# On the Asymptotic Stability of a Satellite with a Gravitational Stabilizer

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**Abstract.** The problem of the influence of the structure of forces on the stability of the relative equilibrium of a controlled satellite with a gravitational stabilizer on the circular orbit is studied. In the space of entered parameters, the regions with different degrees of instability by Poincaré are found. Assuming an instability of a potential system, the problem of the possibility of its stabilization up to asymptotic stability is considered. A parametric analysis of the obtained inequalities with the help of "Mathematica" built-in tools for symbolic-numerical modelling is carried out.

## 1 Introduction

Investigation of stability and stabilization of nonlinear or linearized models of mechanical systems often leads to the problem of "parametric analysis" of the conditions (inequalities) obtained. In the case of parametric analysis, it is important to have a possibility to estimate the domain of values of the parameters under which a desired system's state is provided. Naturally, it is hard to hope for obtaining any readable analytical results for the models which have high dimensions and contain many parameters. At this stage, one can efficiently use software packages of computer algebra (SPCA) as well as the corresponding software elaborated on the basis of these software packages.

The paper considers a problem of stability of the position of relative equilibrium in the orbital coordinate system of a controlled satellite with a gravitational stabilizer. The mechanical system in question is a well-studied model (see, for example, the review [1]). To obtain sufficient stability conditions, the second Lyapunov method and the Barbashin–Krasovskii theorem were applied. As noted in [1], obtaining the necessary stability conditions (by linear equations of perturbed motion) leads to presenting very bulky calculations. In contrast to the passive stabilization and orientation systems, the possibilities of active control of a gravitational stabilizer are investigated in [2], in particular, the optimization of the system by degrees of stability and accuracy.

The application of computer algebra methods and SPCA capabilities to the problems of celestial mechanics has rich history and till today attracts academic attention (see, for example, [3,4]).


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## 2 Description and Construction of a Symbolical Model

The system's mass center moves along the Kepler circular orbit with constant angular velocity $\boldsymbol{\omega}$. For the description of a motion of the system, two right-handed rectangular Cartesian coordinate systems are introduced (the orbital coordinate system (OCS) and the coordinate system rigidly connected to a satellite). To define relative positioning of the axes of these coordinate systems, the directional cosines defined by the angles $\psi$, $\theta$, $\varphi$ of Euler's type, are used (see, for example, [2]). The stabilizer is a rigid rod with point mass at its free end. The rod is connected to the satellite with a 2-degree-of-freedom suspension. The rotation axes of the rod coincide with the direction of the axes of pitch and roll. The system is influenced by a gravitation moment. When moving undisturbed, the system's principal central axes of inertia coincide with the axes of orbital coordinate system, and the rod is oriented along the radius of the orbit. This is the equilibrium position of a satellite with the stabilizer in regard to OCS.

With the help of the developed software [5,6], the following results are obtained in a symbolic form on PC for the system of bodies in question:

- kinetic energy and force function of the approximate Newtonian field of gravitation;
- nonlinear equations of motion in Lagrange form of the 2$^{\text{nd}}$ kind;
- matrices of equations of perturbed motion in the first approximation in the vicinity of equilibrium position;
- coefficients of the system's characteristic equation.

Linearized in the vicinity of the equilibrium position, equations of motion for a satellite with a stabilizer are decomposed into two subsystems. Respectively, a “pitch” subsystem ($\theta$) and a “yaw-and-roll” subsystem ($\psi$, $\varphi$) are:

$$\begin{cases} M_1\,\ddot{q}_1 + K_1\,q_1 = Q_1 \\ M_2\,\ddot{q}_2 + G\,\dot{q}_2 + K_2\,q_2 = Q_2, \end{cases} \tag{1}$$

where all derivatives are calculated on dimensionless time $\tau = \omega t$ ($\omega = |\boldsymbol{\omega}|$ is the module of orbital angular velocity); $q_1 = \begin{pmatrix}\theta\\ \delta\end{pmatrix}$, $q_2 = \begin{pmatrix}\psi\\ \varphi\\ \sigma\end{pmatrix}$; $\delta$, $\sigma$ are rotation angles of the rod with regard to the satellite's body; $Q_1 = \begin{pmatrix}0\\ Q_\delta\end{pmatrix}$, $Q_2 = \begin{pmatrix}0\\ 0\\ Q_\sigma\end{pmatrix}$ are control forces;

$$M_1 = \begin{pmatrix} c & f\\ f & d\end{pmatrix}; \quad K_1 = 3\begin{pmatrix} b-a & f\\ f & f\end{pmatrix}; \quad M_2 = \begin{pmatrix} a & 0 & 0\\ 0 & b & f\\ 0 & f & d\end{pmatrix};$$

$$K_2 = \begin{pmatrix} c-b & 0 & 0\\ 0 & 4(c-a) & 4f\\ 0 & 4f & 3f+d\end{pmatrix}; \quad G = \begin{pmatrix} 0 & c-b-a & 0\\ a+b-c & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Here, we introduce the following notations:

$$a = J_y; \quad b = J_x + m\,r\,(l+r) + \frac{1}{3}m\,l^2 + m_0(l+r)^2; \quad c = b + J_z - J_x;$$
$$d = \Big(\frac{m}{3} + m_0\Big)l^2; \quad f = \Big(\frac{m}{2} + m_0\Big)rl + d; \quad c - b - a = J_z - J_x - J_y,$$

where $m$ and $m_0$ are masses of the rod and the point load at the end, respectively; $l > 0$ is the rod length; $r \geq 0$ is the distance from the system's mass center to the point of attachment of the rod; $J_x,\ J_y,\ J_z;\ a,\ b,\ c$ are principal inertia moments of the satellite and whole system, respectively.

Taking into account the mass distribution in the system and in the ellipsoid of inertia of rigid body, the following inequalities are valid

$$\begin{gathered} b > a > 0, \quad c > a, \quad f > d > 0, \quad c > f, \quad b > f, \\ c + a - b \equiv J_z + J_y - J_x > 0, \quad b + a - c \equiv J_x + J_y - J_z > 0, \end{gathered} \tag{2}$$

Equation (1) may be interpreted as equations of oscillations of a mechanical system influenced by potential (with the matrices $K_1, K_2$) and gyroscopic (with the matrix $G$) forces. These forces are determined by gravitation forces as well as by orbital motion. The matrices $M_1$ and $M_2$ play the role of diagonal blocks of a positive definite matrix of kinetic energy.

## 3 Formulation of the Problem

According to Kelvin–Chetaev's theorems [7], examination of stability of trivial solution begins with the analysis of the matrix of potential forces. Let us write out the conditions of positive definiteness of matrices $K_1, K_2$:

$$b > a + f, \quad c > b, \quad (c-a)(3f+d) - 4f^2 > 0. \tag{3}$$

Let us assume that

(1) for the "pitch" subsystem, the values of the parameters satisfy the condition $a < b < a + f$ (i.e., the first inequality in (3) is violated);
(2) for the "yaw-and-roll" subsystem, the last inequality in (3) or simultaneously the second and third inequalities are changed to the opposite.

Taking into account the assumptions presented, the system is unstable when initial potential forces are in action. The simultaneous stabilization of the two subsystems by additional forces of different nature is required. For this purpose, control forces with the suspension of the rod are added into the right-hand sides of the motion Eq. (1) as it is shown below

$$Q_\delta = \tilde{k}^*_\theta\,\dot{\theta} - \tilde{k}^*_\delta\,\dot{\delta} + \tilde{k}_\theta\theta - \tilde{k}_\delta\delta; \qquad Q_\sigma = \tilde{k}^*_\varphi\,\dot{\varphi} - \tilde{k}^*_\sigma\,\dot{\sigma} + \tilde{k}_\varphi\varphi - \tilde{k}_\sigma\sigma, \tag{4}$$

where $\tilde{k}^*_\theta = \dfrac{k^*_\theta}{\omega}; \quad \tilde{k}^*_\delta = \dfrac{k^*_\delta}{\omega}; \quad \tilde{k}_\theta = \dfrac{k_\theta}{\omega^2}; \quad \tilde{k}_\delta = \dfrac{k_\delta}{\omega^2}; \quad \tilde{k}^*_\varphi = \dfrac{k^*_\varphi}{\omega}; \quad \tilde{k}^*_\sigma = \dfrac{k^*_\sigma}{\omega};$
$\tilde{k}_\varphi = \dfrac{k_\varphi}{\omega^2}; \ \ \tilde{k}_\sigma = \dfrac{k_\sigma}{\omega^2}$ are constant coefficients.

The objective of the paper is to investigate the effect of the structure of forces on the stability of the equilibrium position of system (1). In addition, the problem of the possibility of ensuring the asymptotic stability of the two subsystems by a "reduced" set of forces represented in (4) is formulated.

By splitting the matrices in terms of velocities and coordinates in Eq. (1) into the symmetric and skew-symmetric parts, it is not difficult to write out the structure of the forces affecting the system. For example, concerning the "yaw-and-roll" subsystem, potential (with a matrix $P_2$), non-conservative ($N_2$), dissipative ($D_2$) and gyroscopic ($G_2$) forces are added to the initial potential (with a matrix $K_2$) and gyroscopic (with a matrix $G$) forces, where

$$P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{\tilde{k}_\varphi}{2} \\ 0 & -\frac{\tilde{k}_\varphi}{2} & \tilde{k}_\sigma \end{pmatrix}; \quad N_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\tilde{k}_\varphi}{2} \\ 0 & -\frac{\tilde{k}_\varphi}{2} & 0 \end{pmatrix};$$

$$D_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{\tilde{k}^*_\varphi}{2} \\ 0 & -\frac{\tilde{k}^*_\varphi}{2} & \tilde{k}^*_\sigma \end{pmatrix}; \quad G_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\tilde{k}^*_\varphi}{2} \\ 0 & -\frac{\tilde{k}^*_\varphi}{2} & 0 \end{pmatrix}.$$

## 4 Regions of System's Instability

For the convenience of graphical representation of the regions with different degrees of instability and subsequent parametric analysis, we introduce four dimensionless parameters:

$$\alpha = \frac{c-b}{a} = \frac{J_z - J_x}{J_y}; \quad \gamma = \frac{b-a}{c}; \quad p_1 = \frac{d}{f}; \quad p_2 = \frac{f}{c}. \tag{5}$$

The physically obtainable values of the parameters, taking into account (2), lie within the intervals: $-1 < \alpha < 1$, $0 < \gamma < 1$, $0 < p_1 \leq 1$, $0 < p_2 < 1$. It is not difficult to show that conditions (2) imply $\gamma + \alpha > 0$.

The diagonal blocks of the initial matrix of potential forces (when $Q_\delta = 0$, $Q_\sigma = 0$) in notation (5) have the form:

$$K_1 = 3\begin{pmatrix} \gamma & p_2 \\ 1 & 1 \end{pmatrix}; \qquad K_2 = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 4(\gamma+\alpha) & 4p_2(\alpha+1) \\ 0 & 4 & 3+p_1 \end{pmatrix}.$$

In the space of the outlined parameters, the relations $\gamma = p_2$, $\alpha = 0$, $S \equiv (\gamma+\alpha)(3+p_1) - 4p_2(\alpha+1) = 0$ define the surfaces which separate the regions having different degrees of instability. For example, Fig. 1 shows these regions for the values of the parameters $p_1 = 4/5$, $p_2 = 5/7$.

It is known that if the equilibrium position is unstable at potential forces, Kelvin–Chetaev's theorem [7] of influence of gyroscopic forces tells us that gyroscopic stabilization is possible only for systems with an even degree of instability.

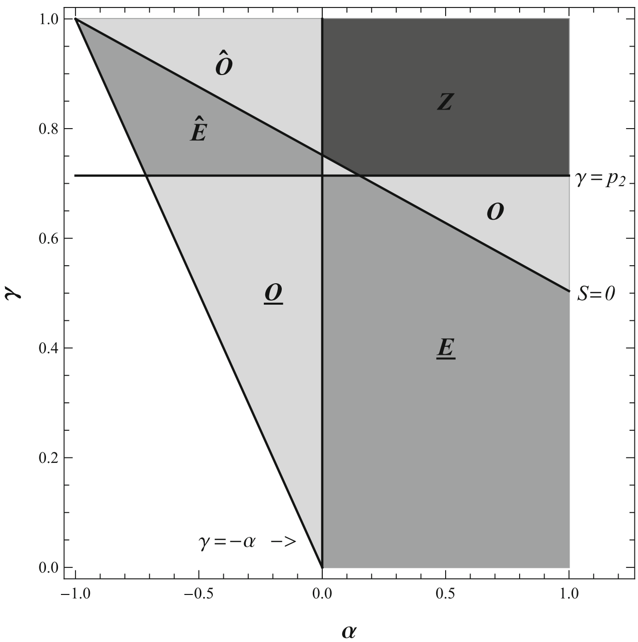


**Fig. 1.** Regions with different degrees of instability.

Here, respectively, instability regions for the entire system have: $\boldsymbol{Z}$ – zero degree; $\widehat{\boldsymbol{E}}$ – an even degree (when $\gamma > p_2$) and $\underline{\boldsymbol{E}}$ (when $\gamma < p_2$); $\boldsymbol{O}$, $\underline{\boldsymbol{O}}$, $\widehat{\boldsymbol{O}}$ – odd degree.

The evenness (or oddness) of the degree of instability according to Poincaré is determined by positivity (or negativity) of the determinant of the matrix of potential forces. It is necessary to emphasize that for the values of the parameters from the regions $\widehat{\boldsymbol{E}}$ and $\underline{\boldsymbol{E}}$, the unstable equilibrium position has an even degree of instability (i.e., $\det K = \det K_1 * \det K_2 > 0$). Thus, under certain conditions, equilibrium can be stabilized due to the influence of gyroscopic forces. Earlier in [8], the author has proved the stabilization of the equilibrium in the needle-shaped part (subregion) of the region $\widehat{\boldsymbol{E}}$ for an uncontrolled satellite. The matrices $K_1$ and $K_2$ are positive definitive in the region $\boldsymbol{Z}$. On the basis of another Kelvin–Chetaev's theorem, the addition to the potential forces of gyroscopic forces preserves the nature of stability of the investigated motion.

The mass distribution in the system in which the initial matrix of potential forces of the system will be positive definitive is usually given for the applied problems of spacecraft dynamics. Further, due to the addition of primarily dissipative forces, the asymptotic stability of motion is ensured by Lyapunov's theorem. However, unstable systems may also be of interest and, besides, "non-standard" situations on the orbit are possible.

Thus, taking into account the assumptions made in the formulation of the problem in Sect. 3, we shall consider the possibility of stabilizing an unstable system in the region $\underline{\boldsymbol{O}}$ (when $\gamma < p_2,\ \alpha < 0,\ (\gamma+\alpha)(3+p_1) - 4p_2(\alpha+1) < 0$) or in the region $\underline{\boldsymbol{E}}$ (when $\gamma < p_2,\ \alpha > 0,\ (\gamma+\alpha)(3+p_1) - 4p_2(\alpha+1) < 0$) to asymptotic stability by additional forces (4).

## 5 Parametric Analysis of Asymptotic Stability Conditions

It is obvious that the characteristic equation of system (1) is factorized: $\Lambda(\lambda) \equiv \Lambda^{(1)} * \Lambda^{(2)} = 0$. After performing elementary transformations with the characteristic matrices (multiplying their rows by positive factors), we obtain the characteristic determinants in notation (5), respectively, in the "pitch" subsystem and in the "yaw-and-roll" subsystem:

$$\Lambda^{(1)} = \begin{vmatrix} \lambda^2 + 3\gamma & p_2(\lambda^2+3) \\ \lambda^2 - \lambda\tilde{k}_\theta^* + (3-\tilde{k}_\theta) & \lambda^2 p_1 + \lambda\tilde{k}_\delta^* + (3+\tilde{k}_\delta) \end{vmatrix} = \sum_{i=0}^{4} w_i\,\lambda^i, \qquad \text{where}$$

$$w_4 \equiv \det M_1 = p_1 - p_2,\ \ w_3 = \tilde{k}_\delta^* + p_2\tilde{k}_\theta^*,\ \ w_2 = 3\,(p_1\gamma - 2p_2 + 1) + \tilde{k}_\delta + p_2\tilde{k}_\theta,$$

$$w_1 = 3\Big(\gamma\,\tilde{k}_\delta^* + p_2\tilde{k}_\theta^*\Big),\ \ w_0 = 3\Big(3\,(\gamma - p_2) + \gamma\,\tilde{k}_\delta + p_2\tilde{k}_\theta\Big)\,;$$

$$\Lambda^{(2)} = \begin{vmatrix} \lambda^2+\alpha & \lambda(\alpha-1) & 0 \\ \lambda(\alpha-1)(\gamma-1) & \lambda^2(1+\gamma\alpha)+4(\alpha+\gamma) & (\lambda^2+4)p_2(\alpha+1) \\ 0 & \lambda^2 - \lambda\tilde{k}_\varphi^* + (4-\tilde{k}_\varphi) & \lambda^2 p_1 + \lambda\tilde{k}_\sigma^* + (3+p_1+\tilde{k}_\sigma) \end{vmatrix} =$$

$$= \sum_{i=0}^{6} v_i\,\lambda^i, \qquad \text{where} \quad v_6 \equiv \det M_2 = (1+\gamma\,\alpha)\,p_1 - (\alpha+1)p_2,$$

$$v_5 = (1+\gamma\,\alpha)\,\tilde{k}_\sigma^* + (\alpha+1)\,p_2\,\tilde{k}_\varphi^*, \quad v_1 = 4\,\alpha\Big((\alpha+\gamma)\,\tilde{k}_\sigma^* + (\alpha+1)\,p_2\,\tilde{k}_\varphi^*\Big)\,,$$

$$v_3 = (1+3\gamma+\alpha(\alpha+2\gamma+3))\,\tilde{k}_\sigma^* + (\alpha+1)(4+\alpha)\,p_2\,\tilde{k}_\varphi^*,$$

$$v_4 = (1+\gamma\alpha)(3+p_1+\tilde{k}_\sigma) + (1+3\gamma+\alpha(3+\alpha+2\gamma))p_1 - (\alpha+1)p_2(8+\alpha-\tilde{k}_\varphi),$$

$$v_2 = (1+3\gamma+\alpha\,(\alpha+2\gamma+3))\Big(3+p_1+\tilde{k}_\sigma\Big) + 4\,\alpha\,p_1\,(\gamma+\alpha) + \\ + (\alpha+1)\,p_2\Big((4+\alpha)\,\tilde{k}_\varphi - 8(\alpha+2)\Big)\,,$$

$$v_0 = 4\,\alpha\Big((\alpha+\gamma)\Big(3+p_1+\tilde{k}_\sigma\Big) + (\alpha+1)\,p_2\Big(\tilde{k}_\varphi - 4\Big)\Big)\,.$$

The principal diagonal minors of the Hurwitz matrix, respectively, for two subsystems

$$\Delta_3^{(1)} = w_1w_2w_3 - w_4w_1^2 - w_0w_3^2; \quad \Delta_3^{(2)} = \begin{vmatrix} v_5 & v_3 & v_1 \\ v_6 & v_4 & v_2 \\ 0 & v_5 & v_3 \end{vmatrix}; \quad \Delta_5^{(2)} = \begin{vmatrix} v_5 & v_3 & v_1 & 0 & 0 \\ v_6 & v_4 & v_2 & v_0 & 0 \\ 0 & v_5 & v_3 & v_1 & 0 \\ 0 & v_6 & v_4 & v_2 & v_0 \\ 0 & 0 & v_5 & v_3 & v_1 \end{vmatrix}$$

are analytically obtained with SPCA “Mathematica” and were used in further calculations, but due to bulkiness, their explicit form is not given here.

The fulfillment of the conditions on the existence of roots with negative real parts for the polynomial $\Lambda(\lambda)$

$$w_i > 0,\ (i = \overline{0,4});\quad \Delta_3^{(1)} > 0, \tag{6}$$

$$v_i > 0,\ (i = \overline{0,6});\quad \Delta_3^{(2)} > 0;\quad \Delta_5^{(2)} > 0 \tag{7}$$

ensures the asymptotic stability of the system’s equilibrium position on the basis of Lyapunov’s theorem on the first approximation.

It is worth noting that the conditions $w_4 > 0$ and $v_6 > 0$ are satisfied by virtue of the positive definiteness of the kinetic energy matrix.

### 5.1 Stabilization in the “Pitch” Subsystem

With the help of “Mathematica” function *Reduce* designed to find the symbolic (analytical) solution of the inequalities systems, the conditions for the control parameters $\tilde{k}^*_\theta$, $\tilde{k}^*_\delta$, $\tilde{k}_\theta$, $\tilde{k}_\delta$ (when $p_1 > p_2, \gamma < p_2$) ensuring the fulfillment of the system of inequalities (6) are obtained. Due to the solution’s bulkiness, its presentation is omitted here. It is worth noting that “extra” forces entail “costs” of their technical implementation.

An analysis of the solution obtained allows us to conclude that it is possible to achieve stabilization of the subsystem to asymptotic stability by a “reduced” set of control forces in Case 1 $Q_\delta = -\tilde{k}^*_\delta\,\dot{\delta} - \tilde{k}_\delta\,\delta$ or Case 2 $Q_\delta = \tilde{k}^*_\theta\,\dot{\theta} + \tilde{k}_\theta\,\theta$. In Case 1, additional dissipative and potential forces make an impact on the subsystem, and in Case 2, all forces (potential, non-conservative, dissipative and gyroscopic) are present. As a result, the following proposition is formulated and proved.

**Proposition 1.** *When choosing control parameters that satisfy the conditions*

$$\tilde{k}^*_\delta > 0,\quad \tilde{k}_\delta > 3\left(\frac{p_2}{\gamma} - 1\right)\ \textit{in Case 1}\quad \textit{or}\ \ \tilde{k}^*_\theta > 0,\quad \tilde{k}_\theta > 3\left(1 - \frac{\gamma}{p_2}\right)\ \textit{in}$$

*Case 2, all the roots of the polynomial $\Lambda^{(1)}(\lambda)$ have negative real parts.*

### 5.2 Stabilization in the “Yaw-and-Roll” Subsystem

We note that the control parameters $\tilde{k}^*_\varphi$ and $\tilde{k}^*_\sigma$ enter only the odd coefficients $v_1$, $v_3$, $v_5$ of the characteristic equation. With the above mentioned *Reduce* function, their positivity is analyzed separately for the regions $\underline{\boldsymbol{O}}$ and $\underline{\boldsymbol{E}}$. For example, for the region $\underline{\boldsymbol{O}}$, the function call and the solution have the following form:

$$\begin{gathered} \textit{Reduce}[\{\,0 < p_2 < p_1 \le 1,\ 0 < \gamma < p_2,\ -\gamma < \alpha < 0,\ S < 0, \\ v_1 > 0,\ v_3 > 0,\ v_5 > 0\,\}, \{\,\tilde{k}^*_\sigma, \tilde{k}^*_\varphi\,\},\ \text{Reals}] \end{gathered}$$

$$p_2 < p_1 \leq 1 \,\wedge\, \gamma < p_2 \,\wedge\, -\gamma < \alpha < 0 \,\wedge$$
$$\wedge\, \tilde{k}^*_\sigma > 0 \,\wedge\, -\frac{(1+3\gamma+\alpha(3+\alpha+2\gamma))\,\tilde{k}^*_\sigma}{(\alpha+1)(\alpha+4)p_2} < \tilde{k}^*_\varphi < -\frac{(\alpha+\gamma)\,\tilde{k}^*_\sigma}{(\alpha+1)p_2}.$$

Looking at the analytical solution of this system of inequalities, we note the positivity of $\tilde{k}^*_\sigma$ and the negativity of $\tilde{k}^*_\varphi$. Therefore, forces (in the matrix $D_2$) can only be dissipative but not accelerating. As a result, the following proposition is formulated and proved.

**Proposition 2.** *It is impossible to ensure the coefficients $v_1, v_3, v_5$ are simultaneously positive for the values of the parameters from the region $\underline{\boldsymbol{O}}$ when $\tilde{k}^*_\sigma = 0$ or $\tilde{k}^*_\varphi = 0$, but in the region $\underline{\boldsymbol{E}}$, this can be done.*

Thus, in order to stabilize the system in the region $\underline{\boldsymbol{O}}$, a complete set of control forces with respect to velocities is required (in contrast to the region $\underline{\boldsymbol{E}}$, where a "reduced" set of forces is sufficient).

It is not possible to obtain an analytical solution for the entire system of inequalities (7) because of the large number of parameters and the complexity of the expressions being analyzed. Therefore, to simplify the analysis, let us move on to symbolic-numerical analysis for fixed values of some parameters.

To start with, we consider the question of the possibility of asymptotic stability for the region $\underline{\boldsymbol{O}}$. Since in this region $v_0\Big|_{\tilde{k}_\sigma=0\,,\,\tilde{k}_\varphi=0} \equiv \det K_2 > 0$, it is possible not to take into account the positional forces in $Q_\sigma$ from (4) (i.e., let us add $\tilde{k}_\sigma = 0$ and $\tilde{k}_\varphi = 0$). When solving the system of inequalities (7) using *Reduce* function for the specific numerical values $\tilde{k}^*_\varphi < 0$, $\tilde{k}^*_\sigma > 0$ (for example, $\tilde{k}^*_\sigma = 1$, $\tilde{k}^*_\varphi = -\gamma/p_2$, $p_1 = 4/5$) we get the answer FALSE (i.e. the system is incompatible). The same answer was received in the case $\tilde{k}_\sigma \neq 0$, $\tilde{k}_\varphi \neq 0$ (i.e. under the action of the whole set of forces $Q_\sigma$). As a result of the analysis, the following proposition can be formulated.

**Proposition 3.** *For the values of the parameters in the region $\underline{\boldsymbol{O}}$ system (1) cannot be stabilized up to the asymptotic stability due to the control forces' effect (4).*

Now, let us consider the question of the possibility of asymptotic stability for the region $\underline{\boldsymbol{E}}$. Taking into account the second part of Proposition 2, we assume that $Q_\sigma = -\tilde{k}^*_\sigma \dot{\sigma} - \tilde{k}_\sigma \sigma$ (that is, additionally only dissipative and potential forces act). In this case, the principal diagonal minors of the third and fifth order Hurwitz matrix do not depend on the second control parameter $\tilde{k}_\sigma$ and have the form:

$$\Delta_3^{(2)} = -p_2(\alpha-1)^2(\alpha+1)(\gamma-1)(\,9\,(1-\gamma)+\alpha\,(6\,(1-\gamma)+\alpha+1)\,)(\tilde{k}^*_\sigma)^2,$$
$$\Delta_5^{(2)} = -144\,p_2^2\,\alpha\,(\alpha-1)^4(\alpha+1)^2(\gamma-1)^3(\tilde{k}^*_\sigma)^3.$$

When solving the system of inequalities (7) (where, as in Fig. 1, $p_1 = 4/5$, $p_2 = 5/7$) in relation to $\tilde{k}^*_\sigma$, $\tilde{k}_\sigma$ using function

$$\mathit{Reduce}[\,\{\,0<\gamma<5/7,\,0<\alpha<1,\,S<0,\,v_6>0,\,v_0>0,\,v_2>0,\,v_4>0,$$
$$v_1>0,\,v_3>0,\,v_5>0,\,\Delta_3^{(2)}>0,\,\Delta_5^{(2)}>0\,\},\{\,\tilde{k}^*_\sigma,\,\tilde{k}_\sigma\,\},\mathit{Reals}],$$

we get the answer:

$$\tilde{k}_\sigma^* > 0 \wedge \tilde{k}_\sigma > \frac{100 - 33\,\alpha - 133\,\gamma}{35\,(\,\alpha + \gamma\,)} \wedge$$
$$\wedge \Big(\Big(0 < \alpha \le \frac{3}{25} \wedge 0 < \gamma < \frac{5}{7}\Big) \vee \Big(\frac{3}{25} < \alpha \le \frac{5}{33} \wedge \frac{25\alpha - 3}{28\alpha} < \gamma < \frac{5}{7}\Big) \vee$$
$$\vee \Big( \frac{5}{33} < \alpha < \frac{19}{44} \wedge \frac{25\,\alpha - 3}{28\,\alpha} < \gamma < \frac{100 - 33\,\alpha}{133} \Big)\Big). \qquad (8)$$

It is not difficult to show that in the region $\underline{\boldsymbol{E}}$, the value $\frac{100-33\,\alpha-133\,\gamma}{35\,(\alpha+\gamma)} > 0$, and, therefore, the parameter $\tilde{k}_\sigma$ in (8) is positive. We note that any positive value of the other parameter $\tilde{k}_\sigma^*$ satisfies solution (8). Thus, in the present case, $Q_\sigma$ are the forces of friction and elasticity.

Let us construct the region of asymptotic stability (8) in the parameter plane $\alpha, \gamma$ using "Mathematica" function *RegionPlot*, designed for a graphical representation of the solution of the system of inequalities, with the next value of the parameter $\tilde{k}_\sigma = 10$. The result obtained is shown with a shaded region in Fig. 2. It has been found that with an increasing (decreasing) value $\tilde{k}_\sigma$, this area expands (narrows) within the limits of the borders found $v_6 = 0$, $S = 0$, $\alpha = 0$, $\gamma = 0$, $\gamma = 5/7$ (see Fig. 2) and disappears at a value $\tilde{k}_\sigma = 0$.

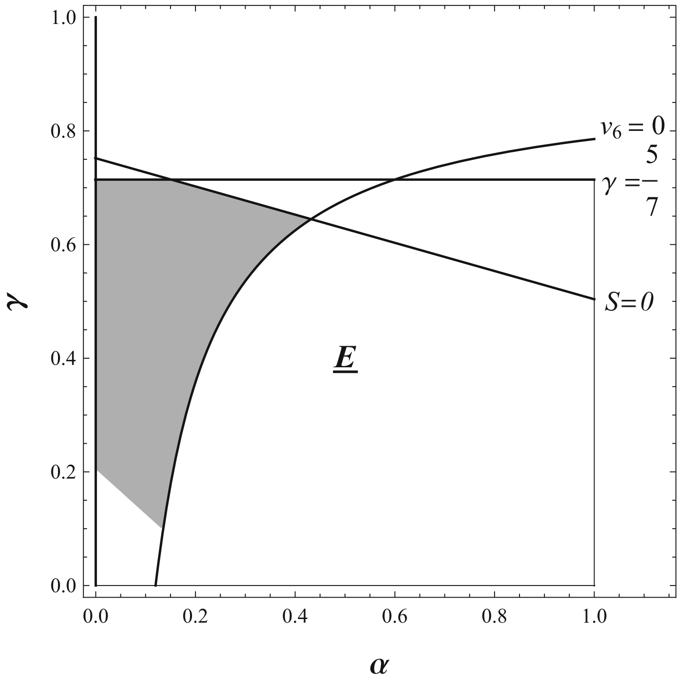


**Fig. 2.** Region of asymptotic stability.

A similar symbolic-numerical analysis has also been carried out for the control forces $Q_\sigma = \tilde{k}^*_\varphi \dot{\varphi} + \tilde{k}_\varphi \varphi$. As a result of the analysis, the following proposition can be formulated.

**Proposition 4.** *For the values of the parameters from the region* $\underline{\boldsymbol{E}}$*, system (1) can be stabilized up to an asymptotic stability thanks to the effect of control forces* $Q_\sigma = -\tilde{k}^*_\sigma \dot{\sigma} - \tilde{k}_\sigma \sigma$ *or* $Q_\sigma = \tilde{k}^*_\varphi \dot{\varphi} + \tilde{k}_\varphi \varphi$.

## 6 Conclusion

Based on the analogy with the parametric analysis presented above, the possibility of asymptotic stability was also investigated for other regions in Fig. 1. The study has shown that replacing the initial parameters $a, b, c, f, d$ with the parameters $\alpha, \gamma, p_1, p_2$ only slightly simplified the symbolic-numerical analysis. But due to the limited values of $\alpha, \gamma, p_1, p_2$, this replacement allowed us to see a qualitative picture of the research. For a future research, the problem of the influence of the structure of forces on system's stability and its stabilization requires a more detailed study.

It is necessary to emphasize the problems of reliability and precision of computations, as well as the problems of explicitness and speeding-up of the process of investigations can be partially solved when SPCA is chosen as a software tool. Along with the application of the SPCA (as "a calculator") for solving a definite problem, the approach, which implies the elaboration of some software for solving a definite class of problems on the basis of the internal programming language of the SPCA (in our case – "Mathematica"), is quite important. Practically, the whole above analysis has been conducted using this software.

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