union

$$Z = \bigcup\{Z_q : q \in Y\} \tag{1}$$

is also a subset of X, so it has a supremum p in B, by assumption. We now show that p is the supremum of Y in B. Certainly, p is above each element in the set Z_q, because p is the supremum of the set Z, which includes the set Z_q. Consequently, p is an upper bound of Z_q, so it must be above the least upper bound of Z_q, which is q. This is true for each q in Y, so p is an upper bound of Y. If t is any other upper bound of Y, then t is above q for each q in Y. Since q is an upper bound of the set Z_q, it follows that t is also an upper bound of Z_q. This is true for each q in Y, so t is an upper bound of the union set Z, by (1). But p is the least upper bound of Z, so $p \leq t$. This argument shows that p is below every upper bound of Y, so p is the least upper bound of Y. □

Fix a Boolean algebra A for the remainder of this section. The set of complete ideals in A is a Boolean algebra in its own right, by Theorem 15.3. Denote this algebra by C. It is not hard to check that C includes a natural copy of A as a regular subalgebra. The following lemma contains the heart of the argument.

Lemma 15.9. *For each element p in a Boolean algebra A, the set*

$$L_p = \{r \in A : r \leq p\}$$

is a complete ideal in A. The following statements are true in the Boolean algebra C of complete ideals in A, whenever p and q are in A.

(i) *The sum of L_p and L_q is L_t, where $t = p + q$.*
(ii) *The product of L_p and L_q is L_t, where $t = p \cdot q$.*
(iii) *The complement of L_p is L_t, where $t = -p$.*

Proof. It is easy to check that the set L_p is a complete ideal in A. Indeed, this set clearly contains zero and is downward closed. To check that it is closed under suprema, consider a subset X of L_p, and suppose that X has a supremum r in A. The element p is an upper bound of the set L_p, so it is also an upper bound of the subset X. As r is the least upper bound of X, it follows that $r \le p$ and consequently r belongs to L_p.

Turn next to the proof of (i). Write X for the complete ideal that is the sum, or join, of the ideals L_p and L_q in C. It is to be shown that

$$X = L_t, \tag{1}$$

where $t = p + q$. For the inclusion from left to right, observe that p and q are both below t, so the ideals L_p and L_q are certainly included in the ideal L_t, by the definitions of these ideals. Consequently, the join X of the two ideals is included in L_t. To establish the reverse inclusion, consider an element r in L_t, and observe that r is below t, by the definition of L_t, and therefore

$$r = r \cdot t = r \cdot (p + q) = r \cdot p + r \cdot q,$$

by Boolean algebra. The elements $r \cdot p$ and $r \cdot q$ belong to the ideals L_p and L_q respectively, by the definitions of these ideals, so the sum r of these two elements belongs to the join of the two ideals, which is X. Thus, L_t is included in X, so (1) holds.

Part (ii) follows from the equation

$$L_t = L_p \cap L_q,$$

where $t = p \cdot q$, since the intersection on the right side of this equation is just the product, or meet, of the ideals L_p and L_q in C (see Theorem 15.3). The equation is easily justified: for each element r in A,

$$\begin{aligned} r \in L_t \qquad & \text{if and only if} \qquad r \le t, \\ & \text{if and only if} \qquad r \le p \text{ and } r \le q, \\ & \text{if and only if} \qquad r \in L_p \text{ and } r \in L_q, \\ & \text{if and only if} \qquad r \in L_p \cap L_q, \end{aligned}$$

by the definition of L_t, the definition of t, the definitions of L_p and L_q, and the definition of intersection.

From (i) and (ii) (with $q = -p$), and Boolean algebra, it follows that for each element p in A, the sum and product of the ideals L_p and L_{-p} in C are the ideals L_1 and L_0 respectively. These last two ideals are the unit and zero of C, so L_{-p} must be the complement of L_p in C. This proves (iii). □

Lemma 15.10. *The function φ defined by*

$$\varphi(r) = L_r$$

for each r in A is a monomorphism from A into C.

Proof. The fact that φ is a homomorphism from A into C is a direct consequence of Lemma 15.9. For example, if r is an element in A, and if t is the complement of r in A, then part (iii) of the lemma ensures that L_t is the complement of L_r in C. Consequently, $\varphi(t)$ is the complement of $\varphi(r)$, by the definition of φ, so φ preserves the operation of complement.

Similarly, if r and s are elements in A, and if t is their sum in A, then part (i) of the lemma ensures that L_t is the sum of the ideals L_r and L_s. Consequently, $\varphi(t)$ is the sum of $\varphi(r)$ and $\varphi(s)$ in C, by the definition of φ, so φ preserves the operation of addition.

If r is a non-zero element in A, then L_r is a non-trivial complete ideal, since it contains the element r. It follows that the function φ maps non-zero elements in A to non-zero elements in C, so it is monomorphism, by the Boolean analogue of Lemma 8.37. □

The Boolean algebra C of complete ideals in a Boolean algebra A is called the *completion embedding algebra* of A, and the mapping φ defined in Lemma 15.10 is called the *completion embedding* of A into C. The subalgebra of C that is the image of A under the mapping φ is an isomorphic copy of A and will be denoted by $\varphi(A)$.

Theorem 15.11. *The completion embedding algebra of a Boolean algebra A is the completion of the image of A under the completion embedding.*

Proof. The Boolean algebra C is complete, by Theorem 15.3, and Lemma 15.10 implies that $\varphi(A)$ is a subalgebra of C. Thus, condition (i) of Definition 15.4 holds. To prove that C is the completion of $\varphi(A)$, it remains to verify condition (ii) of the definition. In other words, it must be shown that $\varphi(A)$ is dense in C.

The non-zero elements in C are the complete ideals X in A that are different from the trivial ideal L_0, while the non-zero elements in $\varphi(A)$ are the principal ideals L_r for non-zero elements r in A (since r always belongs to L_r). A non-trivial complete ideal X must contain a non-zero element r, and the principal ideal L_r is obviously included in X, by the definition of L_r and the downward closure of X. Thus, every non-zero element in C is above a non-zero element in $\varphi(A)$, so $\varphi(A)$ is dense in C. □

Corollary 15.12. *The completion embedding φ is a complete monomorphism from the Boolean algebra A into the completion embedding algebra C, and the image of A under φ is a regular subalgebra of C.*

Proof. The mapping φ is a monomorphism from A to C, and the image of A under φ is a dense subalgebra of C, by Theorem 15.11. Consequently, this image is a regular subalgebra of C, by Lemma 15.6. The monomorphism φ must therefore be complete, by the Boolean version of Lemma 7.9. □

It has been shown that A is embeddable into the algebra C, and that C is the completion of the image of A under this embedding. An application of the Exchange Principle (Theorem 7.15) yields a Boolean algebra that is the completion of A. The following *Existence Theorem* for completions of Boolean algebras has been proved.

Theorem 15.13. *Every Boolean algebra has a completion.*

The preceding completion of A was constructed as an isomorphic copy of the completion embedding algebra C. There is in fact a canonical isomorphism from each completion of A to C. The proof of this assertion begins with a generalization of Lemma 15.9 from elements in A to elements in an arbitrary completion of A.

Lemma 15.14. *Let B be a completion of a Boolean algebra A. For each element p in B, the set*

$$L_p = \{r \in A : r \leq p\}$$

is a complete ideal in A. Moreover, a subset X of A is a complete ideal in A if and only if $X = L_p$, where p is the sum of the set X in B. The following statements are true in the Boolean algebra C whenever p and q are elements in B.

(i) *The sum of L_p and L_q is L_t, where $t = p + q$.*
(ii) *The product of L_p and L_q is L_t, where $t = p \cdot q$.*
(iii) *The complement of L_p is L_t, where $t = -p$.*

Proof. The proof of the first assertion is identical to the proof of the corresponding part of Lemma 15.9, except that the element p is allowed to be in B, and not just in A. The details are left to the reader.

To prove the second assertion, consider a subset X of A, and put

$$p = \sum X \tag{1}$$

(the sum being formed in B). If $X = L_p$, then X is a complete ideal in A, by the first assertion of the lemma.

Assume now that X is a complete ideal in A. Every element r in X belongs to A, by the assumption on X, and r is below p, by (1), so r belongs to the set L_p, by the definition of this set. Consequently, X is included in L_p. To establish the reverse inclusion, consider an arbitrary element r in L_p. Certainly, r belongs to A and is below p, by the definition of the set L_p, so

$$r = r \cdot p = r \cdot (\sum X) = \sum\{r \cdot s : s \in X\}$$

in B, by Boolean algebra. The elements on the right side of this equation are all in A, as is r, so the equality of r with the sum on the right must hold in A as well (see the remark preceding the definition of a regular subalgebra in Section 6.5). Also, each product $r \cdot s$ on the right belongs to X, because X is an ideal in A, and s is in X, and r is in A. The set on the right is therefore a subset of X, and its sum exists in A and is equal to r, by the preceding observations. Since the ideal X is assumed to be complete, it follows that r must be in X, so L_p is included in X. Thus, $X = L_p$, which proves the second assertion of the lemma.

Turn now to the proof of (i). Let p and q be elements in B, and observe that

$$p = \sum L_p \qquad \text{and} \qquad q = \sum L_q, \tag{2}$$

by Lemma 15.5. Put

$$t = p + q \tag{3}$$

(this sum being formed in B). It is to be shown that L_t is the sum of the ideals L_p and L_q in the completion embedding algebra C. Since the elements p and q are both below t, by (3), the ideals L_p and L_q are both included in L_t, by the definitions of these ideals.

Consider now any complete ideal X in A that includes both L_p and L_q, with the goal of proving that X must also include L_t. To this end, let r be any element in L_t, and put

$$Y = \{r \cdot s : s \in L_p\} \qquad \text{and} \qquad Z = \{r \cdot s : s \in L_q\}. \tag{4}$$

The sets Y and Z are subsets of L_p and L_q respectively, since the latter are ideals and are therefore closed under multiplication by elements from A. Consequently, both sets are included in X, by the hypotheses on X. The element r is in L_t and is therefore below t, so

$$\begin{aligned} r = r \cdot t &= r \cdot (p + q) \\ &= r \cdot p + r \cdot q \\ &= r \cdot (\sum L_p) + r \cdot (\sum L_q) \\ &= \sum\{r \cdot s : s \in L_p\} + \sum\{r \cdot s : s \in L_q\} \\ &= \sum Y + \sum Z \\ &= \sum (Y \cup Z) \end{aligned}$$

in B, by Boolean algebra, (3), (2), the complete distributivity of multiplication, and (4). This computation shows that the equation

$$r = \sum (Y \cup Z)$$

holds in B. Since all of the elements involved in this equation belong to A, it follows that the equation holds in A as well. The ideal X is assumed to be complete, and it has been shown that $Y \cup Z$ is a subset of X with supremum r in A. Consequently, r must belong to X, so L_t is included in X.

The preceding argument shows that every complete ideal which includes L_p and L_q must also include L_t. In other words, L_t is the smallest complete ideal that includes the two ideals L_p and L_q. This is just the definition of the sum of these two ideals in C, by Theorem 15.3, so part (i) of the lemma has been proved.

The proofs of parts (ii) and (iii) of the lemma are identical to the proofs of the corresponding parts of Lemma 15.9, except that the elements p and q are assumed to be in B, and not just in A. The details are left to the reader. □

We can now state in a precise way the assertion that every completion of A is canonically isomorphic to the completion embedding algebra C.

Theorem 15.15. *Let B be a completion of a Boolean algebra A. The function ψ defined by*

$$\psi(p) = L_p$$

for each p in B is an isomorphism from B to the completion embedding algebra C, and ψ extends the completion embedding φ.

Proof. Lemma 15.14 implies that ψ is a well-defined bijection from B to C. For example, if X is an element in C—that is to say, if X is a complete ideal in A—then there is an element p in B (namely the sum of the set X) such that $X = L_p$, by the second assertion of the lemma, and therefore ψ maps p to X, by the definition of ψ. Thus, ψ is onto. If $\psi(p) = \psi(q)$ for some elements p and q in B, then $L_p = L_q$, by the definition of ψ, and therefore

$$p = \sum L_p = \sum L_q = q,$$

by Boolean algebra and the lemma. Thus, ψ is one-to-one.

To prove that ψ preserves the operation of addition, let p and q be elements in B, and write $t = p + q$. The sum of the two complete ideals

$$\psi(p) = L_p \qquad \text{and} \qquad \psi(q) = L_q$$

in C is the ideal $\psi(t) = L_t$, by part (i) of Lemma 15.14, so

$$\psi(p + q) = \psi(t) = L_t = \psi(p) + \psi(q).$$

A completely analogous argument, using part (iii) of the lemma, shows that ψ preserves the operation of complement. Consequently, ψ is an isomorphism from B to C. A comparison of the definitions of ψ and the completion embedding φ in Lemma 15.10 immediately leads to the conclusion that ψ is an extension of φ. □

Completions of a Boolean algebra are uniquely determined up to isomorphisms in a strong sense, as the following *Uniqueness Theorem* for completions of Boolean algebras asserts.

Theorem 15.16. *Any two completions of a Boolean algebra A are isomorphic via a mapping that is the identity function on A.*

Proof. Suppose B and D are completions of a Boolean algebra A. Let C be the completion embedding algebra of A, and φ the completion embedding of A into C. Take ψ and ϑ to be the canonical isomorphisms from B and D respectively to C that are defined in Theorem 15.15. Thus,

$$\psi(p) = L_p \qquad \text{and} \qquad \vartheta(q) = L_q \tag{1}$$

for elements p in B and q in D. Define ϱ to be the composition $\vartheta^{-1} \circ \psi$,

$$B \xrightarrow{\ \psi\ } C \xleftarrow{\ \vartheta\ } D \quad .$$

Thus,

$$\varrho(p) = q \qquad \text{if and only if} \qquad L_p = L_q,$$

by (1). A composition of isomorphisms is an isomorphism, so ϱ is an isomorphism from B to D, by Theorem 15.15. Moreover, if p is in A, then

$$\psi(p) = \varphi(p) \qquad \text{and} \qquad \vartheta(p) = \varphi(p), \tag{2}$$

again by Theorem 15.15, and therefore

$$\varrho(p) = \vartheta^{-1}(\psi(p)) = \vartheta^{-1}(\varphi(p)) = \vartheta^{-1}(\vartheta(p)) = p,$$

by the definition of ϱ and (2). Thus, ϱ is the identity function on A. □

Theorem 15.16 justifies the common practice of referring to *the* completion of a Boolean algebra.

15.3 Completions of Boolean algebras with operators

We turn now to the study of completions of Boolean algebras with operators.

Definition 15.17. A *completion* of a Boolean algebra with operators $\mathfrak{A}$ is a Boolean algebra with additional operations $\mathfrak{B}$ (of the same similarity type as $\mathfrak{A}$) that possesses the following properties.

(i) $\mathfrak{B}$ is complete, and $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$.
(ii) Every non-zero element in $\mathfrak{B}$ is above a non-zero element in $\mathfrak{A}$.
(iii) For any elements p and q in $\mathfrak{B}$,

$$p\,;q = \sum\{r\,;s : r \leq p \text{ and } s \leq q \text{ and } r, s \in A\},$$
$$p^{\smile} = \sum\{r^{\smile} : r \leq p \text{ and } r \in A\}.$$

□

This definition does not require the operations $;$ and $^{\smile}$ in a completion of a Boolean algebra with operators $\mathfrak{A}$ to be distributive, much less quasi-completely distributive. The reason is that the quasi-complete distributivity of these operations is derivable from the conditions in the definition whenever the operators in $\mathfrak{A}$ are quasi-completely distributive. Also, the definition does not require a completion of $\mathfrak{A}$ to satisfy the same equations as $\mathfrak{A}$. In particular, it does not require the completion of a relation algebra to be a relation algebra. We shall see later, however, that it is in fact a relation algebra. Conditions (i) and (ii) in the definition coincide with the same conditions in the definition of the completion of a Boolean algebra (see Definition 15.4). Consequently, if $\mathfrak{B}$ is a completion of $\mathfrak{A}$, then the Boolean part of $\mathfrak{B}$ is a completion of the Boolean part of $\mathfrak{A}$. It follows that notions such as density, and results such as Lemmas 15.5, 15.6, 15.8, and Corollary 15.7, apply automatically to Boolean algebras with operators, and we shall use them in this way without further mention.

In order to elaborate on the role of condition (iii) in the preceding definition, it is convenient to recall some notation that was already employed in the chapter on canonical extensions. For any subsets X and Y of a Boolean algebra with operators $\mathfrak{A}$, write

$$X + Y, \qquad X \cdot Y, \qquad X\,;Y, \qquad X^{\smile}$$

for the operations on complexes (subsets) of $\mathfrak{A}$ induced by the operations of $\mathfrak{A}$ (see the remarks following Definition 14.12).

The density property in condition (ii) of Definition 15.17 implies that $p = \sum L_p$ for each element p in the completion, by Lemma 15.5, where L_p is the set of elements in $\mathfrak{A}$ that are below p. In terms of this notation, condition (iii) just says that

$$p\,;q = \textstyle\sum(L_p\,;L_q) \qquad \text{and} \qquad p^{\smile} = \textstyle\sum(L_p^{\smile}).$$

It does not follow automatically that if $p = \sum X$ and $q = \sum Y$ (in $\mathfrak{B}$) for some arbitrary non-empty subsets X and Y of $\mathfrak{B}$, then

$$p\,;q = \textstyle\sum(X\,;Y) \qquad \text{and} \qquad p^{\smile} = \textstyle\sum(X^{\smile}).$$

The role of condition (iii) in Definition 15.17 is to force these equations to hold whenever the operators in $\mathfrak{A}$ are quasi-complete. The proof of this assertion requires two lemmas.

Lemma 15.18. *If $\mathfrak{B}$ is a completion of a Boolean algebra with operators $\mathfrak{A}$, then the operations $;$ and $^{\smile}$ in $\mathfrak{B}$ are monotone.*

Proof. Suppose p, q, r, and s are elements in $\mathfrak{B}$ with $r \le p$ and $s \le q$. These inequalities imply that

$$L_r \subseteq L_p \qquad \text{and} \qquad L_s \subseteq L_q,$$

by the definitions of the sets involved. It follows that

$$L_r \,;L_s \subseteq L_p \,;L_q,$$

by the definition of the complex product of two sets, and therefore

$$\textstyle\sum(L_r \,;L_s) \le \sum(L_p \,;L_q), \tag{1}$$

by Boolean algebra. Condition (iii) in Definition 15.17 implies that

$$r \,;s = \textstyle\sum(L_r \,;L_s) \qquad \text{and} \qquad p \,;q = \sum(L_p \,;L_q). \tag{2}$$

Combine (1) and (2) to conclude that $r \,;s \le p \,;q$. The proof of the corresponding inequality for the operation $^\smile$ is similar, but easier. □

The next lemma says that if the operators in $\mathfrak{A}$ are quasi-complete, then the operations $\,;\,$ and $^\smile$ in a completion of $\mathfrak{A}$ are quasi-complete with respect to sets of elements in $\mathfrak{A}$.

Lemma 15.19. *Suppose $\mathfrak{B}$ is a completion of a Boolean algebra with quasi-complete operators $\mathfrak{A}$. For any non-empty subsets X and Y of $\mathfrak{A}$, if $p = \sum X$ and $q = \sum Y$ (in $\mathfrak{B}$), then*

$$p \,;q = \textstyle\sum(X \,;Y) \qquad \textit{and} \qquad p^\smile = \sum(X^\smile).$$

Proof. Focus on the case of the operation $\,;\,$. If r and s are elements in X and Y respectively, then $r \le p$ and $s \le q$, by the assumptions on p and q, and therefore $r \,;s \le p \,;q$, by the monotony law for the operation $\,;\,$ in $\mathfrak{B}$ (Lemma 15.18). Sum this last inequality over all elements r in X and s in Y to obtain

$$\textstyle\sum(X \,;Y) = \sum\{r \,;s : r \in X \text{ and } s \in Y\} \le p \,;q.$$

To establish the reverse inequality, consider elements u in L_p and v in L_q. We have $u \le p$ and $v \le q$, by the definitions of the sets L_p and L_q, and therefore

$$u = u \cdot p = u \cdot (\textstyle\sum X) = \sum\{u \cdot r : r \in X\}$$

and

$$v = v \cdot q = v \cdot (\textstyle\sum Y) = \sum\{v \cdot s : s \in Y\}$$

in $\mathfrak{B}$, by Boolean algebra. The elements u and v belong to $\mathfrak{A}$, and the sets X and Y are subsets of $\mathfrak{A}$, so the equations

$$u = \sum\{u \cdot r : r \in X\} \qquad v = \sum\{v \cdot s : s \in Y\} \tag{1}$$

involve only elements in $\mathfrak{A}$. The validity of these equations in $\mathfrak{B}$ therefore implies their validity in $\mathfrak{A}$. The operator $\,;\,$ in $\mathfrak{A}$ is assumed to be quasi-complete, and the sets on the right sides of the equations in (1) are non-empty because X in Y are assumed to be non-empty, so

$$\begin{aligned} u\,;v &= (\textstyle\sum\{u \cdot r : r \in X\})\,;(\textstyle\sum\{v \cdot s : s \in Y\}) \\ &= \textstyle\sum\{(u \cdot r)\,;(v \cdot s) : r \in X \text{ and } s \in Y\} \end{aligned} \tag{2}$$

in $\mathfrak{A}$. All sums in $\mathfrak{A}$ are preserved under the passage to a completion, by Corollary 15.7, so (2) must also hold in $\mathfrak{B}$. Since

$$u \cdot r \leq r \qquad \text{and} \qquad v \cdot s \leq s,$$

we may apply the monotony law for $\,;\,$ to obtain

$$(u \cdot r)\,;(v \cdot s) \leq r\,;s. \tag{3}$$

Sum (3) over all r in X and s in Y to arrive at

$$\begin{aligned} &\textstyle\sum\{(u \cdot r)\,;(v \cdot s) : r \in X \text{ and } s \in Y\} \\ &\qquad\leq \textstyle\sum\{r\,;s : r \in X \text{ and } s \in Y\} \end{aligned} \tag{4}$$

in $\mathfrak{B}$. Combine (2) and (4) to obtain

$$u\,;v \leq \sum\{r\,;s : r \in X \text{ and } s \in Y\} \tag{5}$$

in $\mathfrak{B}$. Sum the left side of (5) over all u in L_p and v in L_q to conclude that

$$\sum\{u\,;v : u \in L_p \text{ and } v \in L_q\} \leq \sum\{r\,;s : r \in X \text{ and } s \in Y\}.$$

In the notation of complex products, the preceding inequality may be written as

$$\sum(L_p\,;L_q) \leq \sum(X\,;Y). \tag{6}$$

Condition (iii) in Definition 15.17 implies that the left side of (6) is equal to $p\,;q$. Consequently,

$$p\,;q \leq \sum(X\,;Y),$$

as desired. This completes the proof of the first equation in the conclusion of the lemma. The second equation is established by a similar but easier argument. □

We are now in a position to prove that for a Boolean algebra with quasi-complete operators $\mathfrak{A}$, if density condition (ii) in Definition 15.17 holds, then condition (iii) is equivalent to the requirement that the operations in $\mathfrak{B}$ are quasi-complete.

Lemma 15.20. *Let $\mathfrak{A}$ be a Boolean algebra with quasi-completely distributive operators, and $\mathfrak{B}$ a complete Boolean algebra with additional operations* (*of the same similarity type as $\mathfrak{A}$*)*. If $\mathfrak{A}$ is a dense subalgebra of $\mathfrak{B}$, then the operations $\,;\,$ and $\smile$ in $\mathfrak{B}$ are quasi-completely distributive if and only if condition* (iii) *in Definition 15.17 holds.*

Proof. Focus on the case of the binary operation $\,;\,$. Suppose first that the operation $\,;\,$ in $\mathfrak{B}$ is quasi-complete. If p and q are elements in $\mathfrak{B}$, then

$$p = \sum L_p \qquad \text{and} \qquad q = \sum L_q, \tag{1}$$

by Lemma 15.5. Notice that the sets on the right sides of these equations are non-empty, since they always contain zero. Consequently,

$$\begin{aligned} p\,;q &= (\textstyle\sum L_p)\,;(\textstyle\sum L_q) \\ &= \textstyle\sum\{r\,;s : r \in L_p \text{ and } s \in L_q\} = \textstyle\sum(L_p\,;L_q), \end{aligned}$$

by (1), the assumed quasi-complete distributivity of the operation $\,;\,$, and the definition of the complex product of subsets of $\mathfrak{A}$. Thus, the first equation in Definition 15.17(iii) holds.

Assume now that the first equation in Definition 15.17(iii) holds. Together with the initial hypotheses of the lemma, this assumption implies that $\mathfrak{B}$ is a completion of $\mathfrak{A}$. Let X and Y be arbitrary non-empty subsets of $\mathfrak{B}$ (and not just non-empty subsets of $\mathfrak{A}$, as was the case in Lemma 15.19), and write

$$p = \sum X \qquad \text{and} \qquad q = \sum Y. \tag{2}$$

For each element u in X, we have $u = \sum L_u$, by Lemma 15.5, and therefore

$$p = \sum\{u : u \in X\} = \sum\{\textstyle\sum L_u : u \in X\} = \sum(\textstyle\bigcup_{u\in X} L_u), \tag{3}$$

by (2) and the general associative law for addition. (Warning: the set $\bigcup_{u\in X} L_u$ may be different from the set L_p of all elements that are below p.) Similarly,

$$q = \sum(\bigcup_{v\in Y} L_v). \tag{4}$$

The sets

$$\bigcup_{u\in X} L_u \qquad \text{and} \qquad \bigcup_{v\in Y} L_v \tag{5}$$

are subsets of $\mathfrak{A}$. They are non-empty because the sets X and Y are assumed to be non-empty, and because the sets L_u and L_v always contain the element zero. Apply Lemma 15.19 (with X and Y respectively replaced by the two sets in (5)), in conjunction with (3) and (4), to arrive at

$$\begin{aligned} p\,;q &= [\sum(\bigcup_{u\in X} L_u)]\,;[\sum(\bigcup_{v\in Y} L_v)] \\ &= \sum[(\bigcup_{u\in X} L_u)\,;(\bigcup_{v\in Y} L_v)] \\ &= \sum(\bigcup\{L_u\,;L_v : u\in X \text{ and } v\in Y\}). \end{aligned} \tag{6}$$

(The final equality uses the definition of the complex product of two sets, which implies that this complex multiplication distributes over arbitrary unions.)

On the other hand,

$$u\,;v = \sum(L_u\,;L_v), \tag{7}$$

by condition (iii), so

$$\begin{aligned} \sum(X\,;Y) &= \sum\{u\,;v : u\in X \text{ and } v\in Y\} \\ &= \sum\{\sum(L_u\,;L_v) : u\in X \text{ and } v\in Y\} \\ &= \sum(\bigcup\{L_u\,;L_v : u\in X \text{ and } v\in Y\}), \end{aligned} \tag{8}$$

by the definition of the complex product of two sets, (7), and the general associative law for addition. A comparison of (6) and (8), together with (2), leads to the conclusion that

$$(\sum X)\,;(\sum Y) = p\,;q = \sum(X\,;Y).$$

This proves that the operation $\,;\,$ in $\mathfrak{B}$ is quasi-complete.

The proof that the second equation in Definition 15.17(iii) is equivalent to the quasi-complete distributivity of the operation $\smile$ in $\mathfrak{B}$ is similar but easier. $\square$

The requirement in Lemma 15.20 that the operators of $\mathfrak{A}$ be quasi-complete is really needed. Indeed, if the operators of the completion $\mathfrak{B}$ are quasi-complete, then the operators of $\mathfrak{A}$ must also be quasi-complete, because $\mathfrak{A}$ is a regular subalgebra of $\mathfrak{B}$, by Lemma 15.6 (see Exercise 6.28).

We now take up the question of the existence of completions. Fix a Boolean algebra with quasi-complete operators $\mathfrak{A}$ (of the same similarity type as a relation algebra). Take

$$(B\,,\,+\,,\,-)$$

be the completion of the Boolean part of $\mathfrak{A}$; such a completion exists by Theorem 15.13. Define a binary operation $\,;\,$ and a unary operation $\,^{\smile}$ on B by

$$\begin{aligned} p\,;q &= \sum\{r\,;s : r,s \in A \text{ and } r \leq p,\ s \leq q\},\\ p^{\smile} &= \sum\{r^{\smile} : r \in A \text{ and } r \leq p\}, \end{aligned}$$

for all elements p and q in B. (The operations $\,;\,$ and $\,^{\smile}$ on the right sides of these equations are those of $\mathfrak{A}$, while the ones on the left are those being defined in B. The suprema on the right are formed in B.) These operations are called the *completions* of the corresponding operations in $\mathfrak{A}$. Put

$$\mathfrak{B} = (B\,,\,+\,,\,-\,,\,;\,,\,^{\smile}\,,\,1\text{'}),$$

where 1' is the identity element in $\mathfrak{A}$.

Lemma 15.21. $\mathfrak{A}$ *is a subalgebra of* $\mathfrak{B}$.

Proof. The Boolean part of $\mathfrak{A}$ is certainly a subalgebra of the Boolean part of $\mathfrak{B}$, by condition (i) in Definition 15.4 and the choice of the Boolean part of $\mathfrak{B}$ as the completion of the Boolean part of $\mathfrak{A}$. The element 1' in both algebras is the same, by the definition of $\mathfrak{B}$.

To see that the operation $\,;\,$ in $\mathfrak{A}$ is the restriction of the corresponding operation in $\mathfrak{B}$, consider elements p and q in $\mathfrak{A}$. They are clearly the largest elements in $\mathfrak{A}$ that are below p and q respectively, so the relative product $p\,;q$ formed in $\mathfrak{A}$ is the largest element in the set

$$\{r\,;s : r,s \in A \text{ and } r \leq p,\ s \leq q\}, \tag{1}$$

by the monotony law for the operation $\,;\,$ (Lemma 2.3). It follows that this relative product is the supremum in $\mathfrak{A}$ of the set in (1). Since the

Boolean part of $\mathfrak{B}$ is the completion of the Boolean part of $\mathfrak{A}$, suprema must be preserved under the passage from $\mathfrak{A}$ to $\mathfrak{B}$, by Corollary 15.7. Consequently, the relative product $p\,;q$ formed in $\mathfrak{A}$ is the supremum in $\mathfrak{B}$ of the set in (1). The relative product $p\,;q$ formed in $\mathfrak{B}$ is, by definition, the supremum in $\mathfrak{B}$ of the set in (1). Consequently, the relative product $p\,;q$ in $\mathfrak{B}$ coincides with the relative product $p\,;q$ in $\mathfrak{A}$. A similar argument shows that the values of $p^{\smile}$ in $\mathfrak{A}$ and in $\mathfrak{B}$ coincide. Thus, $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$. □

We are now ready to prove the following *Existence Theorem* for completions of Boolean algebras with quasi-complete operators.

Theorem 15.22. *Every Boolean algebra with quasi-complete operators has a completion with quasi-complete operators.*

Proof. Assume $\mathfrak{A}$ is a Boolean algebra with quasi-complete operators, and let $\mathfrak{B}$ be the algebra constructed in terms of $\mathfrak{A}$ before Lemma 15.21. Thus, the Boolean part of $\mathfrak{B}$ is the completion of the Boolean part of $\mathfrak{A}$, and the additional operations in $\mathfrak{B}$ are defined according to the formulas given in Definition 15.17(iii). It must be verified that $\mathfrak{B}$ satisfies conditions (i)—(iii) of Definition 15.17. The algebra $\mathfrak{B}$ is complete by the choice of the Boolean part of $\mathfrak{B}$ (and Definition 15.4(i)), and $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, by Lemma 15.21, so condition (i) is satisfied. Every non-zero element in $\mathfrak{B}$ is above a non-zero element $\mathfrak{A}$, by the choice of $\mathfrak{B}$ (and Definition 15.4(ii)), so condition (ii) is satisfied. Condition (iii) is satisfied by the very definition of the operations $;$ and $^{\smile}$ in $\mathfrak{B}$. Lemma 15.20 implies that the additional operations of $\mathfrak{B}$ are quasi-complete. □

Just as in the Boolean case, completions are uniquely determined in a very strong sense, as the following *Uniqueness Theorem* for completions of Boolean algebras with quasi-complete operators asserts.

Theorem 15.23. *Any two completions of a Boolean algebra with quasi-complete operators* $\mathfrak{A}$ *are isomorphic via a mapping that is the identity function on* $\mathfrak{A}$.

Proof. Consider two completions of $\mathfrak{A}$, say $\mathfrak{B}$ and $\mathfrak{D}$. The Boolean parts of $\mathfrak{B}$ and $\mathfrak{D}$ are completions of the Boolean part of $\mathfrak{A}$, by the remarks following Definition 15.17, so there must be a Boolean isomorphism ψ from $\mathfrak{B}$ to $\mathfrak{D}$ that is the identity function on $\mathfrak{A}$, by Theorem 15.16. In particular, ψ must map the distinguished constant in $\mathfrak{B}$ to the distinguished constant in $\mathfrak{D}$. Indeed, each of these constants coincides with the distinguished constant 1' in $\mathfrak{A}$, by the assumption that $\mathfrak{A}$ is a subalgebra of both $\mathfrak{B}$ and $\mathfrak{D}$ (condition (i) in Definition 15.17), and ψ maps each element in $\mathfrak{A}$ to itself.

To see that ψ preserves the operation $\,;\,$, consider elements p and q in $\mathfrak{B}$, and let L_p and L_q be the sets of elements in $\mathfrak{A}$ that are below p and q respectively. We have

$$p = \sum L_p \qquad \text{and} \qquad q = \sum L_q, \tag{1}$$

by Lemma 15.5, where these sums are formed in $\mathfrak{B}$. The sets L_p and L_q are subsets of $\mathfrak{A}$, and ψ is the identity function on $\mathfrak{A}$, so ψ is the identity function on each of these sets. In particular,

$$\psi(L_p) = L_p \qquad \text{and} \qquad \psi(L_q) = L_q, \tag{2}$$

where the sets on the left sides of these equations denote the images of the sets L_p and L_q under the mapping ψ. Similarly,

$$p\,;q = \sum(L_p\,;L_q), \tag{3}$$

by condition (iii) in Definition 15.17, where the sum on the right is formed in $\mathfrak{B}$ and

$$L_p\,;L_q = \{r\,;s : r \in L_p \text{ and } L_q\}. \tag{4}$$

The set in (4) is also a subset of $\mathfrak{A}$, so the mapping ψ is the identity function on this set as well. In particular,

$$\psi(L_p\,;L_q) = L_p\,;L_q. \tag{5}$$

From (1), the Boolean isomorphism properties of ψ, and (2) we obtain

$$\psi(p) = \psi(\textstyle\sum L_p) = \sum \psi(L_p) = \sum L_p \tag{6}$$

and

$$\psi(q) = \psi(\textstyle\sum L_q) = \sum \psi(L_q) = \sum L_q, \tag{7}$$

where the first sums in (6) and (7) are formed in $\mathfrak{B}$, and the second and third are formed in $\mathfrak{D}$. Similarly, from (3) and (5) we obtain

$$\psi(p\,;q) = \psi(\textstyle\sum(L_p\,;L_q)) = \sum \psi(L_p\,;L_q) = \sum(L_p\,;L_q). \qquad (8)$$

The operation $;$ in $\mathfrak{D}$ is quasi-completely distributive, by the assumption that $\mathfrak{D}$ is a completion of $\mathfrak{A}$ and by Lemma 15.20. Also, the sets L_p and L_q are non-empty, because each of them contains the element zero. Consequently,

$$\begin{aligned}(\textstyle\sum L_p)\,;(\sum L_q) &= (\textstyle\sum\{r : r \in L_p\})\,;(\sum\{s : s \in L_q\}) \\ &= \textstyle\sum\{r\,;s : r \in L_p \text{ and } s \in L_q\} = \sum(L_p\,;L_q) \qquad (9)\end{aligned}$$

in $\mathfrak{D}$. Combine (6)–(9) to arrive at

$$\psi(p)\,;\psi(q) = (\textstyle\sum L_p)\,;(\sum L_q) = \sum(L_p\,;L_q) = \psi(p\,;q).$$

This shows that ψ preserves the operation $;$. A similar, but easier, argument shows that ψ also preserves the operation $\smile$. Conclusion: ψ is an isomorphism from $\mathfrak{B}$ to $\mathfrak{D}$ that is the identity function on $\mathfrak{A}$. □

The preceding theorem justifies the common practice of referring to *the* completion of a Boolean algebra with quasi-complete operators.

15.4 The preservation theorems

We have seen that every Boolean algebra with quasi-complete operators $\mathfrak{A}$ has a uniquely determined completion with quasi-complete operators, but we don't yet know much about the properties that this completion inherits from $\mathfrak{A}$. In particular, we don't yet know that the completion of a relation algebra is itself a relation algebra. What we need are some general theorems about properties that are preserved under the passage to completions.

In the remainder of the discussion, let $\mathfrak{A}$ be a fixed Boolean algebra with quasi-complete operators, and $\mathfrak{B}$ the completion of $\mathfrak{A}$. Two lemmas concerning positive polynomials are needed. (The notion of a positive polynomial is defined near the beginning of Section 14.4.) The first is the analogue for completions of Lemma 14.29, and its formulation uses some of the notation introduced before that lemma. The proof is identical to the proof of that lemma, and is left as an exercise.

Lemma 15.24. *Every positive polynomial of rank n in $\mathfrak{B}$ is monotone in the sense that $p \leq q$ implies $\gamma(p) \leq \gamma(q)$ for all p, q in B^n.*

The second lemma is the analogue of Lemma 14.31. It's formulation and proof use the following extension of the notation L_p that was introduced in Lemma 15.9. For every sequence p in B^n, put

$$L_p = \{r \in A^n : r \leq p\} = \{r \in A^n : r_i \leq p_i \text{ for } 0 \leq i < n\}.$$

Lemma 15.25. *If γ is a positive polynomial of rank n in $\mathfrak{B}$, then*

$$\gamma(p) = \sum\{\gamma(r) : r \in L_p\}$$

for every p in B^n.

The proof of this lemma is very similar to the proof of Lemma 14.31, but all references in that proof to the set K_p must be replaced by references to the set L_p, and all references to Lemmas 14.29 and 14.26 must be replaced by references to Lemmas 15.24 and 15.20 respectively. The details are left as an exercise.

With the help of Lemma 15.25 it is easy to establish the *First Preservation Theorem* for completions.

Theorem 15.26. *Any positive equation that holds in a Boolean algebra with quasi-complete operators $\mathfrak{A}$ continues to hold in the completion of $\mathfrak{A}$.*

Proof. Consider an arbitrary positive equation ε, say with variables among $v_0, \ldots, v_{n-1}$, and assume that ε is valid in $\mathfrak{A}$. The goal is to show that ε is valid in the algebra $\mathfrak{B}$ that is the completion of $\mathfrak{A}$. To this end, let σ and τ be the (positive) polynomials of rank n in $\mathfrak{B}$ that are induced by the terms on the right and left sides of ε. Since $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, the polynomials of rank n in $\mathfrak{A}$ that are induced by the right and left sides of ε are just the restrictions of σ and τ to $\mathfrak{A}$.

The assumption that ε is valid in $\mathfrak{A}$ means that

$$\sigma(r) = \tau(r)$$

for all sequences r in A^n. In particular, for each sequence p in B^n,

$$\{\sigma(r) : r \in L_p\} = \{\tau(r) : r \in L_p\},$$

since L_p is a subset of A^n. Form the sums in $\mathfrak{B}$ of both sides of this equation, and use Lemma 15.25, to arrive at

$$\sigma(p) = \sum\{\sigma(r) : r \in L_p\} = \sum\{\tau(r) : r \in L_p\} = \tau(p).$$

Thus, σ and τ agree on all sequences of n elements from $\mathfrak{B}$, so ε is valid in $\mathfrak{B}$. $\square$

The only property of the operations $;$ and $\smile$ in the algebra $\mathfrak{A}$ that is needed in the lemmas leading up to Theorem 15.26 is the quasi-complete distributivity of these operations. Consequently, Lemmas 15.18–15.25 and Theorem 15.26 can be extended without difficulty to Boolean algebras with quasi-complete operators of arbitrary ranks. Consider, as an example, a quasi-complete operator O of rank n on the universe of $\mathfrak{A}$. Let $\mathfrak{A}^*$ be the expanded algebra obtained from $\mathfrak{A}$ by adjoining O as a new fundamental operation. The definition of a completion $\mathfrak{B}^*$ of $\mathfrak{A}^*$ is obtained from Definition 15.17 by adjoining to condition (iii) the requirement that the corresponding operation O in $\mathfrak{B}^*$ satisfies the condition

$$O(p) = \sum\{O(r) : r \leq p \text{ and } r \in A^n\}$$

for every sequence p of n elements in $\mathfrak{B}^*$. All the results from Lemma 15.18 through the First Preservation Theorem 15.26 continue to hold with minimal changes in their proofs.

As a concrete example, consider the unary discriminator O on the universe of $\mathfrak{A}$ that was defined and discussed after the First Preservation Theorem 14.32 for canonical extensions. As was pointed out there, this operation is completely distributive. Let $\mathfrak{B}$ be the completion of $\mathfrak{A}$, and define a unary *completion* of the operation O on the universe of $\mathfrak{B}$ by

$$O(p) = \sum\{O(r) : r \leq p \text{ and } r \in A\}$$

for every element p in $\mathfrak{B}^*$ (where the occurrence of O on the right side of this equation denotes the unary discriminator on $\mathfrak{A}$, and the occurrence on the left side denotes the completion of the unary discriminator on $\mathfrak{A}$ that is being defined on $\mathfrak{B}$). The algebra $\mathfrak{B}^*$ obtained by adjoining this operation to $\mathfrak{B}$ is the completion of $\mathfrak{A}^*$, by Theorem 15.22 (in the version applicable to $\mathfrak{A}^*$ and $\mathfrak{B}^*$). It is easy to check that the new operation O in $\mathfrak{B}$ is the unary discriminator on $\mathfrak{B}$, that is to say, it assumes the value 0 on the zero element, and it assumes the value 1 on all non-zero elements in $\mathfrak{B}$.

The remarks in the preceding paragraphs imply that every positive equation true in $\mathfrak{A}^*$ must also be true in $\mathfrak{B}^*$, by the First Preservation Theorem 15.26 applied to the expanded algebras $\mathfrak{A}^*$ and $\mathfrak{B}^*$. By imitating the proof of the Second Preservation Theorem 14.34 for canonical extensions, with all references to canonical extensions replaced by references to completions, and all references to Theorem 14.32 replaced by references to Theorem 15.26, one arrives at a proof of the following analogue of Theorem 14.34, the *Second Preservation Theorem* for completions.

Theorem 15.27. *Let* ε *be any Boolean combination of positive equations of the form* $\varrho = 0$ *(with* ϱ *positive), and* σ *and* τ *any positive terms, in the augmented language of a Boolean algebra with quasi-complete operators* $\mathfrak{A}$*. If the implication*

$$\varepsilon \to (\sigma = \tau)$$

holds in $\mathfrak{A}$*, then it holds in the completion of* $\mathfrak{A}$*.*

15.5 Applications to relation algebras

We have seen that every Boolean algebra with quasi-complete operators has a uniquely determined completion. A relation algebra is a Boolean algebra with operators that are not only quasi-complete, but in fact complete (see Lemmas 4.2 and 4.16). It follows that every relation algebra has a uniquely determined completion. The two preservation theorems imply that this completion is itself a relation algebra.

Theorem 15.28. *Every relation algebra* $\mathfrak{A}$ *has a completion that is uniquely determined up to isomorphisms that are the identity function on* $\mathfrak{A}$*, and this completion is a relation algebra.*

The proof is very similar to the proof that every relation algebra has a uniquely determined canonical extension that is a relation algebra (Theorem 14.35). References in that proof to the Existence Theorem, the Uniqueness Theorem, and the First and Second Preservation Theorems for canonical extensions must be replaced with references to the corresponding theorems for completions. The details are left as an exercise.

As in the case of canonical extensions, it is natural to ask what other properties of elements, sets of elements, and algebras are preserved under the passage to completions. Equationally defined properties of elements are always preserved. For example, a function or an ideal element in a relation algebra $\mathfrak{A}$ remains a function or an ideal element in the completion of $\mathfrak{A}$. A property of elements that is not of this form, but that is still preserved, is the property of being an atom. In fact, a stronger result is true than in the case of canonical extensions.

Lemma 15.29. *If $\mathfrak{B}$ is the completion of a relation algebra $\mathfrak{A}$, then an element in $\mathfrak{B}$ is an atom if and only if it is already an atom in $\mathfrak{A}$. Consequently, $\mathfrak{B}$ is atomic if and only if $\mathfrak{A}$ is atomic.*

Proof. Every element p in the completion $\mathfrak{B}$ is the sum of the elements in $\mathfrak{A}$ that are below p, by Lemma 15.5. This immediately implies that an atom in $\mathfrak{A}$ must remain an atom in $\mathfrak{B}$. On the other hand, if p is an atom in $\mathfrak{B}$, then the set of elements in $\mathfrak{B}$ that are below p must contain exactly one non-zero element, namely p itself, by the definition of an atom. Since p is above a non-zero element in $\mathfrak{A}$, by Definition 15.17(ii), it follows that p must belong to $\mathfrak{A}$ and therefore must be an atom in $\mathfrak{A}$.

Assume now that $\mathfrak{A}$ is atomic. Every non-zero element p in $\mathfrak{B}$ is above a non-zero element r in $\mathfrak{A}$, by Definition 15.17(ii), and every non-zero element r in $\mathfrak{A}$ is above an atom s in $\mathfrak{A}$, by the assumption that $\mathfrak{A}$ is atomic. The atoms in $\mathfrak{A}$ remain atoms in $\mathfrak{B}$, by the remarks of the preceding paragraph, so every non-zero element p in $\mathfrak{B}$ is above an atom s in $\mathfrak{B}$. Consequently, $\mathfrak{B}$ is atomic.

In the reverse direction, if $\mathfrak{B}$ is atomic, then every non-zero element in $\mathfrak{B}$—and in particular, every non-zero element in $\mathfrak{A}$—must be above an atom in $\mathfrak{B}$. The atoms in $\mathfrak{B}$ are precisely the atoms in $\mathfrak{A}$, by the observations of the first paragraph, so every non-zero element in $\mathfrak{A}$ must be above an atom in $\mathfrak{A}$. Therefore, $\mathfrak{A}$ is atomic. □

The preceding argument goes through in part because the property of atomicity is a density property: it says that the set of atoms is dense in the algebra. Similar arguments apply in the case of other density properties. For example, a relation algebra $\mathfrak{A}$ is said to be *functionally dense* if the set of non-zero functions is dense in $\mathfrak{A}$ in the sense that every non-zero element in $\mathfrak{A}$ is above a non-zero function. An argument similar to the one given in the preceding proof shows that a relation algebra is functionally dense if and only if its completion is functionally dense.

Theorem 15.28 and Lemma 15.29 ensure that the completion of an atomic relation algebra $\mathfrak{A}$ is a complete and atomic relation algebraic extension of $\mathfrak{A}$ in which all suprema existing in $\mathfrak{A}$ are preserved. For this reason, in the case of an *atomic* relation algebra $\mathfrak{A}$, it is almost always unnecessary to pass to the canonical extension of $\mathfrak{A}$ in order to analyze $\mathfrak{A}$ itself, since the completion of $\mathfrak{A}$ possesses all of the properties that are required for this analysis.

A complete relation algebra $\mathfrak{A}$ is always its own completion, as is easily verified by checking that the conditions in Definition 15.17 are satisfied in this case (with $\mathfrak{A}$ in place of $\mathfrak{B}$). In particular, every finite relation algebra is its own completion.

This raises the question of the relative size of the completion of a relation algebra $\mathfrak{A}$ in comparison with the size of $\mathfrak{A}$ when $\mathfrak{A}$ is infinite. If $\mathfrak{A}$ has infinite cardinality m, then the completion of $\mathfrak{A}$ has cardinality between m and 2^m. Indeed, $\mathfrak{A}$ is a subalgebra of its completion $\mathfrak{B}$, so the cardinality of $\mathfrak{B}$ is at least m (and it is exactly m when $\mathfrak{A}$ is complete). On the other hand, every element p in $\mathfrak{B}$ corresponds to a uniquely determined subset of $\mathfrak{A}$, namely the set L_p of elements in $\mathfrak{A}$ that are below p, since p is the sum of L_p in $\mathfrak{B}$. For this reason, there can be at most as many element in $\mathfrak{B}$ as their are subsets of $\mathfrak{A}$, and this number is 2^m.

For relation algebras, the properties of being simple and being integral are preserved under the passage to completions. The proof of this assertion is very similar to the proof of the corresponding result for canonical extensions (Theorem 14.36), and is left as an exercise.

Theorem 15.30. *A relation algebra is simple or integral if and only if its completion is simple or integral respectively.*

The part of the preceding theorem that concerns simplicity may be viewed as a statement about the Boolean algebra of ideal elements in a relation algebra $\mathfrak{A}$ and its completion: it says that if this Boolean algebra has exactly two elements in $\mathfrak{A}$, then it has exactly two elements in the completion of $\mathfrak{A}$. In this form, the result can be generalized.

Theorem 15.31. *If* $\mathfrak{B}$ *is the completion of a relation algebra* $\mathfrak{A}$, *then the Boolean algebra of ideal elements in* $\mathfrak{B}$ *is the Boolean completion of the Boolean algebra of ideal elements in* $\mathfrak{A}$.

Proof. Write A_0 and B_0 for the Boolean algebras of ideal elements in $\mathfrak{A}$ and in $\mathfrak{B}$ respectively. It must be shown that B_0 satisfies conditions (i)–(iii) in Definition 15.4 with respect to A_0. The algebra B_0 is complete, by Corollary 8.25. Also, A_0 and B_0 are strongly regular Boolean subalgebras of the Boolean parts of $\mathfrak{A}$ and $\mathfrak{B}$ respectively, by Lemma 8.24, and $\mathfrak{A}$ is a regular subalgebra of its completion $\mathfrak{B}$, by Lemma 15.6 (in its application to $\mathfrak{A}$ and $\mathfrak{B}$), so A_0 is a regular Boolean subalgebra of B_0. Thus, condition (i) holds.

To verify condition (ii), consider a non-zero element p in B_0. There must be a non-zero element r in $\mathfrak{A}$ that is below p, because $\mathfrak{B}$ is the completion of $\mathfrak{A}$ (see condition (ii) in Definition 15.17). The ideal element generated by r in $\mathfrak{A}$, namely $1 ; r ; 1$, is a non-zero element in A_0 that is below p, because $0 < r \leq p$ and therefore

$$0 < r \leq 1 ; r ; 1 \leq 1 ; p ; 1 = p,$$

by Lemma 4.5(iii) and its first dual, the monotony law for relative multiplication, and the assumption that p is in B_0 and is therefore an ideal element. Thus, every non-zero element in B_0 is above a non-zero element in A_0. □

The preceding theorem can be used together with Lemma 15.29 to establish a stronger version of Theorem 14.38 for completions that applies to an arbitrary number of factors, and not just to a finite number of factors.

Theorem 15.32. *If a relation algebra* $\mathfrak{A}$ *has a total decomposition*

$$\mathfrak{A} = \prod_{i \in I} \mathfrak{A}(a_i), \tag{i}$$

then its completion $\mathfrak{B}$ *has the total decomposition*

$$\mathfrak{B} = \prod_{i \in I} \mathfrak{B}(a_i). \tag{ii}$$

Inversely, if $\mathfrak{B}$ *has a total decomposition in* (ii), *and if the system* $(a_i : i \in I)$ *has the supremum property in* $\mathfrak{A}$, *then* $\mathfrak{A}$ *has the total decomposition in* (i).

Proof. Let A_0 and B_0 be the Boolean algebras of ideal elements in $\mathfrak{A}$ and $\mathfrak{B}$ respectively. The algebra B_0 is the completion of the algebra A_0, by Theorem 15.31, so the two algebras have the same (ideal element) atoms, and one of them is atomic if and only if the other is atomic, by the Boolean version of Lemma 15.29. Let

$$(a_i : i \in I) \tag{1}$$

be a list of the distinct atoms in these Boolean algebras.

If $\mathfrak{A}$ has the total direct decomposition given in (i), then A_0 is atomic, by the Total Decomposition Theorem 11.41; therefore, B_0 is atomic, and (1) is a list of its distinct atoms, by the remarks of the first paragraph. Apply the Atomic Decomposition Theorem 11.44 to conclude that $\mathfrak{B}$ has the total direct decomposition given in (ii). Inversely, if $\mathfrak{B}$ has the total direct decomposition given in (ii), then B_0 is atomic, by Theorem 11.41; therefore, A_0 is atomic, and (1) is a list of its distinct atoms, by the remarks of the first paragraph. If, in addition, the system in (1) has the supremum property in $\mathfrak{A}$, then $\mathfrak{A}$ has the total direct decomposition given in (i), by Theorem 11.41. □

The preceding theorem sheds light on a useful application of completions. A given relation algebra $\mathfrak{A}$ may be very close to being totally decomposable, because its Boolean algebra of ideal elements is atomic; but the supremum property may fail to hold for the system of ideal element atoms in $\mathfrak{A}$. By passing to the completion $\mathfrak{B}$, the missing suprema are filled in and we obtain a total decomposition of $\mathfrak{B}$ that is structurally very close to the total decomposition that we would like to have for $\mathfrak{A}$.

When the Boolean algebra of ideal elements is finite, the supremum property automatically holds for the system of ideal element atoms. Consequently, we at once obtain the following analogue of Theorem 14.38.

Corollary 15.33. *A relation algebra* $\mathfrak{A}$ *has a total decomposition into finitely many simple factors*

$$\mathfrak{A} = \mathfrak{A}(a_0) \times \cdots \times \mathfrak{A}(a_{n-1})$$

if and only if its completion $\mathfrak{B}$ *has the total decomposition*

$$\mathfrak{B} = \mathfrak{B}(a_0) \times \cdots \times \mathfrak{B}(a_{n-1}).$$

15.6 Completions of homomorphisms

The discussion so far about the preservation of properties has been limited to properties involving a single algebra and its completion. The properties that involve more than one algebra, for instance the property of one algebra being a subalgebra or a homomorphic image of another, require some extensions of Lemmas 15.18–15.21 in another direction. Instead of just considering quasi-complete operators, one may also consider functions with arguments that are sequences of elements (of some fixed length n) in one algebra $\mathfrak{A}$ and values that are sequences of elements (of some fixed length m) in another algebra $\bar{\mathfrak{A}}$. If φ is such a function, and if $\mathfrak{B}$ is the completion of $\mathfrak{A}$, and $\bar{\mathfrak{B}}$ any complete extension of $\mathfrak{A}$ with quasi-complete operators such that $\bar{\mathfrak{A}}$ is a regular subalgebra of $\bar{\mathfrak{B}}$—note that $\bar{\mathfrak{B}}$ need not be the completion of $\bar{\mathfrak{A}}$—then the *completion* of φ is defined to be the function φ^+ from $\mathfrak{B}$ into $\bar{\mathfrak{B}}$ whose value at each sequence p of n elements in $\mathfrak{B}$ is given by

$$\varphi^+(p) = \sum\{\varphi(r) : r \in A^n \text{ and } r \leq p\} = \sum\{\varphi(r) : r \in L_p\},$$

where the sum is formed coordinatewise in $\bar{\mathfrak{B}}$. The general theorem that one obtains says, roughly speaking, that if φ is quasi-complete, then the completion of φ is quasi-complete and inherits the positively expressible properties of φ. For example, if one composes functions that are quasi-complete, then the completion of the composition of the functions is equal to the composition of the completions of the functions.

Actually, since $\bar{\mathfrak{A}}$ is assumed to be a regular subalgebra of $\bar{\mathfrak{B}}$, all of the suprema that exist in $\bar{\mathfrak{A}}$ are preserved under the passage to $\bar{\mathfrak{B}}$. For that reason, one may dispense with the algebra $\bar{\mathfrak{A}}$ entirely and focus instead on the target algebra $\bar{\mathfrak{B}}$. In other words, one may consider φ to be a function from $\mathfrak{A}$ into $\bar{\mathfrak{B}}$, and φ^+ a function from $\mathfrak{B}$ into $\bar{\mathfrak{B}}$.

We illustrate the ideas with an important concrete example. Let $\mathfrak{A}$ be a Boolean algebra with quasi-complete operators, let $\mathfrak{B}$ the completion of $\mathfrak{A}$, and let $\bar{\mathfrak{B}}$ be any complete Boolean algebra with quasi-complete operators (of the same similarity type as $\mathfrak{A}$). For notational convenience, we shall assume that the similarity type of the algebras is the same as that of relation algebras. Consider a mapping φ from $\mathfrak{A}$ into $\bar{\mathfrak{B}}$, and for each subset X of $\mathfrak{A}$ write

$$\varphi(X) = \{\varphi(r) : r \in X\}.$$

The *completion* of φ is the mapping φ^+ from $\mathfrak{B}$ into $\bar{\mathfrak{B}}$ that is defined at each p in $\mathfrak{B}$ by

$$\varphi^+(p) = \sum \varphi(L_p),$$

where the sum on the right is formed in $\bar{\mathfrak{B}}$. (The completeness of $\bar{\mathfrak{B}}$ is needed in order to ensure that this sum really does exist.) The immediate goal is to prove that if φ is a complete homomorphism, then φ^+ is a complete homomorphism extending φ. The argument involves a series of lemmas that are the analogues of Lemmas 15.18–15.21 above.

We begin with just the assumption that φ is a *quasi-compete* mapping. This means that φ preserves all existing suprema of non-empty sets. In other words, if X is a non-empty subset of $\mathfrak{A}$ for which $p = \sum X$ exists (in $\mathfrak{A}$), then

$$\varphi(p) = \sum \varphi(X).$$

(Since $\bar{\mathfrak{B}}$ is complete, the supremum on the right always exists in $\bar{\mathfrak{B}}$.)

The analogue of Lemma 15.21 says that φ^+ is an extension of φ. Indeed, if p is an element in $\mathfrak{A}$, then p is the largest element in the set L_p, and consequently p is the supremum of L_p in $\mathfrak{A}$. The assumption that φ is quasi-complete therefore implies that $\varphi(p)$ is the supremum of the set $\varphi(L_p)$ in $\bar{\mathfrak{B}}$. The element $\varphi^+(p)$ is, by definition, the supremum of the set $\varphi(L_p)$ in $\bar{\mathfrak{B}}$, so $\varphi^+(p) = \varphi(p)$.

The analogue of Lemma 15.18 says that φ^+ is monotone in the sense that $p \le q$ implies $\varphi^+(p) \le \varphi^+(q)$ for all elements p and q in $\mathfrak{B}$. The analogue of Lemma 15.19 says that if X is a non-empty subset of $\mathfrak{A}$, and if $p = \sum X$ in $\mathfrak{B}$, then

$$\varphi^+(p) = \sum \varphi(X) = \sum\{\varphi(r) : r \in X\},$$

where the sum is formed in $\bar{\mathfrak{B}}$. The analogue of the implication from right to left in Lemma 15.20 says that the mapping φ^+ is quasi-complete. In other words, for every non-empty subset X of $\mathfrak{B}$ (and not just of $\mathfrak{A}$), if $p = \sum X$, then

$$\varphi^+(p) = \sum\{\varphi^+(r) : r \in X\}.$$

The proofs of these lemmas are slightly modified versions of the proofs of the original lemmas, and are left as exercises.

The next step is to prove an analogue of Lemma 14.40 for completions.

Lemma 15.34. *For arbitrary elements p and q in $\mathfrak{B}$,*

(i) $\varphi^+(p \cdot q) = \sum \varphi(L_p \cdot L_q)$,
(ii) $\varphi^+(p) \cdot \varphi^+(q) = \sum(\varphi(L_p) \cdot \varphi(L_q))$,
(iii) $\varphi^+(p \,;\, q) = \sum \varphi(L_p \,;\, L_q)$,
(iv) $\varphi^+(p) \,;\, \varphi^+(q) = \sum(\varphi(L_p) \,;\, \varphi(L_q))$,
(v) $\varphi^+(p^\smile) = \sum \varphi(L_p^\smile)$,
(vi) $\varphi^+(p)^\smile = \sum(\varphi(L_p)^\smile)$.

Proof. Focus on the proofs of (iii) and (iv), and begin with (iii). From Lemma 15.5, we have

$$p = \sum L_p \qquad \text{and} \qquad q = \sum L_q, \tag{1}$$

and from Definition 15.17(iii), we have

$$p \,;\, q = \sum(L_p \,;\, L_q). \tag{2}$$

Observe that the sets on the right sides of these three equations are non-empty subsets of $\mathfrak{A}$, since the first two sets contain the element 0, and the last set contains the element $0 \,;\, 0$. Use (2) and the analogue of Lemma 15.19 for φ^+ (with the set $L_p \,;\, L_q$ in place of X) to arrive at (iii). A similar argument, with (1) in place of (2), yields

$$\varphi^+(p) = \sum \varphi(L_p) \qquad \text{and} \qquad \varphi^+(q) = \sum \varphi(L_q). \tag{3}$$

The quasi-complete distributivity of the operation $\,;\,$ in $\bar{\mathfrak{B}}$, and the definitions of the sets involved, imply that

$$\begin{aligned}(\sum \varphi(L_p)) \,;\, (\sum \varphi(L_q)) &= (\sum\{\varphi(r) : r \in L_p\}) \,;\, (\sum\{\varphi(s) : s \in L_q\}) \\ &= \sum\{\varphi(r) \,;\, \varphi(s) : r \in L_p \text{ and } s \in L_q\} \\ &= \sum(\varphi(L_p) \,;\, \varphi(L_q)). \end{aligned} \tag{4}$$

Combine (3) and (4) to conclude that

$$\varphi^+(p) \,;\, \varphi^+(q) = \sum(\varphi(L_p) \,;\, \varphi(L_q)).$$

This proves (iv). □

Here is the *Existence Theorem* for completions of complete homomorphisms.

Theorem 15.35. *Let $\mathfrak{A}$ be a Boolean algebra with quasi-complete operators, $\mathfrak{B}$ the completion of $\mathfrak{A}$, and $\bar{\mathfrak{B}}$ a complete Boolean algebra with quasi-complete operators. If φ is a complete homomorphism from $\mathfrak{A}$ into $\bar{\mathfrak{B}}$, then φ^+ is a complete homomorphism from $\mathfrak{B}$ into $\bar{\mathfrak{B}}$ that extends φ. If φ is one-to-one, then so is φ^+. If $\bar{\mathfrak{B}}$ is the completion of the image of $\mathfrak{A}$ under φ, then φ^+ is onto.*

Proof. The function φ^+ is an extension of φ, by the analogue of Lemma 15.21 for φ^+, and in particular

$$\varphi^+(0) = \varphi(0) = 0, \quad \varphi^+(1) = \varphi(1) = 1, \quad \varphi^+(1') = \varphi(1') = 1'. \tag{1}$$

To see that φ^+ preserves the operation $\,;\,$, consider elements p and q in $\mathfrak{B}$. The assumed homomorphism properties of φ and the definitions of the sets involved imply that

$$\begin{aligned}\varphi(L_p \,;\, L_q) &= \{\varphi(r \,;\, s) : r \in L_p \text{ and } s \in L_q\} \\ &= \{\varphi(r) \,;\, \varphi(s) : r \in L_p \text{ and } s \in L_q\} \\ &= \varphi(L_p) \,;\, \varphi(L_q).\end{aligned}$$

Consequently,

$$\sum \varphi(L_p \,;\, L_q) = \sum(\varphi(L_p) \,;\, \varphi(L_q)), \tag{2}$$

by Boolean algebra. Combine (2) with parts (iii) and (iv) of Lemma 15.34 to arrive at

$$\varphi^+(p \,;\, q) = \sum \varphi(L_p \,;\, L_q) = \sum(\varphi(L_p)) \,;\, (\varphi(L_q)) = \varphi^+(p) \,;\, \varphi^+(q).$$

The proof that φ^+ preserves the operation of multiplication is almost identical to the preceding argument. One replaces the operation ; everywhere with the operation $\cdot$ and uses parts (i) and (ii) of Lemma 15.34. The proof that φ^+ preserves the operation $^\smile$ is similar, but easier, and uses parts (v) and (vi) of Lemma 15.34. The details are left as an exercise.

The function φ^+ preserves arbitrary non-empty sums, by the analogue of Lemma 15.20 for φ^+, and it preserves the empty sum, by the first equation in (1), so it preserves arbitrary sums. In particular, it preserves the operation of addition. A mapping between Boolean algebras with operators that preserves addition and multiplication, and maps zero to zero, and one to one, is necessarily a Boolean homomorphism (see the remarks following the proof of Lemma 7.6). Conclusion: φ^+ is a complete homomorphism from $\mathfrak{B}$ into $\bar{\mathfrak{B}}$ that extends φ.

Assume now that φ is one-to-one, with the goal of showing that φ^+ is one-to-one. Consider a non-zero element p in $\mathfrak{B}$. There must be a non-zero element r in $\mathfrak{A}$ that is below p, by Definition 15.17(ii). The homomorphism φ is assumed to be one-to-one, so $\varphi(r)$ is not zero. The homomorphism φ^+ is monotone, by the analogue of Lemma 15.18 for φ^+, and it extends φ, by the analogue of Lemma 15.21 for φ^+, so

$$0 < \varphi(r) = \varphi^+(r) \le \varphi^+(p).$$

Thus, every non-zero element in $\mathfrak{B}$ is mapped by φ^+ to a non-zero element in $\bar{\mathfrak{B}}$. It follows that φ^+ is one-to-one, by Lemma 8.37.

Turn now to the final assertion of the theorem. Write $\bar{\mathfrak{A}}$ for the image of the algebra $\mathfrak{A}$ under the homomorphism φ, and suppose that $\bar{\mathfrak{B}}$ is the completion of $\bar{\mathfrak{A}}$. To prove that φ^+ maps $\mathfrak{B}$ onto $\bar{\mathfrak{B}}$, consider an element q in $\bar{\mathfrak{B}}$, and let $\bar{L}_q$ be the set of elements in $\bar{\mathfrak{A}}$ that are below q. Observe that $q = \sum \bar{L}_q$, by Lemma 15.5 applied to $\bar{\mathfrak{B}}$. For each element s in $\bar{L}_q$, there must be an element r_s in $\mathfrak{A}$ that is mapped to s by φ, because φ maps $\mathfrak{A}$ onto $\bar{\mathfrak{A}}$. The set

$$X = \{r_s : s \in \bar{L}_q\}$$

is a subset of $\mathfrak{A}$ that is mapped onto $\bar{L}_q$ by the homomorphism φ. Also, X has a supremum p in $\mathfrak{B}$, because $\mathfrak{B}$ is the completion of $\mathfrak{A}$. Use the analogue of Lemma 15.19 for φ^+, and the preceding observations, to arrive at

$$\varphi^+(p) = \sum \varphi(X) = \sum \bar{L}_q = q.$$

Thus, every element in $\bar{\mathfrak{B}}$ is the image of an element in $\mathfrak{B}$, so φ^+ is onto.

□

It turns out that the completion of a complete homomorphism φ is the only possible complete extension of φ, as the following *Uniqueness Theorem* for completions of complete homomorphisms makes clear.

Theorem 15.36. *Let $\mathfrak{A}$ be a Boolean algebra with quasi-complete operators, $\mathfrak{B}$ the completion of $\mathfrak{A}$, and $\bar{\mathfrak{B}}$ a complete Boolean algebra with quasi-complete operators. A complete homomorphism from $\mathfrak{A}$ into $\bar{\mathfrak{B}}$ has only one extension to a complete homomorphism from $\mathfrak{B}$ to $\bar{\mathfrak{B}}$.*

Proof. A complete homomorphism φ from $\mathfrak{A}$ into $\bar{\mathfrak{B}}$ certainly has one extension to a complete homomorphism from $\mathfrak{B}$ to $\bar{\mathfrak{B}}$, namely the completion φ^+. Let ψ be any complete homomorphism from $\mathfrak{B}$ to $\bar{\mathfrak{B}}$ that extends φ. Each element p in $\mathfrak{B}$ is the sum of the set L_p of elements in $\mathfrak{A}$ that are below p, so

$$\psi(p) = \sum \psi(L_p) = \sum \varphi(L_p) = \varphi^+(p),$$

by the completeness of the homomorphism ψ, the assumption that ψ extends φ, and the definition of the homomorphism φ^+. It follows that ψ and φ^+ agree on all elements in $\mathfrak{B}$, so the two homomorphisms coincide. □

The preceding theorem implies that the completion of the composition of two complete homomorphisms is equal to the composition of the completions of the two homomorphisms.

Corollary 15.37. *Let $\mathfrak{A}$, $\bar{\mathfrak{A}}$, and $\bar{\bar{\mathfrak{A}}}$ be Boolean algebras with quasi-complete operators, and $\mathfrak{B}$, $\bar{\mathfrak{B}}$, and $\bar{\bar{\mathfrak{B}}}$ their respective completions. If φ is a complete homomorphism from $\mathfrak{A}$ into $\bar{\mathfrak{A}}$, and ψ a complete homomorphism from $\bar{\mathfrak{A}}$ into $\bar{\bar{\mathfrak{A}}}$, then the completion of the composition $\psi \circ \varphi$ is just the composition of the completions $\psi^+ \circ \varphi^+$.*

The proof is very similar to the proof of Corollary 14.43, and is left as an exercise.

15.7 Minimality

A Boolean algebra with quasi-complete operations $\mathfrak{A}$ has many complete extensions with quasi-complete operators, that is to say, there are many complete Boolean algebras with quasi-complete operators that include $\mathfrak{A}$ as a subalgebra. For instance, when $\mathfrak{A}$ is infinite, the canonical extension of $\mathfrak{A}$, the canonical extension of the canonical extension of $\mathfrak{A}$, and so on are all distinct complete extensions of $\mathfrak{A}$ with quasi-complete operators. The completion of $\mathfrak{A}$ distinguishes itself from these other complete extensions by its minimality: it is the smallest complete extension of $\mathfrak{A}$ with quasi-complete operators that preserves all suprema existing in $\mathfrak{A}$. The following precise statement of this result is known as the *Minimality Theorem* for completions.

Theorem 15.38. *The completion of a Boolean algebra with quasi-complete operators* $\mathfrak{A}$ *can be completely embedded, via a mapping that is the identity function on* $\mathfrak{A}$, *into any regular extension of* $\mathfrak{A}$ *that is complete with quasi-complete operators.*

Proof. Suppose $\mathfrak{B}$ is the completion of $\mathfrak{A}$, and let $\mathfrak{C}$ be any regular extension of $\mathfrak{A}$ that is complete with quasi-complete operators. Take φ to be the identity automorphism of $\mathfrak{A}$, and observe that φ is a complete monomorphism of $\mathfrak{A}$ into $\mathfrak{C}$, by Lemma 7.9 and the assumption that $\mathfrak{A}$ is a regular subalgebra of $\mathfrak{C}$. The completion of φ is a complete monomorphism from $\mathfrak{B}$ into $\mathfrak{C}$ that extends φ, by the Existence Theorem 15.35 for completions of complete homomorphisms. Since φ is the identity function on $\mathfrak{A}$, it follows that the restriction of φ^+ must also be the identity function on $\mathfrak{A}$. □

Here is another way to think about the minimality of completions.

Lemma 15.39. *If* $\mathfrak{B}$ *is the completion of a Boolean algebra with quasi-complete operators* $\mathfrak{A}$, *then the only complete homomorphism of* $\mathfrak{B}$ *into itself that is the identity function on* $\mathfrak{A}$ *is the identity automorphism of* $\mathfrak{B}$.

Proof. Let φ be the identity automorphism of $\mathfrak{A}$, and φ^+ the completion of φ. Existence Theorem 15.35 (with $\mathfrak{B}$ in place of $\mathfrak{C}$) implies that φ^+ is an automorphism of $\mathfrak{B}$ extending φ. The identity function on $\mathfrak{B}$ is also an automorphism of $\mathfrak{B}$ extending φ. Consequently, φ^+ must be the identity automorphism of $\mathfrak{B}$, by Uniqueness Theorem 15.36. Any other complete homomorphism from $\mathfrak{B}$ into itself that agrees with φ on $\mathfrak{A}$ must coincide with φ^+, by Theorem 15.36, and must therefore be the identity automorphism of $\mathfrak{B}$, by the preceding observation. □

The minimality of the completion may be characterized in yet another way.

Theorem 15.40. *If* $\mathfrak{A}$ *and* $\mathfrak{C}$ *are Boolean algebra with quasi-complete operators, then* $\mathfrak{C}$ *is the completion of* $\mathfrak{A}$ *if and only if* $\mathfrak{C}$ *is a regular extension of* $\mathfrak{A}$ *that is complete, and no proper subalgebra of* $\mathfrak{C}$ *that extends* $\mathfrak{A}$ *is complete with quasi-complete operators.*

Proof. Let $\mathfrak{B}$ be the completion of $\mathfrak{A}$, which exists by the Existence Theorem 15.22 for completions. The first step is to show that $\mathfrak{B}$ has the properties stated in the theorem (with $\mathfrak{B}$ in place of $\mathfrak{C}$). To this end, let $\mathfrak{D}$ be any subalgebra of $\mathfrak{B}$ that extends $\mathfrak{A}$ and is complete with quasi-complete operators. It is to be shown that $\mathfrak{D}$ coincides with $\mathfrak{B}$.

The algebra $\mathfrak{A}$ is a regular subalgebra of $\mathfrak{B}$, because $\mathfrak{B}$ is the completion of $\mathfrak{A}$ (see Lemma 15.6), and $\mathfrak{A}$ is a subalgebra of $\mathfrak{D}$, which in turn is a subalgebra of $\mathfrak{B}$, by assumption. It follows that $\mathfrak{A}$ must be a regular subalgebra of $\mathfrak{D}$, by Lemma 6.16(ii) (with $\mathfrak{A}$, $\mathfrak{D}$, and $\mathfrak{B}$ in place of $\mathfrak{C}$, $\mathfrak{B}$, and $\mathfrak{A}$ respectively). Also, $\mathfrak{D}$ is assumed to be complete with quasi-complete operators. Apply Minimality Theorem 15.38 to obtain a complete embedding ψ of $\mathfrak{B}$ into $\mathfrak{D}$ that is the identity function on $\mathfrak{A}$.

It is not hard to see that ψ is the identity function on $\mathfrak{D}$ as well. Indeed, every element p in $\mathfrak{D}$ belongs to $\mathfrak{B}$ (since $\mathfrak{D}$ is a subalgebra of $\mathfrak{B}$) and is therefore the supremum in $\mathfrak{B}$ of some subset X of $\mathfrak{A}$, by Lemma 15.5 and the assumption that $\mathfrak{B}$ is the completion of $\mathfrak{A}$. Since p belongs to $\mathfrak{D}$, it must also be the supremum in $\mathfrak{D}$ of the set X, by Lemma 6.15. Use this observation, the completeness of the mapping ψ, and the fact that ψ is the identity function on the elements in $\mathfrak{A}$ (and therefore on the elements in X) to arrive at

$$\psi(p) = \psi(\textstyle\sum X) = \sum\{\psi(r) : r \in X\} = \sum X = p,$$

where the first sum is formed in $\mathfrak{B}$, and the second and third sums are formed in $\mathfrak{D}$.

The desired conclusion now follows easily. The function ψ maps $\mathfrak{B}$ into $\mathfrak{D}$, and it maps each element in $\mathfrak{D}$ (which is a subalgebra of $\mathfrak{B}$) to itself. The one-to-oneness of ψ therefore forces $\mathfrak{B}$ to coincide with $\mathfrak{D}$. This establishes the implication from left to right in the theorem.

To establish the reverse implication, assume that $\mathfrak{C}$ has the stated property—namely, it is a regular extension of $\mathfrak{A}$ that is complete, and no proper subalgebra of $\mathfrak{C}$ that extends $\mathfrak{A}$ is complete with quasi-complete operators. The goal is to show that $\mathfrak{B}$ is isomorphic to $\mathfrak{C}$ via a mapping that is the identity function on $\mathfrak{A}$, so that $\mathfrak{C}$ is also a completion of $\mathfrak{A}$. There is a complete embedding ψ of $\mathfrak{B}$ into $\mathfrak{C}$ that is the identity function on $\mathfrak{A}$, by Minimality Theorem 15.38. The image of $\mathfrak{B}$ under ψ is of course a subalgebra of $\mathfrak{C}$ that extends $\mathfrak{A}$, and this image algebra is complete with quasi-complete operators, because $\mathfrak{B}$ has this property. Consequently, this image algebra must coincide with $\mathfrak{C}$, by the assumptions on $\mathfrak{C}$, so ψ is an isomorphism from $\mathfrak{B}$ to $\mathfrak{C}$ that is the identity function on $\mathfrak{A}$. □

A relation algebra is a Boolean algebra with complete (and not just quasi-complete) operators, so for relation algebras the preceding theorem may be formulated in a somewhat simpler way.

Corollary 15.41. *A relation algebra $\mathfrak{C}$ is the completion of a relation algebra $\mathfrak{A}$ if and only if $\mathfrak{C}$ is a regular extension of $\mathfrak{A}$ that is complete, and no proper subalgebra of $\mathfrak{C}$ that extends $\mathfrak{A}$ is complete.*

15.8 Applications to algebraic constructions

The main results concerning the preservation of algebraic constructions under the passage to completions say, roughly speaking, that in the class of Boolean algebras with quasi-complete operators (of a fixed similarity type), the completion of a regular subalgebra is a regular subalgebra (and therefore a complete subalgebra) of the completion, the completion of a complete homomorphic image is a complete homomorphic image of the completion, and the completion of a direct product is the direct product of the completions. Here is a precise statement of the first of these results.

Theorem 15.42. *Let $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ be Boolean algebras with quasi-complete operators. If $\mathfrak{A}$ is a regular subalgebra of $\bar{\mathfrak{A}}$, then the completion of $\mathfrak{A}$ is (up to isomorphisms that are the identity function on $\mathfrak{A}$) a complete subalgebra of the completion of $\bar{\mathfrak{A}}$.*

Proof. Let $\mathfrak{B}$ and $\bar{\mathfrak{B}}$ be the completions of $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ respectively. The relation of being a regular subalgebra is transitive on the class of all Boolean algebras with operators of a fixed similarity type, by Lemma 6.16(i). Since $\mathfrak{A}$ is a regular subalgebra of $\bar{\mathfrak{A}}$, by assumption, and $\bar{\mathfrak{A}}$ is a regular subalgebra of $\bar{\mathfrak{B}}$, by Lemma 15.6, it follows that $\mathfrak{A}$ is a regular subalgebra of $\bar{\mathfrak{B}}$. Also, $\bar{\mathfrak{B}}$ is complete with quasi-complete operators, by Definition 15.17 and Lemma 15.20. Apply Minimality Theorem 15.38 to obtain a complete embedding of $\mathfrak{B}$ into $\bar{\mathfrak{B}}$ that is the identity function on $\mathfrak{A}$. The image of $\mathfrak{B}$ under this complete embedding is complete (because $\mathfrak{B}$ is complete) and a regular subalgebra of $\bar{\mathfrak{B}}$, by Lemma 7.9. Consequently, this image algebra is a complete subalgebra of $\bar{\mathfrak{B}}$ (see the remark preceding Lemma 6.16). Identify $\mathfrak{B}$ with its image in $\bar{\mathfrak{B}}$ under this embedding to obtain the desired conclusion. □

The analogue of Lemma 14.45 does not hold for completions, that is to say, the equality of the completions in Theorem 15.42 does not imply the equality of the algebras $\mathfrak{A}$ and $\bar{\mathfrak{A}}$, even in the case of relation algebras. For example, take $\mathfrak{A}$ to be any incomplete relation algebra, and take $\bar{\mathfrak{A}}$ to be the completion of $\mathfrak{A}$. The two algebras have the same completion, namely $\bar{\mathfrak{A}}$, but $\mathfrak{A}$ is not equal to $\bar{\mathfrak{A}}$. The same example shows that the analogue of Corollary 14.47 does not hold for completions. Indeed, the identity function φ on $\mathfrak{A}$ is a complete embedding of $\mathfrak{A}$ into $\bar{\mathfrak{A}}$, by Lemma 7.9. The completion of φ is a complete monomorphism φ^+ from the completion of $\mathfrak{A}$ into the completion of $\bar{\mathfrak{A}}$ that extends φ, by Existence Theorem 15.35. In other words, φ^+ is a complete monomorphism from $\bar{\mathfrak{A}}$ into $\bar{\mathfrak{A}}$ that extends φ. The identity function on $\bar{\mathfrak{A}}$ is also a complete monomorphism from $\bar{\mathfrak{A}}$ into $\bar{\mathfrak{A}}$ that extends φ, so φ^+ must be the identity function on $\bar{\mathfrak{A}}$, by Uniqueness Theorem 15.36. Thus, φ^+ maps the completion of $\mathfrak{A}$ onto the completion of $\bar{\mathfrak{A}}$, but φ does not map $\mathfrak{A}$ onto $\bar{\mathfrak{A}}$.

The preservation theorem for complete homomorphic images says, roughly speaking, that the completion of a complete homomorphic image is a complete homomorphic image of the completion. It is an almost immediate consequence of Existence Theorem 15.35.

Theorem 15.43. *Let $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ be Boolean algebras with quasi-complete operators. If $\bar{\mathfrak{A}}$ is the image of $\mathfrak{A}$ under a complete homomorphism φ, then the completion of $\bar{\mathfrak{A}}$ is the image of the completion of $\mathfrak{A}$ under a complete homomorphism that extends φ.*

Proof. The function φ is a complete homomorphism from $\mathfrak{A}$ onto $\bar{\mathfrak{A}}$, by assumption. Since $\bar{\mathfrak{A}}$ is a regular subalgebra of its completion, it follows that φ is a complete homomorphism from $\mathfrak{A}$ into the completion of $\bar{\mathfrak{A}}$. The completion of φ is a complete homomorphism φ^+ from the completion of $\mathfrak{A}$ onto the completion of $\bar{\mathfrak{A}}$ that extends φ, by Theorem 15.35. $\square$

There is one more notable difference between canonical extensions and completions: arbitrary direct products are preserved under the passage to completions, not just products that involve only finitely many factors.

Theorem 15.44. *Let* $(\mathfrak{A}_i : i \in I)$ *be a system of Boolean algebras with quasi-complete operators, and* $(\mathfrak{B}_i : i \in I)$ *the corresponding system of completions. The completion of the product* $\prod_i \mathfrak{A}_i$ *is just the product* $\prod_i \mathfrak{B}_i$ *of the completions (up to isomorphisms that are the identity function on* $\prod_i \mathfrak{A}_i$*).*

Proof. One can verify directly that $\prod_i \mathfrak{B}_i$ satisfies the conditions in Definition 15.17 for being the completion of $\prod_i \mathfrak{A}_i$. Instead of taking this direct approach, however, we take an indirect approach by showing that the completion of $\prod_i \mathfrak{A}_i$ is isomorphic to $\prod_i \mathfrak{B}_i$ via a mapping that is the identity function on $\prod_i \mathfrak{A}_i$. It seems notationally easier to use internal products instead of external products. Accordingly, let $\mathfrak{A}$ be the internal product of the algebras $\mathfrak{A}_i$, let $\mathfrak{B}$ be the internal product of the completions $\mathfrak{B}_i$, and let $\mathfrak{C}$ be the completion of $\mathfrak{A}$. The goal is to show that $\mathfrak{C}$ is isomorphic to $\mathfrak{B}$ via a mapping that is the identity function on $\mathfrak{A}$.

The product $\mathfrak{B}$ is complete, by Corollary 11.38. The operators in $\mathfrak{B}$ are performed componentwise, so they inherit the quasi-completeness of the operators in the factor algebras (see Exercise 11.54). Also, $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, by the internal version of Corollary 11.20. To see that $\mathfrak{A}$ is dense in $\mathfrak{B}$, and therefore a regular subalgebra of $\mathfrak{B}$ (Lemma 15.6), consider a non-zero element p in $\mathfrak{B}$, say $p = \sum p_i$ with p_i in $\mathfrak{B}_i$ for each i. There must be an index i such that p_i is not zero. The algebra $\mathfrak{A}_i$ is dense in $\mathfrak{B}_i$, because $\mathfrak{B}_i$ is assumed to be the completion of $\mathfrak{A}_i$, so there must be a non-zero element s in $\mathfrak{A}_i$ that is below p_i. The definition of an internal product ensures that s is a non-zero element in $\mathfrak{A}$ that is below p. Thus, every non-zero element in $\mathfrak{B}$ is above a non-zero element in $\mathfrak{A}$, so $\mathfrak{A}$ is dense in $\mathfrak{B}$. Conclusion: $\mathfrak{B}$ is a regular extension of $\mathfrak{A}$ that is complete with quasi-complete operators.

Apply Minimality Theorem 15.38 to obtain a complete embedding ψ of $\mathfrak{C}$ into $\mathfrak{B}$ that is the identity function on $\mathfrak{A}$. It remains to prove that ψ is onto and therefore an isomorphism. Consider an arbitrary element q in $\mathfrak{B}$, say $q = \sum q_i$ with q_i in $\mathfrak{B}_i$ for each i. Since $\mathfrak{B}_i$ is the completion of $\mathfrak{A}_i$, there must be a subset X_i of $\mathfrak{A}_i$ such that q_i is the supremum of X_i in $\mathfrak{B}_i$, by Lemma 15.5 applied to $\mathfrak{B}_i$. Observe that q_i is also the supremum of X_i in $\mathfrak{B}$, by Corollary 11.38 (with X_i in place of X). The union $X = \bigcup_i X_i$ is a subset of $\mathfrak{A}$, so it must have a supremum p in $\mathfrak{C}$. Use the completeness of the embedding ψ, the fact that ψ is the identity function on $\mathfrak{A}$, the definition of the set X, the general associative law for addition, and the assumptions about q_i and q, to obtain

$$\begin{aligned}\psi(p) = \psi(\textstyle\sum X) = \sum \psi(X) &= \textstyle\sum X \\ &= \textstyle\sum(\bigcup_i X_i) = \sum_i \sum X_i = \sum q_i = q.\end{aligned}$$

Thus, ψ maps $\mathfrak{C}$ onto $\mathfrak{B}$. □

There are two more results concerning the preservation of algebraic constructions under the passage to completions that should be mentioned. The first says that if $\mathfrak{B}$ is the completion of a relation algebra $\mathfrak{A}$, and if e is an equivalence element in $\mathfrak{A}$, then the relativization $\mathfrak{B}(e)$ is the completion of the relativization $\mathfrak{A}(e)$. In other words, the completion of a relativization is the relativization of the completion.

To formulate the second result, we need a definition. A relation algebraic ideal M in $\mathfrak{A}$ is said to be *complete* if it is complete as a Boolean ideal (see Section 15.1). This means that if a subset of M has a supremum in $\mathfrak{A}$, then that supremum belongs to M. The second result says that if $\mathfrak{B}$ is the completion of a relation algebra $\mathfrak{A}$, and if M is a complete relation algebraic ideal in $\mathfrak{A}$, then the set of all suprema in $\mathfrak{B}$ of subsets of M is a complete relation algebraic ideal N in $\mathfrak{B}$, and the quotient $\mathfrak{B}/N$ is isomorphic to the completion of the quotient $\mathfrak{A}/M$ via a function that maps r/N to r/M for each element r in $\mathfrak{A}$. In other words, roughly speaking, the completion of a quotient is the quotient of the completion. The proofs of these two results are left as exercises.

15.9 A characterization of complete homomorphic images

It was shown in Theorem 14.54 that every homomorphic image of a relation algebra $\mathfrak{A}$ is isomorphic to a relativization of $\mathfrak{A}$ to an ideal element from the canonical extension of $\mathfrak{A}$. The corresponding theorem for complete homomorphic images says that every complete homomorphic image of $\mathfrak{A}$ is isomorphic to a relativization of $\mathfrak{A}$ to an ideal element from the completion of $\mathfrak{A}$. There are two possible approaches to proving this theorem, one using complete ideals and the other using complete filters. In the present setting, the first approach seems more natural.

Here are two preliminary observations about complete ideals. Their proofs are easy and are left as exercises. First, every complete ideal in a relation algebra is the kernel of a complete epimorphism, namely the quotient homomorphism onto the corresponding quotient algebra. Second, if φ is a complete homomorphism from a relation algebra $\mathfrak{A}$ onto a relation algebra $\mathfrak{B}$, then the kernel of φ is a complete ideal M in $\mathfrak{A}$, and the quotient $\mathfrak{A}/M$ is isomorphic to $\mathfrak{B}$ via the function that maps r/M to $\varphi(r)$ for every r in $\mathfrak{A}$, by the First Isomorphism Theorem 8.39.

We begin with a characterization of the complete relation algebraic ideals in $\mathfrak{A}$.

Lemma 15.45. *A subset M of a relation algebra $\mathfrak{A}$ is a complete relation algebraic ideal if and only if the supremum of M in the completion of $\mathfrak{A}$ is an ideal element u and $M = L_u$.*

Proof. Let $\mathfrak{B}$ be the completion of $\mathfrak{A}$. Consider a subset M of $\mathfrak{A}$, and let u be the supremum of M in $\mathfrak{B}$. Assume first that u is an ideal element in $\mathfrak{B}$ and $M = L_u$, that is to say, M is the set of elements in $\mathfrak{A}$ that are below u. Lemma 15.14 (with u in place of p) implies that M is a complete Boolean ideal in $\mathfrak{A}$. To show that M is closed under relative multiplication by arbitrary elements from $\mathfrak{A}$, and is therefore a complete relation algebraic ideal (see Definition 8.7), consider an element r in M and an element s in $\mathfrak{A}$. The definition of M, the monotony law for relative multiplication, and the assumption that u is an ideal element imply that

$$r\,;s \leq u\,;1 = u,$$

so $r\,;s$ is in M. A symmetric argument shows that $s\,;r$ is in M.

Assume now that M is a complete relation algebraic ideal in $\mathfrak{A}$. In particular, M is a complete Boolean ideal in $\mathfrak{A}$, so $M = L_u$, by the definition of u and Lemma 15.14 (with u in place of p). Use the definition of u, the complete distributivity of relative multiplication, and the closure of M under relative multiplication by elements from $\mathfrak{A}$, to obtain

$$1 ; u ; 1 = 1 ; (\sum M) ; 1 = \sum\{1 ; r ; 1 : r \in M\} \leq \sum M = u.$$

The reverse inequality holds by Lemma 4.5(iii) and its first dual, so u is an ideal element in $\mathfrak{B}$. □

The lemma easily implies the following analogue of Corollary 14.52, giving a characterization of arbitrary ideal elements in the completion of a relation algebra.

Corollary 15.46. *Suppose $\mathfrak{B}$ is the completion of a relation algebra $\mathfrak{A}$. An element u in $\mathfrak{B}$ is an ideal element if and only if there is a complete relation algebraic ideal M in $\mathfrak{A}$ such that $u = \sum M$ in $\mathfrak{B}$.*

Proof. Consider an arbitrary element u in $\mathfrak{B}$. If M is a complete relation algebraic ideal in $\mathfrak{A}$ such that

$$u = \sum M, \tag{1}$$

then u is an ideal element, by Lemma 15.45. On the other hand, if u is an ideal element, then the set $M = L_u$ is a complete relation algebraic ideal in $\mathfrak{A}$, by Lemma 15.45, and (1) holds, by Lemma 15.5 and the definition of the set L_u.

The next lemma is the analogue of Lemma 14.53.

Lemma 15.47. *Suppose $\mathfrak{B}$ is the completion of a relation algebra $\mathfrak{A}$. If u is the supremum in $\mathfrak{B}$ of a complete relation algebraic ideal M in $\mathfrak{A}$, then the restriction to $\mathfrak{A}$ of the relativization homomorphism $p \longmapsto p \cdot -u$ on $\mathfrak{B}$ is a complete epimorphism from $\mathfrak{A}$ to $\mathfrak{A}(-u)$ with kernel M.*

Proof. Lemma 15.45 implies that u is an ideal element in $\mathfrak{B}$, so the complement $-u$ is also an ideal element, by Lemma 5.39(iv). It therefore makes sense to speak of the relativization homomorphism φ from $\mathfrak{B}$ to $\mathfrak{B}(u)$ that is defined by

$$\varphi(p) = p \cdot -u$$

for p in $\mathfrak{B}$. The kernel of this homomorphism is, by definition, the set K of elements in $\mathfrak{B}$ that are mapped to 0 by φ. Since $p \cdot -u = 0$ if and only if $p \leq u$, we may write

$$K = \{p \in B : p \leq u\}. \tag{1}$$

The restriction of φ to $\mathfrak{A}$ is an epimorphism from $\mathfrak{A}$ to $\mathfrak{A}(u)$, by Lemma 10.12, and in fact it is a complete epimorphism. For the proof, suppose that r is the supremum in $\mathfrak{A}$ of a subset X of $\mathfrak{A}$. Since $\mathfrak{B}$ is the completion of $\mathfrak{A}$, the element r remains the supremum of X in $\mathfrak{B}$, by Corollary 15.7, and therefore

$$\begin{aligned}\varphi(\textstyle\sum X) = \varphi(r) = r \cdot -u = (\textstyle\sum X) \cdot -u \\ = \textstyle\sum\{s \cdot -u : s \in X\} = \textstyle\sum\{\varphi(s) : s \in X\}.\end{aligned}$$

Thus, the restriction of φ to $\mathfrak{A}$ preserves all sums that happen to exist in $\mathfrak{A}$.

The kernel of the restriction of φ to $\mathfrak{A}$ is the set $K \cap A$ (see Exercise 8.23). It is easy to see that

$$M = K \cap A. \tag{2}$$

Indeed, M is a complete relation algebraic ideal in $\mathfrak{A}$, and u is its supremum in $\mathfrak{B}$, by assumption, so M must be the set of all elements in $\mathfrak{A}$ that are below u, by Lemma 15.45. But $K \cap A$ is also the set of all elements in $\mathfrak{A}$ that are below u, by (1) and the definition of intersection, so (2) holds. This completes the proof of the lemma. □

Theorem 15.48. *Every complete homomorphic image of a relation algebra $\mathfrak{A}$ is isomorphic to a relativization of $\mathfrak{A}$ to some ideal element from the completion of $\mathfrak{A}$.*

Proof. Let $\mathfrak{B}$ be the completion of $\mathfrak{A}$. The kernel of a complete epimorphism on $\mathfrak{A}$ is a complete ideal in $\mathfrak{A}$, and the quotient of $\mathfrak{A}$ modulo that complete ideal is isomorphic to the image of $\mathfrak{A}$ under the epimorphism, by the preliminary observations made before Lemma 15.45. Consequently, it suffices to show that every quotient of $\mathfrak{A}$ modulo a complete ideal is isomorphic to the relativization of $\mathfrak{A}$ to some ideal element in $\mathfrak{B}$.

Consider a complete relation algebraic ideal M in $\mathfrak{A}$, and let u be the supremum of M in $\mathfrak{B}$. Lemma 15.45 ensures that u is an ideal element in $\mathfrak{B}$, and Lemma 15.47 ensures that the restriction to $\mathfrak{A}$ of the relativization homomorphism $p \longmapsto p \cdot -u$ on $\mathfrak{B}$ is a complete epimorphism from $\mathfrak{A}$ to $\mathfrak{A}(-u)$ with kernel M. The quotient $\mathfrak{A}/M$ is therefore isomorphic to the relativization $\mathfrak{A}(-u)$, by the First Isomorphism Theorem 8.39. □

15.10 Historical remarks

Stone [127], [128] and Tarski [131] proved that the class of complete ideals in a Boolean algebra is itself a complete Boolean algebra (Theorem 15.3). Dedekind [27] constructed the real numbers as a kind of order completion of the rational numbers, using sets of rational numbers called Dedekind cuts (see the remarks following Lemma 15.2). MacNeille [91] extended Dedekind's method to construct completions of partial orderings, and in particular, completions of Boolean algebras. Existence Theorem 15.13 for completions of Boolean algebras is due to MacNeille [91] and Tarski [131], who also showed that the completion of a Boolean algebra A can be embedded into any complete extension of A via a mapping that is the identity function on A. This implies the uniqueness of the completion up to isomorphisms over A. Tarski [131] also pointed out that a Boolean algebra is atomic if and only if its completion is atomic (see Lemma 15.29).

Piero Mangani [104] constructed the completion of an arbitrary cylindric algebra A, and showed that every complete monomorphism from A into a complete cylindric algebra C can be extended to a monomorphism from the completion of A into C. Monk [112] developed the general theory of completions of Boolean algebras with quasi-complete operators. Definition 15.17 and the results in Section 15.3 are essentially due to him, as are the First Preservation Theorem 15.26 and a somewhat weaker form of the Second Preservation Theorem 15.27 in Section 15.4. He used these results to show that the properties of being a relation algebra, a simple relation algebra, and an integral relation algebra are preserved under the passage to completions (see Theorems 15.28 and 15.30). He also proved a slightly weaker version of Minimality Theorem 15.38.

The Second Preservation Theorem 15.27 in its present form is due to Givant and Yde Venema [43]. The results in Section 15.6 are due to Givant. So, too, are Theorems 15.31 and 15.32, Corollary 15.33, Lemma 15.39, and Theorem 15.40 in Section 15.7, as well as the results in Section 15.8 (including the theorems mentioned at the end of the section and given in Exercises 15.58 and 15.60), and in Section 15.9 (including the result in Exercise 15.65).

Hodkinson [62] (see also [59]) proved the important and difficult theorem that the completion of a set relation algebra need not be isomorphic to a set relation algebra.

Exercises

15.1. Prove that if M is a complete relation algebraic ideal in a relation algebra $\mathfrak{A}$, then the quotient homomorphism from $\mathfrak{A}$ to $\mathfrak{A}/M$ is a complete epimorphism. Conclude that every complete relation algebraic ideal is the kernel of a complete homomorphism.

15.2. Prove that the kernel of a complete homomorphism on a relation algebra is a complete ideal. Conclude that every complete homomorphic image of a relation algebra is isomorphic to a quotient of the algebra modulo a complete ideal.

15.3. By imitating the argument used to establish the distributive law for meet over join in the Boolean algebra of all complete ideals in a Boolean algebra, prove the distributive law for join over meet.

15.4. Derive the distributive law for join over meet from the distributive law for meet over join, using the standard axioms of lattice theory.

15.5. Verify directly, without appealing to Theorem 15.3, that the laws

$$-(X \vee Y) = -X \wedge -Y \qquad \text{and} \qquad -(X \wedge Y) = -X \vee -Y$$

hold in the Boolean algebra of complete ideals in a Boolean algebra (where the complement of a complete ideal is defined to be the annihilator of the ideal).

15.6. Prove that an ideal X in a Boolean algebra is complete if and only if it is the annihilator of the annihilator of X.

15.7. Give an example to show that the operation of join in the Boolean algebra of complete ideals in a Boolean algebra A is not always the same as the operation of join in the lattice of ideals in A.

15.8. Give an example to show that an ideal may not have a complement in the lattice of ideals of a Boolean algebra.

15.9. Fill in the missing details in the proof of Lemma 15.14.

15.10. Complete the proof of Theorem 15.15 by showing that the function ψ preserves the operation of complement.

15.11. Give a direct proof of Uniqueness Theorem 15.16 without making use of Lemma 15.14.

15.12. If two Boolean algebras are isomorphic via a mapping φ prove that their completions are isomorphic via a uniquely determined mapping that extends φ.

15.13. Prove that condition (i) in Definition 15.17 may be weakened to say that $\mathfrak{B}$ is complete, the Boolean part of $\mathfrak{A}$ is a subalgebra of the Boolean part of $\mathfrak{B}$, and the distinguished constant 1' is the same in both algebras.

15.14. Formulate a version of Definition 15.17 that applies to Boolean algebras with operators of arbitrary ranks.

15.15. Complete the proof of Lemma 15.18 by showing that the operation $^{\smile}$ in $\mathfrak{B}$ is monotone.

15.16. Formulate and prove a version of Lemma 15.18 for Boolean algebras with operators of arbitrary ranks.

15.17. Complete the proof of Lemma 15.19 by showing that for every non-empty subset X of $\mathfrak{A}$, if $p = \sum X$, then $p^{\smile} = \sum(X^{\smile})$.

15.18. Formulate and prove a version of Lemma 15.19 for Boolean algebras with quasi-complete operators of arbitrary ranks.

15.19. Complete the proof of Lemma 15.20 by treating the case of the operation $^{\smile}$.

15.20. Formulate and prove a version of Lemma 15.20 for Boolean algebras with quasi-complete operators of arbitrary ranks.

15.21. Complete the proof of Lemma 15.21 by treating the case of the operation $^{\smile}$.

15.22. Formulate a version of the definition preceding Lemma 15.21 that applies to Boolean algebras with quasi-complete operators of arbitrary ranks, and prove the analogues of Lemma 15.21 and Existence Theorem 15.22 for those algebras.

15.23. Complete the proof of Uniqueness Theorem 15.23 by showing that the function ψ preserves the operation $^{\smile}$.

15.24. Prove a version of Uniqueness Theorem 15.23 that applies to Boolean algebras with quasi-complete operators of arbitrary ranks.

15.25. For a Boolean algebra with quasi-complete operators $\mathfrak{A}$, prove that there is at most one isomorphism between two completions of $\mathfrak{A}$ that is the identity function on $\mathfrak{A}$.

15.26. If two Boolean algebras with quasi-complete operators are isomorphic via a mapping φ prove that their completions are isomorphic via a uniquely determined mapping that extends φ.

15.27. Prove Lemma 15.24.

15.28. Prove a version of Lemma 15.24 that applies to Boolean algebras with quasi-complete operators of arbitrary ranks.

15.29. Prove Lemma 15.25.

15.30. Prove a version of Lemma 15.25 that applies to Boolean algebras with quasi-complete operators of arbitrary ranks.

15.31. For relation algebras, the existence of completions (in the sense of Definition 15.17) with quasi-complete operators is a consequence of Theorem 15.22. Assuming this existence, prove directly (without using the lemmas or theorems of Section 15.4) that the relation algebraic axioms (R4)–(R7) and the cycle law (R11) are all valid in a completion.

15.32. Suppose $\mathfrak{A}$ is a Boolean algebra with quasi-complete operators, and $\mathfrak{B}$ its completion. Prove that the completion of the unary discriminator on the universe of $\mathfrak{A}$ is the unary discriminator on the universe of $\mathfrak{B}$.

15.33. Prove the Second Preservation Theorem 15.27.

15.34. Prove Theorem 15.28.

15.35. Formulate and prove a version of Lemma 15.29 that applies to Boolean algebras with quasi-complete operators of arbitrary ranks.

15.36. Prove that the completion of a functionally dense relation algebra is functionally dense.

15.37. Prove that a complete Boolean algebra with quasi-complete operators is always its own completion.

15.38. Prove that the completion of an atomless Boolean algebra with quasi-complete operators is atomless.

15.39. Prove that the completion of a countably infinite relation algebra must have cardinality $2^{\aleph_0}$.

15.40. Assuming Theorem 15.28, prove directly—without using the preservation theorems—that a relation algebra is simple or integral if and only if its completion is simple or integral respectively.

15.41. Prove Theorem 15.30 by imitating the proof of Theorem 14.36.

15.42. Prove that the mapping φ^+ defined in Section 15.6 is monotone.

15.43. Prove the analogue of Lemma 15.19 for the mapping φ^+. In other words, prove that for every non-empty subset X of $\mathfrak{A}$, if $p = \sum X$, then $\varphi^+(p) = \sum\{\varphi(r) : r \in X\}$.

15.44. Prove that the mapping φ^+ defined in Section 15.6 is quasi-complete.

15.45. Prove parts (i), (ii), (v), and (vi) of Lemma 15.34.

15.46. Complete the proof of Theorem 15.35 by showing that the function φ^+ preserves the operations $\cdot$ and $\smile$.

15.47. Let $\mathfrak{B}$ be the completion of a Boolean algebra with quasi-complete operators $\mathfrak{A}$, and suppose φ is a mapping from $\mathfrak{A}$ into a Boolean algebra with operators $\bar{\mathfrak{B}}$. If the completion φ^+ extends φ and is quasi-complete, prove that φ must be quasi-complete.

15.48. Given an example to demonstrate the necessity, even for relation algebras, of the hypothesis in Theorem 15.35 that φ be a complete homomorphism. In other words, give an example of an incomplete homomorphism between two relation algebras that cannot be extended to a complete homomorphism between the completions.

15.49. How would one define the notion of a quasi-complete homomorphism between Boolean algebras with operators? Prove that a quasi-complete homomorphism is necessarily a complete homomorphism. (This explains why the notion of a quasi-complete homomorphism was not introduced in the text, for example, for Theorem 15.35.)

15.50. Prove Corollary 15.37

15.51. Suppose $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ are Boolean algebras with quasi-complete operators, and $\mathfrak{B}$ and $\bar{\mathfrak{B}}$ are their completions. Let φ be a function of two arguments from $\mathfrak{A}$ into $\bar{\mathfrak{A}}$, so that $\varphi(r,s)$ is an element in $\bar{\mathfrak{A}}$ for every pair of elements r and s in $\mathfrak{A}$. The completion of φ is defined to be the function φ^+ of two arguments from $\mathfrak{B}$ into $\bar{\mathfrak{B}}$ that is defined by

$$\begin{aligned}\varphi^+(p,q) &= \sum\{\varphi(r,s) : r,s \in A \text{ and } r \leq p,\ s \leq q\}\\ &= \sum\{\varphi(r,s) : r \in L_p \text{ and } s \in L_q\}.\end{aligned}$$

Prove that if φ is quasi-complete, then so is φ^+. Prove further that φ^+ is the unique quasi-completely distributive function of two arguments from $\mathfrak{B}$ into $\bar{\mathfrak{B}}$ that extends φ.

15.52. Let $\mathfrak{A}_1$, $\mathfrak{A}_2$, and $\mathfrak{A}_3$ be Boolean algebras with quasi-complete operators. Suppose φ and ψ are functions of two arguments from $\mathfrak{A}_1$ into $\mathfrak{A}_2$, and ϑ is a function of two arguments from $\mathfrak{A}_2$ into $\mathfrak{A}_3$. Prove that if these three functions are quasi-complete, then the completion of the composition $\rho = \vartheta(\varphi,\psi)$ (see Exercise 14.60) is equal to the composition of the completion of ϑ with the completions of φ and ψ, that is to say,

$$\rho^+ = \vartheta^+(\varphi^+,\psi^+).$$

15.53. Derive Uniqueness Theorem 15.23 as a corollary of the Existence Theorem 15.35 for completions of complete homomorphisms.

15.54. Derive Uniqueness Theorem 15.23 as a corollary of Minimality Theorem 15.38 and Lemma 15.39.

15.55. Prove the following alternative version of Theorem 15.40 without using that theorem in the proof. If $\mathfrak{A}$ and $\mathfrak{C}$ are Boolean algebra with quasi-complete operators, then $\mathfrak{C}$ is the completion of $\mathfrak{A}$ if and only if $\mathfrak{C}$ is a regular extension of $\mathfrak{A}$ that is complete, and no extension of $\mathfrak{A}$ that is complete is a proper regular subalgebra of $\mathfrak{C}$.

15.56. Give an example to demonstrate the necessity, even for relation algebras, of the hypothesis in Theorem 15.42 that $\mathfrak{A}$ be a regular subalgebra of $\bar{\mathfrak{A}}$. In other words, give an example of relation algebras $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ such that $\mathfrak{A}$ is a subalgebra, but not a regular subalgebra, of $\bar{\mathfrak{A}}$, and the completion of $\mathfrak{A}$ is not a complete subalgebra of the completion of $\bar{\mathfrak{A}}$.

15.57. Prove Theorem 15.44 directly, by showing that the product $\prod_i \mathfrak{B}_i$ of the completions satisfies the conditions in Definition 15.17 for being the completion of the product $\prod_i \mathfrak{A}_i$.

15.58. Suppose $\mathfrak{B}$ is the completion of a relation algebra $\mathfrak{A}$, and e is an equivalence element in $\mathfrak{A}$. Prove that the relativization $\mathfrak{B}(e)$ is the completion of the relativization $\mathfrak{A}(e)$.

15.59. Here is the sketch of another proof of Theorem 15.44 for the case of relation algebras. Fill in the missing details. Let $\mathfrak{B}$ be the completion of $\mathfrak{A}$. The system of units u_i of the relation algebras $\mathfrak{A}_i$ is a partition of unity in $\mathfrak{A}$, so it must also be a partition of unity in $\mathfrak{B}$. Consequently, $\mathfrak{B}$ is the internal product of the relativizations $\mathfrak{B}(u_i)$, for i in I. The relativization $\mathfrak{B}(u_i)$ is the completion of the relativization $\mathfrak{A}(u_i)$, by Exercise 15.58, and $\mathfrak{A}(u_i)$ coincides with $\mathfrak{A}_i$, so $\mathfrak{B}(u_i)$ is the completion of $\mathfrak{A}_i$ for each i.

15.60. Suppose $\mathfrak{B}$ is the completion of a relation algebra $\mathfrak{A}$, and M is a complete relation algebraic ideal in $\mathfrak{A}$. Prove that the set N of all elements in $\mathfrak{B}$ that can be written as sums of subsets of M is a complete ideal in $\mathfrak{B}$, and the quotient $\mathfrak{B}/N$ is isomorphic to the completion of the quotient $\mathfrak{A}/M$ via a function that maps r/N to r/M for each element r in $\mathfrak{A}$.

15.61. Two Boolean algebras with quasi-complete operators are said to be *essentially isomorphic* if their completions are isomorphic. Suppose $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ are atomic Boolean algebras with quasi-complete operators (of the same similarity type as relation algebras), and φ is a bijection from the set of atoms in $\mathfrak{A}$ onto the set of atoms in $\bar{\mathfrak{A}}$ with the property that

$$\begin{aligned} t \le r\,;s &\qquad \text{if and only if} \qquad \varphi(t) \le \varphi(r)\,;\varphi(s),\\ t \le r^{\smile} &\qquad \text{if and only if} \qquad \varphi(t) \le \varphi(r)^{\smile},\\ t \le 1' &\qquad \text{if and only if} \qquad \varphi(t) \le 1', \end{aligned}$$

for all atoms r, s, and t in $\mathfrak{A}$. Prove that $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ are essentially isomorphic, and in fact φ can be extended to an isomorphism from the completion of $\mathfrak{A}$ to the completion of $\bar{\mathfrak{A}}$.

15.62. Formulate and prove a version of Exercise 15.61 that applies to Boolean algebras with quasi-complete operators of arbitrary ranks.

15.63. Formulate and prove a stronger version of Exercise 15.61 that applies to relation algebras.

15.64. Let $\mathfrak{B}$ be the completion of a relation algebra $\mathfrak{A}$. Prove that an element u in $\mathfrak{B}$ is an ideal element if and only if there is a complete relation algebraic filter N in $\mathfrak{A}$ such that $u = \prod N$. Moreover, if such a filter N exists, then $N = \{r \in A : u \leq r\}$, and the restriction of the relativization homomorphism $p \longmapsto p \cdot u$ (from $\mathfrak{B}$ to $\mathfrak{B}(u)$) to the subalgebra $\mathfrak{A}$ has cokernel N. Use these two observations to give a somewhat different proof of Theorem 15.48.

15.65. Prove that the set of complete relation algebraic ideals in a relation algebra $\mathfrak{A}$ is a complete Boolean subalgebra of the Boolean algebra of all complete Boolean ideals in $\mathfrak{A}$.

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