Chapter 2
Non-instantaneous Impulses in Differential Equations with Caputo Fractional Derivatives

2.1 Statement of the Problem

Fractional calculus is the theory of integrals and derivatives of arbitrary non-integer order, which unifies and generalizes the concepts of ordinary differentiation and integration. For more details on geometric and physical interpretations of fractional derivatives and for a general historical perspective we refer the reader to the monographs [42, 45, 101] and the cited references therein.

Impulsive differential equations arise from real world problems to describe the dynamics of processes in which sudden, discontinuous jumps occur. Such processes are natural in biology, physics, engineering, etc.

As it is mentioned in the Introduction there are two popular types of impulses:

– *instantaneous impulses*—the duration of these changes is relatively short compared to the overall duration of the whole process. For ordinary differential equations with impulses we refer the reader to the monographs [79, 104] and the cited references therein. There are also many recent contributions on fractional differential equations with instantaneous impulses (see, for example, [5, 9, 32, 49, 116, 117];
– *non-instantaneous impulses*—an impulsive action, which starts abruptly at a fixed point and its action continues on a finite time interval. This kind of impulse is observed in lasers, and in the intravenous introduction of drugs in the bloodstream. E. Hernandez and D. O'Regan [56] introduced this new class of abstract differential equations where the impulses are not instantaneous and they investigated the existence of mild and classical solutions. For recent work about fractional differential equations and non-instantaneous impulses we refer the reader to [48, 84, 85, 97, 99, 100, 107, 122].

The main goal of this chapter is to introduce non-instantaneous impulses in Caputo fractional differential equations. In the literature there are two main approaches in the interpretation of solutions. Both approaches are discussed and

R. Agarwal et al., *Non-Instantaneous Impulses in Differential Equations*,
DOI 10.1007/978-3-319-66384-5_2

their advantages/disadvantages are illustrated with examples. The existence of non-instantaneous impulsive fractional differential equations and the corresponding sufficient conditions are discussed using both approaches.

2.1.1 Preliminary Notes on Fractional Derivatives and Equations

Fractional calculus generalizes the derivative and the integral of a function to a non-integer order [68, 69, 82, 101]. In engineering, the fractional order q is often less than 1, so we restrict our attention to $q \in (0, 1)$.

The uniform formula of a fractional integral with $q \in (0, 1)$ is defined by

$$ {}_{t_0}\mathcal{D}^{-q}m(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{-q+1}\, m(s)ds, \qquad t \geq t_0 \tag{2.1}$$

where $m(t)$ is an arbitrary integrable function, and $\Gamma(.)$ denotes the Gamma function.

There are several definitions of fractional derivatives and fractional integrals.

1: *The Riemann–Liouville (RL) fractional derivative* of order $q \in (0, 1)$ of $m(t)$ is given by (see, for example, Section 1.4.1.1 [42], or [101])

$$ {}_{t_0}^{RL}D^{q}m(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{-q}\, m(s)ds, \qquad t \geq t_0. $$

2: *The Caputo fractional derivative* of order $q \in (0, 1)$ is defined by (see, for example, Section 1.4.1.3 [42])

$$ {}_{t_0}^{c}D^{q}m(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q}\, m'(s)ds, \qquad t \geq t_0. \tag{2.2}$$

Note the Caputo derivative of a constant is zero, whereas the Riemann-Liouville derivative is ${}_{t_0}^{RL}D^{q}C = \frac{C(t-t_0)^{-q}}{\Gamma(1-q)}$. The properties of the Caputo derivative are quite similar to those of ordinary derivatives. Also, the initial conditions of fractional differential equations with the Caputo derivative have a clear physical meaning and as a result the Caputo derivative is usually used in real applications.

If both the Caputo derivative and Riemann-Liouville derivative of $m(t)$ exist (for example, if $m(t)$ is absolutely continuous function), then from (2.4.2) [68] we have that ${}_{t_0}^{c}D^{q}m(t) = {}_{t_0}^{RL}D^{q}[m(t)-m(t_0)] = {}_{t_0}^{RL}D^{q}m(t) - \frac{m(t_0)(t-t_0)^{-q}}{\Gamma(1-q)}$ holds (see Lemma 3.4 in [45]).

3: *The Grunwald−Letnikov fractional derivative* is given by (see, for example, Section 1.4.1.2 [42])

$$ {}_{t_0}^{GL}D^q m(t) = \lim_{h\to 0} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \; {}_qC_r \, m(t-rh), \quad t \geq t_0, $$

and the Grunwald−Letnikov fractional Dini derivative by

$$ {}_{t_0}^{GL}D_+^q m(t) = \limsup_{h\to 0+} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \; {}_qC_r \, m(t-rh), \quad t \geq t_0, \tag{2.3} $$

where ${}_qC_r = \frac{q(q-1)\ldots(q-r+1)}{r!}$ and $[\frac{t-t_0}{h}]$ denotes the integer part of the fraction $\frac{t-t_0}{h}$.

Proposition 2.1.1 (Theorem 2.25 [45]) *Let* $m \in C^1[t_0, b]$. *Then, for* $t \in (t_0, b]$

$$ {}_{t_0}^{GL}D^q m(t) = {}_{t_0}^{RL}D^q m(t). $$

Also, according to Lemma 3.4 [45] we have

$$ {}_{t_0}^{c}D_t^q m(t) = {}_{t_0}^{RL} D_t^q m(t) - m(t_0)\frac{(t-t_0)^{-q}}{\Gamma(1-q)}. $$

From the relation between the Caputo fractional derivative and the Grunwald−Letnikov fractional derivative using (2.3) we define the Caputo fractional Dini derivative as

$$ {}_{t_0}^{c}D_+^q m(t) = {}_{t_0}^{GL}D_+^q [m(t) - m(t_0)], \tag{2.4} $$

i.e.

$$ \begin{aligned} {}_{t_0}^{c}D_+^q m(t) = \limsup_{h\to 0+} \frac{1}{h^q}\Big[& m(t) - m(t_0) \\ & - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} \; {}_qC_r \Big(m(t-rh) - m(t_0) \Big) \Big]. \end{aligned} \tag{2.5} $$

Proposition 2.1.2 (Lemma 6 [88]) *Let* $q \in (0,1)$ *and function* $m(t)$ *is such that both fractional derivative* ${}_0^cD^q m(t)$ *and* ${}_0^{RL}D^q m(t)$ *exist and* $M(0) \geq 0$. *Then*

$$ {}_0^cD^q m(t) \leq {}_0^{RL}D^q m(t). $$

Definition 2.1.1 ([44]) *We say $m \in C^q([t_0, T], \mathbb{R}^n)$ if $m(t)$ is differentiable (i.e., $m'(t)$ exists), the Caputo derivative ${}_{t_0}^{c}D^q m(t)$ exists and satisfies (2.2) for $t \in [t_0, T]$.*

Remark 2.1.1 *Definition 2.1.1 could be extended to any interval $I \subset \mathbb{R}_+$.*

Remark 2.1.2 *If $m \in C^q([t_0, T], \mathbb{R}^n)$, then ${}_{t_0}^{c}D_+^q m(t) = {}_{t_0}^{c}D^q m(t)$.*
The classical (with one parameter) and generalized (with two parameters) Mittag–Leffler functions are defined by

$$E_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+1)}, \quad E_{q,q}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+q)}. \tag{2.6}$$

Some properties of classical and generalized Mittag–Leffler functions are given in the following Lemma:

Lemma 2.1.1 (Lemma 2 [121]) *The classical and generalized Mittag–Leffler functions are nonnegative and have the following properties:*

(i). For any $\lambda > 0$ and $t \in [0, T]$, $T > 0$ is a given constant,

$$E_q(-t^q\lambda) \le 1, \quad E_{q,q}(-t^q\lambda) \le \frac{1}{\Gamma(q)};$$

(ii). For any $\lambda > 0$ and $t_1, t_2 \in [0, T]$

$$E_q(-t_1^q\lambda) \to E_q(-t_2^q\lambda) \quad \text{as } t_1 \to t_2,$$

$$E_{q,q}(-t_1^q\lambda) \to E_{q,q}(-t_2^q\lambda) \quad \text{as } t_1 \to t_2;$$

Or rather,

$$|E_q(-t_1^q\lambda) - E_q(-t_2^q\lambda)| = O(|t_1 - t_2|^q) \quad \text{as } t_1 \to t_2,$$

$$|E_{q,q}(-t_1^q\lambda) - E_{q,q}(-t_2^q\lambda)| = O(|t_1 - t_2|^q) \quad \text{as } t_1 \to t_2;$$

(iii). For any $\lambda > 0$, $t_1, t_2 \in [0, T]$ and $t_1 \le t_2$

$$E_q(-t_1^q\lambda) \ge E_q(-t_2^q\lambda), \qquad E_{q,q}(-t_1^q\lambda) \ge E_{q,q}(-t_2^q\lambda).$$

Lemma 2.1.2 ([21]) *Let $x \in C^q([\tau_0, \infty), \mathbb{R})$. Then for any $t \ge \tau_0$ the inequality*

$${}_{\tau_0}^{c}D^q\Big(x(t)\Big) \le 2\, x(t)\, {}_{\tau_0}^{c}D^q x(t)$$

holds.

2.1.2 Ordinary Differential Equations Versus Caputo Fractional Differential Equations

We compare some properties of ordinary differential equations (ODE) and Caputo fractional differential equations (FrDE). Following two equivalent approaches to the solutions of the initial value problem of ordinary differential equations we will present two approaches to the solutions of the initial value problem of Caputo fractional differential equations.

I. Ordinary differential equations.

Consider the ODE

$$x'(t) = f(t, x) \quad \text{for } t \geq \tau, \tag{2.7}$$

with the initial condition

$$x(\tau) = \tilde{x}_0, \tag{2.8}$$

where $\tilde{x_0} \in \mathbb{R}^n$.

Denote the solution of the initial value problem (IVP) for ODE (2.7), (2.8) by $x(t; \tau, \tilde{x}_0)$.

Now consider ODE (2.7) with different initial time $\tau_1 > \tau$, i.e.

$$x(\tau_1) = \tilde{u}_0, \tag{2.9}$$

where $\tilde{u}_0 \in \mathbb{R}^n$.

Denote the solution of the initial value problem (IVP) for ODE (2.7), (2.9) by $x(t; \tau_1, \tilde{x}_0)$.

We will assume that the ODE (2.7) has a unique solution for any given initial value and initial point. Then $x(t; \tau, \tilde{x}_0) = x(t; \tau_1, \tilde{u}_0)$ for $t \geq \tau_1$.

Remark 2.1.3 *For the IVP for ODE (2.7), (2.9) note the right side part $f(t, x)$ has to be defined only for $t \geq \tau_1$.*

We can look at the solutions of both IVPs for ODE (2.7), (2.8) and (2.7), (2.9) in two equivalent ways:

(A1 for ODE.) Let $c \in \mathbb{R}^n$ be an arbitrary and $x(t; \tau, c)$ be the solution of the IVP for ODE (2.7), (2.8) with $\tilde{x}_0 = c$. Choose the constant vector $c = c_1$ such that $x(\tau_1; \tau, c_1) = \tilde{u}_0$ where $\tilde{u}_0 \in \mathbb{R}^n$ is initially given. Then we call the function $x(t; \tau, c_1)$ a solution of the IVP for ODE (2.7), (2.9) for $t \geq \tau_1$. The solution $x(t) = x(t; \tau, c_1)$ of the IVP for ODE (2.7), (2.9) will satisfy the integral equality

$$x(t) = x(\tau_1; \tau, c_1) + \int_{\tau_1}^{t} f(s, x(s))ds$$
$$= \tilde{u}_0 - \int_{\tau}^{\tau_1} f(s, x(s; \tau, c_1))ds + \int_{\tau}^{t} f(s, x(s; \tau, c_1))ds, \quad t \geq \tau_1. \tag{2.10}$$

(A2 for ODE.) Let $\tilde{u}_0 \in \mathbb{R}^n$ be the same as in (A1 for ODE). Denote by $x(t; \tau, \tilde{u}_0)$ the solution of the IVP for ODE (2.7), (2.9). The solution of IVP for ODE (2.7), (2.9) will satisfy the following integral equality

$$x(t; \tau, \tilde{u}_0) = \tilde{u}_0 + \int_{\tau_1}^{t} f(s, x(s; \tau, \tilde{u}_0))ds, \quad t \geq \tau_1. \tag{2.11}$$

Remark 2.1.4 *Note approach (A1 for ODE) and approach (A2 for ODE) are equivalent in the general case and give one and the same solution of the IVP for ODE (2.7), (2.8).*

II. Caputo fractional differential equations.

Consider the Caputo fractional differential equation (FrDE)

$${}_{\tau}^{c}D^{q}x(t) = f(t, x) \quad \text{for } t \geq \tau \tag{2.12}$$

with initial condition

$$x(\tau) = \tilde{x}_0, \tag{2.13}$$

where $\tilde{x}_0 \in \mathbb{R}^n$.

The fractional Volterra integral equation corresponding to the IVP for FrDE (2.12), (2.13) is given by

$$x(t) = \tilde{x}_0 + \frac{1}{\Gamma(q)} \int_{\tau}^{t} (t-s)^{q-1} f(s, x(s))ds, \quad t \geq \tau. \tag{2.14}$$

Change the initial time to $\tau_1 > \tau$ and consider the FrDE (2.12) with the following initial condition

$$x(\tau_1) = \tilde{u}_0, \tag{2.15}$$

where $\tilde{u}_0 \in \mathbb{R}^n$.

Both approaches to the solutions of ODE are equivalent. The case of fractional derivatives is totally different. Now, based on the above presented both approaches for ODE we will present two different approaches to the solution of the IVP for the Caputo FrDE:

(A1 for FrDE.) Let $c \in \mathbb{R}^n$ be an arbitrary and $x(t;\tau,c)$ be the solution of the IVP for FrDE (2.12), (2.13) with $\tilde{x}_0 = c$. Choose the constant vector $c = c_1$ such that $x(\tau_1;\tau,c_1) = \tilde{u}_0$ where $\tilde{u}_0 \in \mathbb{R}^n$ is initially given. Then we call the function $x(t;\tau,c_1)$ a solution of the IVP for FrDE (2.12), (2.15) for $t \geq \tau_1$. Using (2.14) it follows the solution $x(t) = x(t;\tau,c_1)$ of the IVP for FrDE (2.12), (2.15) will satisfy the integral equality

$$x(t) = c_1 + \frac{1}{\Gamma(q)} \int_{\tau}^{t} (t-s)^{q-1} f(s, x(s;\tau,c_1))ds, \quad t \geq \tau_1. \tag{2.16}$$

Also, from the choice of c_1 and (2.14) it follows

$$\tilde{u}_0 = x(\tau_1;\tau,c_1) = c_1 + \frac{1}{\Gamma(q)} \int_{\tau}^{\tau_1} (\tau_1-s)^{q-1} f(s, x(s;\tau,c_1))ds \tag{2.17}$$

and therefore, the solution of the IVP for the Caputo FrDE (2.12), (2.9) satisfies the fractional integral equation

$$\begin{aligned} x(t) = \tilde{u}_0 &+ \frac{1}{\Gamma(q)} \int_{\tau}^{t} (t-s)^{q-1} f(s, x(s;\tau,c_1))ds \\ &- \frac{1}{\Gamma(q)} \int_{\tau}^{\tau_1} (\tau-s)^{q-1} f(s, x(s;\tau,c_1))ds \quad \text{for } t \geq \tau. \end{aligned} \tag{2.18}$$

(compare Eq. (2.18), $q \in (0,1)$ with Eq. (2.10), $q = 1$).

In the case $f(t,x) = h(t)$ the formula (2.18) is proved in Lemma 3.2 [120].

Note K. Diethelm (Section 6 [45]) pointed out that the problem consisting of Eqs. (2.12) and (2.9) is more closely related to a boundary value problem than to an initial value problem. This is in contrast to the situation observed for first-order ordinary differential equations (see I. Ordinary Differential Equations).

Remark 2.1.5 *Note in Eq. (2.17) the right side part $f(t,x)$ has to be defined for all $t \geq \tau$.*

Remark 2.1.6 *Using (A1 for FrDE) we keep one of the basic properties of ODEs, namely, $x(t;\tau_1,x(\tau_1;\tau,c)) = x(t;\tau,c)$ for $t \geq \tau_1$.*

(A2 for FrDE.) Let $\tilde{u}_0 \in \mathbb{R}^n$ be an arbitrary point and let $x(t;\tau,\tilde{u}_0)$ be the solution of the IVP for FrDE (2.12), (2.15). Using (2.14) the solution of IVP for FrDE (2.12), (2.15) will satisfy the following integral equality

$$x(t) = \tilde{u}_0 + \frac{1}{\Gamma(q)} \int_{\tau_1}^{t} (t-s)^{q-1} f(s, x(s))ds, \quad t \geq \tau_1. \tag{2.19}$$

(Compare Eq. (2.19), $q \in (0,1)$ with Eq. (2.11), $q = 1$).

The fractional integral equation (2.19) is equivalent to the following Caputo fractional differential equation

$$ {}_{\tau_1}^{c}D^q x(t) = f(t,x) \quad \text{for } t \geq \tau_1 \tag{2.20}$$

with initial condition (2.15).

Note ${}_{\tau_1}^{c}D^q x(t) = \frac{1}{\Gamma(1-q)} \int_{\tau_1}^{t} (t-s)^{-q} x'(s)ds \neq \frac{1}{\Gamma(1-q)} \int_{\tau}^{t} (t-s)^{-q} x'(s)ds = {}_{\tau}^{c}D^q x(t)$.

Therefore, the change of the initial time leads to a change in the Caputo fractional derivative of the unknown function in the differential equation (compare (2.7) with (2.20)).

Remark 2.1.7 *Using (A2 for FrDE) we lose one of the basic properties of ODEs, namely,* $x(t;\tau_1, x(\tau_1;\tau,c)) \neq x(t;\tau,c)$ *for* $t > \tau_1$ *(compare with Remark 2.1.6).*

Remark 2.1.8 *In (A2 for FrDE) the right side part* $f(t,x)$ *of the IVP (2.12), (2.9) has to be defined only for* $t \geq \tau_1$ *(compare with Remark 2.1.3 for ODEs and Remark 2.1.5 for the approach (A1 for FrDE)).*

Remark 2.1.9 *Differently than the ordinary case* ($q = 1$) *in the fractional case* ($q \in (0,1)$) *both approaches (A1 for FrDE) and (A2 for FrDE) differ and in the general case they give different solutions to the FrDE (2.12).*

Example 2.1.2.1 Let $n = 1$, $\tau_1 = 1$ and

$$f(t,x) \equiv h(t) = \begin{cases} 0 & t \in [0,1] \\ 1-t & t \geq 1 \end{cases}$$

Case 1. (*Approach (A1 for FrDE)*). Consider the IVP for the FrDE

$$ {}_{0}^{c}D^q x(t) = f(t,x) \quad \text{for } t \geq 1, \quad x(1) = 0. \tag{2.21}$$

Using formula (2.17) we get

$$\begin{aligned} x(t) &= 0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)ds - \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} h(s)ds \\ &= \frac{1}{\Gamma(q)} \int_1^t (t-s)^{q-1}(1-s)ds, \quad t \geq 1. \end{aligned} \tag{2.22}$$

Case 2. (*Approach (A2 for FrDE)*). Consider the IVP for the FrDE

$$ {}_{1}^{c}D^q x(t) = f(t,x) \quad \text{for } t \geq 1, \quad x(1) = 0. \tag{2.23}$$

The solution of IVP for FrDE (2.23) applying (2.19) is

$$x(t) = 0 + \frac{1}{\Gamma(q)} \int_1^t (t-s)^{q-1}(1-s)ds, \quad t \geq 1. \tag{2.24}$$

In this particular case both solutions coincide.

Now let $f(t,x) = 1-t, \quad t \in [0,1]$. This will not change the solution obtained by (A2 for FrDE).

The application of (A1 for FrDE) gives

$$x(t) = 0 - \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1}(1-s)ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}(1-s)ds, \quad t \geq 1, \tag{2.25}$$

and note (2.25) differs from (2.22).

Therefore, the definition of the function $f(t,x)$ to the left of the initial point has no influence in (A2 for FrDE) (similar to the ODE situation) but it has a huge influence in (A1 for FrDE).

□

Remark 2.1.10 *Note (A1 for FrDE) is similar in some sense to a boundary value problem, whereas (A2 for FrDE) is close to the idea of initial value problems defined and studied in the classical books [45, 101] (the initial time coincides with the lower limit of the Caputo fractional derivative).*

Example 2.1.2.2 Let $n = 1, f(t,x) \equiv 1$.

Using Eq. (2.14) we obtain the solution of IVP for FrDE (2.12), (2.13) given by $x(t;\tau,\tilde{x_0}) = \tilde{x_0} + \frac{1}{q\Gamma(q)}\Big(t-\tau\Big)^q$.

Approach (A1 for FrDE): Using (2.18) we get the solution of IVP for FrDE (2.12), (2.15), namely, $x(t;\tau_1,\tilde{u_0}) = \tilde{u_0} + \frac{1}{q\Gamma(q)}\Big((t-\tau)^q - (\tau_1-\tau)^q\Big), \ t \geq \tau_1$. Since $\tau < \tau_1$ could be zero, then $x(t;\tau_1,\tilde{u_0}) = \tilde{u_0} + \frac{1}{q\Gamma(q)}\Big(t^q - \tau_1^q\Big), \ t \geq \tau_1$.

Using (A2 for FrDE) the solution of IVP for FrDE (2.12), (2.15) (or the equivalent (2.20), (2.9)) is $x(t;\tau_1,\tilde{u_0}) = \tilde{u_0} + \frac{1}{q\Gamma(q)}\Big(t-\tau_1\Big)^q$.

In this particular case both solutions differ.

□

Example 2.1.2.3 Let $n = 1, f(t,x) = x, \tau_1 > 0$.

Using (2.14) we obtain the solution of IVP for FrDE (2.12), (2.13) given by $x(t;\tau,\tilde{x_0}) = \tilde{x_0}E_q((t-\tau)^q), \ t \geq \tau$, where $E_q(z)$ is the Mittag-Leffler function with one parameter q.

Now we will apply both approaches to obtain the solution of the scalar linear fractional differential equation.

(A1 for FrDE): Choose the constant c_1 such that $x(\tau_1;\tau,c_1) = \tilde{u}_0$, or $c_1E_q((\tau_1 - \tau)^q) = \tilde{u}_0$. Therefore, $c_1 = \frac{\tilde{u}_0}{E_q((\tau_1-\tau)^q)}$. Then from (2.17) we get the solution of IVP for FrDE (2.12), (2.15) $x(t;\tau_1,\tilde{u}_0) = \frac{\tilde{u}_0}{E_q((\tau_1-\tau)^q)}E_q((t-\tau)^q)$.

Since $\tau < \tau_1$ we can choose $\tau = 0$ and obtain $x(t;\tau_1,\tilde{u}_0) = \tilde{u}_0 \frac{E_q(t^q)}{E_q(\tau_1^q)}$.

In this case integral equality (2.18) is reduced to

$$
\begin{aligned}
x(t) = \tilde{u}_0 &+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} c_1 E_q(s^q) ds \\
&- \frac{1}{\Gamma(q)} \int_0^{\tau_1} (\tau_1 - s)^{q-1} c_1 E_q(s^q) ds \quad \text{for } t \geq \tau.
\end{aligned} \tag{2.26}
$$

Now (2.26) does not give us an explicit form of the solution.

(A2 for FrDE): the solution of IVP for FrDE (2.12), (2.9) (or the equivalent (2.20), (2.9)) is $x(t;\tau_1,\tilde{u}_0) = \tilde{u}_0 E_q((t-\tau_1)^q)$.

In this case the integral equality (2.19) is reduced to

$$
x(t) = \tilde{u}_0 + \frac{1}{\Gamma(q)} \int_{\tau_1}^t (t-s)^{q-1} x(s) ds, \quad t \geq \tau_1 \tag{2.27}
$$

which solution coincides with the solution obtained above by application of (A2 for FrDE).

Solutions obtained by both approaches differ.

Consider the case $q = 1$, i.e., the scalar ODE $x' = x, \ \ x(\tau_1) = \tilde{u}_0$ which solution is $x(t) = \tilde{u}_0 e^{t-\tau}$. This solution coincides with the solutions obtained by (A1 for FrDE) and (A2 for FrDE) for $q = 1$ ($E_q(z) = e^z$ for $q = 1$).

□

Remark 2.1.11 *Both approaches described above usually differ and give different solutions in the general case.*

2.1.3 Non-instantaneous Impulses in Caputo Fractional Differential Equations

We start with the case of instantaneous impulses in Caputo fractional differential equations. We will begin with a brief overview of its statements and later we will compare it with the case of non-instantaneous impulses.

Case I. Instantaneous impulses.

Let an increasing sequence of points $\{t_i\}_{i=1}^{\infty}$ be given such that $0 < t_i < t_{i+1}$, $i = 1, 2, \dots, \lim_{k\to\infty} t_k = \infty$.

Let $t_0 \in R_+, \ t \neq t_k, k = 1, 2, \dots,$ be a given arbitrary point. Without loss of generality we will assume that $t_0 \in [0, t_1)$.

Consider the IVP for Caputo impulsive fractional differential equation (IFrDE)

$$
\begin{aligned}
&{}_{t_0}^{c}D^q x(t) = f(t,x) \text{ for } t \neq t_k,\ k = 1,\ldots,\\
&x(t_k+0) = x(t_k-0) + I_k(x(t_k-0)) \quad \text{for } k = 1,2,\ldots,\\
&x(t_0) = x_0,
\end{aligned}
\tag{2.28}
$$

where $x_0 \in \mathbb{R}^n, f : [t_0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n, I_k : \mathbb{R}^n \to \mathbb{R}^n, (k = 1,2,3,\ldots)$.
Let $J \subset \mathbb{R}_+$ and introduce

$$
IPC(J,\mathbb{R}^n) = \{u : J \to \mathbb{R}^n :\ u \in C^q((t_k,t_{k+1}] \cap J,\mathbb{R}^n),\ k = 0,1,\ldots,\\
\text{and } \lim_{t\uparrow t_k} = u(t_k-0) = u(t_k),\ \lim_{t\downarrow t_k} = u(t_k+0) = u(t_k+0) < \infty\}.
$$

Fractional derivatives can create different interpretations of the solutions of the IVP for IFrDE (2.28). We will present the two main approaches.

(A1 for IFrDE.) We will use the following definition:

Definition 2.1.2 ([120]) *A function $x \in IPC([t_0,T],\mathbb{R}^n)$, $T > t_0, t \le \infty$, is called a solution of the IVP for the IFrDE (2.28) if*

- *$x(t) = x_k(t)$ for $t \in (t_k,t_{k+1}] \cap [t_0,T]$ where $x_k \in C^q([t_0,t_{k+1}],\mathbb{R}^n)$, $k = 0,1,2,\cdots :\ t_{k+1} \le T$, satisfies ${}_{t_0}^{c}D^q x_k(t) = f(t,x_k(t))$ a.e. on (t_0,t_{k+1}) with the restriction of $x_k(t_k) = x_{k-1}(t_k) + I_k(x_{k-1}(t_k))$ (in the case $k = 0$ we have $x_{-1}(t_0 + I_0(x_{-1}(t_0)) = x_0$;*
- *$x(t_0) = x_0$.*

In the special case $n = 1$ and $f(t,x) = h(t)$ an implicit formula for the solution of the IVP for IFrDE ((2.28) is obtained in [120] (Lemma 3.3):

$$
x(t) = x_0 + \sum_{i=1}^{k} I_i(x(t_i-0)) + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}h(s)ds, \text{ for } t \in (t_k,t_{k+1}] \cap J.
\tag{2.29}
$$

Example 2.1.3.1 Let us consider the IVP for IFrDE ((2.28) in the special case $n = 1$, $t_0 = 0$, and $f(t,x) = 1$. Then according to Definition 2.1.2 the solution $x(t) = x_0(t) = x_0 + \frac{t^q}{q\Gamma(q)}$ on $[0,t_1]$. Let $t \in (t_1,t_2]$. Then $x(t) = x_1(t)$ where $x_1(t) = c + \frac{t^q}{q\Gamma(q)}$, $t \in [0,t_2]$, $x_1(t) = x_0(t) = x_0 + \frac{t^q}{q\Gamma(q)}$, $t \in [0,t_1]$ and $x_1(t_1) = x_0(t_1) + I_1(x_0(t_1))$, i.e., $c + \frac{t_1^q}{q\Gamma(q)} = x_0 + \frac{t_1^q}{q\Gamma(q)} + I_1(x_0 + \frac{t_1^q}{q\Gamma(q)})$. Therefore,

$$
x(t;0,x_0) = \begin{cases}
x_0 + \frac{t^q}{q\Gamma(q)} & t \in (0,t_1]\\
x_0 + I_1(x_0 + \frac{t_1^q}{q\Gamma(q)}) + \frac{t^q}{q\Gamma(q)} & t \in (t_1,t_2]\\
x_0 + I_1(x_0 + \frac{t_1^q}{q\Gamma(q)}) + \frac{t^q}{q\Gamma(q)} & \\
\quad +I_2(x_0 + I_1\Big(x_0 + \frac{t_1^q}{q\Gamma(q)}) + \frac{t_2^q}{q\Gamma(q)}\Big), & t \in (t_2,t_3]\\
\ldots\ldots\ldots &
\end{cases}
$$

The solution satisfies the integral equation (2.29). □

Example 2.1.3.2 Let us consider the IVP for IFrDE (2.28) in the special case $n = 1$, $t_0 = 0$, and $f(t, x) = x$, i.e., formula (2.29) cannot be applied. Then according to Definition 2.1.2 the solution $x(t) = x_0(t) = x_0 E_q(t^q)$ on $[0, t_1]$. Let $t \in (t_1, t_2]$. Then $x(t) = x_1(t)$ where $x_1(t) = cE_q(t^q),\ t \in [0, t_2], x_1(t) = x_0(t) = x_0 E_q(t^q),\ t \in [0, t_1]$ and $x_1(t_1) = x_0(t_1) + I_1(x_0(t_1))$, i.e., $cE_q(t_1^q) = x_0 E_q(t_1^q) + I_1(x_0 E_q(t_1^q))$. Therefore,

$$x(t;0,x_0) = \begin{cases} x_0 E_q(t^q) & t \in (0, t_1] \\ \left(x_0 + \frac{I_1(x_0 E_q(t_1^q))}{E_q(t_1^q)}\right) E_q(t^q) & t \in (t_1, t_2] \\ \Bigg(x_0 + \frac{I_1(x_0 E_q(t_1^q))}{E_q(t_1^q)} & \\ \quad + \frac{I_2\left((x_0 + \frac{I_1(x_0 E_q(t_1^q))}{E_q(t_1^q)}) E_q(t_2^q)\right)}{E_q(t_2^q)}\Bigg) E_q(t^q), & t \in (t_2, t_3] \\ \ldots\ldots\ldots & \end{cases}$$

□

We define *a mild solution* of the IVP for IFrDE (2.28). Following Definition 2.5 [118] we introduce the following definition:

Definition 2.1.3 *The function* $x(t; t_0, x_0) \in IPC([t_0, T], \mathbb{R}^n)$ *is called* a mild solution *of IVP for IFrDE (2.28) if it satisfies the following integral equation*

$$\begin{aligned} x(t; t_0, x_0) &= x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s; t_0, x_0)) ds \\ &\quad + \sum_{i:\ t_0 < t_i < t} I_i(x(t_i; t_0, x_0)), \quad t \in [t_0, T]. \end{aligned} \tag{2.30}$$

Example 2.1.3.3 Consider the IVP for IFrDE (2.28) in the special case $n = 1$, $t_0 = 0, f(t, x) = x$, and $I_k(x) = a_k x,\ k = 1, \ldots,\ a_k = const$. Then according to Definition 2.1.3 the mild solution is

$$\begin{aligned} x(t; 0, x_0) &= x_0 E_q(t^q) + \sum_{i=1}^{k} a_k x(t_i; 0, x_0) E_q((t - t_i)^q), \\ & t \in (t_k, t_{k+1}],\ k = 0, 1, 2, \ldots. \end{aligned} \tag{2.31}$$

Indeed, for $t \in [0, t_1]$ it is obvious that the integral equation (2.31) is satisfied.

Let $t \in (t_1, t_2]$. Then $(t; 0, x_0) = x_0 E_q(t^q) + a_1 x(t_1; 0, x_0) E_q((t - t_1)^q)$. Denoting $m(t) = x(t; 0, x_0)$ and using

$$x_0 E_q(t^q) = x_0 + x_0 \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} E_q(s^q) ds,$$

$$E_\alpha((t - t_1)^q) = E_q(0) + \frac{1}{\Gamma(q)} \int_{t_1}^{t} (t-s)^{\alpha-1} E_q((s - t_1)^q) ds$$

we get

$$
\begin{aligned}
& x_0 + a_1(t_1) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} m(s) ds \\
& = x_0 + a_1 m(t_1) + \frac{1}{\Gamma(q)} \int_0^{t_1} (t-s)^{q-1} x_0 E_q(s^q) ds \\
& + \frac{1}{\Gamma(q)} \int_{t_1}^t (t-s)^{q-1} \Big(x_0 E_q(s^q) + a_1 m(t_1) E_q((s-t_1)^q) \Big) ds \\
& = x_0 + x_0 \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} E_q(s^q) ds \\
& + a_1 m(t_1) \Big(1 + \frac{1}{\Gamma(q)} \int_{t_1}^t (t-s)^{q-1} E_q((s-t_1)^q) ds \Big) \\
& = x_0 E_q(t^q) + a_1 m(t_1) E_\alpha((t-t_1)^q) = m(t)
\end{aligned}
$$

which proves the validity of (2.31) on $(0, t_2]$.

Similarly, we prove equality (2.31) holds on each interval $(0, t_k],\ k = 1, 2, \ldots$.

At the same time, the function $m(t) = x(t; 0, x_0)$ is not a solution of the IVP for IFrDE (2.28) according to Definition 2.1.2. Indeed, for $t \in (t_1, t_2]$ it has to be satisfied

$$
{}_{t_0}^{c}D^q m(t) = {}_{t_0}^{c}D^q \Big(x_0 E_q(t^q) + a_1 m(t_1) E_q((t-t_1)^q) \Big) = x_0 E_q(t^q) + a_1 m(t_1) E_q((t-t_1)^q)
$$

However ${}_{t_0}^{c}D^q E_q((t-t_1)^q) \neq E_q((t-t_1)^q)$. □

Remark 2.1.12 *In the application of approach (A1 for IFrDE) we should indicate whether Definition 2.1.2 or Definition 2.1.3 is used.*

(A2 for IFrDE.) We will use the following definition:

Definition 2.1.4 *A function $x \in IPC([t_0, T], \mathbb{R}^n)$ is called a solution of the IVP for the IFrDE (2.28) if for any $t \in (t_k, t_{k+1}],\ k = 0, 1, \cdots : \ t_{k+1} \le T$, the equality ${}_{t_k}^{c}D^q x(t) = f(t, x(t))$ holds with $x(t_k + 0) = x(t_k - 0) + I_k(x(t_k - 0)),\ k = 1, 2, \ldots,\ t_k \in J$ and $x(t_0) = x_0$.*

The solution of the IVP for the IFrDE (2.28) defined by Definition 2.1.4 is satisfying the following integral equalities

$$
x(t; t_0, x_0) = \begin{cases} x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s, x(s; t_0, x_0)) ds \\ \qquad \text{for } t \in [t_0, t_1] \\ x(t_k - 0; t_0, x_0) + I_k(x(t_k - 0; t_0, x_0)) \\ \quad + \frac{1}{\Gamma(q)} \int_{t_k}^t (t-s)^{q-1} f(s, x(s; t_0, x_0)) ds \\ \qquad \text{for } t \in (t_k, t_{k+1}] \cap [t_0, T],\ k = 1, 2, \ldots. \end{cases}
$$

or

$$x(t;t_0,x_0) = x_0 + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, x(s;t_0,x_0))ds$$
$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f(s, x(s;t_0,x_0))ds$$
$$+ \sum_{0<t_k<t} I_k(x(t_k - 0;t_0,x_0)), \quad t \in [t_0, T].$$

Example 2.1.3.4 Consider the IVP for IFrDE (2.28) with $n = 1$, $f(t,x) = x$, $I_k(x) = a_k x$, $a_k = const \neq 0$.

According to approach (A2 for IFrDE) and Definition 2.1.4 the solution is given by

$$x_{(A2)}(t;t_0,x_0) = x_0\Big(\prod_{i=1}^{k} a_k E_q((t_i - t_{i-1})^q)\Big)E_q((t-t_k)^q)$$
$$\text{for } t \in (t_k, t_{k+1}],\ k = 0, 1, 2, \dots. \tag{2.32}$$

Let $A = -1$, $q = 0.3, x_0 = 0.5$. For $a = 0.7$ the solutions $x_{(A1)}(t;t_0,0.5)$ and $x_{(A2)}(t;t_0,0.5)$ obtained by both approaches have similar behavior; for example both solutions approach zero (see Figure 2.1). For $a = 2$ only the solution obtained by (A2 for IFrDE) approaches zero, the other one is increasing without bound (Figure 2.2). □

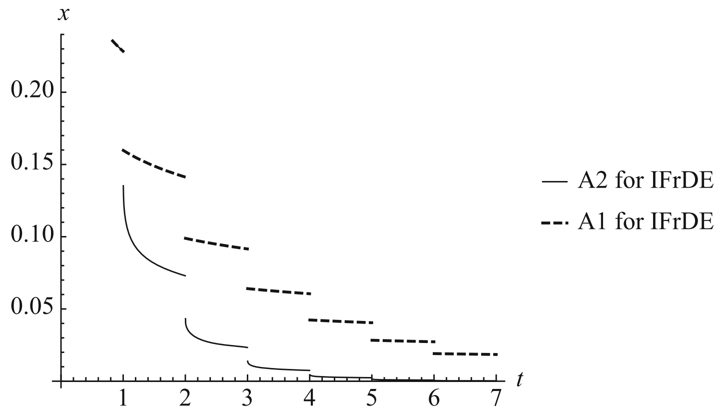

Fig. 2.1 Graphs of solutions of IVP for the IFrDE (2.28) with $A = -1$, $q = 0.3, x_0 = 0.5$, $a = 0.7$.

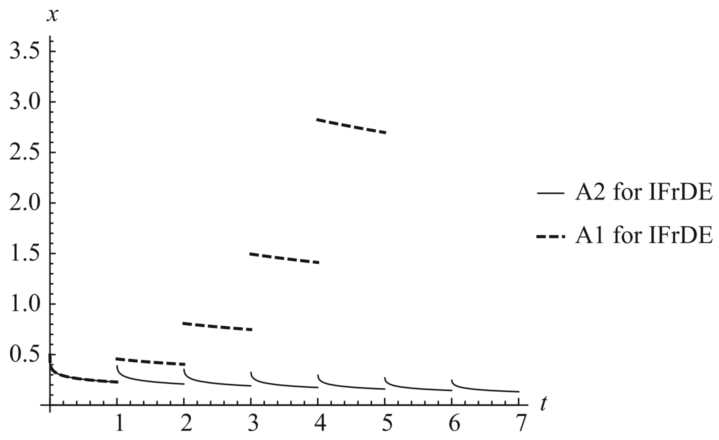

Fig. 2.2 Graphs of solutions of IVP for the IFrDE (2.28) with $A = -1, q = 0.3, x_0 = 0.5, a = 2$.

Case II. Non-instantaneous impulses.

In this book we will assume two increasing sequences of points $\{t_i\}_{i=1}^{\infty}$ and $\{s_i\}_{i=0}^{\infty}$ are given such that $0 < s_0 < t_i \le s_i < t_{i+1}$, $i = 1, 2, \dots,$ and $\lim_{k\to\infty} t_k = \infty$.

Let $t_0 \in [0, s_0) \bigcup \cup_{i=1}^{\infty} [t_k, s_k)$ and $T > t_T \le \infty$ be given points and $p = \min\{k : T < s_k\}$ (in case $t = \infty$ we denote $p = \infty$).

Consider the initial value problem (IVP) for the nonlinear *Caputo fractional differential equation with non-instantaneous impulses* (NIFrDE)

$$\begin{aligned} {}_{t_0}^{c}D^q x(t) &= f(t,x) \text{ for } t \in [t_0, T] \cap \Big([0, s_0] \cap_{k=1}^{\infty} (t_k, s_k]\Big), \\ x(t) &= \phi_k(t, x(t), x(s_{k-1} - 0)) \quad \text{for } t \in [t_0, T] \cap (s_k, t_{k+1}],\ k = 0, 1, 2, \dots, \\ x(t_0) &= x_0, \end{aligned} \tag{2.33}$$

where $x_0 \in \mathbb{R}^n$, $f : t \in [t_0, T] \cap \Big([0, s_0] \cap_{k=1}^{\infty} (t_k, s_k]\Big) \times \mathbb{R}^n \to \mathbb{R}^n$, $\phi_k : [s_k, t_{k+1}] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ for all $k :\ [t_k, T] \cap [s_k, t_{k+1}] \ne$.

Definition 2.1.5 *For NIFrDE (2.33) the intervals $(s_k, t_{k+1}]$, $k = 0, 1, 2, \dots$ are called intervals of non-instantaneous impulses and the corresponding functions $\phi_k(t, x, y)$ are called non-instantaneous impulsive functions.*

Remark 2.1.13 *If $t_k = s_{k-1}$, $k = 1, 2, \dots,$ then the IVP for NIFrDE (2.33) reduces to an IVP for impulsive fractional differential equations.*

We introduce the following classes of functions

$$NPC([t_0, T], \mathbb{R}^n) = \{u : [t_0, T] \to \mathbb{R}^n :\ u \in C([t_0, T] \cap \Big([0, s_0] \cup_{k=1}^{\infty} (t_k, s_k]\Big), \mathbb{R}^n) :$$

$$
\begin{aligned}
& u(s_k) = u(s_k - 0) = \lim_{t \uparrow s_k} u(t) < \infty,\ u(s_k + 0) \\
& = \lim_{t \downarrow s_k} u(t) < \infty \text{ for all } k:\ s_k \in [t_0, T]\}, \\
NPC^q([t_0,T],\mathbb{R}^n) &= \{u : [t_0,T] \to \mathbb{R}^n :\ u \in NPC([t_0,T],\mathbb{R}^n), \\
& u \in C^q([t_k,s_k],\mathbb{R}^n)) \text{ for all } k:\ \text{such that } [t_k,s_k] \subset [t_0,T]\} \qquad (2.34) \\
PC([t_0,T],\mathbb{R}^n) &= \{u : [t_0,T] \to \mathbb{R}^n :\ u \in NPC([t_0,T],\mathbb{R}^n), \\
& u \in C([t_0,T]/\{s_k\}_{k=0}^{\infty},\mathbb{R}^n)\}, \\
PC^q([t_0,T],\mathbb{R}^n) &= \{u : [t_0,T] \to \mathbb{R}^n :\ u \in NPC([t_0,T],\mathbb{R}^n),\ u \in C^q([t_0,T],\mathbb{R}^n)\}.
\end{aligned}
$$

We give a brief description of the solution of IVP for NIFrDE (2.33) following both approaches to the solutions of IVP for FrDE.

(A1 for NIFrDE.) Let $f(t,x)$ be defined for $t \in [t_0,T]$, $x \in \mathbb{R}^n$. Following the approach (A1 for IFrDE) and Definition 2.1.2 we introduce the following definition:

Definition 2.1.6 *A function $x \in PC^q([t_0,T],\mathbb{R}^n)$ is called a solution of the IVP for NIFrDE (2.33) if*

- *$x(t) = x_k(t)$ for $t \in (t_k,s_k], k : s_k \in [t_0,T]$, where $x_k \in C^q([t_0,s_k],\mathbb{R}^n)$, satisfies ${}_{t_0}^{c}D^q x_k(t) = f(t,x_k(t))$ a.e. on (t_0,s_k) with the restriction of $x_k(t_k) = \tilde{x}_k(t_k)$, $k \geq 1$ (in the case $k = 0$ we have $\tilde{x}_k(t_0) = x_0$);*
- *$x(t) = \tilde{x}_k(t)$ for $t \in (s_k,t_{k+1}], k : t_{k+1} \in [t_0,T]$, where $\tilde{x}_k : [s_k,t_{k+1}] \to \mathbb{R}^n$ satisfies $\tilde{x}_k(t) = \phi_k(t,\tilde{x}_k(t),x_k(s_k))$ for $t \in [s_k,t_{k+1}]$;*
- *$x(t_0) = x_0$.*

Following Definition 2.1.6 and the presentation (2.18) of the solution of the Caputo FrDE (2.12) we obtain the solution $x(t;t_0,x_0)$ of the IVP for NIFrDE (2.33):

Let $t \in (t_0,s_0]$. Consider the solution $x_0(t;t_0,x_0)$ of the IVP for FrDE (2.12), (2.13) with $\tau = t_0, \tilde{x}_0 = x_0$. Then we let $x(t;t_0,x_0) = x_0(t;t_0,x_0)$ and therefore,

$$x(t;t_0,x_0) = x_0 + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}f(s,x(s;t_0,x_0))ds, \quad t \in (t_0,s_0].$$

Let $t \in (s_0,t_1]$. Consider the function $\tilde{x}_0(t)$ such that the equality $\tilde{x}_0(t) = \phi_0(t,\tilde{x}_0(t),x_0(s_0;t_0,x_0))$ holds for $t \in [s_0,t_1]$ and let $x(t;t_0,x_0) = \tilde{x}_0(t)$ for $t \in (s_0,t_1]$. Therefore, $x(t;t_0,x_0) = \phi_0(t,x(t;t_0,x_0),x(s_0-0;t_0,x_0))$.

Let $t \in (t_1,s_1]$ and $c \in \mathbb{R}^n$ be an arbitrary. Consider the solution $x_1(t;t_0,c)$ of the IVP for FrDE (2.12), (2.13) with $\tau = t_0, \tilde{x}_0 = c$. Choose the constant vector $c = c_1$ such that $x_1(t_1;t_0,c_1) = \tilde{x}_0(t_1)$, i.e., $x_1(t_1;t_0,c_1) = \phi_0(t_1,x_1(t_1;t_0,c_1),x_0(s_0;t_0,x_0))$. Then we call the function $x_1(t;t_0,c_1)$ a solution of the IVP for FrDE (2.33) for $t \in (t_1,s_1]$. Therefore, the solution $x(t) = x(t;t_0,x_0)$ of the IVP for NIFrDE (2.33) satisfies the fractional integral equation

$$x(t) = \phi_0(t_1, x(t_1), x(s_0 - 0)) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s)) ds$$
$$- \frac{1}{\Gamma(q)} \int_{t_0}^{t_1} (t_1 - s)^{q-1} f(s, x(s)) ds \quad \text{for } t \in (t_1, s_1].$$

Following the above arguments we obtain the solution of the IVP for NIFrDE (2.33) satisfies the following algebraic–integral equalities

$$x(t) = x(t; t_0, x_0) =$$
$$\begin{cases} x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s)) ds, & t \in (t_0, s_0] \\ \phi_k(t, x(t), x(s_k - 0)), & t \in (s_k, t_{k+1}], k = 0, 1, 2, \ldots, \\ \phi_{k-1}(t_k, x(t_k), x(s_{k-1} - 0)) & \\ \quad - \frac{1}{\Gamma(q)} \int_{t_0}^{t_k} (t_k - s)^{q-1} f(s, x(s)) ds & \\ \quad + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s)) ds, & t \in (t_k, s_k], \; k = 1, 2, \ldots. \end{cases} \tag{2.35}$$

Remark 2.1.14 *In the special case of $f(t, x) = f(t)$ and $\phi_k(t, x, y) = g_k(t)$ the formula (2.35) is obtained in Lemma 2.7 [124]. In the special case $\phi_k(t, x, y) = g_k(t, x)$ the formula (2.35) is obtained in Theorem 4.2 [122] and Section 9 [124].*

Remark 2.1.15 *The approach (A1 for NIFrDE) is applied in [48, 116] for periodic solutions, and in [49, 122, 124] for existence results.*

(A2 for NIFrDE.) We will use the following definition:

Definition 2.1.7 *A function $x \in NPC^q([t_0, T], \mathbb{R}^n)$ is called a solution of the IVP for NIFrDE (2.33) if*

- *$x(t) = x_k(t)$ for $t \in (t_k, s_k], k : s_k \in [t_0, T]$, where $x_k \in C^q([t_k, s_k], \mathbb{R}^n)$, satisfies ${}_{t_k}^{c}D^q x_k(t) = f(t, x_k(t)),\ t \in (t_k, s_k]$ with the restriction of $x_k(t_k) = \tilde{x}_k(t_k)$, $k \geq 1$ (in the case $k = 0$ we define $\tilde{x}_0(t_0) = x_0$);*
- *$x(t) = \tilde{x}_k(t)$ for $t \in (s_k, t_{k+1}], k : t_{k+1} \in [t_0, T]$, where $\tilde{x}_k : [s_k, t_{k+1}] \to \mathbb{R}^n$ satisfies $\tilde{x}_k(t) = \phi_k(t, \tilde{x}_k(t), x_k(s_k))$ for $t \in [s_k, t_{k+1}]$;*
- *$x(t_0) = x_0$.*

Let $\tau \geq 0$ and consider the IVP for the system of Caputo *fractional differential equations* (FrDE)

$${}_{\tau}^{c}D^q x = f(t, x) \quad \text{for } t \in [\tau, s_p] \quad \text{with } x(\tau) = \tilde{x}_0 \tag{2.36}$$

where $p = \min\{k : \tau < s_k\}$, $x \in \mathbb{R}^n$. Assume (2.36) has a solution $x(t) = x(t; \tau, \tilde{x}_0) \in C^q([\tau, s_p], \mathbb{R}^n)$. Some sufficient conditions for global existence of solutions of (2.36) are given in [30, 82].

If $x \in C^q([\tau, s_k] \times \mathbb{R}^n, \mathbb{R}^n)$ satisfies the IVP for FrDE (2.36), then it also satisfies the fractional integral equation (see (2.8) in [44])

$$x(t) = \tilde{x}_0 + \frac{1}{\Gamma(q)} \int_{\tau}^{t} (t-s)^{q-1} f(s, x(s)) ds \quad \text{for} \quad t \in [\tau, s_k]. \tag{2.37}$$

Following Definition 2.1.7 the solution $x(t; t_0, x_0)$, $t \geq t_0$ of the IVP for NIFrDE (2.33) is given by

$$x(t; t_0, x_0) = \begin{cases} X_k(t) \quad \text{for} \ t \in (t_k, s_k], \ k = 0, 1, 2, \dots, \\ \phi_k(t, X_k(t), X_k(s_k - 0)) \quad \text{for} \ t \in (s_k, t_{k+1}], \ k = 0, 1, 2, \dots, \end{cases}$$

where

- $X_0(t)$ is the solution of IVP for FrDE (2.36) for $k = 0$, $\tau = t_0$, $t \in [t_0, s_0]$, $\tilde{x}_0 = x_0$, and $X_0(t)$ satisfies (2.37) on $[t_0, s_0]$;
- $X_1(t)$ is the solution of IVP for FrDE (2.36) for $k = 1$, $\tau = t_1$, $t \in [t_1, s_1]$, $\tilde{x}_0 = \phi_0(t_1 X_0(t_1), X_0(s_0 - 0))$, and $X_1(t)$ satisfies (2.37) on $[t_1, s_1]$;
- $X_2(t)$ is the solution of IVP for FrDE (2.36) for $k = 2$, $\tau = t_2$, $t \in [t_2, s_2]$, $\tilde{x}_0 = \phi_1(t_2, X_1(t_2), X_1(s_1 - 0))$, and $X_2(t)$ satisfies (2.37) on $[t_2, s_2]$;

and so on.

Also, the solution $x(t) = x(t; t_0, x_0)$, $t \geq t_0$ of the IVP for NIFrDE (2.33) satisfies the following algebraic–integral equalities

$$x(t) = x(t; t_0, x_0) =$$

$$\begin{cases} x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s)) ds, & t \in (t_0, s_0], \\ \phi_k(t, x(t), x(s_k - 0)), & t \in (s_k, t_{k+1}], \ k = 0, 1, 2, \dots, \\ \phi_{k-1}(t_k, x(t_k), x(s_{k-1} - 0)) & \\ \quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f(s, x(s)) ds, & t \in (t_k, s_k], \ k = 1, 2, \dots. \end{cases} \tag{2.38}$$

Remark 2.1.16 *Note in the general case both approaches (A1 for NIFrDE) and (A2 for NIFrDE) as well as the presentations (2.35) and (2.38) of the solution of the IVP for NIFrDE (2.33) differ. In the study of the properties of the solutions of the IVP for NIFrDE (2.33) is important which point of view is applied.*

Now we discuss the statement of problem (2.33) and the type of impulsive functions with examples.

Example 2.1.3.5 Consider the IVP for NIFrDE (2.33) with $f(t, x) = G(t)$, $t_0 = 0$, and $\phi_k(t, x, y) = g_k(t)$, $k = 0, 1, 2, \dots$.

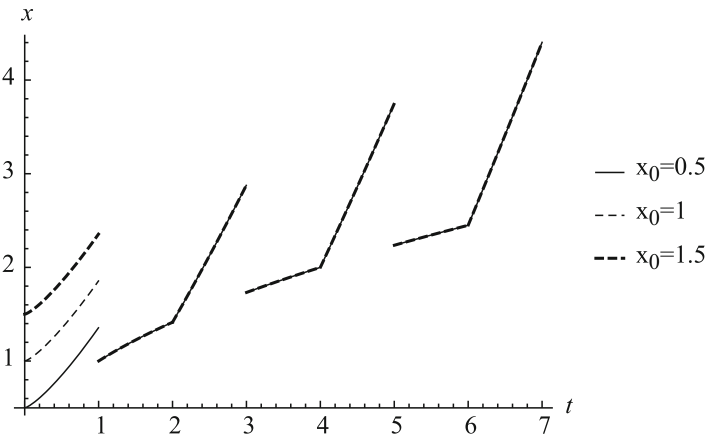

Fig. 2.3 Graphs of solutions of IVP for the IFrDE (2.28) with $t_k = 2k,\ s_k = 2k+1,\ k = 0,1,2,\dots,\ q = 0.3, f(t,x) = t$ and $g_k(t) = \sqrt{t}$, approach (A1 for NIFrDE).

Case 1.1. Approach (A1 for NIFrDE). From formula (2.35) the solution is

$$x(t) = x(t;0,x_0) =$$
$$\begin{cases} x_0 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}G(s)ds & t \in (0,s_0] \\ g_k(t) & t \in (s_k, t_{k+1}], k = 0,1,2,\dots, \\ g_k(t_k) - \frac{1}{\Gamma(q)}\int_0^{t_k}(t_k-s)^{q-1}G(s)ds & \\ \quad +\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}G(s)ds, & t \in (t_k, s_k],\ k = 1,2,\dots, \end{cases} \tag{2.39}$$

i.e., the solution depends on the initial value x_0 only on the interval $[0, s_0]$.

Case 1.2. Approach (A2 for NIFrDE). From formula (2.38) it follows as in Case 1.1 that the solution depends on the initial value x_0 only on the interval $[0, s_0]$.

The special case of $t_k = 2k,\ s_k = 2k+1,\ k = 0,1,2,\dots, q = 0.3,\ G(t) = t$ and $g_k(t) = t$ the graphs of the solutions are given in Figure 2.3 for approach (A1 for NIFrDE) and in Figure 2.4 for approach (A2 for NIFrDE).

□

Example 2.1.3.6 Consider the IVP for NIFrDE (2.33) with $f(t,x) \equiv 1$, $t_0 = 0$ and $\phi_k(t,x,y) = g_k(t),\ k = 0,1,2,\dots$. We obtain for (A1 for NIFrDE).

$$x(t;0,x_0) = \begin{cases} g_k(t) & t \in (s_k, t_{k+1}],\ k = 0,1,2,3,\dots \\ x_0 + \frac{t^q}{q\Gamma(q)} & t \in (0,s_0] \\ g_{k-1}(t_k) + \frac{t^q - t_k^q}{q\Gamma(q)} & t \in (t_k, s_k],\ \ k = 1,2,\dots, \end{cases}$$

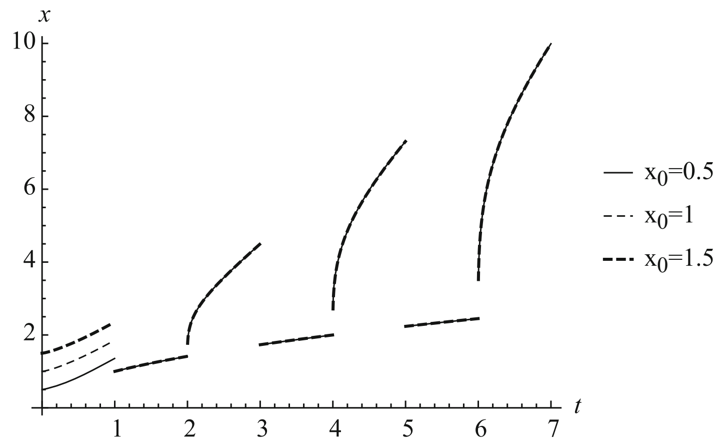

Fig. 2.4 Graphs of solutions of IVP for the IFrDE (2.28) with $t_k = 2k,\ s_k = 2k+1,\ k = 0, 1, 2, \ldots, q = 0.3, f(t,x) = t$ and $g_k(t) = \sqrt{t}$, approach (A2 for NIFrDE).

and for (A2 for NIFrDE):

$$x(t;0,x_0) = \begin{cases} x_0 + \frac{t^q}{q\Gamma(q)} & t \in (0, s_0] \\ g_k(t) & t \in (s_{k-1}, t_k],\ k = 1, 2, 3, \ldots \\ g_k(t_k) + \frac{(t-t_k)^q}{q\Gamma(q)} & t \in (t_k, s_k],\ \ k = 1, 2, \ldots, \end{cases}$$

This particular case shows the solution $x(t;0,x_0)$ in both approaches depends on the initial value only on the interval $(0, s_0]$. □

Example 2.1.3.7 Consider the IVP for NIFrDE (2.33) with $f(t,x) = G(t)$, $t_0 = 0$ and $\phi_k(t,x,y) = g_k(t,x),\ k = 1, 2, \ldots$. Let, for example $g_k(t,x) = t + 2x,\ k = 1, 2, \ldots$. Then on each intervals of non-instantaneous impulses the equality $x(t) = t + 2x(t)$ is satisfied, i.e., $x(t) = -t$.

Case 1.1. Approach (A1 for NIFrDE). From formula (2.35) the solution is given by

$$x(t) = x(t;0,x_0) =$$

$$\begin{cases} x_0 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} G(s)ds & t \in (0, s_0] \\ -t & t \in (s_k, t_{k+1}], k = 0, 1, 2, \ldots, \\ -t_k - \frac{1}{\Gamma(q)}\int_0^{t_k} (t_k - s)^{q-1} G(s)ds & \\ \quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} G(s)ds, & t \in (t_k, s_k],\ k = 1, 2, \ldots, \end{cases} \tag{2.40}$$

i.e., the solution depends on the initial value x_0 only on the interval $[0, s_0]$.

Case 1.2. Approach (A2 for NIFrDE). From formula (2.38) we obtain for the solution

$$x(t) = x(t;0,x_0) =$$
$$\begin{cases} x_0 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}G(s)ds & t\in(0,s_0] \\ -t & t\in(s_{k-1},t_k], k=1,2,\dots, \\ -t_k \\ \quad +\frac{1}{\Gamma(q)}\int_{t_k}^t (t-s)^{q-1}G(s)ds, & t\in(t_k,s_k],\ k=1,2,\dots. \end{cases} \tag{2.41}$$

As in Case 1.1 the solution depends on the initial value x_0 only on the interval $[0,s_0]$. □

Example 2.1.3.8 Consider the IVP for NIFrDE (2.33) with $f(t,x) = G(t)$, $t_0 = 0$ and the impulsive functions $\phi_k(t,x,y)$ depend on their third argument, i.e., on $x(s_k-0)$, $k=1,2,\dots$. Let, for example $\phi_k(t,x,y) = \frac{y}{t}$, $k=1,2,\dots$.

Case 1.1. Approach (A1 for NIFrDE). From formula (2.35) the solution is given by

$$x(t) = x(t;0,x_0) =$$
$$\begin{cases} x_0 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}G(s)ds & t\in(0,s_0] \\ t^{-1}x(s_{k-1}-0) & t\in(s_k,t_{k+1}], k=0,1,2,\dots, \\ t_k^{-1}x(s_{k-1}-0) - \frac{1}{\Gamma(q)}\int_0^{t_k}(t_k-s)^{q-1}G(s)ds \\ \quad +\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}G(s)ds, & t\in(t_k,s_k],\ k=1,2,\dots, \end{cases}$$

i.e., the solution depends on the initial value x_0 for all $t\geq 0$.

In the special case $G(t) \equiv 1$ we obtain for (A1 for NIFrDE) the solution

$$x(t;0,x_0) = \begin{cases} x_0 + \frac{t^q}{q\Gamma(q)} & t\in(0,s_0] \\ t^{-1}(x_0 + \frac{s_0^q}{q\Gamma(q)}) & t\in(s_0,t_1] \\ t_1^{-1}(x_0+\frac{s_0^q}{q\Gamma(q)}) + \frac{t^q-t_1^q}{q\Gamma(q)} & t\in(t_1,s_1] \\ t^{-1}\Big(t_1^{-1}\big(x_0+\frac{s_0^q}{q\Gamma(q)}) + \frac{s_1^q-t_1^q}{q\Gamma(q)}\Big) & t\in(s_1,t_2] \\ \dots\dots\dots \end{cases}$$

The graphs of the solutions with $t_k = 2k,\ s_k = 2k+1,\ k=0,1,2,dots$ and different initial values are given in Figure 2.5.

Case 1.2. Approach (A2 for NIFrDE). From formula (2.38) we obtain for the solution

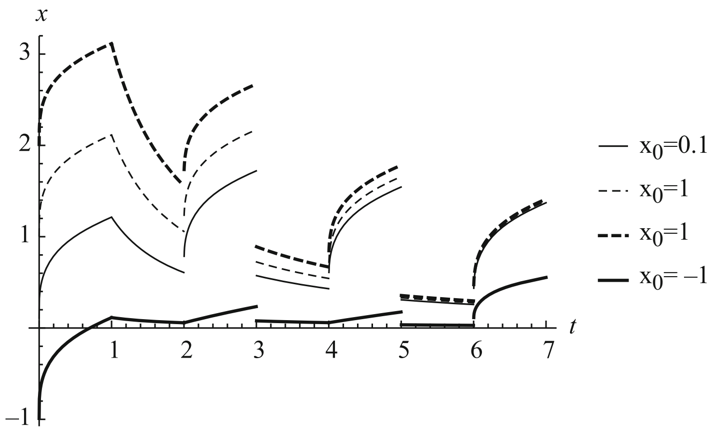

Fig. 2.5 Graphs of solutions of IVP for the IFrDE (2.28) with $t_k = 2k,\ s_k = 2k+1,\ k = 0, 1, 2, \ldots, q = 0.3, f(t,x) \equiv 1$ and $\phi_k(t,x,y) = \frac{y}{t}$, approach (A1 for NIFrDE).

$$x(t) = x(t; 0, x_0) =$$
$$\begin{cases} x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} G(s) ds & t \in (0, s_0] \\ t^{-1} x(s_{k-1} - 0) & t \in (s_{k-1}, t_k], k = 1, 2, \ldots, \\ t_k^{-1} x(s_{k-1} - 0) & \\ \quad + \frac{1}{\Gamma(q)} \int_{t_k}^t (t-s)^{q-1} G(s) ds, & t \in (t_k, s_k],\ k = 1, 2, \ldots. \end{cases}$$

As in Case 1.1 the solution depends on the initial value x_0 for all $t > 0$.

The graphs of the solutions in the partial case $G(t) \equiv 1, t_k = 2k,\ s_k = 2k+1,\ k = 0, 1, 2, \ldots$ and different initial values are given in Figure 2.6.

Now the solution depends on the initial value x_0 for all $t \geq 0$. The same happens with the application of (A2 for NIFrDE). □

Example 2.1.3.9 Consider the IVP for NIFrDE (2.33) with $f(t,x) = \tan(t)$, $t_0 = 0$, $t_k = -\frac{\pi}{4} + k\pi$, $k = 1, 2, \ldots$ and $s_k = \frac{\pi}{4} + k\pi$, $k = 0, 1, 2, \ldots$. Then the approach (A1 for NIFrDE) is not applicable since the function $f(t,x)$ is not continuous on the whole interval $[0, \infty)$ and the integral $\int_0^t (t-s)^{q-1} \tan(s) ds$ in Eq. (2.38) is not convergent for all $t > 0$.

There is no problem with the direct application of Eq. (2.38) in approach (A2 for NIFrDE) because the function $f(t,x)$ is defined and continuous on $\bigcup_{k=0}^{\infty} [t_k, s_k]$ and the integrals $\int_{t_k}^t (t-s)^{q-1} \tan(s) ds,\ t \in (t_k, s_k]$ are convergent. □

Remark 2.1.17 *In the approach (A1 for NIFrDE) the right side part $f(t,x)$ of the IVP for NIFrDE (2.33) has to be defined and integrable on the whole interval of consideration.*

In the approach (A2 for NIFrDE) the function $f(t,x)$ has to be defined and integrable only on $\cup_{k=0}^{\infty} [t_k, s_k]$.

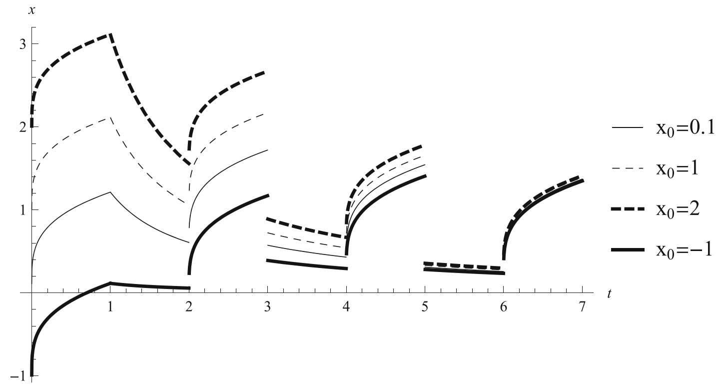

Fig. 2.6 Graphs of solutions of IVP for the IFrDE (2.28) with $t_k = 2k,\ s_k = 2k + 1,\ k = 0, 1, 2, \ldots, q = 0.3, f(t, x) \equiv 1$ and $\phi_k(t, x, y) = \frac{y}{t}$, approach (A2 for NIFrDE).

Example 2.1.3.10 Consider the IVP for NIFrDE (2.33) with various types of non-instantaneous impulsive functions.

Case 1. Let $\phi_k(t, x, y) = a_k x + b_k$ for $t \in [s_{k-1}, t_k]$, $(k = 1, 2, 3, \ldots)$ where a_k, b_k are constants.

If $a_k = 1,\ b_k = 0$, then any function $x(t)$ will satisfy the impulsive condition $x(t) = x(t)$ for $t \in [s_{k-1}, t_k]$, $(k = 1, 2, 3, \ldots)$ and obviously the IVP for NIFrDE (2.33) will have an infinite number of solutions.

If $a_k = 1,\ b_k \neq 0$, then no function $x(t)$ will satisfy the impulsive condition $x(t) = x(t) + b$ for $t \in [s_{k-1}, t_k]$, $(k = 1, 2, 3, \ldots)$ and obviously the IVP for NIFrDE (2.33) will have no solution.

If $a_k \neq 1,\ b_k = 0$, then the only function $x(t)$ that satisfies the impulsive condition $x(t) = ax(t)$ for $t \in [s_{k-1}, t_k]$, $(k = 1, 2, 3, \ldots)$ is the zero function, and therefore any solution of IVP for NIFrDE (2.33) will be zero on $(s_{k-1}, t_k]$, $(k = 1, 2, 3, \ldots)$

If $a_k \neq 1,\ b_k \neq 0$, then there will be a unique function $x(t)$ that satisfies the impulsive condition $x(t) = ax(t) + b$ for $t \in [s_{k-1}, t_k]$, $(k = 1, 2, 3, \ldots)$ and we can talk about uniqueness of the solution IVP for NIFrDE (2.33).

Case 2. Let $\phi_1(t, x, y) = t + x + y$ for $t \in [s_0, t_1]$. On the interval $(s_0, t_1]$ the solution $x(t)$ of IVP for NIFrDE (2.33) satisfies the equation $x(t) = t + x(t) + x(s_0 - 0)$ or $x(s_0 - 0) = -t$ which is not true in the general case. Therefore, the IVP for NIFrDE (2.33) has no solution in both approaches (A1 for NIFrDE) and (A2 for NIFrDE).

Case 3. Let $\phi_1(t, x, y) = x^2 t y$ for $t \in [s_0, t_k]$. On the interval $(s_0, t_1]$ the solution $x(t)$ of IVP for NIFrDE (2.33) satisfies the equation $x(t) = \Big(x(t)\Big)^2 t x(s_0 - 0)$ which has two solutions $x(t) \equiv 0$ and $x(t) = \frac{1}{tx(s_0-0)}$. Therefore, the IVP for NIFrDE (2.33) has a nonunique solution in both approaches (A1 for NIFrDE) and (A2 for NIFrDE).

Case 4. Let $\phi_1(t, x, y) = y\sqrt[3]{t - \frac{t_1+s_0}{2}}$ for $t \in [s_0, t_1]$. On the interval $(s_0, t_1]$ the solution $x(t)$ of IVP for NIFrDE (2.33) satisfies the equality $x(t) =$

$x(s_0-0)\sqrt[3]{t-\frac{t_1+s_0}{2}}$. Therefore, in the application of the above approaches we obtain a continuous solution on $[s_0,t_1]$ which is not differentiable (i.e., which is not a Caputo fractional derivative).

□

Example 2.1.3.11 Consider the IVP for the scalar NIFrDE

$$\begin{aligned}
&{}_{t_0}^{c}D^q x(t) = Ax \text{ for } t\in(t_k,s_k],\ k=0,1,2,\dots,\\
&x(t) = a_k(t)x(s_k-0) \text{ for } \ t\in(s_k,t_{k+1}],\ k=0,1,2,\dots,\\
&x(t_0)=x_0,
\end{aligned}\tag{2.42}$$

where $x_0\in\mathbb{R}$, A is a constant, $a_k\in C([s_k,t_{k+1}],\mathbb{R}^n)$, $k=1,2,\dots$.

We will use both approaches to obtain the solution. Using any of the two approaches we obtain

$$x(t)=\begin{cases} x_0E_q(A(t-t_0)^q) & \text{for } t\in[t_0,s_0],\\ a_0(t)x_0E_q(A(s_0-t_0)^q) & \text{for } t\in(s_0,t_1].\end{cases}\tag{2.43}$$

(A1 for NIFrDE) In this case (2.35) reduces to the form

$$x(t)=\begin{cases} x_0E_q(A(t-t_0)^q) & \text{for } t\in[t_0,s_0],\\ a_0(t)x_0E_q(A(s_0-t_0)^q) & \text{for } t\in(s_0,t_1],\\ a_{k-1}(t_k)x(s_{k-1}-0)) & \\ \quad -A\frac{1}{\Gamma(q)}\int_{t_0}^{t_k}(t_k-s)^{q-1}x(s)ds & \\ \quad +A\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}x(s)ds & \text{for } t\in(t_k,s_k],\ k=1,2,\dots,\\ a_k(t)x(s_k-0) & \text{for } t\in(s_k,t_{k-1}],\ k=1,2,\dots.\end{cases}\tag{2.44}$$

Now (2.1.3.11) does not give us the explicit form of the solution.

Now we will use the concept of the solution in (A1 for NIFrDE) and the solution $\tilde{x}_0E_q((t-\tau)^q)$ of the IVP for FrDE (2.12), (2.13) with $f(t,x)=Ax$.

On the interval $[t_1,s_1]$ the solution of the IVP for FrDE (2.12), (2.13) with $\tilde{x}_0=c$, $f(t,x)=Ax$ is $cE_q(A(t-t_0)^q)$ with $c\in\mathbb{R}^n$ being an arbitrary. Choose the constant vector $c=c_1$ such that $c_1E_q(A(t_1-t_0)^q)=a_0(t_1)x_0E_q(A(s_0-t_0)^q)$. Then the solution of the IVP for FrDE (2.48) is

$$x(t)=a_0(t_1)x_0\frac{E_q(A(s_0-t_0)^q)}{E_q(A(t_1-t_0)^q)}E_q(A(t-t_0)^q),\quad t\in[t_1,s_1].$$

On the interval $[t_2, s_2]$ the solution of the IVP for FrDE (2.12), (2.13) with $\tilde{x}_0 = c$, $f(t,x) = Ax$ is $cE_q(A(t-t_0)^q)$ with $c \in \mathbb{R}^n$ being an arbitrary. Choose the constant vector $c = c_2$ such that

$$c_2 E_q(A(t_2-t_0)^q) = a_1(t_2)a_0(t_1)x_0 \frac{E_q(A(s_0-t_0)^q)}{E_q(A(t_1-t_0)^q)} E_q(A(s_1-t_0)^q).$$

Then the solution of the IVP for FrDE (2.33) is

$$x(t) = a_0(t_1)a_1(t_2)x_0 \frac{E_q(A(s_0-t_0)^q)E_q(A(s_1-t_0)^q)}{E_q(A(t_1-t_0)^q)E_q(A(t_2-t_0)^q)} E_q(A(t-t_0)^q),\ t \in [t_2, s_2].$$

Inductively we get the solution of IVP for NIFrDE

$$x(t) = \begin{cases} x_0 E_q(A(t-t_0)^q) & \text{for } t \in [t_0, s_0], \\ a_0(t)x_0 E_q(A(s_0-t_0)^q) & \text{for } t \in (s_0, t_1], \\ x_0\Big(\prod_{i=0}^{k-1} a_i(t_{i+1}) \frac{E_q(A(s_i-t_0)^q)}{E_q(A(t_{i+1}-t_0)^q)}\Big) E_q(A(t-t_0)^q) & \\ \quad \text{for } t \in (t_k, s_k],\ k = 1, 2, \dots, & \\ a_k(t)x_0\Big(\prod_{i=0}^{k-1} a_i(t_{i+1}) \frac{E_q(A(s_i-t_0)^q)}{E_q(A(t_{i+1}-t_0)^q)}\Big) E_q(A(s_k-t_0)^q) & \\ \quad \text{for } t \in (s_k, t_{k+1}],\ k = 1, 2, \dots. & \end{cases} \tag{2.45}$$

(A2 for FrDE) From (2.38) we get the solution

$$x(t) = \begin{cases} x_0 E_q(A(t-t_0)^q) & \text{for } t \in [t_0, s_0], \\ a_0(t)x_0 E_q(A(s_0-t_0)^q) & \text{for } t \in (s_0, t_1], \\ x_0\Big(\prod_{i=0}^{k-1} a_i(t_{i+1}) E_q(A(s_i-t_i)^q)\Big) E_q(A(t-t_k)^q) & \\ \quad \text{for } t \in (t_k, s_k],\ k = 1, 2, \dots, & \\ a_k(t)x_0\Big(\prod_{i=0}^{k-1} a_i(t_{i+1}) E_q(A(s_i-t_i)^q)\Big) E_q(A(s_k-t_k)^q) & \\ \quad \text{for } t \in (s_k, t_{k+1}],\ k = 1, 2, \dots. & \end{cases} \tag{2.46}$$

The solutions obtained by the two approaches differ.

In the special case $q = 1$ both (2.45) and (2.46) coincide. Also they coincide with (1.9) giving the solution of the linear scalar differential equation with non-instantaneous impulses.

In the special case $A = 0$ applying any of the two approaches we obtain

$$x(t; 0, x_0) = \begin{cases} x_0 & \text{for } t \in [t_0, s_0], \\ x_0 a_k(t) \prod_{i=0}^{k-1} a_i(t_{i+1}) & \text{for } t \in (s_k, t_{k+1}],\ \ k = 0, 1, 2, \dots, \\ x_0 \prod_{i=0}^{k-1} a_i(t_{i+1}) & \text{for } t \in (t_k, s_k],\ k = 1, 2, \dots. \end{cases} \tag{2.47}$$

□

IMPORTANT NOTE. In connection with the application of any of the above approaches to NIFrDE and for it to be easy to distinguish which one is applied we will use two different notations for the NIFrDE:

– in the application of approach (A1 for NIFrDE) we will use (2.33).
– in the application of approach (A2 for NIFrDE) we will use the equation (2.33) written in form

$$
\begin{aligned}
&{}_{t_k}^{c}D^q x(t) = f(t,x) \text{ for } t \in (t_k, s_k],\ k = 0,1,\dots,\\
&x(t) = \phi_k(t, x(t), x(s_k - 0)) \quad \text{for } t \in (s_k, t_{k-1}],\ k = 0,1,2,\dots, \qquad (2.48)\\
&x(t_0) = x_0,
\end{aligned}
$$

where $x_0 \in \mathbb{R}^n$, $f : \bigcup_{k=0}^{\infty}[t_k, s_k] \times \mathbb{R}^n \to \mathbb{R}^n$, $\phi_k : [s_k, t_{k+1}] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $(k = 0,1,2,3,\dots)$.

2.2 Existence Results for Caputo Fractional Differential Equations with Non-instantaneous Impulses

In this section we consider the scalar case of IVP for NFrDE (2.33) (respectively (2.48)) on a finite interval $J = [0,T]$, $T < \infty$ is a fixed positive number, i.e., $n = 1$, and $0 = t_0 < s_0 < t_1 < \dots < t_m < s_m = T$.

We introduce Ulam type stability for Caputo fractional differential equations with non-instantaneous impulses. Applying both approaches we prove existence and Ulam-Hyers-Rassias stability results for (2.48) on a compact interval.

Since 1940, the Ulam type stability problem [113] was studied by many researchers. We refer the reader to the monographs [66, 67].

Case I. Existence by the application of approach (A1 for NIFrDE)

In [122] the NIFrDE (2.33) is studied for the scalar case on finite interval when the impulsive functions do not depend on the value of the solution before the impulse, i.e., $\phi_k(t,x,y) = g_k(t,x)$, $k = 1,2,\dots,m$.

We will change the impulsive functions to $\phi_k(t,x,y) = g_k(t,y)$, $k = 1,2,\dots,m$, i.e., the impulsive functions will depend on the value of the unknown function before the impulse and we will consider the scalar nonlinear Caputo non-instantaneous impulsive fractional differential equation

$$
\begin{aligned}
&{}_{t_0}^{c}D^q x(t) = f(t,x) \text{ for } t \in (t_k, s_k],\ k = 0,1,m,\\
&x(t) = g_k(t, x(s_k - 0)) \quad \text{for } t \in (s_k, t_{k+1}],\ k = 0,1,2,\dots,m-1, \qquad (2.49)\\
&x(0) = x_0,
\end{aligned}
$$

where $x_0 \in \mathbb{R}$, $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$, $g_k \in C([s_k, t_{k+1}] \times \mathbb{R}, \mathbb{R})$, $(k = 0,1,2,3,\dots,m-1)$.

Let $\Psi > 0$, $\varphi \in PC(J, \mathbb{R}_+)$ and consider the fractional non-instantaneous impulsive differential inequalities

$$\begin{aligned} &|{}_{t_0}^{c}D^q y(t) - f(t, y)| \le \varphi(t) \text{ for } t \in (t_k, s_k],\ k = 0, 1, \ldots, m \\ &|y(t) - g_k(t, y(s_k - 0))| \le \Psi \quad \text{for } t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \ldots, m - 1. \end{aligned} \tag{2.50}$$

Definition 2.2.1 *[122] NIFrDE (2.49) is generalized Ulam-Hyers-Rassias stable with respect to the couple (φ, Ψ) if there exists $c_{f,q,g,\phi} > 0$ such that for each solution $y \in PC(J, \mathbb{R})$ of the inequalities (2.50) there exists a solution $x \in PC^q(J, \mathbb{R})$ of the NIFrDE (2.33) with $|y(t) - x(t)| \le c_{f,q,g,\phi}(\phi(t) + \Psi),\ \ t \in J$.*

Remark 2.2.1 *A function $y \in PC(J, \mathbb{R})$ is a solution of the inequalities (2.50) iff there is $G \in PC(J, \mathbb{R})$ and a sequence $G_k,\ k = 1, 2, \ldots, m$ (which depend on y) such that*

(i) $|G(t)| \le \phi(t),\ t \in J,$ *and* $\|G_k\| \le \Psi,\ k = 1, 2, \ldots, m;$
(ii) ${}_{t_0}^{c}D^q y(t) = f(t, y) + G(t),\ \ t \in (t_k, s_k],\ k = 0, 1, \ldots, m;$
(iii) $y(t) = g_k(t, y(s_k - 0)) + G_k,\quad t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \ldots, m - 1.$

Theorem 2.2.1 *Let the following conditions be satisfied:*

1. *The function $f \in C(J \times \mathbb{R}, \mathbb{R})$ and there exists a positive constant L_f such that $|f(t, x) - f(t, y)| \le L_f|x - y|$ for each $t \in J,\ x, y \in \mathbb{R}$.*
2. *For all $k = 0, 1, \ldots, m - 1$ the functions $g_k(t, x) \in C([s_k, t_{k+1}] \times \mathbb{R}, \mathbb{R})$ and there exist constants L_{g_k} such that $|g_k(t, x_1) - g_k(t, x_2)| \le L_{g_k}|x_1 - x_2|$ for each $t \in [s_k, t_{k+1}],\ x_1, x_2 \in \mathbb{R}$.*
3. *The function $y(t)$ satisfies the fractional non-instantaneous impulsive differential inequalities (2.50) with the constant $\Psi > 0$, and the function $\varphi \in C(J, \mathbb{R})$ that is a nondecreasing function in $\bigcup_{i=0}^{m}[t_i, s_i]$ such that there exists a constant C_φ with*

$$\Big(\int_0^t (\varphi(s))^{\frac{1}{p}} ds\Big)^p \le C_\varphi \varphi(t) \ \textit{ for } t \in J.$$

Then there exists a unique solution $y_0(t)$ of the IVP for NIFrDE (2.48) with $x_0 = y(t_0)$ such that it satisfies the integral-algebraic equations

$$y_0(t) = \begin{cases} x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^t (t - s)^{q-1} f(s, y_0(s)) ds, & t \in (0, s_0] \\ g_k(t, y_0(s_k - 0)), & t \in (s_k, t_{k+1}],\ k = 0, 1, \ldots, m - 1, \\ g_{k-1}(t_k, y_0(s_{k-1} - 0)) & \\ \quad - \frac{1}{\Gamma(q)} \int_0^{t_k} (t_k - s)^{q-1} f(s, y_0(s)) ds & \\ \quad + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s, y_0(s)) ds, & t \in (t_k, s_k],\ k = 1, 2, \ldots, m. \end{cases} \tag{2.51}$$

and

$$|y(t)-y_0(t)| \le \frac{\frac{2C_\varphi}{\Gamma(q)}\left(\frac{1-p}{q-p}\right)^{1-p}T^{q-p}+1}{1-M}(\varphi(t)+\Psi) \tag{2.52}$$

for all $t\in J$ *provided that* $0<p<q<1$ *where* $M=max\{M_1,M_2\}<1$, *with*

$$M_1 = max\{L_{g_k}+\frac{L_f C_\varphi}{\Gamma(q)}\left(\frac{1-p}{q-p}\right)^{1-p}(s_k^{q-p}+t_k^{q-p}),\ k=0,1,2\dots,m\}<1 \tag{2.53}$$

and

$$M_2 = max\{L_{g_k}+\frac{L_f}{\Gamma(q+1)}(t_k^q+s_k^q)|\ k=1,2\dots,m\}<1. \tag{2.54}$$

Proof Consider the space of piecewise continuous functions $PC(J,\mathbb{R})$ endowed with the generalized metric for any $H_1,h_2\in PC(J,\mathbb{R})$ defined by

$$D(h_1,h_2)=inf\{C_1+C_2\in[0,\infty):\ |h_1(t)-h_2(t)|\le(C_1+C_2)(\phi(t)+\Psi)\ \text{ for all } t\in J\}$$

where

$$C_1\in\{C\in[0,\infty):\ |h_1(t)-h_2(t)|\le C\phi(t) \text{ for all }\ t\in(t_k,s_k],\ k=0,1,2,\dots,m\}$$

$$C_2\in\{C\in[0,\infty):\ \ |h_1(s_k-0)-h_2(s_k-0)|\le C\Psi \text{ for all }\ k=0,1,\dots,m-1\}.$$

It is easy to verify that $(PC(J,\mathbb{R}),D)$ is a complete generalized metric space.

Define the operator $\Lambda: PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ by (compare with formula (2.35))

$$(\Lambda x)(t)=\begin{cases} x_0+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s,x(s))ds, & t\in(0,s_0]\\ g_k(t,x(s_k-0)), & t\in(s_k,t_{k+1}],\ k=0,1,\dots,m-1,\\ g_{k-1}(t_k,x(s_{k-1}-0)) & \\ \quad-\frac{1}{\Gamma(q)}\int_0^{t_k}(t_k-s)^{q-1}f(s,x(s))ds & \\ \quad+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s,x(s))ds, & t\in(t_k,s_k],\ k=1,2,\dots,m.\end{cases} \tag{2.55}$$

We will prove Λ is a strictly contractive on $PC(J,\mathbb{R})$. From the definition of $(PC(J,\mathbb{R}),d)$, for any $g,h\in PC(J,\mathbb{R})$ it is possible to find $C_1,C_2\ge 0$ such that

$$|g(t)-h(t)|\le\begin{cases} C_2\Psi, & t\in(s_k,t_{k+1}],\ k=0,1,\dots,m-1,\\ C_1\varphi(t), & t\in(t_k,s_k],\ k=0,1,2,\dots,m.\end{cases} \tag{2.56}$$

We will use induction to prove the claim.

Case 1. Let $t \in [0, s_0]$. Then we get

$$\begin{aligned}|(\Lambda g)(t) - (\Lambda h)(t)| &\le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, g(s)) - f(s, h(s))| ds \\ &\le \frac{L_f}{\Gamma(q)} \int_0^t (t-s)^{q-1} |g(s) - h(s)| ds \\ &\le \frac{L_f C_1}{\Gamma(q)} \Big(\int_0^t (t-s)^{\frac{q-1}{1-p}} ds \Big)^{1-p} \Big(\int_0^t (\varphi(t))^{\frac{1}{p}} ds \Big)^p \\ &\le \frac{L_f C_1}{\Gamma(q)} \Big(\frac{1-p}{q-p} \Big)^{1-p} s_0^{q-p} C_\varphi \varphi(t) < C_1 \varphi(t).\end{aligned} \tag{2.57}$$

Case 2. Let $t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \ldots, m-1$. Then we get

$$|(\Lambda g)(t) - (\Lambda h)(t)| \le L_{g_k} |g(s_k - 0) - h(s_k - 0)| \le L_{g_k} C_2 \Psi.$$

Case 3. Let $t \in [t_k, s_k],\ k = 1, 2, \ldots, m$. Then we obtain

$$\begin{aligned}|(\Lambda g)(t) - (\Lambda h)(t)| \le\ & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, g(s)) - f(s, h(s))| ds \\ & + \frac{1}{\Gamma(q)} \int_0^{t_k} (t-s)^{q-1} |f(s, g(s)) - f(s, h(s))| ds \\ & + |g_{k-1}(t_k, g(s_{k-1} - 0)) - g_{k-1}(t_k, h(s_{k-1} - 0))| \\ \le\ & \Big[\frac{L_f C_\varphi}{\Gamma(q)} \Big(\frac{1-p}{q-p} \Big)^{1-p} (s_k^{q-p} + t_k^{q-p}) + L_{g_k} \Big] (C_1 + C_2)(\varphi(t) + \Psi).\end{aligned}$$

From above, we have

$$|(\Lambda g)(t) - (\Lambda h)(t)| \le M(C_1 + C_2)(\varphi(t) + \Psi),$$

i.e., $d(\Lambda g, \Lambda h) \le M d(g, h)$ for any $g, h \in PC(J, \mathbb{R})$.

Let $g_0 \in PC(J, \mathbb{R})$. From the piecewise continuous properties of g_0 and Λg_0, the boundedness of f, g_k, g_0 on J and $\phi(.) + \Psi > 0$ it follows there exists a constant $G_1 \in (0, \infty)$ such that $|(\Lambda g_0)(t) - g_0(t)| \le G_1(\phi(t) + \Psi),\ t \in J$ or $d(\Lambda g_0, g_0) < \infty$.

The Banach fixed point theorem guarantees that there exists a function $y_0 \in PC(J, \mathbb{R})$ such that $\Lambda^n g_0 \to y_0$ in $(PC(J, \mathbb{R}), d)$ as $n \to \infty$ and $\Lambda y_0 = y_0$, that is, y_0 satisfies Eq. (2.51) for $t \in J$.

Next, we check that $\{g \in PC(J, \mathbb{R}) : d(g, g_0) < \infty\} = PC(J, \mathbb{R})$. For any $g \in PC(J, \mathbb{R})$, since g and g_0 are bounded on J and $\min_{t \in J}(\phi(t) + \Psi) > 0$, there exists a constant $C_g \in (0, \infty)$ such that $|g_0(t) - g(t)| \le C_g(\phi(t) + \Psi),\ t \in J$. Hence, we have $D(g_0, g) < \infty$ for all $g \in PC(J, \mathbb{R})$, that is, $\{g \in PC(J, \mathbb{R}) : D(g, g_0) < \infty\} = PC(J, \mathbb{R})$. Thus, we conclude that y_0 is the unique piecewise continuous function with property (2.51). According to condition 3 and Remark 2.2.1 it follows that

$$D(y, \Lambda y) \leq \frac{2C_\varphi}{\Gamma(q)} \Big(\frac{1-p}{q-p}\Big)^{1-p} T^{q-p} + 1.$$

Summarizing, we get

$$D(y, y_0) \leq \frac{d(y, \Lambda y)}{1-M} \leq \frac{\frac{2C_\varphi}{\Gamma(q)} \Big(\frac{1-p}{q-p}\Big)^{1-p} T^{q-p} + 1}{1-M},$$

which proves inequality (2.52) is true for $t \in J$. □

Remark 2.2.2 *Note the condition $M_1, M_2 < 1$ concerning constants M_1, M_2 requires conditions on the impulsive points t_k, s_k and on the Lipschitz constants. Also this condition does not allow the result to be generalized to the infinite interval $[t_0, \infty)$.*

Example 2.2.1 We consider the scalar Caputo fractional differential equation with non-instantaneous impulse given in [122] (but we will change the impulsive condition on $(1, 2]$), i.e., consider

$$\begin{aligned} &{}_0^cD^{0.5}x(t) = \frac{|x(t)|}{8 + e^t + t^2} \text{ for } t \in (0, 1] \cup (2, 3] \\ &x(t) = \frac{|x(1-0)|}{(3+t^2)(1 + |x(1-0)|)} \quad \text{for } t \in (1, 2], \\ &x(0) = x_0, \end{aligned} \tag{2.58}$$

and the corresponding scalar Caputo fractional differential inequalities with non-instantaneous impulse on $(1, 2]$

$$\begin{aligned} &\Big|\, {}_0^cD^{0.5}y(t) - \frac{|y(t)|}{8 + e^t + t^2} \Big| \leq e^t \text{ for } t \in (0, 1] \cup (2, 3] \\ &\Big|\, y(t) - \frac{|y(1-0)|}{(3+t^2)(1 + |y(1-0)|)} \Big| \leq 1 \quad \text{for } t \in (1, 2], \\ &y(0) = x_0, \end{aligned} \tag{2.59}$$

where $x_0 \in \mathbb{R}$.

In this particular case $J = [0, 3]$, $q = 0.5$, $p = \frac{1}{3}$, $t_0 = 0 < s_0 = 1 < t_1 = 2 < s_1 = 3 = T$, $m = 1$, $f(t, x) = \frac{|x|}{8+e^t+t^2} \in C([0, 3] \times \mathbb{R}, \mathbb{R})$ with $L_f = \frac{1}{9}$ and $g_1(t, x) = \frac{|x|}{(3+t^2)(1+|x|)} \in C([1, 2], \mathbb{R})$ with $L_{g_1} = 0.25$. Denote $\varphi(t) = e^t$, $\psi = 1$ and $C_\varphi = 1$. Then $\int_0^t e^{3s} ds \leq e^t$.

Let $M_1 = \max\{\frac{1}{9\sqrt{\pi}} 4^{\frac{2}{3}}, \frac{1}{9\sqrt{\pi}} 4^{\frac{2}{3}} (3^{\frac{1}{6}} + 2^{\frac{1}{6}}) + \frac{1}{4}\}$, $M_2 = \max\{\frac{1}{9\Gamma(1.5)} (3^{\frac{1}{2}} + 2^{\frac{1}{2}})\}$. Then $M < 1$.

From Theorem 2.2.1 there exists a unique solution $y_0(t)$ of NIFrDE (2.58) such that

$$y_0(t) = \begin{cases} x_0 + \frac{1}{\Gamma(0.5)} \int_0^t (t-s)^{-0.5} \frac{|y_0(s)|}{8+e^s+s^2} ds & \text{for } t \in [0,1], \\ \frac{|y_0(1-0)|}{(3+t^2)(1+|y_0(1-0)|)} \\ \quad \text{for } t \in (1,2], \\ \frac{|y_0(1-0)|}{7(1+|y_0(1-0)|)} + \frac{1}{\Gamma(0.5)} \int_0^t (t-s)^{-0.5} \frac{|y_0(s)|}{8+e^s+s^2} ds \\ \quad -\frac{1}{\Gamma(0.5)} \int_0^2 (2-s)^{-0.5} \frac{|y_0(s)|}{8+e^s+s^2} ds \quad \text{for } t \in (2,3] \end{cases} \tag{2.60}$$

with

$$|y(t) - y_0(t)| \leq \frac{\frac{2}{\sqrt{\pi}4^{\frac{2}{3}}3^{\frac{1}{6}}+1}}{1-0.7}(e^t+1), \quad t \in [0,3].$$

$\square$

Case II. Existence by the application of approach (A2 for NIFrDE)

Consider the IVP for NIFrDE (2.48) with general impulsive functions $\phi_k(t,x,y) \in C([s_k, t_{k+1}] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $k = 0, 1, 2, \dots, m-1$.

In our study we will use the following result for FrDE (2.12):

Lemma 2.2.1 (Theorem 3.1 [119]) *Let the following conditions be satisfied:*

1. *The function $f \in C(I, \mathbb{R})$, $I = [\tau, T]$ and there exists a positive constant L such that $|f(t,x) - f(t,y)| \leq L|x-y|$, $t \in I$, $x, y \in \mathbb{R}$.*
2. *The function $y \in C^1(I, \mathbb{R})$ satisfies the fractional differential equation*

$$|{}_{\tau}^{c}D^q y(t) - f(t, y(t))| \leq \varpi(t), \quad t \in I$$

where the function $\varpi \in C(I, \mathbb{R})$ is such that

$$\frac{1}{\Gamma(q)} \int_{\tau}^{t} (t-s)^{q-1} \varpi(s) ds \leq K \varpi(t), \quad t \in I$$

with $0 < KL < 1$.

Then there exists a unique function $x(t) \in C(I, \mathbb{R})$ such that

$$x(t) = y(\tau) + \frac{1}{\Gamma(q)} \int_{\tau}^{t} (t-s)^{q-1} f(s, x(s) ds, \quad t \in I \tag{2.61}$$

and

$$|y(t) - x(t)| \leq \frac{K}{1-KL} \varpi(t) \quad t \in I. \tag{2.62}$$

Now we give sufficient conditions for existence of the NIFrDE (2.48) by the application of (A2 for NIFrDE) for the interpretation of the solution.

Let $\Psi_k > 0, \varphi_k \in C([s_k, t_{k+1}], \mathbb{R}),\ k = 0, 1, \dots, m-1$ and consider the fractional non-instantaneous differential inequalities

$$\begin{aligned} &|{}_{t_k}^{c}D^q y(t) - f(t, y)| \le \varphi_k(t) \text{ for } t \in (t_k, s_k],\ k = 0, 1, \dots, m \\ &|y(t) - \phi_k(t, y(t), y(s_k - 0))| \le \Psi_k \quad \text{for } t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \dots, m-1. \end{aligned} \tag{2.63}$$

Remark 2.2.3 *Note if $y(t)$ is a solution of the fractional non-instantaneous differential inequalities (2.63), then it satisfies the integral–algebraic inequalities*

$$\begin{cases} |y(t) - y(t_k) + \frac{1}{\Gamma(q)} \int_{t_k}^t (t-s)^{q-1} f(s, y(s)) ds| \le \frac{1}{\Gamma(q)} \int_{t_k}^t (t-s)^{q-1} \varphi_k(s) ds, \\ \qquad t \in (t_k, s_k],\ k = 0, 1, 2, \dots, m \\ |y(t) - \phi_k(t, y(t), y(s_k - 0))| \le \Psi_k, \qquad t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \dots, m-1. \end{cases}$$

Theorem 2.2.2 (By A2 for NIFrDE) *Let the following conditions be satisfied:*

1. *The function $f \in C(\cup_{k=0}^m [t_k, s_k] \times \mathbb{R}, \mathbb{R})$ and there exist positive constants $L_k = L_k(f)$ such that $|f(t, x) - f(t, y)| \le L_k |x - y|$ for each $t \in [t_k, s_k],\ x, y \in \mathbb{R},\ k = 0, 1, \dots, m$.*
2. *The functions $\phi_k(t, x, y) \in C([s_k, t_{k+1}] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}),\ k = 0, 1, 2, \dots, m-1$ are such that for any $t \in [s_k, t_{k+1}]$ and $y \in \mathbb{R}$ there exists a unique solution $x(t, y)$ of the algebraic equation $x = \phi_k(t, x, y)$ and there exist constants $l_k = l_k(\phi_k) \in (0, 1),\ k = 0, 1, 2, \dots, m-1$ such that $|\phi_k(t, x_1, y_1) - \phi_k(t, x_2, y_2)| \le l_k(|x_1 - x_2| + |y_1 - y_2|)$ for each $t \in [s_k, t_{k+1}],\ x_1, x_2, y_1, y_2 \in \mathbb{R},\ k = 0, 1, \dots, m-1$.*
3. *The functions $\varphi_k \in C([t_k, s_k], \mathbb{R}),\ k = 0, 1, \dots, m$ are nondecreasing functions and there exist constants $C_k = C_k(\varphi_k) > 0,\ \ L_k C_k < 1,\ k = 0, 1, \dots, m$ such that*

$$\frac{1}{\Gamma(q)} \int_{t_k}^t (t-s)^{q-1} \varphi_k(s) ds \le C_k \varphi_k(t), \quad t \in [t_k, s_k]. \tag{2.64}$$

Then for each solution $y(t) \in PC([t_0, T], \mathbb{R})$ of the fractional differential inequality (2.63) there exists a solution $x \in NPC^q([t_0, T], \mathbb{R}$ of the IVP for NIFrDE (2.48) with $x_0 = y(t_0)$ such that

$$|y(t) - x(t)| \le \begin{cases} F_0(t), \qquad t \in (t_0, s_0], \\ \frac{C_k}{1 - C_k L_k} \varphi_k(t) + \frac{1}{1 - l_{k-1}} \frac{1}{1 - C_k L_k} \Big(\Psi_{k-1} + l_{k-1} F_{k-1}(s_{k-1})\Big), \\ \qquad t \in (t_k, s_k],\ k = 1, 2, \dots, m \\ \frac{1}{1 - l_k} \Big(\Psi_k + l_k F_k(s_k)\Big), \qquad t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \dots, m-1. \end{cases} \tag{2.65}$$

where

$$F_0(t) = \frac{C_0}{1 - C_0 L_0}\varphi_0(t) \ \ t \in (t_0, s_0],$$

and

$$F_k(t) = \frac{C_k}{1 - C_k L_k}\varphi_k(t) + \frac{1}{1 - l_{k-1}}\frac{1}{1 - C_k L_k}\Big(\Psi_{k-1} + l_{k-1}F_{k-1}(s_{k-1})\Big),$$
$$t \in (t_k, s_k], \ k = 1, 2, \ldots, m.$$

Proof We will use induction to prove the claim.

Let $t \in [t_0, s_0]$. According to Lemma 2.2.1 with $\tau = t_0, T = s_0, L = L_0, K = C_0$ and $\varpi(t) = \varphi_0(t)$ there exists a solution $x_0(t) \in C([t_0, s_0]$ satisfying the integral equality

$$x_0(t) = x_0 + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t - s)^{q-1}f(s, x_0(s))ds \tag{2.66}$$

and the inequality

$$|y(t) - x_0(t)| \le \frac{C_0}{1 - C_0 L_0}\varphi_0(t) = F_0(t), \ \ t \in [t_0, s_0]. \tag{2.67}$$

Let $t \in (s_0, t_1]$. Consider the algebraic equation $x = \phi_0(t, x, x_0(s_0 - 0))$ which has a solution $\tilde{x}_0(t)$ and

$$\begin{aligned} &|y(t) - \tilde{x}_0(t)| \\ &\le |y(t) - \phi_0(t, y(t), y(s_0 - 0))| + |\phi_0(t, y(t), y(s_0 - 0)) - \phi_0(t, \tilde{x}_0(t), x_0(s_0 - 0))| \\ &\le \Psi_0 + l_0(|y(t) - \tilde{x}_0(t)| + |y(s_0 - 0) - x_0(s_0 - 0)|) \\ &\le \Psi_0 + l_0|y(t) - \tilde{x}_0(t)| + l_0 F_0(s_0) \end{aligned} \tag{2.68}$$

or

$$|y(t) - \tilde{x}_0(t)| \le \frac{1}{1 - l_0}\Big(\Psi_0 + l_0 F_0(s_0)\Big), \quad t \in (s_0, t_1]. \tag{2.69}$$

Let $t \in (t_1, s_1]$. Define the function $\tilde{y}(t) = y(t) - y(t_1) + \phi_0(t_1, \tilde{x}_0(t_1), x_0(s_0))$. Then

$$\begin{aligned} &|{}_{t_1}^{c}D^q\tilde{y}(t) - f(t, \tilde{y}(t))| \le |{}_{t_1}^{c}D^q y(t) - f(t, y(t))| + |f(t, \tilde{y}(t) - f(t, y(t))| \\ &\le \varphi_1(t) + L_1|\tilde{y}(t) - y(t)|. \end{aligned} \tag{2.70}$$

From Remark 2.2.3, condition 2, and inequalities (2.67), (2.69) we obtain

$$
\begin{aligned}
|\tilde{y}(t)-y(t)| &\le |y(t_1)-\phi_0(t_1,y(t_1),y(s_0-0))| \\
&\quad + |\phi_0(t_1,y(t_1),y(s_0-0))-\phi_0(t_1,\tilde{x}_0(t_1),x_0(s_0))| \\
&\le \Psi_0 + l_0|y(t_1)-\tilde{x}_0(t_1)| + l_0|y(s_0-0)-x_0(s_0)| \\
&\le \Psi_0 + \frac{l_0}{1-l_0}\Psi_0 + \frac{l_0 l_0}{(1-l_0)}F_0(s_0) + l_0F_0(s_0) \\
&= \frac{1}{1-l_0}\Psi_0 + \frac{l_0}{1-l_0}F_0(s_0).
\end{aligned} \tag{2.71}
$$

From (2.70), (2.71) we get

$$
|{}_{t_1}^{c}D^q\tilde{y}(t)-f_1(t,\tilde{y}(t)| \le \varphi_1(t) + \frac{L_1}{1-l_0}\Psi_0 + \frac{L_1 l_0}{1-l_0}F_0(s_0). \tag{2.72}
$$

According to Lemma 2.2.1 with $\tau = t_1, T = s_1, L = L_1, K = C_1, y(t) = \tilde{y}(t)$, and $\varpi(t) = \varphi_1(t) + \frac{L_1}{1-l_0}\Psi_0 + \frac{L_1 l_1}{1-l_0}F_0(s_0)$ there exists a solution $x_1(t) \in C([t_1,s_1],\mathbb{R})$ satisfying the integral equation

$$
x_1(t) = \phi_0(t_1,\tilde{x}_0(t_1),x_0(s_0)) + \frac{1}{\Gamma(q)}\int_{t_1}^{t}(t-s)^{q-1}f_1(s,x_1(s))ds, \quad t \in (t_1,s_1] \tag{2.73}
$$

and

$$
|\tilde{y}(t)-x_1(t)| \le \frac{C_1}{1-C_1L_1}\Big(\varphi_1(t)+\frac{L_1}{1-l_0}\Psi_0+\frac{L_1l_0}{1-l_0}F_0(s_0)\Big) \quad t \in (t_1,s_1]. \tag{2.74}
$$

Using inequalities (2.71) and (2.74) we get

$$
\begin{aligned}
&|y(t)-x_1(t)| \le |\tilde{y}(t)-x_1(t)| + |\tilde{y}(t)-y(t)| \\
&\le \frac{C_1}{1-C_1L_1}\Big(\varphi_1(t) + \frac{L_1}{1-l_0}\Psi_0 + \frac{L_1l_0}{1-l_0}F_0(s_0)\Big) + \frac{1}{1-l_0}\Psi_0 + \frac{l_0}{1-l_0}F_0(s_0) \\
&\le \frac{C_1}{1-C_1L_1}\varphi_1(t) + \frac{1}{1-l_0}\frac{1}{1-C_1L_1}\Psi_0 + \frac{l_0}{1-l_0}\frac{1}{1-C_1L_1}F_0(s_0)
\end{aligned} \tag{2.75}
$$

i.e.

$$
\begin{aligned}
|y(t)-x_1(t)| &\le \frac{C_1}{1-C_1L_1}\varphi_1(t) + \frac{1}{1-l_0}\frac{1}{1-C_1L_1}\Big(\Psi_0 + l_0F_0(s_0)\Big) \\
&= F_1(s), \quad t \in (t_1,s_1].
\end{aligned}
$$

Following this inductive process we construct the function

$$x(t) = \begin{cases} x_k(t), & t \in (t_k, s_k],\ k = 0, 1, 2, \dots, m \\ \tilde{x}_k(t), & t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \dots, m-1 \end{cases} \tag{2.76}$$

which is a solution of IVP for the NIFrDE (2.48) with $x_0 = y(t_0)$ and satisfies (2.65). $\square$

Remark 2.2.4 *Note in (2.65) for the solution in Theorem 2.2.2 the points t_k, s_k are not included (compare with (2.52) in Theorem 2.2.1). This allows the result of Theorem 2.2.2 to be generalized to the infinite interval $[t_0, \infty)$ for appropriate values of the constants L_k, l_k, C_k (for example, $l_k : \prod_{i=0}^{\infty}(1 - l_i) < K_1 < \infty$, $\prod_{i=0}^{\infty} \frac{l_i}{1-l_i} < K_2 < \infty$, $C_k : \prod_{i=0}^{\infty}(1 - C_i L_i) < K_3 < \infty$ and $\prod_{i=0}^{\infty} \frac{C_i}{1-C_i L_i} < K_4 < \infty$).*

Example 2.2.2 Consider the IVP for NIFrDE (2.48) with $n = 1$, $0 = t_0 < s_0 = 1 < t_1 = 2 < s_1 = 4 < t_2 = 5 < s_2 = 7 < t_3 = 9 < s_3 = 10$, and $q = 0.1$, i.e.

$$\begin{aligned} &{}_{t_k}^{c}D^{0.1}x(t) = 0.2x\tan(t) \text{ for } t \in (0, 1] \cup (2, 3] \cup (5, 7] \cup (9, 10], \\ &x(t) = \frac{1}{k+2}\Big(x(t) + x(s_k - 0)\Big) \quad \text{for } t \in (1, 2] \cup (3, 5] \cup (7, 9], \\ &x(0) = 1. \end{aligned} \tag{2.77}$$

The function $f(t, x) = 0.2x\tan(t)$ is not defined and continuous on the whole interval [0, 10]. Therefore the conditions of Theorem 2.2.1 are not satisfied for (2.77) and approach (A1 for NIFrDE) and Theorem 2.2.1 does not guarantee the existence.

Consider $f(t, x) = 0.2x\tan(t),\ t \in [t_k, s_k],\ k = 0, 1, 2, 3$. Note the function $f \in C([t_k, s_k] \times \mathbb{R}, \mathbb{R}),\ k = 0, 1, 2, 3$ and there exist positive constants $L_0 = 0.312, L_1 = 0.44, L_2 = 0.68, L_3 = 0.13$, i.e., condition 1 of Theorem 2.2.2 is satisfied. Let $\phi_k(t, x, y) = \frac{1}{k+2}(x + y),\ k = 0, 1, 2$. Then condition 2 of Theorem 2.2.2 is satisfied with $l_k = \frac{1}{k+2},\ k = 0, 1, 2$.

Consider the function $y(t) \equiv 1, t \in [0, 10]$ which satisfies the inequalities (2.63) with $\varphi_k(t) \equiv L_k,\ t \in (t_k, s_k],\ k = 0, 1, 2, 3$ and $\Psi_k = |\frac{k}{k+2}|,\ k = 0, 1, 2$. Then, $\frac{L_k}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} ds = \frac{(t-t_k)^q}{q\Gamma(q)} L_k \le C_k \varphi_k(t),\quad t \in [t_k, s_k]$ with $C_k = \frac{(s_k - t_k)^q}{q\Gamma(q)},\ k = 0, 1, 2, 3$. That is, $C_0 = 1.052,\ C_1 = 1.13,\ C_2 = 1.13,\ C_3 = 1.13$. Condition 3 of Theorem 2.2.2 is satisfied.

According to Theorem 2.2.2 there exists a solution $x(t)$ of the IVP for NIFrDE (2.77) for which the inequality (2.65) holds. $\square$

2.3 Stability of Caputo Fractional Differential Equations with Non-instantaneous Impulses

The question of stability is of interest in physical and biological systems, such as the fractional Duffing oscillator [137], fractional predator-prey and rabies models [22], etc. and stability theory of FDEs is widely applied to chaos and chaos synchronization [87] because of its potential applications in control processing and secure communication. There are several approaches in the literature to study stability, one of which is the Lyapunov approach. As it is mentioned in [112] there are several difficulties encountered when one applies the Lyapunov technique to fractional differential equations.

2.3.1 Lyapunov Functions and Their Derivatives for Caputo Fractional Differential Equations

One approach to study various stability properties of solutions of nonlinear Caputo fractional differential equations is based on using Lyapunov like functions. A basic question which arises is the definition of the derivative of the Lyapunov like function along the given fractional equation. In this section we will give a brief overview of fractional derivatives of Lyapunov functions.

I. Lyapunov functions for Caputo fractional differential equations.

We will give a brief overview of the applications of Lyapunov functions to study stability properties of the solutions of nonlinear Caputo fractional differential equations.

Consider the nonlinear Caputo fractional differential equation

$$ {}_{\tau_0}^{c}D^q x(t) = f(t,x) \quad \text{for } t \geq \tau_0 \tag{2.78}$$

with initial condition

$$x(\tau_0) = \tilde{x}_0, \tag{2.79}$$

where $\tilde{x_0} \in \mathbb{R}^n$, $\tau_0 \in \mathbb{R}_+$.

Definition 2.3.1 *Let $J = [\tau_0, T)$, $T \leq \infty$, be a given interval, and $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$ be a given set. We will say that the function $V(t,x) : J \times \Delta \to \mathbb{R}_+$ belongs to the class $\Lambda^C(J, \Delta)$ if $V(t,x)$ is continuous on $J \times \Delta$ and it is locally Lipschitzian with respect to its second argument.*

There are three types of derivatives of Lyapunov functions from the class $\Lambda^C(J, \Delta)$ used in the literature to study stability properties of solutions of Caputo fractional differential equations:

– **Caputo fractional derivative of Lyapunov function** – this derivative is given by

$$ {}_{\tau_0}^{c}D^q V(t, x(t)) = \frac{1}{\Gamma(1-q)} \int_{\tau_0}^{t} (t-s)^{-q} \frac{d}{ds}\Big(V(s, x(s))\Big) ds, \qquad t \in [\tau_0, T) \tag{2.80} $$

where $x(t)$ is the solution of the studied IVP for the FrDE (2.78), (2.79) (see, for example, [86, 89]). This type of derivative is applicable for continuously differentiable Lyapunov functions.

– **Dini fractional derivative of Lyapunov function**– it is defined by (see, for example, [81, 82]):

$$ D^{+}_{(2.78)} V(t, x) = \limsup_{h \to 0} \frac{1}{h^q} \Big[V(t, x) - V(t-h, x - h^q f(t, x)) \Big], \quad t \in [\tau_0, T). \tag{2.81} $$

This definition is based on the definition of Dini derivative of the Lyapunov function $V(t, x)$ among the ordinary differential equation $x' = f(t, x)$ given by

$$ DV(t, x) = \limsup_{h \to 0} \frac{1}{h} \Big[V(t, x) - V(t-h, x - hf(t, x)) \Big]. \tag{2.82} $$

The operator defined by (2.81) has no memory. Also, it depends neither on the fractional order q of the FrDE (2.78) nor on the initial time τ_0 which is very important for fractional derivatives.

This type of derivative is applicable for continuous Lyapunov functions.

– **Caputo fractional Dini derivative of Lyapunov function** – this derivative is defined by:

$$ \begin{aligned} &{}_{(2.78)}^{c}D^q_+ V(t, x; \tau_0, \tilde{x_0}) \\ &= \limsup_{h \to 0+} \frac{1}{h^q} \Bigg\{ V(t, x) - V(\tau_0, \tilde{x_0}) \\ &\quad - \sum_{r=1}^{[\frac{t-\tau_0}{h}]} (-1)^{r+1} {}_qC_r \Big[V(t - rh, x - h^q f(t, x)) - V(\tau_0, \tilde{x_0}) \Big] \Bigg\}, \\ &\qquad \text{for } t \in [\tau_0, T), \ x \in \Delta \end{aligned} \tag{2.83} $$

where there exists $h_1 > 0$ such that $t - h \in [\tau, s_p)$, $x - h^q f(t, x) \in \Delta$ for $0 < h \leq h_1$.

This definition is based on the definition of the Caputo fractional Dini derivative of a function $m(t)$ given by (2.5).

This type of derivative is applicable for continuous Lyapunov functions.

Formula (2.83) could be reduced to

$$\begin{aligned}
&{}^{c}_{(2.78)}D^{q}_{+}V(t,x;\tau_0,\tilde{x_0})\\
&\quad=\limsup_{h\to 0+}\frac{1}{h^q}\Big\{V(t,x)-\sum_{r=1}^{[\frac{t-\tau_0}{h}]}(-1)^{r+1}\,{}_qC_rV(t-rh,x-h^qf(t,x))\Big\}\\
&\qquad-V(\tau_0,\tilde{x_0})\frac{(t-\tau_0)^{-q}}{\Gamma(1-q)}\quad\text{for } t\in[\tau_0,T),\ x\in\Delta.
\end{aligned} \tag{2.84}$$

Note the Caputo fractional Dini derivative ${}^{c}_{(2.78)}D^{q}_{+}V(t,x;\tau_0,\tilde{x_0})$ given by (2.83) or its equivalent (2.84) depends significantly on both the fractional order q and the initial data of the NIFrDE.

Example 2.3.1.1 Let $V(t,x)=x^2$, with $x\in\mathbb{R}$.

Case 1. Caputo fractional derivative. Let $x(t)$ be a solution of the IVP for the FrDE (2.78), (2.79). Then ${}^{c}_{\tau_0}D^qV(t,x(t))={}^{c}_{\tau_0}D^q(x(t))^2$. According to Lemma 2.1.2 we get ${}^{c}_{\tau_0}D^q(x(t))^2\le 2\,x(t)\,f(t,(x(t))$ which easily allows for the application of Caputo fractional derivative of the scalar quadratic function for studying stability properties for the solutions of FrDE (2.78).

Case 2. Dini fractional derivative. From (2.81) we obtain

$$D^{+}_{(2.78)}V(t,x)=\limsup_{h\to 0}\frac{1}{h^q}\Big[x^2-(x-h^qf(t,x))^2\Big]=2xf(t,x). \tag{2.85}$$

In this case the formula (2.85) coincides with the corresponding Dini derivative $DV(t,x)=2xf(t,x)$ in the ordinary case $q=1$ obtained by (2.82) and it is easy to apply.

Case 3. Caputo fractional Dini derivative. Use (2.83) and obtain

$${}^{c}_{(2.78)}D^{q}_{+}V(t,x;\tau_0,\tilde{x_0})=2xf(t,x)+\Big(x^2-(x_0)^2\Big)\frac{(t-\tau_0)^{-q}}{\Gamma(1-q)},\quad t\in(\tau_0,T),\ x\in\mathbb{R}. \tag{2.86}$$

Since often the object of investigation is the stability of zero solution, let $\tau_0=0,\tilde{x_0}=0$. Then ${}^{c}_{(2.78)}D^{q}_{+}V(t,x;0,0)=2xf(t,x)+x^2\frac{t^{-q}}{\Gamma(1-q)}$. This formula differs from the Dini derivative $DV(t,x)$ obtained by (2.82) ($q=1$). □

Example 2.3.1.2 Let $V(t,x)=sin^2(t)\,x^2$, with $x\in\mathbb{R}$ and $t\in(\tau_0,T]$.

Case 1. Caputo fractional derivative. Let $x(t)$ be a solution of the IVP for the FrDE (2.78), (2.79). In this case the fractional derivative ${}^{c}_{\tau_0}D^qV(t,x(t))={}^{c}_{\tau_0}D^q\Big(sin^2(t)\,x^2(t)\Big)$ which is difficult to obtain.

Case 2. Dini fractional derivative. From (2.81) we obtain

$$\begin{aligned}
D^{+}_{(2.78)}V(t,x) &= \limsup_{h\to 0} \frac{1}{h^q}\Big[\, sin^2(t)\, x^2 - \, sin^2(t-h)(x-h^q f(t,x))^2\Big] \\
&= \limsup_{h\to 0} \frac{1}{h^q}\Big[\, sin^2(t)\Big(x^2-(x-h^q f(t,x))^2\Big) \\
&\quad + \Big(sin^2(t)-sin^2(t-h)\Big)(x-h^q f(t,x))^2\Big] \\
&= 2x\, sin^2(t) f(t,x) \\
&\quad + \limsup_{h\to 0} h^{1-q}\Big(x^2-(x-h^q f(t,x))^2\Big)\frac{sin^2(t)-sin^2(t-h)}{h}\Big] \\
&= 2x\, sin^2(t) f(t,x).
\end{aligned} \tag{2.87}$$

In the integer case ($q = 1$) from (2.82) we obtain for the derivative of Lyapunov function $DV(t,x) = 2x\, sin^2(t) f(t,x) + x^2\Big(sin^2(t)\Big)'$ which differs significantly from $D^{+}_{(2.78)}V(t,x)$.

Case 3. Caputo fractional Dini derivative. Use (2.83) and obtain

$$\begin{aligned}
&{}^{c}_{(2.78)}D^{q}_{+}V(t,x;\tau_0,x_0) \\
&= \limsup_{h\to 0+} \frac{1}{h^q}\Big\{x^2(sin^2 t) - \sum_{r=1}^{[\frac{t-\tau_0}{h}]}(-1)^{r+1}\, {}_qC_r\big(x-h^q f(t,x)\big)^2(sin^2(t-rh))\Big\} \\
&\qquad - (x_0)^2 sin^2(\tau_0)\frac{(t-\tau_0)^{-q}}{\Gamma(1-q)} \\
&= \limsup_{h\to 0+} \frac{1}{h^q}\Big\{(sin^2 t)\big(x^2-(x-h^q f(t,x))^2\big) \\
&\qquad + \big(x-h^q f(t,x)\big)^2\sum_{r=0}^{[\frac{t-\tau_0}{h}]}(-1)^{r}\, {}_qC_r(sin^2(t-rh))\Big\} \\
&\qquad - (x_0)^2 sin^2(\tau_0)\frac{(t-\tau_0)^{-q}}{\Gamma(1-q)} \\
&= 2x(sin^2 t) f(t,x) + x^2\, {}^{RL}_{\tau_0}D^q\Big(sin^2 t\Big) - (x_0)^2 sin^2(\tau_0)\frac{(t-\tau_0)^{-q}}{\Gamma(1-q)}.
\end{aligned} \tag{2.88}$$

Since often the object of investigation is the stability of zero solution, let $\tau_0 = 0, x_0 = 0$. Then from (2.88) the Caputo fractional Dini derivative is given by

$${}^{c}_{(2.78)}D^{q}_{+}V(t,x;0,0) = 2x\, sin^2 t\, f(t,x) + x^2\, {}^{C}_{0}D^q\Big(sin^2 t\Big).$$

This formula is similar to the obtained by Eq. (2.82) the Dini derivative in the ordinary case ($q = 1$), i.e., to

$$DV(t,x) = 2x\, sin^2(t) f(t,x) + x^2 \Big(sin^2(t) \Big)'.$$

$\square$

II. Lyapunov functions for Caputo fractional differential equations with non-instantaneous impulses

We now introduce the class Λ of Lyapunov-like functions which will be used to investigate the stability of Caputo fractional differential equations with non-instantaneous impulses.

Definition 2.3.2 *Let $J = [\tau_0, T)$, $0 < T \leq \infty$, be a given interval, and $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$ be a given set. We will say that the function $V(t,x) : J \times \Delta \to \mathbb{R}_+$ belongs to the class $\Lambda(J, \Delta)$ if*

1. *The function $V(t,x)$ is continuous on $J/\{s_k \in J\} \times \Delta$ and it is locally Lipschitzian with respect to its second argument;*
2. *For each $s_k \in Int(J)$ and $x \in \Delta$ there exist finite limits*

$$V(s_k - 0, x) = \lim_{t \uparrow s_k} V(t,x) < \infty, \quad \textit{and} \;\; V(s_k + 0, x) = \lim_{t \downarrow s_k} V(t,x) < \infty$$

and the following equalities are valid

$$V(s_k - 0, x) = V(s_k, x).$$

Remark 2.3.1 *Note the quadratic Lyapunov function $V(t,x) = x^2$ belongs to the class $\Lambda(J, \Delta)$ for any J and $x \in \Delta$.*

Remark 2.3.2 *Based on the idea of the non-instantaneous impulses in differential equations and the fact that on each interval without impulses the unknown function is defined by a differential equation, it is natural to consider the derivatives of the Lyapunov functions only on the intervals without impulses.*

2.3.1.1 Approach (A1 for NIFrDE) and Lyapunov Functions

Consider the IVP for NIFrDE (2.33).

Three types of derivatives of Lyapunov functions from the class $\Lambda(J, \Delta)$ can be used:

- **Caputo fractional derivative of Lyapunov function** – ${}^{c}_{\tau_0}D^q V(t, x(t))$ where $x(t)$ is the unknown solution of the studied fractional differential equation (2.33). We require the Lyapunov function be at least piecewise continuously differentiable function on $[\tau_0, T)$.

– **Dini fractional derivative** of the function $V(t,x) \in \Lambda(J,\Delta)$, along trajectories of solutions of the system FrDE (2.33) as follows

$$
\begin{gathered}
{}_{(2.33)}\mathcal{D}_{+}^{q} V(t,x) = \limsup_{h \to 0} \frac{1}{h^q} \Big[V(t,x) - V(t-h, x - h^q f(t,x)) \Big], \\
t \in J \bigcap \Big([0, s_0) \cup_{k=1}^{\infty} (t_k, s_k) \Big).
\end{gathered}
\tag{2.89}
$$

where $x \in \Delta$, and for any $k = 0, 1, 2, \dots$ and $t \in (t_k, s_k) \cap J$ there exists $h_t > 0$ such that $t - h \in (t_k, s_k) \cap J$, $x - h^q f(t,x) \in \Delta$ for $0 < h \leq h_t$ (in the case of $k = 0$ we have the interval $(0, s_0)$).

– **Caputo fractional Dini derivative** of the function $V(t,x) \in \Lambda(J,\Delta)$, $T > 0$, along trajectories of solutions of the system FrDE (2.33) as follows:

$$
\begin{aligned}
&{}^{c}_{(2.33)}D_{+}^{q} V(t,x;\tau_0,x_0) \\
&= \limsup_{h \to 0+} \frac{1}{h^q} \Bigg\{ V(t,x) - V(\tau_0,x_0) \\
&\quad - \sum_{r=1}^{[\frac{t-\tau_0}{h}]} (-1)^{r+1} {}_{q}C_r \Big[V(t - rh, x - h^q f(t,x)) - V(\tau_0,x_0) \Big] \Bigg\}, \\
&\qquad\qquad \text{for } t \in J \bigcap \Big((0, s_0) \cup_{k=1}^{\infty} (t_k, s_k) \Big)
\end{aligned}
\tag{2.90}
$$

where $x, x_0 \in \Delta$, and there exists $h_1 > 0$ such that $t - h \in J$, $x - h^q f(t,x) \in \Delta$ for $0 < h \leq h_1$.

Example 2.3.1.3 Let $V(t,x) = x^2$, $x \in \mathbb{R}$. Consider the NIFrDE (2.33) with $\tau_0 = t_0 = 0$, $s_0 = 1$, $t_1 = 3$, $s_1 = 4$, $f(t,x) = -x$, and $\phi_0(t,x,y) = y\Big(t-2\Big)^{\frac{1}{3}}$ for $t \in [1,3]$. Then using the approach (A1 for NIFrDE) on the interval $[0,4]$ the solution of NIFrDE (2.33) is

$$
x(t) = \begin{cases}
x_0 E_q(-t^q), & t \in (0,1], \\
x_0 E_q(-1)\Big(t-2\Big)^{\frac{1}{3}}, & t \in (1,3], \\
x_0 \frac{E_q(-1)}{E_q(-3^q)} E_q(-t^q), & t \in (3,4],
\end{cases}
$$

and

$$
V(t,x(t)) = \begin{cases}
\Big(x_0 E_q(-t^q)\Big)^2, & t \in (0,1], \\
\Big(x_0 E_q(-1)\Big(t-2\Big)^{\frac{1}{3}}\Big)^2, & t \in (1,3], \\
\Big(x_0 \frac{E_q(-1)}{E_q(-3^q)} E_q(-t^q)\Big)^2, & t \in (3,4].
\end{cases}
$$

On $(0, 3]$ the function $V(t, x(t))$ is continuous but its derivative at $t = 2$ is infinity. Therefore the Caputo fractional derivative ${}_0^cD^qV(t, x(t))$ exists for $t \in [0, 1]$ but it does not exist on $[0, 4]$.

Note the application of both the Dini fractional derivative and the Caputo fractional Dini derivative of the quadratic function do not cause problems because they are continuous. □

If we use the direct Lyapunov method to study stability properties of the solutions and the approach (A1 for NIFrDE) to the system of NIFrDE (2.48), then we need to apply the fractional derivative of the Lyapunov function $V(t, x)$, i.e., ${}_0^cD^qV(t, x(t))$ for $t \geq 0$ with the solution $x(t) \in C^q(\cup_{k=0}^{\infty}[t_k, s_k], \mathbb{R}^n)$ of (2.48). This Caputo fractional derivative might not exist.

2.3.1.2 Approach (A2 for NIFrDE) and Lyapunov Functions

Consider the IVP for NIFrDE (2.48).

Three types of derivatives of Lyapunov functions from the class $\Lambda(J, \Delta)$ could be used:

- **Caputo fractional derivative of Lyapunov function**. In connection with the interpretation (A2 for NIFrDE) we consider the following Caputo fractional derivative of Lyapunov function ${}_{t_k}^cD^qV(t, x(t))$ for $t \in [t_k, s_k]$ for all $k: \ t_k, s_k \in J$. Here $x(t)$ is the unknown solution of the fractional differential equation (2.48). We require the Lyapunov function be at least continuously differentiable function on $[t_k, s_k], \ k = 0, 1, 2, \dots$ (in the case $k = 0$ we have the interval $[0, s_0]$).
- **Dini fractional derivative** of the function $V(t, x) \in \Lambda(J, \Delta)$, along trajectories of solutions of the system FrDE (2.48) defined by (2.89).
- **Caputo fractional Dini derivative** –let $V(t, x) \in \Lambda(J, \Delta)$ and $\tau \in J \bigcap \Big([0, s_0) \cup_{k=1}^{\infty} [t_k, s_k)\Big)$, $y_0 \in \Delta$. Then we define

$$
\begin{aligned}
&{}_{(2.33)}^{\;\;c}D_+^qV(t, x; \tau, y_0) \\
&= \limsup_{h \to 0+} \frac{1}{h^q} \Bigg\{ V(t, x) - V(\tau, y_0) \\
&\quad - \sum_{r=1}^{[\frac{t-\tau}{h}]} (-1)^{r+1} {}_qC_r \Big[V(t - rh, x - h^q f(t, x)) - V(\tau, y_0) \Big] \Bigg\}, \\
&\qquad\qquad \text{for } \ t \in J \cap (\tau, s_p], \ x \in \Delta
\end{aligned}
\tag{2.91}
$$

where $p = \inf\{k : \ \tau < s_k\}$, and there exists $h_1 > 0$ such that $t - h \in J \cap (\tau, s_p]$, $x - h^q f(t, x) \in \Delta$ for $0 < h \leq h_1$.

The formula (2.91) could be reduced to

$$
\begin{aligned}
&{}^{c}_{(2.48)}D^{q}_{+}V(t,x;\tau,y_0)\\
&= \limsup_{h\to 0+}\frac{1}{h^q}\Big\{V(t,x)-\sum_{r=1}^{[\frac{t-\tau}{h}]}(-1)^{r+1}\,{}_qC_r V(t-rh,x-h^q f(t,x))\Big\} \qquad (2.92)\\
&- V(\tau,y_0)\frac{(t-\tau)^{-q}}{\Gamma(1-q)}\ \text{ for } t\in J\cap(\tau,s_p),\ x\in\Delta.
\end{aligned}
$$

Formulas (2.91) and (2.92) are introduced by Agarwal et al. [7] and used for studying various stability properties of Caputo fractional differential equations [8, 10], Caputo fractional non-instantaneous impulsive differential equations [13, 15, 16, 18], differential equations with random non-instantaneous impulses [11, 12, 14, 17], for stability with respect to initial time difference for Caputo fractional differential equations [20]. For a survey of the application of Lyapunov functions to fractional equations see [9].

Remark 2.3.3 *Note in the special case of $s_k = t_{k+1}$, $k = 0, 1, 2, \dots$, i.e., the case of instantaneous impulses the above formulas for the derivatives of Lyapunov functions are applicable with slight changes (see [9]).*

Example 2.3.1.4 Let $V(t,x) = x^2, x \in \mathbb{R}$. Consider the NIFrDE (2.48) with $t_0 = 0$, $s_0 = 1$, $t_1 = 3$, $s_1 = 4$, $f(t,x) = -x$, and $\phi_0(t,x,y) = y\Big(t-2\Big)^{\frac{1}{3}}$ for $t \in [1,3]$, $(k = 1, 2, 3, \dots)$. Then using the approach (A2 for NIFrDE) on the interval $[0,4]$ the solution of NIFrDE (2.48) is

$$
x(t) = \begin{cases} x_0E_q(-t^q), & t\in(0,1],\\ x_0E_q(-1)\Big(t-2\Big)^{\frac{1}{3}}, & t\in(1,3],\\ x_0E_q(-1)E_q(-(t-3)^q), & t\in(3,4],\end{cases}
$$

and

$$
V(t,x(t)) = \begin{cases} \Big(x_0E_q(-t^q)\Big)^2, & t\in(0,1],\\ \Big(x_0E_q(-1)\Big(t-2\Big)^{\frac{1}{3}}\Big)^2, & t\in(1,3],\\ \Big(x_0E_q(-1)E_q(-(t-3)^q)\Big)^2, & t\in(3,4].\end{cases}
$$

The function $V(t, x(t))$ is continuous on $(0, 4]$. Then the Caputo fractional derivative ${}_0^cD^qV(t, x(t))$ exists for $t \in [0, 1]$ and ${}_3^cD^qV(t, x(t))$ exists on the interval $[3, 4]$ (compare with Example 2.3.1.3).

The application of both the Dini fractional derivative and the Caputo fractional Dini derivative of the quadratic function do not cause problems because they are continuous and they do not depend on the solutions. □

Example 2.3.1.5 Let $V \in \Lambda(\mathbb{R}_+, \mathbb{R})$ be given by $V(t, x) = m(t)g(x)$ where the function $m \in C^1((0, s_0) \cup_{i=1}^{\infty} (t_k, s_k), \mathbb{R}_+)$, $g : \mathbb{R}^n \to \mathbb{R}_+$ is a locally Lipschitz function such that the limit $\lim_{h\to 0+} \sup \frac{g(x)-g(x-h^qf(t,x))}{h^q}$ exists for $x \in \mathbb{R}^n$.

First we apply the formula (2.89) to obtain the Dini fractional derivative of the considered Lyapunov function. We obtain

$$
\begin{aligned}
{}_{(2.48)}\mathcal{D}_+^q V(t, x) &= \lim_{h\to 0+} \sup \frac{1}{h^q}\Big[m(t)g(x) - m(t-h)g\big(x - h^q f(t, x)\big)\Big] \\
&= m(t) \lim_{h\to 0+} \sup \frac{g(x) - g(x - h^q f(t, x))}{h^q} \\
&\quad + \Big(\lim_{h\to 0+} \sup \frac{m(t) - m(t-h)}{h}\Big)\Big(\lim_{h\to 0+} \sup h^{1-q} g\big(x - h^q f(t, x)\big)\Big) \\
&= m(t)\ lim_{h\to 0+} \sup \frac{g(x) - g(x - h^q f(t, x))}{h^q} \quad \text{for} \quad t \in (t_k, s_k),\ k = 0, 1, 2, \dots .
\end{aligned}
\tag{2.93}
$$

Note above in the case $k = 0$ we let $t_0 = 0$.

Note ${}_{(2.48)}^{\,c}\mathcal{D}_+^q V(t, x)$ does not depend on the order q of the fractional differential equation. The behavior of solutions of fractional differential equations depends significantly on the order q. For example, let us consider the simple fractional differential equation ${}_0^cD^q x + x(t) = 1$, $x(0) = 0$ whose solution is given by $x(t) = t^q E_{q,1+q}(-t^q)$. From Figure 2.7 it can be seen $\lim_{t\to\infty} x(t) = a$ where a is different for different values of the order q of fractional differential equation.

Consider the special case $V(t, x) = m(t)x^2$. Then from (2.93) we obtain the Dini fractional derivative

$$
{}_{(2.48)}\mathcal{D}_+^q V(t, x) = 2m(t)xf(t, x) \quad \text{for} \quad t \in (t_k, s_k),\ k = 0, 1, 2, \dots . \tag{2.94}
$$

Next we use (2.92) to obtain the Caputo fractional Dini derivative of the function $V(t, x) = m(t)g(x)$. Let $\tau \in [0, s_0) \cup_{k=1}^{\infty} [t_k, s_k)$. Apply the equalities ${}_{\tau}^{RL}D^q 1 = \frac{(t-\tau)^{-q}}{\Gamma(1-q)}$, and $\lim_{h\to 0+} \sup \frac{1}{h^q} \sum_{r=0}^{[\frac{t-\tau}{h}]} (-1)^r \ {}_qC_r\, m(t - rh) = {}_{\tau}^{RL}D^q\Big(m(t)\Big)$ to (2.92) and obtain

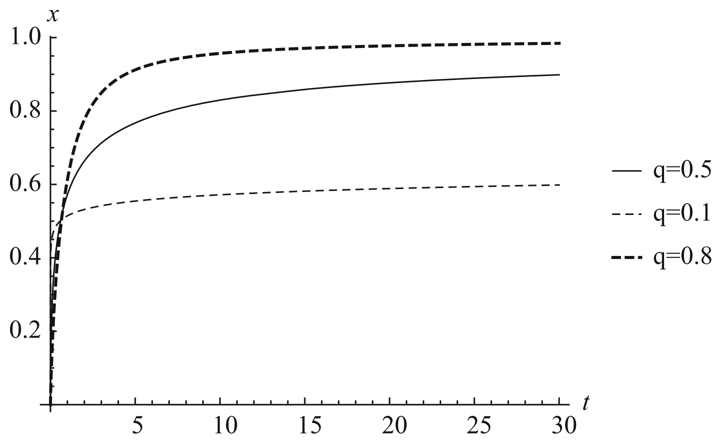

Fig. 2.7 Example 2.3.1.5. Graphs of solutions of ${}_0^cD^qx + x(t) = 1,\ x(0) = 0$ for different q.

$$
\begin{aligned}
&{}_{(2.48)}^{\;c}D_+^q V(t,x;\tau,y_0)\\
&= \lim_{h\to 0^+}\sup \frac{1}{h^q}\Big[m(t)g(x) + g(x-h^qf(t,x))\sum_{r=1}^{[\frac{t-\tau}{h}]}(-1)^r\ {}_qC_r\, m(t-rh)\Big]\\
&\quad - m(\tau)g(y_0)\frac{(t-\tau)^{-q}}{\Gamma(1-q)}\\
&= m(t)\lim_{h\to 0^+}\sup\frac{g(x)-g(x-h^qf(t,x))}{h^q}\\
&\quad + \lim_{h\to 0} g(x-h^qf(t,x))\lim_{h\to 0^+}\sup\frac{1}{h^q}\sum_{r=0}^{[\frac{t-\tau}{h}]}(-1)^r\ {}_qC_r\, m(t-rh)\Big]\\
&\quad - m(\tau)g(y_0)\frac{(t-\tau)^{-q}}{\Gamma(1-q)}\\
&= m(t)\lim_{h\to 0^+}\sup\frac{g(x)-g(x-h^qf(t,x))}{h^q} + g(x)\,{}_\tau^{RL}D^q\Big(m(t)\Big)\\
&\quad - m(\tau)g(y_0)\frac{(t-\tau)^{-q}}{\Gamma(1-q)},\ \text{for}\quad t\in(\tau,s_p),
\end{aligned}
\tag{2.95}
$$

or

$$\begin{aligned}
&{}^{c}_{(2.48)}D^{q}_{+}V(t,x;\tau,y_0)\\
&= m(t)\lim_{h\to 0+}\sup\frac{g(x)-g(x-h^q f(t,x))}{h^q}+g(x)\,{}^{C}_{\tau}D^{q}\Big(m(t)\Big)\\
&\quad+\Big(g(x)-g(y_0)\Big)m(\tau)\frac{(t-\tau)^{-q}}{\Gamma(1-q)},\ \text{for}\quad t\in(\tau,s_p).
\end{aligned}\tag{2.96}$$

Note the Caputo fractional Dini derivative ${}^{c}_{(2.48)}D^{q}_{+}V(t,x;t_0,x_0)$ depends significantly not only on the order q of the fractional differential equation but also on the initial data (t_0,x_0).

In the special case $g(x)=x^2$ we obtain

$$\begin{aligned}
&{}^{c}_{(2.48)}D^{q}_{+}V(t,x;\tau,y_0)=\\
&=2xm(t)f(t,x)+x^2\,{}^{C}_{\tau}D^{q}\Big(m(t)\Big)+\Big(x^2-y_0^2\Big)m(\tau)\frac{(t-\tau)^{-q}}{\Gamma(1-q)},\\
&\qquad \text{for}\quad t\in(\tau,s_p).
\end{aligned}\tag{2.97}$$

If $m(\tau)=0$, then we get

$$\begin{aligned}
&{}^{c}_{(2.48)}D^{q}_{+}V(t,x;0,y_0)=2xm(t)f(t,x)+x^2\,{}^{C}_{\tau}D^{q}\Big(m(t)\Big),\\
&\qquad t\in(\tau,s_p).
\end{aligned}\tag{2.98}$$

The derivative of the Lyapunov function in the well-known case for first order non-instantaneous impulsive differential equations ($q=1$) is (see Section 1.3 and Eq. (1.29))

$$D_{+}V(t,x)=2x\,m(t)f(t,x)+x^2\frac{d}{dt}\Big[m(t)\Big],\quad t\in(\tau,s_p).\tag{2.99}$$

Formulas (2.98) and (2.99) differ only by the type of derivatives. At the same time, both (2.98) and (2.99) are totally different than (2.93). Therefore, the Caputo fractional Dini derivative given by formula (2.92) seems to be the natural generalization of the derivative of Lyapunov functions for ordinary differential equations (with or without impulses) especially when approach (A2 for NIFrDE) is applied. □

Remark 2.3.4 *Note that if in (2.91) (respectively (2.92)) instead of a point $x \in \mathbb{R}^n$ we use the value $y(t)$ of the function $y : \Big([0, s_0) \cup_{k=1}^{\infty} (t_k, s_k)\Big) \cap J \to \mathbb{R}^n : \; y(\tau) = y_0$, then*

$$
\begin{aligned}
{}^{c}_{(2.48)}D^q_+ V(t, y(t); \tau, x_0) &= \limsup_{h\to 0+} \frac{1}{h^q}\Bigg\{ V(t, y(t)) - V(\tau, y_0) \\
&- \sum_{r=1}^{[\frac{t-\tau}{h}]} (-1)^{r+1}_q C_r \Big[V(t - rh, y(t) - h^q f(t, y(t))) - V(\tau, y_0) \Big] \Bigg\}, \; t \in (\tau, s_p)
\end{aligned} \tag{2.100}
$$

Example 2.3.1.6 Consider the scalar case, i.e., let $V(t, x) = x^2$ where $x \in \mathbb{R}$.

Initially, let $\tau \in \Big([0, s_0) \cup_{k=1}^{\infty} [t_k, s_k)\Big) \cap J$ and $y_0 \in \mathbb{R}$. Then we have from (2.91)

$$
\begin{aligned}
{}^{c}_{(2.48)}D^q_+ V(t, x; \tau, y_0) &= \limsup_{h\to 0+} \frac{1}{h^q}\Big[\big(x^2 - y_0^2\big) - (x - h^q f(t, x))^2 + (y_0)^2 \Big] \\
&- \limsup_{h\to 0+} \frac{1}{h^q}\Big[(x - h^q f(t, x))^2 - (y_0)^2 \Big] \Bigg\{ \sum_{r=0}^{[\frac{t-\tau}{h}]} (-1)^{r+1}_q C_r \Bigg\} \\
&= \limsup_{h\to 0+} \frac{1}{h^q}\Big[h^q f(t, x)(2x - h^q f(t, x)) \Big] \\
&- \limsup_{h\to 0+} \frac{1}{h^q}\Big[- h^q f(t, x)\big(2x - h^q f(t, x)\big) + x^2 - (y_0)^2 \Big] \Bigg\{ \sum_{r=0}^{[\frac{t-\tau}{h}]} (-1)^{r+1}_q C_r \Bigg\} \\
&= f(t, x) \limsup_{h\to 0+} (2x - h^q f(t, x)) + f(t, x) \limsup_{h\to 0+} \Bigg\{ \sum_{r=0}^{[\frac{t-\tau}{h}]} (-1)^{r+1}_q C_r \Bigg\} \\
&- (x^2 - (y_0)^2) \limsup_{h\to 0+} \frac{1}{h^q} \Bigg\{ \sum_{r=0}^{[\frac{t-\tau}{h}]} (-1)^{r+1}_q C_r \Bigg\} \\
&= 2x f(t, x) - (x^2 - (y_0)^2)\, {}^{GL}_{\tau}D^q_+ 1 = 2x f(t, x) - (x^2 - (y_0)^2) \frac{(t - \tau)^{-q}}{\Gamma(1 - q)}.
\end{aligned} \tag{2.101}
$$

Now consider a scalar function $y(t) : \Big([0, s_0] \cup_{k=1}^{\infty} (t_k, s_k]\Big) \cap J \to \mathbb{R}$ such that $y(\tau) = y_0$ where $\tau \in \Big([0, s_0) \cup_{k=1}^{\infty} [t_k, s_k)\Big) \cap J$ and $y_0 \in \mathbb{R}$. Then we have from (2.100)

$$
\begin{aligned}
&{}_{(2.48)}^{\;\;c}D_{+}^{q}V(t,y(t);\tau,x_0)\\
&=\limsup_{h\to 0+}\frac{1}{h^q}\Big[\big((y(t))^2-y_0^2\big)-(y(t)-h^qf(t,y(t)))^2+(y_0)^2\Big]\\
&\quad-\limsup_{h\to 0+}\frac{1}{h^q}\Big[(y(t)-h^qf(t,y(t)))^2-(y_0)^2\Big]\Bigg\{\sum_{r=0}^{[\frac{t-\tau}{h}]}(-1)_q^{r+1}C_r\Bigg\}\\
&=f(t,y(t))\limsup_{h\to 0+}(2x-h^qf(t,y(t)))+f(t,y(t))\limsup_{h\to 0+}\Bigg\{\sum_{r=0}^{[\frac{t-\tau}{h}]}(-1)_q^{r+1}C_r\Bigg\}\\
&\quad-((y(t))^2-(y_0)^2)\limsup_{h\to 0+}\frac{1}{h^q}\Bigg\{\sum_{r=0}^{[\frac{t-\tau}{h}]}(-1)^{r+1}\binom{q}{r}\Bigg\}\\
&=2y(t)f(t,y(t))-((y(t))^2-(y_0)^2)\;{}_{\tau}^{GL}D_{+}^{q}1\\
&=2y(t)f(t,y(t))-((y(t))^2-(y_0)^2)\frac{(t-\tau)^{-q}}{\Gamma(1-q)},\quad t\in[\tau,s_p)\cap J.
\end{aligned}
\tag{2.102}
$$

$\square$

2.3.2 *Comparison Results for Caputo Fractional Differential Equations with Non-instantaneous Impulses and Lyapunov Functions*

We will obtain some comparison results for NIFrDE (2.48) using the given above definitions for a derivative of Lyapunov-like function.

We will use the followings sets:

$$
\begin{aligned}
\mathcal{K}&=\{a\in C[\mathbb{R}_+,\mathbb{R}_+]: a \text{ is strictly increasing and } a(0)=0\},\\
S(A)&=\{x\in\mathbb{R}^n: ||x||\le A\},\quad A>0.
\end{aligned}
$$

I. Comparison results for FrDE and Lyapunov functions

We will give some comparison results for the IVP for Caputo fractional differential equations of the type

$$
{}_{\tau_0}^{\;c}D^q x(t)=f(t,x(t)),\quad t\in[\tau_0,\tau_0+T]\quad\text{with } x(\tau_0)=\tilde{x}_0 \tag{2.103}
$$

where $\tau_0\ge 0,\ T>0, f\in C([\tau_0,\tau_0+T]\times\Delta,\Delta)$, where $\Delta\subset\mathbb{R}^n,\ 0\in\Delta$. These results will be used to prove comparison results for NIFrDE (2.48).

We will use the three types of derivatives of Lyapunov functions to obtain some comparison results.

Lemma 2.3.1 (Fractional Comparison Principle) *Let* $z, y \in C([\tau_0, \tau_0 + T] \times \Omega, \mathbb{R}^n)$, $\Omega \subset \mathbb{R}^n$, $z(\tau_0) \geq y(\tau_0)$ *and* ${}^{c}_{\tau_0}D^q z(t) \geq {}^{c}_{\tau_0}D^q y(t)$ *for* $t \in [\tau_0, \tau_0 + T]$, *where* $q \in (0, 1)$.

Then $z(t) \geq y(t)$ *for* $t \in [\tau_0, \tau_0 + T]$.

Proof It follows that there exists a nonnegative function $M(t) : [\tau_0, \tau_0 + T] \to \mathbb{R}_+$ such that

$$ {}^{c}_{\tau_0}D^q z(t) = M(t) + {}^{c}_{\tau_0}D^q y(t). \tag{2.104}$$

Taking a Laplace transform of (2.104) we obtain

$$ s^q Z(s) - s^{q-1} z(\tau_0) = M(s) + s^q Y(s) - s^{q-1} y(\tau_0), \tag{2.105}$$

where $M(s) = \mathcal{L}\{m(t)\}$.

There exists a constant $C \geq 0$ such that $z(\tau_0) + C = y(\tau_0)$. Then from (2.105) we get

$$ s^q Z(s) - s^{q-1} z(\tau_0) = M(s) + s^q Y(s) - s^{q-1} x(\tau_0) - s^{q-1} C. \tag{2.106}$$

or

$$ Z(s) = s^{-q} M(s) + Y(s) - s^{-1} C. \tag{2.107}$$

Apply the inverse Laplace transform to (2.107) and obtain

$$ z(t) = {}_{\tau_0}\mathcal{D}^{-q} M(t) + y(t) - C \tag{2.108}$$

where ${}_{\tau_0}\mathcal{D}^{-q} M(t)$ is defined by (2.1).

It follows from $m(t) \geq 0$ and (2.108) the inequality $z(t) \geq y(t)$. □

Remark 2.3.5 *The fractional comparison principle of Lemma 2.3.1 was proved in Lemma 10 [88] if* $\tau_0 = 0$ *and* $x(0) = y(0)$ *and in Lemma 3.1. [109] if* $\tau_0 = 0$ *and* $x(0) \geq y(0)$.

Lemma 2.3.2 *Let* $m \in C([t_0, T], \mathbb{R})$ *and suppose that there exists* $t^* \in (t_0, T]$, *such that* $m(t^*) = 0$ *and* $m(t) < 0$ *for* $t_0 \leq t < t^*$. *Then if the Caputo fractional Dini derivative (2.5) of m exists at* t^*, *then the inequality* ${}^{c}_{t_0}D^q_+ m(t^*) > 0$ *holds.*

Proof From (2.3) (note $m(t^*) = 0$, $r - q > 0$ for $r = 1, 2, \ldots,$ and $0 < q < 1$) we obtain

$$
\begin{aligned}
{}_{t_0}^{GL}D_+^q m(t^*) &= \limsup_{h\to 0+} \frac{1}{h^q} \sum_{r=0}^{[\frac{t^*-t_0}{h}]} (-1)^r \,{}_qC_r\, m(t^*-rh) \\
&= m(t^*) + \limsup_{h\to 0+} \frac{1}{h^q} \sum_{r=1}^{[\frac{t^*-t_0}{h}]} (-1)^r \frac{q(q-1)\dots(q-r+1)}{r!} m(t^*-rh) \\
&= \limsup_{h\to 0+} \frac{1}{h^q} \sum_{r=1}^{[\frac{t^*-t_0}{h}]} \frac{q(1-q)\dots(r-1-q)}{r!} \Big(-m(t^*-rh)\Big).
\end{aligned}
$$

Since all the terms of the above series are positive we obtain ${}_{t_0}^{c}D_+^q m(t^*) \geq 0$. From (2.4) and Remark 2.1.2 we get

$$
{}_{t_0}^{c}D_+^q m(t^*) = {}_{t_0}^{GL}D_+^q m(t^*) - \frac{m(t_0)(t^*-t_0)^{-q}}{\Gamma(1-q)}. \tag{2.109}
$$

Now $m(t_0) < 0$, $t^* > t_0$, $\Gamma(1-q) > 0$ and (2.109) completes the proof. $\square$

Now we present a comparison result applying as a comparison equation the following scalar Caputo fractional differential equation

$$
{}_{\tau_0}^{c}D^q u(t) = g(t, u(t)), \quad t \in [\tau_0, \tau_0 + T]. \tag{2.110}
$$

Note (2.110) with $u(\tau_0) = u_0$ is called the initial value problem (2.110). We will assume that the function $g : [\tau_0, \tau_0 + T] \times \mathbb{R} \to \mathbb{R}$ is such that for any initial data $(\tau_0, u_0) \in [\tau_0, \tau_0 + T] \times \mathbb{R}$ the scalar FrDE (2.110) has a solution $u(t; \tau_0, u_0) \in C^q([\tau_0, \tau_0 + T], \mathbb{R})$. Also, we assume that for any compact subset $I \subset [\tau_0, \tau_0 + T]$ there exists a small enough number $L_1 > 0$ such that the corresponding FrDE ${}_{\tau_0}^{c}D^q u(t) = g(t, u(t)) + \eta$ with $\eta \in (0, L_1]$ has a solution $u(t; \tau_0, u_0, \eta) \in C^q(I \cap [\tau_0, \tau_0 + T], \mathbb{R})$ where $(\tau_0, u_0) \in I \times \mathbb{R}$. Note some existence results for (2.110) are given in [30, 45, 82].

Lemma 2.3.3 (Caputo Fractional Dini Derivative)
Assume the following conditions are satisfied:

1. *The function $x^*(t) = x(t; \tau_0, x_0) \in C^q([\tau_0, \tau_0 + T], \Delta)$, is a solution of the IVP for the FrDE (2.103), where $x_0 \in \Delta$.*
2. *The function $g \in C([\tau_0, \tau_0 + T] \times \mathbb{R}_+, \mathbb{R})$.*
3. *The function $V \in \Lambda^C([\tau_0, \tau_0 + T], \Delta)$ and the inequality*

$$
{}_{(2.103)}^{c}D_+^q V(t, x; \tau_0, x_0) \leq g(t, V(t, x)) \ \ for \ \ (t, x) \in [\tau_0, \tau_0 + T] \times \Delta
$$

holds.

4. *The function $u^*(t) = u(t; \tau_0, u_0) \in C^q([\tau_0, \tau_0 + T], \mathbb{R})$ is the maximal solution of the IVP (2.110).*

Then the inequality $V(\tau_0, x_0) \le u_0$ *implies* $V(t, x^*(t)) \le u^*(t)$ *for* $t \in [\tau_0, \tau_0 + T]$.

Proof Let $\eta > 0$ be an arbitrary number and consider the initial value problem for the scalar FrDE

$$ {}^{c}_{\tau_0}D^q u(t) = g(t, u(t)) + \eta, \quad \text{for } t \in [\tau_0, \tau_0 + T], \quad u(\tau_0) = u_0 + \eta, \tag{2.111}$$

where η is enough small (i.e., $\eta \le L_{[\tau_0, \tau_0+T]}$ as described after (2.110)). The function $u(t, \eta)$ is a solution of the scalar fractional differential equation (2.111) iff it satisfies the Volterra fractional integral equation (Lemma 6.2 [45])

$$ u(t, \eta) = u_0 + \eta + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \Big(g(s, u(s, \eta)) + \eta \Big) ds, \quad t \in [\tau_0, \tau_0 + T]. \tag{2.112}$$

Let the function $m(t) \in C([\tau_0, \tau_0 + T], \mathbb{R}_+)$ be $m(t) = V(t, x^*(t))$. We now prove that

$$ m(t) < u(t, \eta) \quad \text{for} \ \ t \in [\tau_0, \tau_0 + T]. \tag{2.113}$$

Note that the inequality (2.113) holds for $t = \tau_0$ since $m(\tau_0) = V(\tau_0, x_0) \le u_0 < u(\tau_0, \eta)$. Assume that inequality (2.113) is not true. Then there exists a point $t^* \in (\tau_0, \tau_0 + T]$ such that $m(t^*) = u(t^*, \eta), \quad m(t) < u(t, \eta)$ for $t \in [\tau_0, t^*)$. Now Lemma 2.3.2 (applied to $m(t) - u(t, \eta)$) yields ${}^{c}_{\tau_0}D^q_+ (m(t^*) - u(t^*, \eta)) > 0$, i.e.

$$ {}^{c}_{\tau_0}D^q_+ m(t^*) > g(t^*, u(t^*, \eta)) + \eta > g(t^*, m(t^*)). \tag{2.114}$$

From condition 1 of Lemma 2.3.3 the function $x^*(t)$ satisfies the following initial value problem for the system FrDE

$$ {}^{c}_{\tau_0}D^q_+ x(t) = f(t, x(t)), \ \ x(\tau_0) = x_0, \quad t \in [\tau_0, \tau_0 + T]. \tag{2.115}$$

Then for $t \in (\tau_0, \tau_0 + T)$ the equality

$$ \limsup_{h \to 0+} \frac{1}{h^q} \Big[x^*(t) - x_0 - S(x^*(t), h) \Big] = f(t, x^*(t)) $$

holds, where

$$ S(x^*(t), h) = \sum_{r=1}^{[\frac{t-\tau_0}{h}]} (-1)^{r+1} \, {}_qC_r \Big[x^*(t - rh) - x_0 \Big]. $$

Therefore,

$$ S\big(x^*(t), h\big) = x^*(t) - x_0 - h^q f(t, x^*(t)) - \Lambda(h^q) $$

or

$$x^*(t) - h^q f(t, x^*(t)) = S\left(x^*(t), h\right) + x_0 + \Lambda(h^q) \tag{2.116}$$

with $\frac{\Lambda(h^q)}{h^q} \to 0$ as $h \to 0$. Then for any $t \in (\tau_0, \tau_0 + T]$ we obtain

$$\begin{aligned}
&m(t) - m(\tau_0) - \sum_{r=1}^{[\frac{t-\tau_0}{h}]} (-1)^{r+1}\ {}_qC_r\Big[m(t-rh) - m(\tau_0)\Big] \\
&= \Big\{V(t, x^*(t)) - V(\tau_0, x_0) \\
&\quad - \sum_{r=1}^{[\frac{t-\tau_0}{h}]} (-1)^{r+1}\ {}_qC_r\Big[V(t-rh, x^*(t) - h^q f(t, x^*(t)) - V(\tau_0, x_0)\Big]\Big\} \\
&\quad + \sum_{r=1}^{[\frac{t-\tau_0}{h}]} (-1)^{r+1}\ {}_qC_r\Big\{\Big[V(t-rh, S\left(x^*(t), h\right) + x_0 + \Lambda(h^q)) - V(\tau_0, x_0)\Big] \\
&\quad - \Big[V(t-rh, x^*(t-rh)) - V(\tau_0, x_0)\Big]\Big\}.
\end{aligned} \tag{2.117}$$

Since V is locally Lipschitzian in its second argument with a Lipschitz constant $L > 0$ we obtain

$$\begin{aligned}
&\sum_{r=1}^{[\frac{t-\tau_0}{h}]} (-1)^{r+1}\ {}_qC_r\Big\{V(t-rh, S\left(x^*(t), h\right) + x_0 + \Lambda(h^q)) - V(t-rh, x^*(t-rh))\Big\} \\
&\leq L|| \sum_{r=1}^{[\frac{t-\tau_0}{h}]} (-1)^{r+1}\ {}_qC_r \sum_{j=1}^{[\frac{t-\tau_0}{h}]} (-1)^{j+1}\ {}_{-q}C_j\Big(x^*(t-jh) - x_0\Big) \\
&\quad - \sum_{r=1}^{[\frac{t-\tau_0}{h}]} (-1)^{j+1}\ {}_{-q}C_j\Big((x^*(t-rh) - x_0)\Big)|| + L\,\Lambda(h^q) \sum_{r=1}^{[\frac{t-\tau_0}{h}]}\ {}_qC_r \\
&= L||\Big(\sum_{r=0}^{[\frac{t-\tau_0}{h}]} (-1)^{r+1}\ {}_qC_r\Big)\Big(\sum_{j=1}^{[\frac{t-\tau_0}{h}]} (-1)^{j+1}\ {}_{-q}C_j\Big(x^*(t-jh) - x_0\Big)\Big)|| \\
&\quad + L\,\Lambda(h^q) \sum_{r=1}^{[\frac{t-\tau_0}{h}]}\ {}_qC_r.
\end{aligned} \tag{2.118}$$

Using $\lim_{N\to\infty} \sum_{r=0}^{N} (-1)^r\ {}_qC_r = 0$, where N is a natural number, and $lim_{h\to 0+}[\frac{t-t_0}{h}] = \infty$ we obtain

$$\lim_{h\to 0+}\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r\,{}_qC_r = -1 \tag{2.119}$$

Substitute (2.118) in (2.117), divide both sides by h^q, take the limit as $h \to 0^+$, use (2.119) and $\sum_{r=0}^{\infty} {}_qC_r z^r = (1+z)^q$ if $|z| \le 1$, and we obtain for any $t \in (\tau_0, \tau_0+T]$ the inequality

$$\begin{aligned}
{}_{\tau_0}^{c}D_+^q m(t^*) &\le {}_{(2.103)}^{\ c}D_+^q V(t, x^*(t); \tau_0, x_0) + L\lim_{h\to 0+}\frac{\Lambda(h^q)}{h^q}\lim_{h\to 0+}\sum_{r=1}^{[\frac{t-\tau_0}{h}]} {}_qC_r \\
&+ L\lim_{h\to 0+}\sup\left\|\left(\sum_{r=0}^{[\frac{t-\tau_0}{h}]}(-1)^{r+1}\,{}_qC_r\right)\left(\frac{1}{h^q}\sum_{j=1}^{[\frac{t-\tau_0}{h}]}(-1)^{j+1}\,{}_{-q}C_j\Big(x^*(t-jh)-x_0\Big)\right)\right\| \\
&= {}_{(2.103)}^{\ c}D_+^q V(t, x^*(t); \tau_0, \tilde{x}_0) \le g(t, V(t, x^*(t))) = g(t, m(t)).
\end{aligned} \tag{2.120}$$

Now (2.120) with $t = t^*$ contradicts (2.114). Therefore (2.113) holds.

We now show if $\eta_2 < \eta_1$, then

$$u(t, \eta_2) < u(t, \eta_1) \quad \text{for } \ t \in [t_0, T]. \tag{2.121}$$

Note that the inequality (2.121) holds for $t = t_0$. Assume that inequality (2.121) is not true. Then there exists a point t^* such that $u(t^*, \eta_2) = u(t^*, \eta_1)$ and $u(t, \eta_2) < u(t, \eta_1)$ for $t \in [t_0, t^*)$. Now Lemma 2.3.2 (applied to $u(t, \eta_2) - u(t, \eta_1)$) yields ${}_{\tau_0}^{c}D_+^q(u(t^*, \eta_2) - u(t^*, \eta_1))) > 0$. However

$$\begin{aligned}
{}_{\tau_0}^{c}D_+^q(u(t^*, \eta_2) - u(t^*, \eta_1))) &= g(t^*, u(t^*, \eta_2)) + \eta_2 - [g(t^*, u(t^*, \eta_1)) + \eta_1] \\
&= \eta_2 - \eta_1 < 0,
\end{aligned}$$

a contradiction. Thus (2.121) is true.

Recall $0 < \eta \le L_{[\tau_0, \tau_0+T]}$. Now (2.113) and (2.121) guarantee that the family of solutions $\{u(t,\eta)\}, t \in [\tau_0, \tau_0+T]$ of (2.111) is uniformly bounded, i.e., there exists $K > 0$ with $|u(t, \eta)| \le K$ for $(t, \eta) \in [\tau_0, \tau_0+T] \times [0, L_{[\tau_0, \tau_0+T]}]$. Let $M = \sup\{|g(t,x)| : (t,x) \in [\tau_0, \tau_0+T] \times [-K, K]\}$. Take a decreasing sequence of positive numbers $\{\eta_j\}_{j=0}^{\infty}$, $\eta_0 \le L_{[\tau_0, \tau_0+T]}$, such that $\lim_{j\to\infty}\eta_j = 0$ and consider the sequence of functions $u(t; \eta_j)$. Now for $t_1, t_2 \in [\tau_0, \tau_0+T]$, $t_1 < t_2$, we have

$$\begin{aligned}
&|u(t_2, \eta_j) - u(t_1, \eta_j)| \\
&\le \frac{1}{\Gamma(q)}\left|\int_{\tau_0}^{t_1}\Big((t_2-s)^{q-1} - (t_1-s)^{q-1}\Big)\Big(g(s, u(s, \eta_j) + \eta_j\Big)ds\right. \\
&\quad \left.+\int_{t_1}^{t_2}((t_2-s)^{q-1})(g(s, u(s, \eta_j)) + \eta_j)ds\right| \le 2\frac{[M+1]}{q\Gamma(q)}(t_2-t_1)^q.
\end{aligned} \tag{2.122}$$

Thus the family $\{u(t;\eta_j)\}$ is equicontinuous on $[\tau_0, \tau_0 + T]$. The Arzela-Ascoli theorem guarantees that there exists a subsequence, $\{u(t;\eta_{j_k})\}$ and a $w \in C[\tau_0, \tau_0 + T]$ with $u(t;\eta_{j_k}) \to w$ in $C[\tau_0, \tau_0 + T]$ as $k \to \infty$. Take the limit in (2.112) as $k \to \infty$ and we see that $w(t)$ satisfies the initial value problem (2.110) for $t \in [\tau_0, \tau_0 + T]$. Now from (2.113) we have $m(t) \le w(t) \le u^*(t)$ on $[\tau_0, \tau_0 + T]$. □

The result is true in the case of reversed inequalities:

Lemma 2.3.4 (Caputo Fractional Dini Derivative) *Assume the following conditions are satisfied:*

1. *The function* $x^*(t) = x(t;\tau_0, x_0) \in C^q([\tau_0, \tau_0 + T], \Delta)$, *is a solution of the IVP for the FrDE (2.103), where* $x_0 \in \Delta$.
2. *The function* $g \in C([\tau_0, \tau_0 + T] \times \mathbb{R}_+, \mathbb{R})$.
3. *The function* $V \in \Lambda^C([\tau_0, \tau_0 + T], \Delta)$ *and the inequality*

$$ {}_{(2.103)}^{\;\;c}D_+^q V(t, x; \tau_0, x_0) \ge g(t, V(t, x)) \;\; for \;\; (t, x) \in [\tau_0, \tau_0 + T] \times \Delta $$

holds.

4. *The function* $u^*(t) = u(t;\tau_0, u_0) \in C^q([\tau_0, \tau_0 + T], \mathbb{R})$ *is the minimal solution of the IVP (2.110).*

Then the inequality $V(\tau_0, x_0) \ge u_0$ *implies* $V(t, x^*(t)) \ge u^*(t)$ *for* $t \in [\tau_0, \tau_0 + T]$.

Proof The proof of Lemma 2.3.4 is similar to the one of Lemma 2.3.3 with slight changes and we omit it. □

If $g(t, x) \equiv 0$ in Lemma 2.3.3 we obtain the following result:

Corollary 2.3.1 *Assume the following conditions are satisfied:*

1. *The function* $x^*(t) = x(t;t_0, x_0)$, $x^* \in C^q([\tau_0, \tau_0 + T], \Delta)$, *is a solution of the FrDE (2.103) where* $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$.
2. *The function* $V \in \Lambda^C([\tau_0, \tau_0 + T], \Delta)$ *and for any points* $t \in [\tau_0, \tau_0 + T]$ *and* $x \in \Delta$ *the inequality*

$$ {}_{(2.103)}^{\;\;c}D_+^q V(t, x; \tau_0, x_0) V(t, x) \le 0 $$

holds.

Then for $t \in [\tau_0, \tau_0 + T]$ *the inequality* $V(t, x^*(t)) \le V(\tau_0, x_0)$ *holds.*

Proof The proof follows from the fact that the Caputo fractional differential equation ${}_{\tau_0}^{c}D^q x(t) = 0$ has a constant solution. Apply Lemma 2.3.3 with $u_0 = V(t_0, x_0)$. □

In the case of a linear function $g(t, x)$ we obtain the following comparison result:

Corollary 2.3.2 *Assume the following conditions are satisfied:*

1. *The function* $x^*(t) = x(t;\tau_0, x_0) \in C^q([\tau_0, \tau_0 + T], \Delta)$, *is a solution of the IVP for the FrDE (2.103), where* $x_0 \in \Delta$.

2. *The function* $V \in \Lambda^C([\tau_0, \tau_0 + T], \Delta)$ *and the inequality*

$$ {}^{\ c}_{(2.103)}D^q_+ V(t, x; \tau_0, x_0) \le -\alpha V(t, x) \ \ for \ \ (t, x) \in [\tau_0, \tau_0 + T] \times \Delta $$

holds.

Then the inequality $V(\tau_0, x_0) \le u_0$ *implies* $V(t, x^*(t)) \le u_0 E_q(-(t - \tau_0)^q)$ *holds for* $t \in [\tau_0, \tau_0 + T]$.

Proof The proof follows from Lemma 2.3.3 with $g(t, u) = -\alpha u$ and the fact that the equation of the IVP for scalar FrDE (2.110) in this case is $u(t) = u_0 E_q(-(t - \tau_0)^q)$. □

Remark 2.3.6 *Some comparison results for Dini fractional derivative of the Lyapunov functions are proved in [81].*

Example 2.3.2.1 Let $n = 1$, $\tau_0 = 0, f(t, x) = -\frac{x}{t^q \Gamma(1-q)}$ and $V : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ be given by $V(t, x) = x^2$ as in Example 2.3.1.1. From (2.88) we get

$$ {}^{\ c}_{(2.103)}D^q_+ V(t, x; 0, x_0) = 2xf(t, x) + \Big(x^2 - (x_0)^2\Big) \frac{t^{-q}}{\Gamma(1-q)} \le 0, \quad t \ge 0. \tag{2.123} $$

From Corollary 2.3.1 the inequality $|x(t)| \le |x_0|$, $t \ge t_0$, holds for any solution of (2.103). □

The result of Lemma 2.3.3 is also true on the half line. The idea is to fix $T > t_0$ and once again we have (2.112) and (2.113). Take the limit in (2.112) as $k \to \infty$ and we see that $\lim_{k\to\infty} u(t; \eta_{j_k}))$ satisfies the initial value problem (2.110) for $t \in [\tau_0, \tau_0 + T]$. We can do this argument for each $T < \infty$. This yields the following result.

Corollary 2.3.3 *Assume the following conditions are satisfied:*

1. *The function* $x^*(t) = x(t; \tau_0, x_0)$, $x^* \in C^q([\tau_0, \infty), \Delta)$, *is a solution of the FrDE (2.103) where* $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$.
2. *The function* $g \in C([\tau_0, \infty) \times \mathbb{R}, \mathbb{R})$.
3. *The function* $V \in \Lambda^C([\tau_0, \infty), \Delta)$ *and for any points* $t \ge \tau_0$ *and* $x \in \Delta$ *the inequality*

$$ {}^{\ c}_{(2.103)}D^q_+ V(t, x) \le g(t, V(t, x)) $$

holds.

4. *The function* $u^*(t) = u(t; \tau_0, u_0)$, $u^* \in C^q([t_0, \infty), \mathbb{R})$ *is the maximal solution of the initial value problem (2.110).*

Then the inequality $V(\tau_0, x_0) \le u_0$ *implies* $V(t, x^*(t)) \le u^*(t)$ *for* $t \ge \tau_0$.

If the derivative of the Lyapunov function is negative, the following result is true.

Lemma 2.3.5 (Negative Caputo Fractional Dini Derivative) *Let the following conditions be satisfied:*

1. *The function* $x^*(t) = x(t;\tau_0,x_0)$, $x^* \in C^q([\tau_0,\infty),\Delta)$, *is a solution of the FrDE (2.103) where* $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$, $T > 0$ *is a given constant.*
2. *The function* $V \in \Lambda^C([\tau_0,\tau_0+T],\Delta)$ *is such that for any points* $t \in [\tau_0,\tau_0+T]$, $x \in \Delta$ *the inequality*

$$ {}_{(2.103)}^{\ \ c}D_+^q V(t,x;\tau_0,x_0) \le -c(||x||) $$

holds where $c \in \mathcal{K}$.

Then for $t \in [\tau_0,\tau_0+T]$ *the inequality*

$$ V(t,x^*(t)) \le V(\tau_0,x_0) - \frac{1}{\Gamma(q)}\int_{\tau_0}^t (t-s)^{q-1} c(||x^*(s)||)ds \tag{2.124} $$

holds.

Proof Define the function $m(t) \in C([\tau_0,\tau_0+T],\mathbb{R}_+)$ by $m(t) = V(t,x^*(t))$ and the function $p \in C([\tau_0,\tau_0+T],\mathbb{R}_+)$ by $p(t) = c(||x^*(t)||)$. As in the proof of (2.120) we have

$$ {}_{\tau_0}^{c}D_+^q m(t) \le {}_{(2.103)}^{\ \ c}D_+^q V(t,x;\tau_0,x_0)\Big|_{x=x^*(t)} \le -c(||x^*(t)||) = -p(t), \quad t \in [\tau_0,\tau_0+T]. \tag{2.125} $$

Let $\eta > 0$ be arbitrary. Consider the following initial value problem for the scalar FrDE

$$ {}_{\tau_0}^{c}D^q u(t) = -p(t), \quad t \ge \tau_0, \quad u(\tau_0) = m(\tau_0) + \eta. $$

Its solution satisfies the following fractional integral equation

$$ u(t) = m(\tau_0) - \frac{1}{\Gamma(q)}\int_{\tau_0}^t (t-s)^{q-1} p(s)ds + \eta. \tag{2.126} $$

We now prove that

$$ m(t) < u(t), \quad t \in [\tau_0,\tau_0+T]. \tag{2.127} $$

Assume the contrary and let $t^* \in (\tau_0,\tau_0+T]$ be such that

$$ m(t^*) = u(t^*), \quad \text{and} \quad m(t) < u(t) \ \text{ for } t \in [\tau_0,t^*). $$

From Lemma 2.3.2 (applied to $m(t) - u(t)$) we obtain

$$ {}_{\tau_0}^{c}D_+^q m(t^*) > {}_{\tau_0}^{c}D_+^q u(t^*) = {}_{\tau_0}^{c}D^q u(t^*) = p(t^*), \tag{2.128} $$

and this contradicts (2.125). Therefore (2.127) is satisfied. From (2.126) and (2.127) since $\eta > 0$ is arbitrary we obtain (2.124). □

II. Comparison results for NIFrDE and Lyapunov functions

Now we will prove some comparison results for non-instantaneous impulsive Caputo fractional differential equations. Keeping in mind the discussions in Section 2.4.1 about the two approaches to the solutions of NIFrDE (2.48) and the derivatives of Lyapunov functions we will use approach (A2 for NIFrDE) and both the Caputo fractional derivative and the Caputo fractional Dini derivative of Lyapunov functions.

We will use the following condition:

(H2.3.2.1) The function $\phi_k: \ [s_k, t_{k+1}] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is such that the equation $x = \phi_k(t, x, y)$ has a unique solution $x = \psi_k(t, y),\ t \in [s_k, t_{k+1}]$ and $\psi_k \in C([s_k, t_{k+1}] \times \mathbb{R}^n, \mathbb{R}^n)$.

We will use as a comparison equation the following initial value problem for scalar Caputo fractional differential equations with non-instantaneous impulses (NIFrDE)

$$\begin{aligned}
&{}_{t_0}^{c}D^q u = g\,(t, u) \ \text{ for } t \in [t_0, T] \bigcap \Big([0, s_0] \cup_{k=1}^{\infty} (t_k, s_k]\Big) \\
&u(t) = \Psi_k(t, u(s_k - 0)) \quad \text{for } t \in [t_0, T] \cap (s_k, t_{k+1}],\ k = 0, 1, \ldots, \\
&u(t_0) = u_0
\end{aligned} \tag{2.129}$$

where $u, u_0 \in \mathbb{R}$, $t_0 \in [0, s_0) \cup_{k=1}^{\infty} [t_k, s_k)$, $T > t_0$, $g : \Big([0, s_0] \cup_{k=1}^{\infty} (t_k, s_k]\Big) \bigcap [t_0, T] \times \mathbb{R} \to \mathbb{R}$, $\Psi_k : [s_k, t_{k+1}] \times \mathbb{R} \to \mathbb{R}$, $(k \in \{j :\ [t_0, T] \cap (s_j, t_{j+1}) \neq \emptyset\}$.

We also will use the corresponding IVP for the scalar Caputo *fractional differential equations* (FrDE)

$$ {}_{\tau}^{c}D^q u = g(t, u) \quad \text{for } \ t \in [\tau, s_p] \quad \text{with } \ u(\tau) = \tilde{u}_0 \tag{2.130}$$

where $\tilde{u}_0 \in \mathbb{R}$, $\tau \geq 0,\ p = \{k :\ \tau < s_k\}$.

We will use minimal/maximal solutions of the IVP for FrDE (2.130).

Definition 2.3.3 *We say the function $u^*(t) \in C^q([\tau, s_p], \mathbb{R})$ is a minimal/maximal solution of the scalar IVP for FrDE (2.130) if it is a solution of (2.130) and for any other solution $u(t) \in C^q([\tau, s_p], \mathbb{R})$ of (2.130) the inequality $u^*(t) \leq (\geq) u(t),\ t \in [\tau, s_p]$ holds.*

We will use the following conditions:

(H2.3.2.2.) The function $g \in C(J \times \mathbb{R}, \mathbb{R})$, $J \subset [0, s_0] \bigcup \cup_{k=1}^{\infty} [t_k, s_k]$ is such that $g(t, 0) = 0,\ t \in J$ and for any initial point $(\tau, \tilde{u}_0) :\ \tau \in [t_k, s_k) \cap J$, $k = 0, 1, 2, \ldots,$ and $\tilde{u}_0 \in \mathbb{R}$ the IVP for FrDE (2.130) has a maximal solution $\tilde{u}(t; \tau, \tilde{u}_0)$ defined on $[\tau, s_k]$ (in the case of $k = 0$ the interval $[t_k, s_k)$ is replaced by $[0, s_0)$).

(H2.3.2.3.) The function $g \in C(J \times \mathbb{R}, \mathbb{R})$, $J \subset [0, s_0] \bigcup \cup_{k=1}^{\infty} [t_k, s_k]$ is such that $g(t, 0) = 0,\ t \in J$ and for any initial point $(\tau, \tilde{u}_0) :\ \tau \in [t_k, s_k) \cap J$,

$k = 0, 1, 2, \ldots,$ and $\tilde{u}_0 \in \mathbb{R}$ the IVP for FrDE (2.130) has a minimal solution $\tilde{u}(t; \tau, \tilde{u}_0)$ defined on $[\tau, s_k]$ (in the case of $k = 0$ the interval $[t_k, s_k)$ is replaced by $[0, s_0)$).

(H2.3.2.4.) The function $\Psi_k \in C([s_k, t_{k+1}] \times \mathbb{R}, \mathbb{R})$, $\Psi_k(t, 0) = 0$ for $t \in [s_k, t_{k+1}]$ and $\Psi_k(t, u) \leq \Psi_k(t, v)$ for $u \leq v,\ t \in [s_k, t_{k+1}]$.

Definition 2.3.4 *Let m be a natural number and $T \in (t_m, s_m]$ be a given number. The function $u^*(t)$ will be called a maximal solution (minimal solution) of the IVP for NIFrDE (2.129) on the interval $[t_0, T]$ if*

- *it is a solution of the IVP for NIDE (2.129) on $[t_0, T]$ (according to approach (A2 for NIFrDE));*
- *for any $k = 0, 1, 2, \ldots, m$ and any solution $u(t) \in C^1([t_k, s_k], \mathbb{R})$ of IVP for FrDE (2.130) with $\tau = t_k$, $\tilde{u}_0 = u^*(t_k)$ the inequalities*

$$u^*(t) \geq (\leq) u(t) \ \ \textit{for } t \in [t_k, s_k] \cap [t_0, T]$$

and for any $k = 0, 1, 2, \ldots, m-1$

$$\Psi_k(t, u^*(s_k - 0)) \geq (\leq) \Psi_k(t, u(s_k)) \ \ \textit{for } t \in (s_k, t_{k+1}]$$

hold.

Lemma 2.3.6 (Existence of a Maximal Solution of NIFrDE) *Let $t_0 \in [0, s_0)$, $T = s_m]$ and:*

1. *Condition (H2.3.2.2.) be satisfied for $J = \cup_{k=0}^{m} (t_k, s_k]$.*
2. *Condition (H2.3.2.4) be satisfied for all $k = 0, 1, 2, \ldots, m-1$.*

Then there exists a maximal solution of IVP for NIFrDE (2.129) on the interval $[t_0, T]$.

Proof We will use induction to prove the claim.

Let $t \in [t_0, s_0]$. According to condition (H2.3.2.2) there exists a maximal solution $u_0^*(t)$ of IVP for FrDE (2.130) with $\tau = t_0$ and $\tilde{u}_0 = u_0$.

Let $t \in (s_0, t_1]$. According to condition (H2.3.2.4) for the function $\Psi_0(t, u)$ the inequality $\Psi_0(t, u_0^*(s_0)) \geq \Psi_0(t, u(s_0))$ for $t \in (s_0, t_1]$ holds where $u(t)$ is any solution of IVP for FrDE (2.130) with $\tau = t_0$, $\tilde{u}_0 = u_0$ which exists on $[t_0, s_0]$.

Let $t \in (t_1, s_1]$. According to condition (H2.3.2.2) there exists a maximal solution $u_1^*(t)$ of IVP for FrDE (2.130) with $\tau = t_1$ and $\tilde{u}_0 = \Psi_0(t_1, u_0^*(s_0))$.

Let $t \in (s_1, t_2]$. According to condition (H2.3.2.4) for Ψ_1 the inequality $\Psi_1(t, u_1^*(s_1)) \geq \Psi_1(t, u(s_1))$ for $t \in (s_1, t_2]$ holds where $u(t)$ is any solution of IVP for FrDE (2.130) with $\tau = t_1$, $\tilde{u}_0 = \Psi_0(t_1, u_0^*(s_0)) = u_1^*(t_1)$ which exists on $[t_1, s_1]$.

Following the same idea we construct the function

$$u^*(t; t_0, u_0) = \begin{cases} u_k^*(t) & \text{for } t \in (t_k, s_k],\ k = 0, 1, 2, \ldots, m \\ \Psi_k(t, u_k^*(s_k - 0)) & \text{for } t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \ldots, m-1, \end{cases}$$

where $u_k^*(t)$ is the maximal solution of the IVP for FrDE (2.130) on $[t_k, s_k]$ with $\tau = t_k$ and $\tilde{u}_0 = \Psi_{k-1}(t_k, u_{k-1}^*(s_{k-1}))$ (in the case $k = 0$ it is denoted $\Psi_{-1}(t_0, u_{-1}^*(s_{-1})) = u_0$).

According to Definition 2.3.4 the function $u^*(t; t_0, u_0)$ is a maximal solution of IVP for NIFrDE (2.129).

□

Lemma 2.3.7 (Existence of a Minimal Solution of NIFrDE) *Let:*

1. *Condition (H2.3.2.3) be satisfied for* $J = \cup_{k=0}^m (t_k, s_k]$.
2. *Condition (H2.3.2.4) be satisfied for all* $k = 0, 1, 2, \ldots, m-1$.

Then there exists a minimal solution of IVP for NIFrDE (2.129) on the interval $[t_0, s_m]$.

The proof of Lemma 2.3.7 is similar to the one of Lemma 2.3.6 and we omit it.

We will prove a comparison result for the IVP for NIFrDE (2.48) on the interval $[t_0, T]$. Without loss of generality we can assume $t_0 \in [0, s_0)$ and $T = s_m$.

Lemma 2.3.8 (Lower Comparison Result for NIFrDE by Caputo Fractional Dini Derivative) *Let:*

1. *The function* $x^*(t) = x(t; t_0, x_0) \in NPC^q([t_0, T], \Delta)$ *is a solution of the NIFrDE (2.48) where* $\Delta \subset \mathbb{R}^n$, $x_0 \in \Delta$.
2. *For any* $k = 0, 1, 2, \ldots, m-1$ *the condition (H2.3.2.4) is satisfied.*
3. *The condition (H2.3.2.2) is satisfied for* $J = \cup_{k=0}^m (t_k, s_k]$.
4. *The function* $u^*(t) = u(t; t_0, x_0) \in PC^q([t_0, T], \mathbb{R})$ *is the maximal solution of the IVP for scalar NIFrDE (2.129).*
5. *The function* $V \in \Lambda([t_0, T], \Delta)$ *and*

   (i) for any $k = 0, 1, 2, \ldots, m$ *the inequality*

$$ {}_{(2.48)}^{\ c}D_+^q V(t, x; t_k, y_0) \le G(t, V(t, x)) \quad \textit{for} \ \ t \in (t_k, s_k), \quad x, y_0 \in \Delta$$

   holds;

   (ii) for any $k = 0, 1, 2, \ldots, m-1$ *the inequalities*

$$V(t, \psi_k(t, x)) \le \Psi_k(t, V(s_k - 0, x)) \ \ \textit{for}\ t \in (s_k, t_{k+1}], \quad x \in \Delta$$

   hold.

Then the inequality $V(t_0, x_0) \le u_0$ *implies* $V(t, x^*(t)) \le u^*(t)$ *on* $[t_0, T]$.

Proof Note the existence of the maximal solution $u^*(t)$ follows from conditions (H2.3.2.2), (H2.3.2.4) and Lemma 2.3.6.

We use induction to prove our result.

Let $t \in [t_0, s_0]$. The function $x^*(t) \in C^q([t_0, s_0], \mathbb{R}^n)$, satisfies the FrDE (2.103) with $\tau_0 = t_0$ and from Lemma 2.3.3 (with $\tau_0 = t_0,\ T = s_0 - t_0$) the inequality

$$V(t, x^*(t)) \le u^*(t), \quad t \in [t_0, s_0] \tag{2.131}$$

holds.

Let $t \in (s_0, t_1]$. From conditions 3, 5(ii) and inequality (2.131) for $t = s_0$ we get

$$\begin{aligned} V(t, x^*(t)) &= V(t, \psi_0(t, x^*(s_0 - 0))) \le \Psi(t, V(s_0 - 0, x^*(s_0 - 0))) \\ &\le \Psi(t, u^*(s_0 - 0)) = u^*(t), \quad t \in (s_0, t_1] \end{aligned} \tag{2.132}$$

Let $t \in (t_1, s_1]$. Consider the function $\bar{x}_1(t) = x^*(t)$ for $t \in (t_1, s_1]$ and $\bar{x}_1(t_1) = x^*(t_1) = \psi_0(t_1, x^*(s_0 - 0))$. The function $\bar{x}_1(t) \in C^q([t_1, s_1], \mathbb{R}^n)$ and satisfies IVP for FrDE (2.103) with $\tau_0 = t_1$, $x_0 = x^*(t_1)$, and $T = s_1 - t_1$. Using condition 5(i) for $k = 1$, Lemma 2.3.3 for the function $\bar{x}_1(t)$ with $\tau_0 = t_1,\ T = s_1 - t_1$ and (2.132) for $t = t_1$ we obtain

$$V(t, x^*(t)) = V(t, \bar{x}_1(t)) \le u^*(t), \quad t \in (t_1, s_1]. \tag{2.133}$$

Continue this process and an induction argument proves the claim of Lemma 2.3.8 is true for $t \in [t_0, T]$. □

The claim of Lemma 2.3.8 is true in the case of reversed inequalities.

Lemma 2.3.9 (Upper Comparison Result for NIFrDE by Caputo Fractional Dini Derivative) *Let:*

1. *The function* $x^*(t) = x(t; t_0, x_0) \in NPC^q([t_0, T], \Delta)$ *is a solution of the NIFrDE (2.48) where* $\Delta \subset \mathbb{R}^n$, $x_0 \in \Delta$.
2. *For any* $k = 0, 1, 2, \dots, m-1$ *the condition (H2.3.2.4) is satisfied.*
3. *The condition (H2.3.2.3) is satisfied for* $J = \cup_{k=0}^{m} (t_k, s_k]$.
4. *The function* $u^*(t) = u(t; t_0, x_0) \in PC^q([t_0, T], \mathbb{R})$*is the minimal solution of the IVP for scalar NIFrDE (2.129).*
5. *The function* $V \in \Lambda([t_0, T], \Delta)$ *and*

   (i) for any $k = 0, 1, 2, \dots, m$ *the inequality*

$${}^{c}_{(2.48)}D^q_+ V(t, x; t_k, y_0) \ge g(t, V(t, x)) \quad \textit{for} \ \ t \in (t_k, s_k), \quad x, y_0 \in \Delta$$

   holds;

(ii) *for any* $k = 0, 1, 2, \ldots, m-1$ *the inequalities*

$$V(t, \psi_k(t,x)) \geq \Psi_k(t, V(s_k - 0, x)) \ \ for\ t \in (s_k, t_{k+1}], \ \ x \in \Delta$$

hold.

Then the inequality $V(t_0, x_0) \geq u_0$ *implies* $V(t, x^*(t)) \geq u^*(t)$ *on* $[t_0, T]$.

Proof Note the existence of the minimal solution $u^*(t)$ follows from conditions (H2.3.2.3), (H2.3.2.4) and Lemma 2.3.7.

The proof of Lemma 2.3.9 is similar to the one of Lemma 2.3.8 with the application of Lemma 2.3.4 instead of Lemma 2.3.3 and we omit it. □

Remark 2.3.7 *The result of Lemma 2.3.8 could be extended to the interval* $[t_0, \infty)$ *if conditions are satisfied on this interval and Corollary 2.3.3 is applied instead of Lemma 2.3.3.*

Remark 2.3.8 *The results of Lemma 2.3.8 will be similar with slight changes in condition 5(ii) if the initial time* t_0 *is in an interval of non-instantaneous impulses, i.e.,* $t_0 \in \cup_{k=1}^{\infty}(s_k, t_{k+1}]$.

Lemma 2.3.10 (Caputo Fractional Dini Derivative) *Let:*

1. *The condition (H2.3.2.1) is satisfied for* $k = 0, 1, 2, \ldots, m-1$.
2. *The function* $x^*(t) = x(t; t_0, x_0) \in NPC^q([t_0, T], \Delta)$ *is a solution of the NIFrDE (2.48) where* $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$, $x_0 \in \Delta$.
3. *The function* $V \in \Lambda([t_0, T], \Delta)$ *and*

   (i) *for any* $k = 0, 1, 2, \ldots, m$ *and* $y_0 \in \Delta$ *the inequality*

$$ {}^{c}_{(2.48)}D^{q}_{+} V(t, x; t_k, y_0) \leq 0 \ \ for\ \ t \in (t_k, s_k), \ \ x \in \Delta \qquad (2.134)$$

   holds;

   (ii) *for any* $k = 0, 1, 2, \ldots, m-1$ *the inequalities*

$$V(t, \psi_k(t,x)) \leq V(s_k - 0, x) \ \ for\ t \in (s_k, t_{k+1}], \ \ x \in \Delta$$

   hold.

Then the inequality $V(t, x^*(t)) \leq V(t_0, x_0)$ *holds on* $[t_0, T]$.

Proof The proof of Lemma 2.3.10 follows from Lemma 2.3.8 with $G(t,u) \equiv 0$ and $\Psi(u) = u$. □

Lemma 2.3.11 (Comparison Result for NIFrDE, Negative Generalized Caputo Fractional Dini Derivative) *Assume the following conditions are satisfied:*

1. *The condition (H2.3.2.1) is satisfied.*
2. *The function* $x^*(t) = x(t; t_0, x_0) \in NPC^q([t_0, T], \Delta)$ *is a solution of the NIFrDE (2.48) where* $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$, $x_0 \in \Delta$.

3. *The function* $V \in \Lambda([t_0, T], \Delta)$ *and*

   (i) *for any* $k = 0, 1, 2, \ldots, m$ *and* $y_0 \in \Delta$ *the inequality*

$$ {}^{c}_{(2.48)}D^q_+ V(t, x, t_k, y_0) \le -c(||x||), \quad \text{for } t \in (t_k, s_k),\ x \in \Delta $$

   holds where $c \in \mathcal{K}$;

   (ii) *for any* $k = 0, 1, 2, \ldots, m-1$ *the inequalities*

$$ V(t, \psi_k(t, x)) \le V(s_k - 0, x) \ \text{ for } t \in (s_k, t_{k+1}], \quad x \in \Delta $$

   hold.

   Then for $t \in [t_0, T]$ *the inequality*

$$ V(t, x^*(t)) \le \begin{cases} V(t_0, x_0) - \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} c(||x^*(s)||) ds, \\ \qquad\qquad t \in [t_0, s_0] \\ V(t_0, x_0) - \frac{1}{\Gamma(q)} \Big(\sum_{i=0}^{k-1} \int_{t_i}^{s_i} (s_i - s)^{q-1} c(||x^*(s)||) ds \\ \qquad - \int_{t_k}^{t} (t-s)^{q-1} c(||x^*(s)||) ds \Big), \\ \qquad\qquad t \in (t_k, s_k],\ k = 1, 2, \ldots, m \\ V(t_0, x_0) - \frac{1}{\Gamma(q)} \sum_{i=0}^{k} \int_{t_i}^{s_i} (s_i - s)^{q-1} c(||x^*(s)||) ds, \\ \qquad\qquad t \in (s_k, t_{k+1}],\ k = 0, 1, \ldots, m-1 \end{cases} $$

holds.

Proof Let $t \in [t_0, s_0]$. The function $x^*(t) \in C^q([t_0, s_0], \Delta)$ and satisfies the IVP for FrDE (2.103) for $\tau_0 = t_0, T = s_0 - t_0$. From Lemma 2.3.5 the inequality

$$ V(t, x^*(t)) \le V(t_0, x_0) - \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} c(||x^*(s)||) ds, \quad t \in [t_0, s_0] \tag{2.135} $$

holds.

Let $t \in (s_0, t_1]$. From condition 3(ii) and (2.135) for $t = s_0$ we get

$$ \begin{aligned} &V(t, x^*(t)) = V(t, \psi_0(t, x^*(s_0 - 0)))) \le V(s_0 - 0, x^*(s_0 - 0)) \\ &= V(s_0, x^*(s_0 - 0)) \le V(t_0, x_0) - \frac{1}{\Gamma(q)} \int_{t_0}^{s_0} (s_0 - s)^{q-1} c(||x^*(s)||) ds, \quad t \in (s_0, t_1]. \end{aligned} \tag{2.136} $$

Let $t \in (t_1, s_1]$. Consider the function $\bar{x}_1(t) = x^*(t)$ for $t \in (t_1, s_1]$ and $\bar{x}_1(t_1) = x^*(t_1) = \psi_0(t_1, x^*(s_0 - 0))$. The function $\bar{x}_1(t) \in C^q([t_1, s_1], \mathbb{R}^n)$ and satisfies IVP for FrDE (2.103) with $\tau_0 = t_1$, $x_0 = x^*(t_1)$ and $T = s_1 - t_1$. Using condition 3(i), Lemma 2.3.5 for the function $\bar{x}_1(t)$, and (2.136) for $t = t_1$ we obtain

$$
\begin{aligned}
V(t, x^*(t)) &= V(t, \bar{x}_1(t)) \\
&\le V(t_1 + 0, \bar{x}_1(t_1)) - \frac{1}{\Gamma(q)} \int_{t_1}^{t} (t-s)^{q-1} c(||x^*(s)||) ds \\
&= V(t_1, x^*(t_1)) - \frac{1}{\Gamma(q)} \int_{t_1}^{t} (t-s)^{q-1} c(||x^*(s)||) ds \\
&\le V(t_0, x_0) - \frac{1}{\Gamma(q)} \int_{t_0}^{s_0} (s_0-s)^{q-1} c(||x^*(s)||) ds \\
&\quad - \frac{1}{\Gamma(q)} \int_{t_1}^{t} (t-s)^{q-1} c(||x^*(s)||) ds \quad t \in (t_1, s_1].
\end{aligned}
\tag{2.137}
$$

Let $t \in (s_1, t_2]$. From condition 3(ii) and (2.136) for $t = s_1$ we obtain

$$
\begin{aligned}
&V(t, x^*(t)) = V(t, \psi_1(t, x^*(s_1 - 0))) \le V(s_1 - 0, x^*(s_1 - 0)) = V(s_1, x^*(s_1)) \\
&\le V(t_0, x_0) - \frac{1}{\Gamma(q)} \int_{t_0}^{s_0} (s_0-s)^{q-1} c(||x^*(s)||) ds - \frac{1}{\Gamma(q)} \int_{t_1}^{s_1} (s_1-s)^{q-1} c(||x^*(s)||) ds.
\end{aligned}
$$

Continue this process and an induction argument proves the claim is true for $t \in [t_0, T]$. □

2.3.3 *Mittag–Leffler Stability for NIFrDE*

Consider the IVP for the NIFrDE (2.48). In this section we will use approach (A2 for NIFrDE) to the system of NIFrDE (2.48). We will extend the definition of Mittag-Leffler stability to NIFrDE (2.48). Note the Mittag-Leffler stability is defined and studied for fractional differential equations via Lyapunov functions in [88] and [89].

In Sections 2.3.3–2.3.6 we assume both sequences $\{t_k\}_{k=1}^{\infty},\ \{s_k\}_{k=0}^{\infty} :\ 0 \le s_k < t_{k+1} \le s_{k+1}$, $\lim_{k\to\infty} t_k = \infty$ are given.

Definition 2.3.5 *The zero solution of NIFrDE (2.48) is called* Mittag-Leffler stable with respect to non-instantaneous impulses *if there exist a constant* $\beta \in (0,1)$ *and positive constants* a, b, M, μ *such that for any initial time* $t_0 \in [0, s_0] \bigcup \cup_{k=1}^{\infty} [t_k, s_k)$ *the inequalities*

$$
x(t) \le \begin{cases} M||x_0||^b \Big(\big(\prod_{i=0}^{k-1} E_\beta(-\mu(s_i - t_i)^\beta) \big) E_\beta(-\mu(t - t_k)^\beta) \Big)^a, \\ \qquad t \in [t_k, s_k], k = 0, 1, \ldots \\ M||x_0||^b \Big(\big(\prod_{i=0}^{k-1} E_\beta(-\mu(s_i - t_i)^\beta) \big) E_\beta(-\mu(s_k - t_k)^\beta) \Big)^a, \\ \qquad t \in (s_k, t_{k+1}], k = 0, 1, \ldots. \end{cases}
$$

hold, where $x(t) = x(t; t_0, x_0)$ *is a solution of (2.48) and* $E_\beta(z)$ *is the Mittag-Leffler function with one parameter* β.

Remark 2.3.9 *Note the parameter β of the Mittag-Leffler function could be different than the fractional order q of the given NIFrDE (2.48).*

Remark 2.3.10 *If the zero solution of NIFrDE (2.48) is Mittag-Leffler stable with respect to non-instantaneous impulses, then because of the inequality $E_\beta(-\mu(\xi_2 - \xi_1)^\beta)\big)E_\beta(-\mu(\xi-\xi_2)^\beta) \le E_\beta(-\mu(\xi-\xi_1)^\beta)\big)$, $\xi_1 < \xi_2 < \xi$ the inequality $||x(t)|| \le M||x_0||^b\Big(E_\beta(-\mu(t-t_0)^\beta)\Big)^b$, $t \ge t_0$ holds.*

Remark 2.3.11 *As it is mentioned in [89] for fractional differential equations the Mittag-Leffler stability implies asymptotic stability. The same is true with NIFrDE.* We will obtain some sufficient conditions for Mittag-Leffler stable with respect to non-instantaneous impulses applying different types of the derivatives of Lyapunov functions.

If we use the direct Lyapunov method to study stability properties of the solutions and the approach (A1 for NIFrDE) to the system of NIFrDE (2.48), then we need to apply the fractional derivative of the Lyapunov function $V(t,x)$, i.e., ${}_0^cD^\beta V(t,x(t))$ for $t \ge 0$ with the solution $x(t) \in C^q(\cup_{k=0}^\infty[t_k,s_k],\mathbb{R}^n)$ of (2.48). This Caputo fractional derivative might not exist. In connection with this we will use approach (A2 for NIFrDE) to the system of NIFrDE (2.48).

2.3.3.1 Caputo Fractional Derivatives of Lyapunov Functions

We will use the following result for the IVP for FrDE (2.78) which is similar to Theorem 5 in [88]:

Lemma 2.3.12 *Let $f(t,0) = 0$ for $t \ge 0$ and let $V(t,x) : \mathbb{R}_+ \times D \to \mathbb{R}_+$, $D \subset \mathbb{R}^n$, $0 \in D$, be a continuously differentiable function and locally Lipschitz w.r.t. x, $V(t,0) = 0$, $t \in \mathbb{R}_+$ and*

(i)

$$\alpha_1||x||^a \le V(t,x) \le \alpha_2||x||^{ab} \quad \text{for } t \ge \tau,\ x \in D, \tag{2.138}$$

(ii)

$$ {}_\tau^cD^\beta V(t,x(t)) \le -\alpha_3||x(t)||^{ab} \quad \text{for } t \in [\tau, s_p] \tag{2.139}$$

where $\tau \in [t_p,s_p)$, $p \ge 0$ is an integer, $\beta \in (0,1)$, $\alpha_1,\alpha_2,\alpha_3,a,b$ are arbitrary constants, and $x(t) = x(t;\tau,\tilde{x}_0) \in C^q([\tau,s_p],D)$ is a solution of FrDE (2.78).

Then

$$V(t,x(t)) \le V(\tau,x(\tau))E_\beta(-\frac{\alpha_3}{\alpha_2}(t-\tau)^\beta),\ t \in [\tau,s_p] \tag{2.140}$$

and

$$||x(t;\tau,\tilde{x_0})|| \leq ||\tilde{x_0}||^b \sqrt[a]{\frac{\alpha_2}{\alpha_1} E_\beta(-\frac{\alpha_3}{\alpha_2}(t-\tau)^\beta)},\ t \geq \tau_0 \tag{2.141}$$

holds.

Proof The proof is similar to the proof of Theorem 5 in [88] with a slight modification.

It follows from (2.138) and (2.139) that ${}_{\tau}^{c}D^\beta V(t,x(t)) \leq -\frac{\alpha_3}{\alpha_2} V(t,x(t)),\ t \in [\tau, s_p]$. There exists a function $M(t) \in C([\tau, s_p], [0,\infty))$ such that

$${}_{\tau}^{c}D^\beta V(t,x(t)) + M(t) = -\frac{\alpha_3}{\alpha_2} V(t,x(t)),\quad t \in [\tau, s_p]. \tag{2.142}$$

Denote $w(t) = V(t,x(t)), t \in [\tau, s_p]$, and taking the Laplace transform of (2.142) we obtain

$$s^\beta W(s) - W(\tau)s^{\beta-1} + M(s) = -\frac{\alpha_3}{\alpha_2} W(s) \tag{2.143}$$

with nonnegative constant $W(\tau) = V(\tau,\tilde{x}_0)$ and $W(s) = \mathcal{L}\{w(t)\} = \mathcal{L}\{V(t,x(t))\}$. It follows that $W(s) = \frac{W(\tau)s^{\beta-1} - M(s)}{s^\beta + \frac{\alpha_3}{\alpha_2}}$. If $\tilde{x}_0 = 0$, then $W(\tau) = 0$, the solution to FrDE (2.78) is $x(t) \equiv 0$. If $\tilde{x}_0 \neq 0$, then $W(\tau) > 0$. Because $V(t,x(t))$ is locally Lipschitz with respect to x it follows from the fractional inequalities and an existence theorem [101] and the inverse Laplace transform that the unique solution of (2.142) is

$$w(t) = w(\tau)E_\beta(-\frac{\alpha_3}{\alpha_2}(t-\tau)^\beta) - M(t) * \Big[(t-\tau)^{\beta-1} E_{\beta,\beta}(-\frac{\alpha_3}{\alpha_2}(t-\tau)^\beta)\Big].$$

Since $(t-\tau)^{\beta-1} \geq 0$ and $E_{\beta,\beta}(-\frac{\alpha_3}{\alpha_2}(t-\tau)^\beta) \geq 0$ it follows that

$$w(t) \leq w(\tau)E_\beta(-\frac{\alpha_3}{\alpha_2}(t-\tau)^\beta) = V(\tau,x(\tau))E_\beta(-\frac{\alpha_3}{\alpha_2}(t-\tau)^\beta)$$

or

$$V(t,x(t)) \leq \alpha_2 ||x(\tau_0)||^{ab} E_\beta(-\frac{\alpha_3}{\alpha_2}(t-\tau)^\beta). \tag{2.144}$$

Inequality (2.144), $x(\tau_0) = \tilde{x}_0$ and condition (i) prove the claim. □

Remark 2.3.12 *The fractional order β of the Caputo fractional derivative in condition (ii) could be different than the fractional order q of Caputo fractional derivative in the FrDE (2.78).*

We say condition (H2.3.3) is satisfied if:

(H2.3.3.1) The function $f \in C([0,s_0] \cup_{k=1}^{\infty} [t_k,s_k] \times \mathbb{R}^n, \mathbb{R}^n)$, $f(t,0) \equiv 0$ for $t \in \cup_{k=0}^{\infty}[t_k,s_k]$ is such that for any initial point $(\tilde{t}_0,\tilde{x}_0) \in [0,s_0) \cup_{k=1}^{\infty} [t_k,s_k) \times \mathbb{R}^n$ the IVP for the system of FrDE (2.78) with $\tau_0 = \tilde{t}_0$ has a solution $x(t;\tilde{t}_0,\tilde{x}_0) \in C^q([\tilde{t}_0,s_p],\mathbb{R}^n)$ where $p = \min\{k : \tilde{t}_0 < s_k\}$.

(H2.3.3.2) The functions $\phi_k : [s_k,t_{k+1}] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are such that for any $k = 0,1,2,\ldots$ the equation $x = \phi_k(t,x,y)$ has a unique solution $x = \psi_k(t,y)$, $t \in [s_k,t_{k+1}]$.

(H2.3.3.3) The functions $\psi_k \in C([s_k,t_{k+1}] \times \mathbb{R}^n, \mathbb{R}^n)$, $\psi_k(t,0) \equiv 0$ for $t \in [s_k,t_{k+1}]$, $k = 0,1,2,\ldots$.

Remark 2.3.13 *Condition (H2.3.3) guarantees the existence of a solution $x(t;t_0,x_0)$ of NIFrDE (2.48) from $C^q(\cup_{k=0}^{\infty}[t_k,s_k],\mathbb{R}^n)$ for any initial data (t_0,x_0). If conditions (H2.3.3.1) and (H2.3.3.3) are satisfied, then NIFrDE (2.48) has a zero solution.*

Theorem 2.3.1 *Let the following conditions be satisfied:*

1. *Condition (H2.3.3) is satisfied.*
2. *The function $V(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is a continuously differentiable function on $[0,s_0] \cup_{k=1}^{\infty} [t_k,s_k] \times \mathbb{R}^n$ and locally Lipschitz w.r.t. $x \in \mathbb{R}^n$, $V(t,0) = 0$ for $t \geq 0$ and such that*
   - *(i) $\alpha_1||x||^a \leq V(t,x) \leq \alpha_2||x||^{ab}$ for $t \geq 0$, $x \in \mathbb{R}^n$, where α_1,α_2,a,b are positive numbers, $\alpha_2 \leq 1$;*
   - *(ii) for any $\tau \in [0,s_0) \cup_{k=1}^{\infty} [t_k,s_k)$ and any solution $x(t) \in C^q([\tau,s_p],\mathbb{R}^n)$ of FrDE (2.78) the inequality*
   $$ {}_{\tau}^{c}D^{\beta} V(t,x(t)) \leq -\alpha_3||x(t)||^{ab} \quad \textit{for} \quad t \in (\tau,s_p] $$
   holds, where $p = \min\{k : \tau < s_k\}$, $\beta \in (0,1)$, α_3 is a positive constant;
   - *(iii) for any $k = 0,1,2,\ldots$ the inequality*
   $$ V(t,\psi_k(t,x)) \leq \alpha_4||x||^a \quad \textit{for} \quad t \in (s_k,t_{k+1}],\ x \in \mathbb{R}^n $$
   holds where α_4 is a positive constant such that $\alpha_4 \leq \alpha_1$.

Then the zero solution of NIFrDE (2.48) is Mittag-Leffler stable with respect to non-instantaneous impulses.

Proof Let $t_0 \in [0,s_0) \cup_{k=1}^{\infty} [t_k,s_k)$ be an arbitrary initial time. Without loss of generality we can assume $t_0 \in [0,s_0)$. Consider the solution $x(t;t_0,x_0)$ of NIFrDE (2.48) with arbitrary given $x_0 \in \mathbb{R}^n$. We will prove the claim by induction.

Let $t \in [t_0,s_0]$. According to Lemma 2.3.12 with $\tau = t_0$, and $\tilde{x}_0 = x_0$ we have

$$ ||x(t;t_0,x_0)|| \leq ||x_0||^b \sqrt[a]{\frac{\alpha_2}{\alpha_1}E_\beta(-\frac{\alpha_3}{\alpha_2}(t-t_0)^\beta)} \leq ||x_0||^b \sqrt[a]{\frac{\alpha_2}{\alpha_1}E_\beta(-\alpha_3(t-t_0)^\beta)}. \tag{2.145} $$

Let $t \in (s_0, t_1]$.From inequality (2.145) for $t = s_0 - 0$, conditions (i) and (iii) we have

$$\begin{aligned}\alpha_1||x(t;t_0,x_0)||^a &\leq V(t,x(t;t_0,x_0)) = V(t,\psi_0(t,x(s_0-0;t_0,x_0)) \\ &\leq \alpha_4||x(s_0-0;t_0,x_0)||^a \leq \alpha_4||x_0||^{ab}\frac{\alpha_2}{\alpha_1}E_\beta(-\frac{\alpha_3}{\alpha_2}(t-t_0)^\beta) \\ &\leq ||x_0||^{ab}\alpha_2 E_\beta(-\alpha_3(t-t_0)^\beta)\end{aligned} \tag{2.146}$$

or

$$||x(t;t_0,x_0)|| \leq ||x_0||^b \sqrt[a]{\frac{\alpha_2}{\alpha_1}E_\beta(-\alpha_3(s_0-t_0)^\beta)}, \quad t \in (s_0,t_1]. \tag{2.147}$$

Let $t \in [t_1, s_1]$. Since we use approach (A2 for NIFrDE) the function $X_1(t) = x(t;t_0,x_0)$, $t \in [t_1,s_1]$ is a solution of the FrDE (2.78) with $\tau = t_1$ and $\tilde{x}_0 = x(t_1;t_0,x_0)$. From inequality (2.147) for $t = t_1$ we get $||\tilde{x}_0|| \leq ||x_0||^b\sqrt[a]{\frac{\alpha_2}{\alpha_1}E_\beta(-\alpha_3(s_0-t_0)^\beta)}$. From conditions (i) and (ii) it follows that ${}_{t_1}^{c}D^\beta V(t,X_1(t)) \leq -\alpha_3||X_1(t)||^{ab} \leq -\frac{\alpha_3}{\alpha_2}V(t,X_1(t))$ for $t \in [t_1,s_1]$. From Lemma 2.3.12 and inequality (2.140) with $\tau = t_1$, $\tilde{x}_0 = X_1(t_1)$, $\alpha_3 = \frac{\alpha_3}{\alpha_2}$ and inequality $V(t_1,X_1(t_1) = V(t_1,x(t;t_0,x_0)) = V(t_1,\psi_0(t_1,x(s_0-0;t_0,x_0)) \leq \alpha_4||x(s_0-0;t_0,x_0)||^a \leq \alpha_4||x_0||^{ab}\frac{\alpha_2}{\alpha_1}E_\beta(-\alpha_3(s_0-t_0)^\beta)$ we have

$$\begin{aligned}V(t,X_1(t)) &\leq V(t_1,X_1(t_1))E_\beta(-\frac{\alpha_3}{(\alpha_2)^2}(t-t_1)^\beta) \\ &\leq ||x_0||^{ab}\alpha_2 E_\beta(-\alpha_3(s_0-t_0)^\beta)E_\beta(-\alpha_3(t-t_1)^\beta).\end{aligned}$$

Using condition (i) we obtain

$$||X_1(t)|| \leq ||x_0||^b\sqrt[a]{\frac{\alpha_2}{\alpha_1}E_\beta(-\alpha_3(s_0-t_0)^\beta)E_\beta(-\alpha_3(t-t_1)^\beta)}, \quad t \in [t_1,s_1],$$

or

$$||x(t;t_0,x_0)|| \leq ||x_0||^b\sqrt[a]{\frac{\alpha_2}{\alpha_1}E_\beta(-\alpha_3(s_0-t_0)^\beta)E_\beta(-\alpha_3(t-t_1)^\beta)}, \quad t \in [t_1,s_1].$$

Let $t \in (s_1, t_2]$. From conditions (i) and (iii) we have

$$\begin{aligned}\alpha_1||x(t;t_0,x_0)||^a &\leq V(t,x(t;t_0,x_0)) = V(t,\psi_1(t,x(s_1-0;t_0,x_0)) \\ &\leq \alpha_4||x(s_1-0;t_0,x_0)||^a\end{aligned}$$

or

$$||x(t;t_0,x_0)|| \le ||x_0||^b \sqrt[a]{\Big(\frac{\alpha_2}{\alpha_1} E_\beta(-\alpha_3(s_0-t_0)^\beta) E_\beta(-\alpha_3(s_1-t_1)^\beta)}.$$

Following the above procedure we obtain the zero solution of NIFrDE (2.48) is Mittag-Leffler stable with respect to non-instantaneous impulses with $M = \sqrt[a]{\frac{\alpha_2}{\alpha_1}}$ and $\mu = \alpha_3$. □

Remark 2.3.14 *From the proof of Theorem 2.3.1 it follows that condition (iii) could be replaced by*

(iii) for any* $k = 0, 1, 2, \dots$ *the inequality*

$$V(t, \psi_k(t,x)) \le V(t,x) \textit{ for } t \in (s_k, t_{k+1}],\ x \in \mathbb{R}^n$$

holds.

Corollary 2.3.4 *Let the conditions of Theorem 2.3.1 be satisfied with condition (ii) replaced by*

(ii) for any* $\tau \in [0, s_0) \cup_{k=1}^{\infty} [t_k, s_k)$ *and any solution* $x(t) \in C^q([\tau, s_p], \mathbb{R}^n)$ *of FrDE (2.78) the inequality*

$${}_{\tau}^{c}D^{\beta} V(t,x(t)) \le -\alpha_3 V(t,x(t)) \quad \textit{for} \ \ t \in (\tau, s_p]$$

holds, where $p = \min\{k : \tau < s_k\}$, $\beta \in (0,1)$, α_3 *is a positive constant.*

Then the zero solution of NIFrDE (2.48) is Mittag-Leffler stable with respect to non-instantaneous impulses.

Corollary 2.3.5 *Let the conditions of Theorem 2.3.1 be satisfied with condition (ii) replaced by*

*(ii**) for any* $\tau \in [0, s_0) \cup_{k=1}^{\infty} [t_k, s_k)$ *and any solution* $x(t) \in C^q([\tau, s_p], \mathbb{R}^n)$ *of FrDE (2.78) the inequality*

$${}_{\tau}^{RL}D_t^q V(t,x(t)) \le -\alpha_3 ||x(t)||^{ab} \textit{ for } t \in (\tau, s_p]$$

holds where $p = \min\{k : \tau < s_k\}$.

Then the zero solution of NIFrDE (2.48) is Mittag-Leffler stable with respect to non-instantaneous impulses.

Proof This follows from Lemma 6 in [88], i.e., ${}_{\tau}^{c}D^{\beta} M(t) \le {}_{\tau}^{RL}D_t^q M(t)$ where $\beta \in (0,1)$, $M(\tau) \ge 0$; note ${}_{\tau}^{c}D^{\beta}$ and ${}_{\tau}^{RL}D_t^q$ are the Caputo fractional derivative and the Riemann-Liouville fractional derivatives, respectively.

2.3.3.2 Caputo Fractional Dini Derivatives of Lyapunov Functions

We will give some sufficient conditions for the Mittag-Leffler stability with respect to non-instantaneous impulses of the zero solution of NIFrDE (2.48) by the application of the Caputo fractional Dini derivatives.

Theorem 2.3.2 *Let condition (H2.3.3) be satisfied and there exists a function $V(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ locally Lipschitz w.r.t. $x \in \mathbb{R}^n$, $V(t,0) = 0$ for $t \geq 0$ and such that*

(i) $\alpha_1 ||x||^a \leq V(t,x) \leq \alpha_2 ||x||^{ab}$ *for* $t \geq 0,\ x \in \mathbb{R}^n$, *where* α_1, α_2, a, b *are positive numbers,* $\alpha_2 \leq 1$*;*

(ii) *for any* $\tau \in [0, s_0) \cup_{k=1}^{\infty} [t_k, s_k)$ *and* $\tilde{x}_0 \in \mathbb{R}^n$ *the inequality*

$$ {}_{(2.48)}^{\ c}D_+^q V(t, x; \tau, \tilde{x}_0) \leq -\alpha_3 ||x||^{ab} \quad \text{for } t \in (\tau, s_p], x \in \mathbb{R}^n $$

holds where $p = \min\{k :\ \tau < s_k\}$*,* α_3 *is a positive constant;*

(iii) *for any* $k = 0, 1, 2, \dots$ *the inequality*

$$ V(t, \psi_k(t,x)) \leq \alpha_4 ||x||^a \quad \text{for } t \in (s_k, t_{k+1}],\ x \in \mathbb{R}^n $$

holds where α_4 *is a positive constant such that* $\alpha_4 \leq \alpha_1$.

Then the zero solution of NIFrDE (2.48) is Mittag-Leffler stable with respect to non-instantaneous impulses.

Proof Let $t_0 \in [0, s_0) \cup_{k=1}^{\infty} [t_k, s_k)$ be an arbitrary initial time. Without loss of generality we can assume $t_0 \in [0, s_0)$. Consider the solution $x(t; t_0, x_0)$ of NIFrDE (2.48) with arbitrary given $x_0 \in \mathbb{R}^n$. We will prove the claim by induction.

Let $t \in [t_0, s_0]$. According to Corollary 2.3.2 with $\tau_0 = t_0$, $T = s_0 - t_0$, $\alpha = \alpha_3$ and condition (i) we have

$$ V(t,\ x(t; t_0, x_0)) \leq V(t_0, x_0) E_q(-\alpha_3 (t - t_0)^q) \leq \alpha_2 ||x_0||^{ab} E_q(-\alpha_3 (t - t_0)^q). \tag{2.148} $$

From condition (i) and inequality (2.148) we get for $\beta = q$

$$ ||x(t; t_0, x_0)|| \leq ||x_0||^b \sqrt[a]{\frac{\alpha_2}{\alpha_1} E_\beta(-\alpha_3 (t - t_0)^\beta)}, \quad t \in [t_0, s_0]. \tag{2.149} $$

Let $t \in (s_0, t_1]$. From inequality (2.149) for $t = s_0 - 0$, conditions (i) and (iii) we have

$$ \begin{aligned} \alpha_1 ||x(t; t_0, x_0)||^a &\leq V(t, x(t; t_0, x_0)) = V(t, \psi_0(t, x(s_0 - 0; t_0, x_0)) \\ &\leq \alpha_4 ||x(s_0 - 0; t_0, x_0)||^a \leq ||x_0||^{ab} \alpha_2 E_\beta(-\alpha_3 (t - t_0)^\beta) \end{aligned} \tag{2.150} $$

or

$$||x(t;t_0,x_0)|| \leq ||x_0||^b \sqrt[a]{\frac{\alpha_2}{\alpha_1} E_\beta(-\alpha_3(s_0-t_0)^\beta)}, \quad t \in (s_0, t_1]. \tag{2.151}$$

Let $t \in [t_1, s_1]$. Since we use approach (A2 for NIFrDE) the function $X_1(t) = x(t;t_0,x_0)$, $t \in [t_1, s_1]$ is a solution of the FrDE (2.78) with $\tau = t_1$ and $\tilde{x}_0 = x(t_1;t_0,x_0)$. From Corollary 2.3.2 with $x^*(t) = X_1(t)$, $\tau_0 = t_1$, $x_0 = X_1(t_1)$, $T = s_1 - t_1$, and $\alpha = \alpha_3$, condition (iii) and inequality (2.149) for $t = s_0$ we get

$$\begin{aligned} V(t, X_1(t)) &\leq V(t_1, X_1(t_1))E_\beta(-\alpha_3(t-t_1)^\beta) = V(t_1, x(t_1;t_0,x_0))E_\beta(-\alpha_3(t-t_1)^\beta) \\ &= V(t_1, \psi_0(t_1, x(s_0-0;t_0,x_0))E_\beta(-\alpha_3(t-t_1)^\beta) \\ &\leq \alpha_4||x(s_0-0;t_0,x_0)||^a E_\beta(-\alpha_3(t-t_1)^\beta) \\ &\leq \alpha_4||x_0||^{ab}\frac{\alpha_2}{\alpha_1}E_\beta(-\alpha_3(s_0-t_0)^\beta)E_\beta(-\alpha_3(t-t_1)^\beta) \\ &\leq ||x_0||^{ab}\alpha_2 E_\beta(-\alpha_3(s_0-t_0)^\beta)E_\beta(-\alpha_3(t-t_1)^\beta). \end{aligned} \tag{2.152}$$

From inequality (2.152) and condition (i) we obtain

$$||X_1(t)|| \leq ||x_0||^b \sqrt[a]{\frac{\alpha_2}{\alpha_1} E_\beta(-\alpha_3(s_0-t_0)^\beta)E_\beta(-\alpha_3(t-t_1)^\beta)}, \quad t \in [t_1, s_1],$$

or

$$||x(t;t_0,x_0)|| \leq ||x_0||^b \sqrt[a]{\frac{\alpha_2}{\alpha_1} E_\beta(-\alpha_3(s_0-t_0)^\beta)E_\beta(-\alpha_3(t-t_1)^\beta)}, \quad t \in [t_1, s_1].$$

Following the above procedure we obtain the zero solution of NIFrDE (2.48) is Mittag-Leffler stable with respect to non-instantaneous impulses with $M = \sqrt[a]{\frac{\alpha_2}{\alpha_1}}$ and $\mu = \alpha_3$. □

2.3.4 *Stability, Uniform Stability, and Asymptotic Stability of NFrDE*

The goal of the section is to study the stability properties of NIFrDE (2.48). We will use the approach (A2 for NIFrDE) and Lyapunov functions which derivatives will be considered only on the intervals without impulses.

In the definition below we denote by $x(t;t_0,x_0) \in NPC^q([t_0,\infty), \mathbb{R}^n)$ any solution of (2.48).

Definition 2.3.6 *The zero solution of the IVP for NIFrDE (2.48) is said to be*

- stable *if for every* $\epsilon > 0$ *and* $t_0 \in [0, s_0] \bigcup \cup_{k=1}^{\infty} [t_k, s_k)$ *there exist* $\delta = \delta(\epsilon, t_0) > 0$ *such that for any* $x_0 \in \mathbb{R}^n$ *the inequality* $||x_0|| < \delta$ *implies* $||x(t; t_0, x_0)|| < \epsilon$ *for* $t \geq t_0$*;*
- uniformly stable *if for every* $\epsilon > 0$ *there exist* $\delta = \delta(\epsilon) > 0$ *such that for any initial point* $t_0 \in [0, s_0] \bigcup \cup_{k=1}^{\infty} [t_k, s_k)$ *and any initial value* $x_0 \in \mathbb{R}^n$ *with* $||x_0|| < \delta$ *the inequality* $||x(t; t_0, x_0)|| < \epsilon$ *holds for* $t \geq t_0$*;*
- uniformly attractive *if for* $\beta > 0$: *for every* $\epsilon > 0$ *there exists* $T = T(\epsilon) > 0$ *such that for any initial point* $t_0 \in [0, s_0] \bigcup \cup_{k=1}^{\infty} [t_k, s_k)$ *and any initial value* $x_0 \in \mathbb{R}^n$ *with* $||x_0|| < \beta$ *the inequality* $||x(t; t_0, x_0)|| < \epsilon$ *holds for* $t \geq t_0 + T$*;*
- uniformly asymptotically stable *if the zero solution is uniformly stable and uniformly attractive.*

Remark 2.3.15 *For any* $t_0 \in [0, s_0] \bigcup \cup_{k=1}^{\infty} [t_k, s_k)$ *there exists* $p \in \{1, \dots\}$ *with* $t_0 \in [t_p, s_p)$ *or* $t \in [0, s_0]$. *Without loss of generality assume* $t_0 \in [0, s_0]$.

Example 2.3.4.1 Consider the scalar NIFrDE (2.48) with $f(t, x) = Ax$, $A \leq 0$ and $\phi_k(t, x, y) = a_k(t)y$, $a_k : [s_k, t_{k+1}] \to \mathbb{R}$, $k = 0, 1, 2, 3, \dots$ are such that $\sup_{t \in [s_k, t_{k+1}]} |a_k(t)| \leq M_k$, $\prod_{i=1}^{\infty} M_i < \infty$ where $M_k > 0$ are constants. From (2.46) and the inequality $0 < E_q(A(T - \tau)^q) \leq 1$ for $T \geq \tau$ there exists a constant $M > 0$ such that

$$|x(t; t_0, x_0)| \leq M\,|x_0| \quad \text{for } t \geq t_0. \tag{2.153}$$

Inequality (2.153) guarantees that the zero solution of (2.48) in this particular case is uniformly stable. □

Theorem 2.3.3 (Stability) *Let the following conditions be satisfied:*

1. *Condition (H2.3.3) is satisfied.*
2. *There exists a function* $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ *such that* $V(t, 0) = 0$ *and*

   (i) for any for any $\tau \in [0, s_0) \cup_{k=1}^{\infty} [t_k, s_k)$ *and* $y_0 \in \mathbb{R}^n$ *the inequality*

   $${}^{c}_{(2.48)}D^q_+ V(t, x; \tau, y_0) \leq 0 \qquad \text{for } t \in (\tau, s_p], \ x \in \mathbb{R}^n$$

   holds where $p = \min\{k : \tau < s_k\}$*;*

   (ii) for $k = 0, 1, 2, 3, \dots$ *the inequality*

   $$V(t, \psi_k(t, x)) \leq V(s_k - 0, x), \quad x \in \mathbb{R}^n, \ t \in (s_k, t_{k+1}]$$

   holds;

   (iii) $b(||x||) \leq V(t, x)$ *for* $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, *where* $b \in \mathcal{K}$.

Then the zero solution of the NIFrDE (2.48) is stable.

Proof Let $\epsilon > 0$ and $t_0 \in [0, s_0) \cup_{k=1}^{\infty} [t_k, s_k)$ be arbitrary. Without loss of generality we can assume $t \in [0, s_0)$.

Since $V(t_0, 0) = 0$ there exists $\delta_1 = \delta_1(t_0, \epsilon) > 0$ such that $V(t_0, x) < b(\epsilon)$ for $||x|| < \delta_1$. Let $x_0 \in \mathbb{R}^n$ with $||x_0|| < \delta_1$. Then $V(t_0, x_0) < b(\epsilon)$. Consider the solution $x^*(t) = x(t; t_0, x_0) \in PC^q([t_0, \infty), \mathbb{R}^n)$ of NIFrDE (2.48). From condition 2(i) it follows that

$$ {}_{(2.48)}^{\ c}D_+^q V(t, x^*(t); t_0, x_0) \le 0 \ \text{ for } t \in \cup_{k=0}^{\infty}(t_k, s_k), $$

i.e., condition 3(i) of Lemma 2.3.10 with $T = \infty$ is satisfied.

From Lemma 2.3.10 applied to the solution $x^*(t)$ with $T = \infty$ (see Remark 2.3.7) and condition 2(*iii*) we obtain

$$ b(||x^*(t)||) \le V(t, x^*(t)) \le V(t_0, x_0) < b(\epsilon), $$

so the result follows. □

Theorem 2.3.4 (Uniform Stability) *Let the following conditions be satisfied:*

1. *Condition (H2.3.3) is satisfied.*
2. *There exists a function $V \in \Lambda(\mathbb{R}_+, S(\lambda))$, $\lambda > 0$ is given, such that*
   - *(i) for any $\tau \in [0, s_0) \cup_{k=1}^{\infty} [t_k, s_k)$ and $y_0 \in S(\lambda)$ the inequality*

     $$ {}_{(2.48)}^{\ c}D_+^q V(t, x; \tau, y_0) \le 0 \qquad \text{for } \ t \in (\tau, s_p), \ \ x \in S(\lambda) $$

     holds where $p = \min\{k : \ \tau < s_k\}$;
   - *(ii) for $k = 0, 1, 2, 3, \dots$ the inequality*

     $$ V(t, \psi_k(t, x)) \le V(s_k - 0, x), \quad x \in S(\lambda), \ t \in (s_k, t_{k+1}] $$

     holds;
   - *(iii) $b(||x||) \le V(t, x) \le a(||x||)$ for $t \in \mathbb{R}_+$, $x \in S(\lambda)$, where $a, b \in \mathcal{K}$.*

Then the zero solution of NIFrDE (2.48) is uniformly stable.

Proof Let $\epsilon \in (0, \lambda]$ and $t_0 \in [0, s_0) \cup_{k=1}^{\infty} [t_k, s_k)$ be arbitrary. Without loss of generality we can assume $t \in [0, s_0)$.

Let $\delta_1 < \min\{\epsilon, b(\epsilon)\}$. From $a \in \mathcal{K}$ there exists $\delta_2 = \delta_2(\epsilon) > 0$ so if $s < \delta_2$, then $a(s) < \delta_1$. Let $\delta = \min(\epsilon, \delta_2)$. Choose the initial value $x_0 \in \mathbb{R}^n$ such that $||x_0|| < \delta$ and let $x^*(t) = x(t; t_0, x_0) \in PC^q([t_0, \infty), \mathbb{R}^n)$ be a solution of the IVP for NIFrDE (2.48). We now prove that

$$ ||x^*(t)|| < \epsilon, \quad t \ge t_0. \tag{2.154} $$

Assume inequality (2.154) is not true and let $t^* = inf\{t > t_0 : \ ||x^*(t)|| \ge \epsilon\}$. Then

$$ ||x^*(t)|| < \epsilon \quad \text{for } t \in [t_0, t^*) \quad \text{and } ||x^*(t^*)|| = \epsilon. \tag{2.155} $$

Assume there exists a nonnegative integer m such that $t^* = s_m$. If $||x^*(s_m - 0)|| < \epsilon$, $||x^*(s_m + 0)|| \geq \epsilon$, then according to Lemma 2.3.10 for $T = s_m$ and $\Delta = S(\lambda)$ we obtain $V(t, x^*(t)) \leq V(t_0, x_0)$ for $t \in [t_0, s_m]$. Then from condition 2(iii) we get $b(\epsilon) \leq b(||x^*(s_m + 0)) = b(||\phi_m(s_m + 0, x^*(s_m - 0))||) \leq V(s_m + 0, \phi_m(s_m + 0, x^*(s_m - 0))) \leq V(s_m - 0, x^*(s_m - 0)) \leq V(t_0, x_0) \leq a(\delta) < \delta_1 < b(\epsilon)$. The obtained contradiction proves this case is not possible. If $||x^*(s_m - 0)|| = ||x^*(t^*)|| \leq \epsilon$ and $||x^*(s_m + 0)|| > \epsilon$, then we obtain again a contradiction with the choice of t^*.

Therefore, $t^* \neq s_k,\ k = 0, 1, 2, \ldots$ and $x(t) \in S(\lambda)$ for $t \in [t_0, t^*]$ and $||x^*(t^*)|| = \epsilon$.

Then conditions 3(i) and 3(ii) of Lemma 2.3.10 are satisfied on $[t_0, t^*]$. From Lemma 2.3.10 applied to the solution $x^*(t)$ with $T = t^*$ and $\Delta = S(\lambda)$ we get $V(t, x^*(t)) \leq V(t_0, x_0)$ on $[t_0, t^*]$. Then applying condition 2 (iii) of Theorem 2.3.4 we obtain $b(\epsilon) = b(||x^*(t^*)||) \leq V(t^*, x^*(t^*)) \leq V(t_0, x_0) \leq a(\delta) < \delta_1 < b(\epsilon)$. The contradiction proves (2.154) and therefore, the zero solution of NIFrDE (2.48) is uniformly stable. □

Theorem 2.3.5 (Uniform Asymptotic Stability) *Let the following conditions be satisfied:*

1. *Condition (H2.3.3) is satisfied.*
2. *There exists a positive constant $M < \infty$ such that $\sum_{i=0}^{\infty}(t_{i+1} - s_i) \leq M$.*
3. *There exists a function $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ such that*

   (i) for any $\tau \in [0, s_0) \cup_{k=1}^{\infty} [t_k, s_k)$ and $y_0 \in S(\lambda)$ the inequality

$$
{}_{(2.48)}^{\;\;c}D_+^q V(t, x; \tau, y_0) \leq -c(||x||) \qquad \text{for } \ t \in (\tau, s_p], \ \ x \in S(\lambda)
$$

   holds where $p = \min\{k : \ \tau < s_k\}$, $\lambda > 0$ is a given number, $c \in \mathcal{K}$;

   (ii) for any $k = 0, 1, 2, 3, \ldots$ the inequality

$$
V(t, \psi_k(t, x)) \leq V(s_k - 0, x), \qquad t \in (s_k, t_{k+1}], \ \ x \in S(\lambda)
$$

   holds;

   (iii) $b(||x||) \leq V(t, x) \leq a(||x||)$ for $t \in \mathbb{R}_+,\ x \in \mathbb{R}^n$ where $a, b \in \mathcal{K}$.

Then the zero solution of NIFrDE (2.48) is uniformly asymptotically stable.

Proof From Theorem 2.3.4 the zero solution of the NIFrDE (2.48) is uniformly stable. Therefore, for the number λ there exists $\alpha = \alpha(\lambda) \in (0, \lambda)$ such that for any $\tilde{t}_0 \in [0, s_0] \bigcup \cup_{k=1}^{\infty}[t_k, s_{k+1})$ and $\tilde{x}_0 \in \mathbb{R}^n$ the inequality $||\tilde{x}_0|| < \alpha$ implies

$$
||x(t; \tilde{t}_0, \tilde{x}_0)|| < \lambda \quad \text{for } \ t \geq \tilde{t}_0 \tag{2.156}
$$

where $x(t; \tilde{t}_0, \tilde{x}_0)$ is any solution of the NIFrDE (2.48) (with initial data $(\tilde{t}_0, \tilde{x}_0)$).

Now we will prove that the zero solution of the fractional differential equations (2.48) is uniformly attractive. Consider the constant $\beta \in (0, \alpha]$ such that $a(\beta) \leq b(\alpha)$. Let $\epsilon \in (0, \lambda]$ and $t_0 \in [0, s_0] \bigcup \cup_{k=1}^{\infty} [t_k, s_{k+1})$ be arbitrary given numbers (assume $t_0 \in [0, s_0]$).

Let the point $x_0 \in \mathbb{R}^n$, $||x_0|| < \beta$ and $x^*(t) = x(t; t_0, x_0)$ be the solution of (2.48). Then $b(||x_0||) \leq a(||x_0||) < a(\beta) < b(\alpha)$, i.e., $||x_0|| < \alpha$ and according to (2.156) the inequality

$$||x^*(t)|| < \lambda \quad \text{for} \ \ t \geq t_0 \tag{2.157}$$

holds, i.e., the solution $x^*(t) \in S(\lambda)$ on $[t_0, \infty)$.

Choose a constant $\gamma = \gamma(\epsilon) \in (0, \epsilon]$ such that $a(\gamma) < b(\epsilon)$. Let $T > \sqrt[q]{a(\alpha)\frac{q\Gamma(q)}{c(\gamma)}} + M$ and m be a natural number such that $t_m < t_0 + T \leq s_m$. Note T depends only on ϵ but not on t_0. We now prove that

$$||x^*(t)|| < \epsilon \ \ \text{for} \ \ t \geq t_0 + T. \tag{2.158}$$

Assume

$$||x^*(t)|| \geq \gamma \ \ \text{for every} \ t \in [t_0, t_0 + T]. \tag{2.159}$$

Then from Lemma 2.3.11 (applied to the interval $[t_0, t_0 + T]$ and $\Delta = S(\lambda)$), conditions 2 and 3 (ii) of Theorem 2.3.5, inequality $a^q + b^q \geq (a + b)^q$ for $a, b > 0$ and the choice of T we get

$$\begin{aligned}
&V(t_0 + T, x^*(t_0 + T)) \\
&\leq V(t_0, x_0) - \frac{1}{\Gamma(q)}\Big(\sum_{i=0}^{m}\int_{t_i}^{s_i}(s_i - s)^{q-1}c(||x^*(s)||)ds \\
&\qquad\qquad + \int_{t_m}^{t_0+T}(t_0 + T - s)^{q-1}c(||x^*(s)||)ds\Big) \\
&\leq a(||x_0||) - \frac{c(\gamma)}{\Gamma(q)}\Big(\sum_{i=0}^{m}\int_{t_i}^{s_i}(s_i - s)^{q-1}ds + \int_{t_m}^{t_0+T}(t_0 + T - s)^{q-1}ds\Big) \\
&< a(\alpha) - \frac{c(\gamma)}{q\Gamma(q)}\Big((s_0 - t_0)^q + \sum_{i=1}^{m}(s_i - t_i)^q + (T + t_0 - t_m)^q\Big) \\
&\leq a(\alpha) - \frac{c(\gamma)}{q\Gamma(q)}\Big((s_0 - t_0) + \sum_{i=1}^{m}(s_i - t_i) + (T + t_0 - t_m)\Big)^q \\
&= a(\alpha) - \frac{c(\gamma)}{q\Gamma(q)}\Big(-\sum_{i=0}^{m-1}(t_{i+1} - s_i) + T\Big)^q \leq a(\alpha) - \frac{c(\gamma)}{q\Gamma(q)}\Big(-M + T\Big)^q < 0.
\end{aligned}$$

The above contradiction proves there exists $t^* \in [t_0, t_0+T]$ such that $||x^*(t^*)|| < \gamma$. Let the natural number p be such that $s_{p-1} \le t^* < s_p$.

Case 1. Let $t \in [t^*, s_p]$.

If $t_p < t^* < s_p$, then for $t \in [t^*, s_p]$ the function $x^*(t) \in C^q([t^*, s_p], \mathbb{R}^n)$ and from Lemma 2.3.5 we get $V(t, x^*(t)) \le V(t^*, x^*(t^*)) - \frac{1}{\Gamma(q)} \int_{t^*}^{t} (t-s)^{q-1} c(||x^*(s)||) ds \le V(t^*, x^*(t^*))$.

If $s_{p-1} < t^* \le t_p$, then for $t \in [t^*, s_p]$ the function $x^*(t) \in PC^q([t^*, s_p], \mathbb{R}^n)$ and from Lemma 2.3.11 we get $V(t, x^*(t)) \le V(t^*, x^*(t^*))$.

Case 2. For any $t > t^*,\ t \in (t_k, s_k],\ k = p+1, p+2\dots,$ from Lemma 2.3.11 for $\Delta = S(\lambda)$ we obtain

$$\begin{aligned}
&V(t, x^*(t)) \le V(t^*, x^*(t^*)) \\
&- \frac{1}{\Gamma(q)} \Big(\int_{t^*}^{s_p} (s_p - s)^{q-1} c(||x^*(s)||) ds - \sum_{i=p+1}^{k-1} \int_{t_i}^{s_i} (s_i - s)^{q-1} c(||x^*(s)||) ds \\
&\qquad - \int_{t_k}^{t} (t-s)^{q-1} c(||x^*(s)||) ds \Big) \le V(t^*, x^*(t^*)).
\end{aligned}$$

Case 3. For any $t > t^*,\ t \in (s_k, t_k],\ k = p, p+1, \dots,$ from Lemma 2.3.11 for $\Delta = \mathbb{R}^n$ we obtain

$$\begin{aligned}
&V(t, x^*(t)) \\
&\le V(t^*, x^*(t^*)) - \frac{1}{\Gamma(q)} \Big(\int_{t^*}^{s_p} (s_p - s)^{q-1} c(||x^*(s)||) ds \\
&\qquad - \sum_{i=p+1}^{k} \int_{t_i}^{s_i} (s_i - s)^{q-1} c(||x^*(s)||) ds \Big) \le V(t^*, x^*(t^*)).
\end{aligned}$$

Therefore, for $t \ge t^*$ the following inequality is satisfied:

$$V(t, x^*(t)) \le V(t^*, x^*(t^*)). \tag{2.160}$$

Then for any $t \ge t^*$ applying (2.160), condition 3(iii) and inequality (2.157) we get the inequalities

$$b(||x^*(t)||) \le V(t, x^*(t)) \le V(t^*, x^*(t^*)) \le a(||x^*(t^*)||) \le a(\gamma) < b(\epsilon).$$

Therefore, the inequality (2.158) holds for all $t \ge t^*$ (hence for $t \ge t_0 + T$). □

Remark 2.3.16 *If the initial time t_0 is in an interval of non-instantaneous impulses, i.e., $t_0 \in \cup_{k=0}^{\infty} (s_k, t_{k+1}]$, then the results of Theorems 2.3.3, 2.3.4 and 2.3.5 will be similar with slight changes in Definition 2.3.6 and condition 2(ii) (Theorems 2.3.3, 2.3.4) or condition 3(ii) (Theorem 2.3.5).*

When approach (A2 for NIFrDE) is applied, then the above sufficient conditions are true if the Caputo fractional derivative is replaced by the Dini fractional derivative defined by (2.89). We will set up only the results because the proofs are similar to the proof of Theorems 2.3.3, 2.3.4 and 2.3.5.

Theorem 2.3.6 (Stability) *Let the conditions of Theorem 2.3.3 be satisfied where the condition 2(i) is replaced by*

2.(i^*) *for all* $k = 0, 1, 2, \dots$ *the inequalities*

$$_{(2.48)}\mathcal{D}^q_+ V(t,x) \le 0 \qquad \text{for } t \in (t_k, s_k], \ \ x \in \mathbb{R}^n \tag{2.161}$$

hold (in the case of $k = 0$ *we consider the interval* $(0, s_0]$ *instead of* $(t_k, s_k]$*).*

Then the zero solution of the NIFrDE (2.48) is stable.

Theorem 2.3.7 (Uniform Stability) *Let the conditions of Theorem 2.3.4 be satisfied where the condition 2(i) is replaced by*

2.(i^{**}) *for all* $k = 0, 1, 2, \dots$ *the inequalities*

$$_{(2.48)}\mathcal{D}^q_+ V(t,x) \le 0 \qquad \text{for } t \in (t_k, s_k], \ \ x \in S(\lambda) \tag{2.162}$$

hold (in the case of $k = 0$ *we consider the interval* $(0, s_0]$ *instead of* $(t_k, s_k]$*).*

Then the zero solution of the IFrDE (2.48) is uniformly stable.

Example 2.3.4.2 Consider the scalar NIFrDE (2.48) with $n = 1, f(t,x) = Ax$, $A \le 0$, and $\psi_k(t,y) = a_k(t)y$, $a_k : [s_k, t_{k+1}] \to \mathbb{R}$, $k = 0, 1, 2, 3, \dots$ are such that $\sup_{t\in[s_k,t_{k+1}]} |a_k(t)| \le 1$. Consider the quadratic Lyapunov function $V(x) = x^2$. We will apply the Dini fractional derivative of the quadratic Lyapunov function and Theorem 2.3.7. Apply (2.85) and obtain $_{(2.48)}\mathcal{D}^q_+ V(t,x) = 2Ax^2 \le 0$ for $x \in S(\lambda$ and all $t \ge 0$. Therefore, the inequality (2.162) holds.

Also, $\Big(\psi_k(t,x)\Big)^2 = \Big(a_k(t)\Big)^2 x^2 \le x^2$ for $t \in [s_k, t_{k+1}]$, i.e., condition (ii) of Theorem 2.3.4 is satisfied.

From Theorem 2.3.7 the zero solution of the scalar NIFrDE (2.48) in this particular case is uniformly stable. □

Example 2.3.4.3 Consider the scalar non-instantaneous impulsive Caputo fractional differential equation

$$\begin{aligned}
&{}^c_{t_k}D^q x = -a(t)x(1+x^2) \ \text{ for } t \in (t_k, s_k], \ k = 0, 1, 2, \dots, \\
&x(t) = c_k(t)x(s_k - 0) \ \text{ for } t \in (s_k, t_{k+1}], \quad k = 0, 1, 2, 3, \dots, \\
&x(0) = x_0
\end{aligned} \tag{2.163}$$

where $x \in \mathbb{R}$, $a(t) \in C(\cup_{k=0}^{\infty}(t_k, s_k], \mathbb{R}_+)$, $c_k(t) \in C([s_k, t_{k+1}], [-1, 1])$, $k = 0, 1, 2, \dots, t_0 = 0$.

Consider the function $V(t,x) = x^2$. Then $xf(t,x) = -a(t)x^2(1+x^2) \le 0$. Therefore, the inequality (2.162) holds. Also, $\Big(c_k(t)x\Big)^2 = (c_k(t)x)^2 \le x^2$ for $t \in (s_k, t_k]$, $k = 0,1,2,3,\dots$, i.e., condition (ii) of Theorem 2.3.4 is satisfied.

From Theorem 2.3.7 the trivial solution of NIFrDE (2.163) is uniformly stable. □

In the case of more general Lyapunov functions we will apply its Caputo fractional Dini derivative.

Example 2.3.4.4 Let the points $t_k = (4k+1)\frac{\pi}{2}, s_k = (4k-1)\frac{\pi}{2}, k = 1,2,\dots,$ $t_0 = 0$. Consider the following initial value problem for the scalar non-instantaneous impulsive Caputo fractional differential equation

$$\begin{aligned} &{}_{t_k}^{c}D^q x(t) = xf(t), \quad t \in (t_k, s_k],\ k = 0,1,2,\dots, \\ &x(t) = c_k(t)x(s_k - 0), \qquad t \in [s_k, t_{k+1}],\ k = 0,1,2,\dots, \\ &x(0) = x_0, \end{aligned} \tag{2.164}$$

where $x \in \mathbb{R}$, $q \in (0,1)$, $c_k \in C([s_k, t_{k+1}], [-1,1])$, $f(t) = 0.5\frac{\frac{-2}{\sqrt{t\pi}} + \sqrt{t}E_{2,1.5}(-t^2)}{2-sin(t)}$, $k = 0,1,2,\dots$.

Let $V(t,x) = x^2$. Then $x\big(xf(t)\big) = x^2 f(t)$. The sign of the function $f(t)$ changes for some $q \in (0,1)$ (see Figure 2.9). Therefore, for $q = 0.1$ (for example) (see Figure 2.8) Theorem 2.3.7 and the quadratic Lyapunov function can be applied to the NIFrDE (2.164). But for $q = 0.5$, for example, the sign of $f(t)$ is changeable and Theorem 2.3.7 and the quadratic Lyapunov function are not applicable to the NIFrDE (2.164).

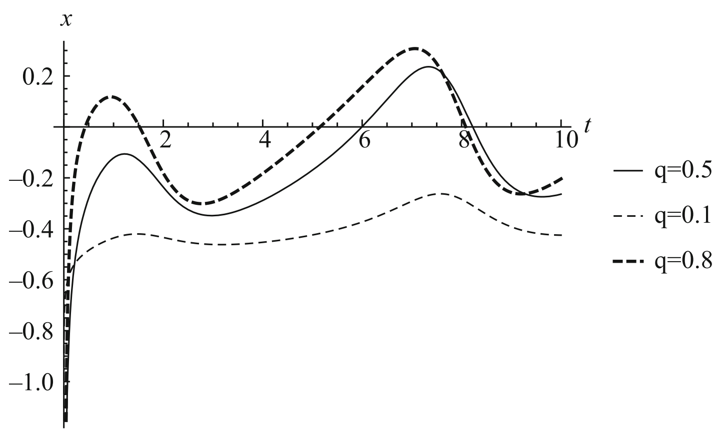

Fig. 2.8 Example 2.3.4.4. Graphs of the function $f(t)$ for various q.

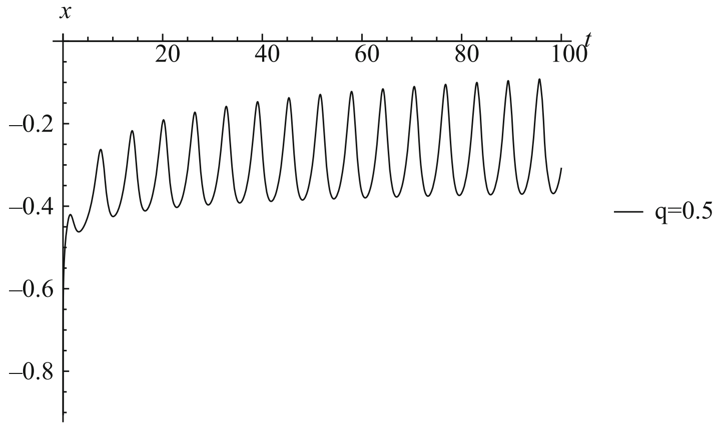

Fig. 2.9 Example 2.3.4.4. Graph of the function $f(t)$ for $q = 0.1$.

Let $V(t,x) = (2 - \sin(t))x^2$.

Then for the Dini fractional derivative given by (2.89), according to (2.94) and (2.85) we get

$$_{(2.164)}\mathcal{D}_+^{0.5}V(t,x) = 2x^2\ (2 - \sin(t))f(t).$$

The sign of both the function $f(t)$ and the derivative ${}^{c}_{(2.164)}\mathcal{D}_+^{0.5}V(t,x)$ is changeable. Therefore, the application of fractional Dini derivative (2.89) does not give us a conclusion about stability properties of NIFrDE (2.164).

Now apply Caputo fractional Dini derivative to the considered Lyapunov function. According to Eq. (2.97) and $\Gamma(0.5) = \sqrt{\pi}$ we obtain

$$\begin{aligned} &{}^{c}_{(2.164)}D_+^{0.5}V(t,x;\tau,y_0) \\ &= 2x^2(2-\sin(t))f(t) + x^2\ {}^{RL}_{\tau}D^{0.5}(2-\sin(t)) - y_0^2(2-\sin(t))\frac{(t-\tau)^{-0.5}}{\Gamma(0.5)} \end{aligned} \tag{2.165}$$

where $\tau \in [t_k, s_k)$, $k \geqq 0$ is an arbitrary integer.

In the partial case $\tau = 0$ we get ${}^{C}_{(2.164)}D_+^{0.5}V(t,x;0,y_0) \leqq$

Also, for $t \in [s_k, t_{k+1}]$, $k = 0,1,2,\ldots$ we get $V(t, c_k(t)x) = \big(2 - \sin(t)\big)(c_k(t)x)^2 \leq \big(2-\sin(t)\big)x^2 \leq \big(2-\sin(s_k)\big)x^2 = \big(2-\sin((4k-1)\frac{\pi}{2})\big)x^2 = V(s_k-0, x)$, i.e., condition 2(ii) of Theorem 2.3.3 is satisfied.

According to Theorem 2.3.3 the zero solution of (2.164) is stable. □

2.3.5 Practical Stability for NIFrDE

In [83] the authors pointed out that stability and even asymptotic stability themselves are neither necessary nor sufficient to ensure practical stability. The desired state of a system may be mathematically unstable, but, however, the system may oscillate sufficiently close to the desired state, and its performance is deemed acceptable. Practical stability is neither weaker nor stronger than the usual stability and an equilibrium can be stable in the usual sense, but not practically stable, and vice versa.

First we define some types of practical stability of the zero solution of fractional differential equations with non-instantaneous impulses. In the definition below we will denote by $x(t; t_0; x_0) \in PC^q([t_0, \infty), \mathbb{R}^n)$ any solution of the IVP for NIFrDE (2.48).

In the definition below we assume the point $t_0 \in [0, s_0] \bigcup \cup_{k=1}^{\infty} [t_k, s_k)$.

Definition 2.3.7 *Let positive constants $\lambda, A,\ \lambda < A$ be given. The zero solution of the system of NIFrDE (2.48) is said to be*

(S1) practically stable with respect to (λ, A) *if there exists t_0 such that for any $x_0 \in \mathbb{R}^n$ the inequality $||x_0|| < \lambda$ implies $||x(t; t_0, x_0)|| < A$ for $t \geq t_0$;*
(S2) uniformly practically stable with respect to (λ, A) *if (S1) holds for all $t_0 \in [0, s_0] \bigcup \cup_{k=1}^{\infty} [t_k, s_k)$;*
(S3) practically quasi stable with respect to (λ, A, T) *if there exists t_0 such that for any $x_0 \in \mathbb{R}^n$ the inequality $||x_0|| < \lambda$ implies $||x(t; t_0, x_0)|| < A$ for $t \geq t_0 + T$, where the positive constant T is given;*
(S4) uniformly practically quasi stable with respect to (λ, A, T) *if (S3) holds for all $t_0 \in [0, s_0] \bigcup \cup_{k=1}^{\infty} [t_k, s_k)$;*
(S5) strongly practically stable with respect to (λ, A, B, T) *if there exists an initial time t_0 such that for any $x_0 \in \mathbb{R}^n$ the inequality $||x_0|| < \lambda$ implies $||x(t; t_0, x_0)|| < A$ for $t \geq t_0$ and $||x(t; t_0, x_0)|| < B$ for $t \geq t_0 + T$, where the positive constants $B, T :\ B < \lambda$ are given;*
(S6) uniformly strongly practically stable with respect to (λ, A, B, T) *if (S5) holds for all $t_0 \in [0, s_0] \bigcup \cup_{k=1}^{\infty} [t_k, s_k)$.*

Example 2.3.5.1 Consider the scalar NIFrDE (2.48) with $f(t, x) = Ax,\ A \leq 0$, $\phi_k(t, x, y) = a_k(t)y$ and the functions $a_k : [s_k, t_{k+1}] \to \mathbb{R},\ k = 0, 1, 2, 3, \dots$ are such that $\sup_{t \in [s_k, t_{k+1}]} |a_k(t)| \leq M_k$, $\prod_{i=1}^{\infty} M_i < \infty$ where $M_k > 0$ are constants. From (2.46) and the inequality $0 < E_q(A(T - \tau)^q) \leq 1$ for $T \geq \tau$ there exists a constant $M > 0$ such that

$$|x(t; t_0, x_0)| \leq M\, |x_0| \quad \text{for } \ t \geq t_0. \tag{2.166}$$

Inequality (2.166) guarantees that the zero solution of (2.48) is uniformly stable. Also, if $M < 1$, then the zero solution of (2.48) is uniformly practically stable w.r.t. any couple (λ, A), $\lambda < A$. However if $M > 1$, then the zero solution of (2.48) is not practically stable w.r.t. the couple $(\frac{1}{M}, M)$.

If $A = -1,\ q = 0.5,\ t_k = 2k, s_k = 2k + 1,\ k = 0, 1, \dots,\ a_1(t) \equiv 3,\ a_k(t) = 1 + e^{-\frac{1}{k}},\ k = 2, 3, \dots,\ t_0 = 0$ the solution is given by

$$x(t) = \begin{cases} x_0\big(E_q(-1)\big)^k \big(\prod_{i=1}^{k} a_i\big) E_q(-(t - 2k)^q) \ \text{ for } t \in [2k, 2k + 1],\ k = 0, 1, 2, \dots \\ \qquad\qquad x_0\big(E_q(-1)\big)^k \big(\prod_{i=1}^{k} a_i\big) \ \text{ for } t \in (2k - 1, 2k],\ k = 1, 2, \dots. \end{cases}$$

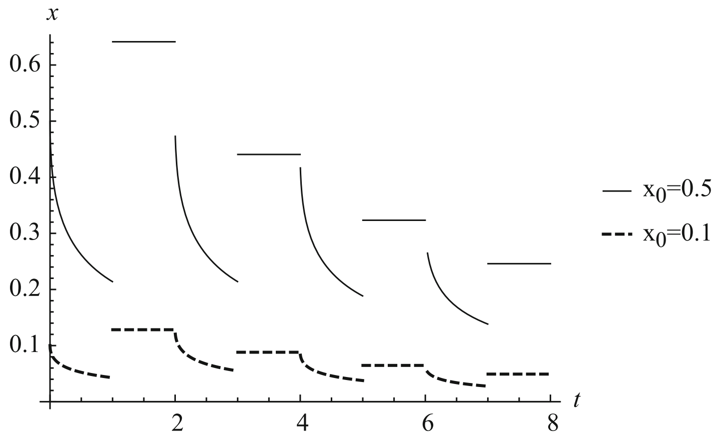

Fig. 2.10 Example 2.3.5.1. Graphs of the solution of (2.48) for various initial values.

The graphs of the solution of (2.48) for various initial values x_0 are given in Figure 2.10. It can be seen that the zero solution of (2.48) is not practically stable w.r.t. the couple (0.5, 0.6) □

Definition 2.3.8 *Let the positive constants* $\lambda, A,\ \lambda < A$ *be given. The solution* $x^*(t) = x(t;t_0,x_0) \in PC^q([t_0,\infty),\mathbb{R}^n)$ *of the system of NIFrDE (2.48) is said to be*

(S7) practically stable with respect to (λ, A) *if for any* $y_0 \in \mathbb{R}^n$ *the inequality* $||y_0 - x_0|| < \lambda$ *implies* $||x(t;t_0,y_0) - x^*(t)|| < A$ *for* $t \geq t_0$, *where* $x(t;t_0,y_0)$ *is a solution of the IVP for NIFrDE (2.48) with* $x_0 = y_0$*;*

(S8) practically quasi stable with respect to (λ, A, T) *if for any* $y_0 \in \mathbb{R}^n$ *the inequality* $||y_0 - x_0|| < \lambda$ *implies* $||x(t;t_0,y_0) - x^*(t)|| < A$ *for* $t \geq t_0 + T$, *where the positive constant* T *is given;*

(S9) strongly practically stable with respect to (λ, A, B, T) *if for any* $y_0 \in \mathbb{R}^n$ *the inequality* $||y_0 - x_0|| < \lambda$ *implies* $||x(t;t_0,y_0) - x^*(t)|| < A$ *for* $t \geq t_0$ *and* $||x(t;t_0,y_0) - x^*(t)|| < B$ *for* $t \geq t_0 + T$, *where the positive constants* $B, T,\ B < \lambda$ *are given.*

We obtain sufficient conditions for various types of practical stability of the system NIFrDE (2.48). As a comparison equation we will use the IVP for the scalar NIFrDE (2.129) with $m = \infty$, $T = \infty$ and $\lim_{k\to\infty} t_k = \infty$.

In this paper we will study the connection between the practical stability of the system NIFrDE (2.48) and the practical stability of the scalar NIFrDE (2.129).

Example 2.3.5.2 Consider IVP for scalar NIFrDE (2.129) with $G(t,u) \equiv 0,\ t \in \cup_{k=0}^{\infty}[t_k,s_k]$ and $\Psi_k \in C([s_k,t_{k+1}],\mathbb{R}),\ k = 0,1,2,\ldots$. The solution is given by

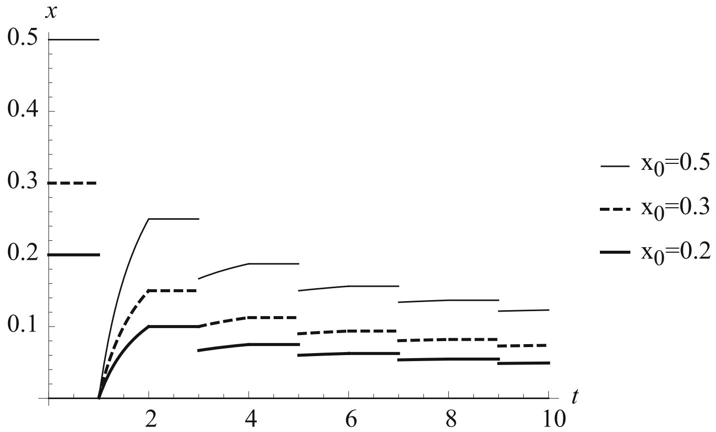

Fig. 2.11 Example 2.3.5.2. *Case 1.*

$$u(t;t_0,u_0)=\begin{cases} u_0 & \text{for } t\in[t_0,s_0],\\ \Psi_0(t,u_0) & \text{for } t\in(s_0,t_1],\\ \Psi_0(t_1,u_0) & \text{for } t\in(t_1,s_1],\\ \Psi_1(t,\Psi_0(t_1,u_0)) & \text{for } t\in(s_1,t_2],\\ \Psi_1(t_2,\Psi_0(t_1,u_0)) & \text{for } t\in(t_2,s_2],\\ \dots\dots\dots\dots\dots & \end{cases}$$

We will consider several cases.

Case 1. Suppose for all $k=0,1,2\dots$ the inequalities $|\Psi_k(t,u)|\le g_k(t)|u|$ hold for $t\in[s_k,t_{k+1}], u\in\mathbb{R}$ with $g_k(t)\in C([s_k,t_{k+1}],[0,1])$. Then the zero solution with $G(t,u)\equiv 0$ is uniformly stable. Also, the zero solution with $G(t,u)\equiv 0$ is uniformly practically stable w.r.t. any couple (λ,A), $\lambda<A$. If $t_k=2k, s_k=2k+1$, $k=0,1,2,\dots$, $\Psi_k(t,u)=u(1-\frac{1}{t})$ for $t\in[2k+1,2k+2], u\in\mathbb{R}$ for all $k=0,1,2,\dots$ the graphs of solutions for various initial values x_0 are given in Figure 2.11.

Case 2. Let $t_k=2k, s_k=2k+1$, $k=0,1,2,\dots$ and $\Psi_k(t,u)=u(1+\frac{1}{t})$ for $t\in[2k+1,2k+2], u\in\mathbb{R}$ for all $k=1,2,\dots$. The graphs of solutions for various initial values x_0 are given in Figure 2.12. In this case $g_k(t)=1+\frac{1}{t}\in(1,2)$ and the zero solution is not stable. Also, because $\lim_{n\to\infty}\prod_{i=1}^{n}(1+\frac{1}{2i})=\infty$, the zero solution is not practically stable w.r.t. any couple.

Case 3. Let $t_k=2k, s_k=2k+1$, $k=0,1,2,\dots$ and $\Psi_1(t,u)=u(1+\frac{1}{t})$, $\Psi_k(t,u)=u(1-\frac{1}{t})$ for $t\in[2k+1,2k+2], u\in\mathbb{R}$ for all $k=0,1,2,\dots$. The graphs of solutions for various initial values x_0 are given in Figure 2.13. In this case, the zero solution is practically stable w.r.t. the couple $(0.5,1)$, but it is not practically stable w.r.t. the couple $(0.6,1)$. The zero solution is practically quasi stable w.r.t. $(0.6,1,5)$. The zero solution is strongly practically stable w.r.t. $(0.5,1,0.4,10)$. □

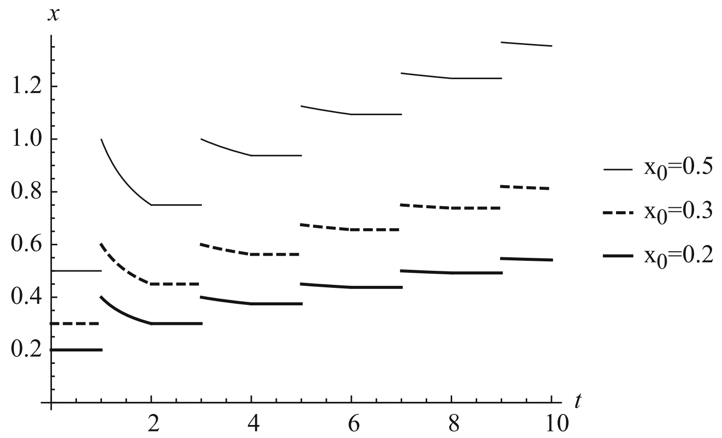

Fig. 2.12 Example 2.3.5.2. *Case 2*.

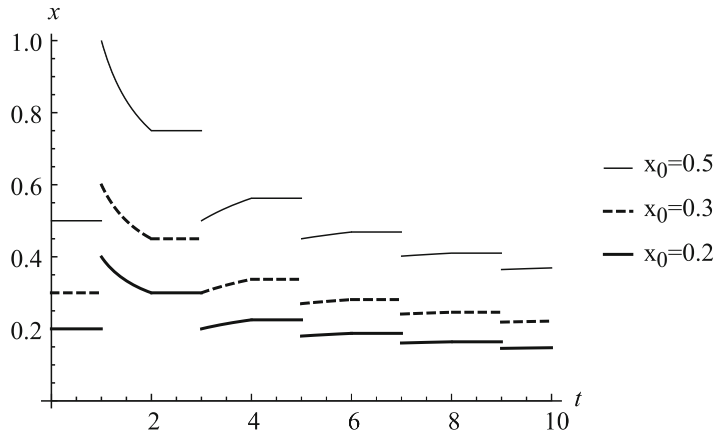

Fig. 2.13 Example 2.3.5.2. *Case 3*.

Theorem 2.3.8 *Let the following conditions be fulfilled:*

1. *Condition (H2.3.3) is satisfied.*
2. *The functions* $\Psi_k \in C([s_k, t_{k+1}] \times \mathbb{R}, \mathbb{R})$, *(*$k = 0, 1, 2, \dots$*), are such that* $\Psi_k(t, 0) = 0$ *and* $\Psi_k(t, u) \le \Psi_k(t, v)$ *for* $u \le v, \ \ t \in [s_k, t_{k+1}]$.
3. *The function* $G \in C(\cup_{k=0}^{\infty}[s_k, t_{k+1}] \times \mathbb{R}, \mathbb{R})$, $G(t, 0) = 0$ *and for any initial point* $(t_0, u_0) \in \Big([0, s_0) \cup_{k=1}^{\infty} [t_k, s_k)\Big) \times \mathbb{R}$ *the IVP for the scalar NIFrDE (2.129) with* $m = \infty$ *has a maximal solution* $u^*(t) = u(t; t_0, u_0) \in PC^q([t_0, \infty), \mathbb{R})$.
4. *The zero solution of scalar NIFrDE (2.129) with* $m = \infty$ *is practically stable w.r.t.* $(a(\lambda), b(A))$ $\Big($*uniformly practically stable w.r.t. the couple* $(a(\lambda), b(A))\Big)$, *where the constants* $\lambda, A, \ 0 < \lambda < A, \ a(\lambda) < b(A)$ *are given and the functions* $a, b \in \mathcal{K}$.

5. *There exists a function* $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ *such that*
   (i) *for any* $k = 0, 1, 2, \dots$ *and* $y_0 \in S(\lambda)$ *the inequality*

$$ {}_{(2.48)}^{\;c}D_+^q V(t, x; t_k, y_0) \le g(t, V(t, x)) \text{ for } t \in (t_k, s_k),\; x \in S(\lambda) $$

   holds;
   (ii) *for any* $k = 0, 1, 2, \dots$ *the inequality*

$$ V(t, \phi_k(t, x)) \le \Psi_k(t, V(s_k - 0, x)) \text{ for } t \in (s_k, t_{k+1}],\; x \in S(\lambda) $$

   holds;
   (iii) $b(||x||) \le V(t, x) \le a(||x||)$ *for* $t \in \mathbb{R}_+,\;\; x \in \mathbb{R}^n$.

Then the zero solution of the system of FrDE (2.48) is practically stable w.r.t. (λ, A) $\Big($*uniformly practically stable w.r.t.* $(\lambda, A)\Big)$.

Proof We will prove only the practical stability since the proof for uniform practical stability is similar.

From condition 4 there exists a point t_0 such that the inequality $|\tilde{u}_0| < a(\lambda)$ implies

$$ |u(t; t_0, \tilde{u}_0)| < b(A) \quad \text{ for } t \ge t_0, \tag{2.167} $$

where $u(t; t_0, \tilde{u}_0)$ is the maximal solution of NIFrDE (2.129) with $m = \infty$ and $u_0 = \tilde{u}_0$.

Choose a point $x_0 \in \mathbb{R}^n$ with $||x_0|| < \lambda$ and let $x(t) = x(t; t_0, x_0)$ be a solution of the IVP for NIFrDE (2.48) for the chosen x_0 and the above t_0. Let $u_0 = V(t_0, x_0)$. According to condition 5(iii) and the choice of x_0 we obtain $u_0 < a(\lambda)$. Therefore the maximal solution $u^*(t) = u(t; t_0, u_0)$ of NIFrDE (2.129) with $m = \infty$ satisfies inequality (2.167).

We now prove

$$ ||x(t; t_0, x_0)|| < A \;\text{ for } t \ge t_0. \tag{2.168} $$

Assume inequality (2.168) is not true and let $t^* = inf\{t > t_0 : \; ||x(t)|| \ge A\}$. We first show

$$ ||x(t)|| < A \quad \text{for } t \in [t_0, t^*) \quad \text{and } ||x(t^*)|| = A. \tag{2.169} $$

If $t^* \ne t_k,\; k = 1, 2, \dots$ or if $t^* = t_p$ for some natural number p and $||x(t_p - 0)|| = A$, then (2.169) is true. If for a natural number p we have $t^* = t_p$ and $||x(t_p - 0)|| < A$, then according to Lemma 2.3.8 for $T = t_p$ and $\Delta = S(\lambda)$ we obtain $V(t, x(t)) \le u^*(t)$ for $t \in [t_0, t_p]$. Then for all $t \in (t_p, s_p]$ from condition 5(iii) we get $b(||x^*(t)||) \le V(t, x(t)) = V(t, \phi_p(t, x(t_p - 0))) \le \psi_p(t, V(t_p - 0, x(t_p - 0))) \le$

$\psi_p(t, u^*(t_p - 0)) = u^*(t)$. Thus using (2.167) we obtain $||x^*(t)|| \le b^{-1}(u^*(t)) < A$ for $t \in (t_p, s_p]$, and this contradicts the choice of t^*. Therefore, (2.169) holds.

Then, $x(t) \in S(\lambda)$ on $[t_0, t^*]$ and conditions (i) and (ii) of Lemma 2.3.8 are satisfied on $[t_0, t^*]$. From Lemma 2.3.8 applied to the solution $x(t)$ with $T = t^*$ and $\Delta = S(\lambda)$ we get $V(t, x(t)) \le u^*(t)$ on $[t_0, t^*]$. Then applying condition 5 (iii) of Theorem 2.3.8 we obtain $b(A) = b(||x(t^*)||) \le V(t^*, x(t^*)) \le u^*(t^*)) < b(A)$. The contradiction proves (2.168) and therefore, the zero solution of NIFrDE (2.48) is practically stable w.r.t. (λ, A). $\square$

Theorem 2.3.9 *Let the following conditions be fulfilled:*

1. *Conditions 1, 2, 3 of Theorem 2.3.8 are satisfied.*
2. *The zero solution of scalar NIFrDE (2.129) with $m = \infty$ is practically quasi stable w.r.t. $(a(\lambda), b(A), T)$ $\Big($uniformly practically quasi stable w.r.t. $(a(\lambda), b(A), T)\Big)$ where the positive constants $T, \lambda, A\ :\ \lambda < A,\ a(\lambda) < b(A)$ are given and the functions $a, b \in \mathcal{K}$.*
3. *There exists a function $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ such that*
   - *(i) for any $k = 0, 1, 2, \dots$ and $y_0 \in S(\lambda)$ the inequality*

$$ {}^{c}_{(2.48)}D^q_+ V(t, x; t_k, y_0) \le G(t, V(t, x)) \text{ for } t \in (t_k, s_k),\ x \in \mathbb{R}^n $$

   holds;
   - *(ii) for any $k = 0, 1, 2, 3, \dots$ the inequality*

$$ V(t, \phi_k(t, x)) \le \Psi_k(t, V(s_k - 0, x)) \text{ for } t \in (s_k, t_{k+1}],\ x \in \mathbb{R}^n $$

   holds;
   - *(iii) $b(||x||) \le V(t, x) \le a(||x||)$ for $t \in \mathbb{R}_+,\ \ x \in \mathbb{R}^n$, where a, b are defined in condition 2.*

Then the zero solution of the system of NIFrDE (2.48) is practically quasi stable w.r.t. (λ, A, T) $\Big($uniformly practically quasi stable w.r.t. $(\lambda, A, T)\Big)$.

Proof We will prove only the practical quasi stability since the proof of the uniform practical quasi stability is similar.

From condition 2 there exists a point t_0 such that the inequality $|\tilde{u}_0| < a(\lambda)$ implies

$$ |u(t; t_0, \tilde{u}_0)| < b(A) \quad \text{for } t \ge t_0 + T, \tag{2.170} $$

where $u(t; t_0, \tilde{u}_0)$ is the maximal solution of NIFrDE (2.129) with $m = \infty$.

Choose a point $x_0 \in \mathbb{R}^n$ with $||x_0|| < \lambda$ and let $x(t; t_0, x_0)$ be a solution of the IVP for NIFrDE (2.48) for the chosen x_0 and the above t_0. Let $u_0^* = V(t_0, x_0)$. According to condition 3(iii) and the choice of x_0 we obtain $u_0^* < a(\lambda)$ and therefore the solution $u(t; t_0, u_0^*)$ of (2.129) with $m = \infty$ satisfies (2.170).

From Lemma 2.3.8 and Remark 2.3.7 applied to the solution $x(t; t_0, x_0)$ with $\Delta = \mathbb{R}^n$ we get

$$V(t, x(t; t_0, x_0)) \leq u(t; t_0, u_0^*) \quad \text{for} \ \ t \geq t_0. \tag{2.171}$$

From condition 3(iii) and inequalities (2.170) with $\tilde{u}_0 = u_0$ and (2.171) we obtain for any $t \geq t_0+T$ the inequalities $b(||x(t; t_0, x_0)||) \leq V(t, x(t; t_0, x_0)) \leq u(t; t_0, u_0^*) < b(A)$, i.e., the inequality

$$||x(t; t_0, x_0)|| < A \ \ \text{for } t \geq t_0 + T$$

holds. Thus, the zero solution of the system of NIFrDE (2.48) is practically quasi stable w.r.t. (λ, A, T). □

Theorem 2.3.10 *Let the following conditions be fulfilled:*

1. *Conditions 1, 2, 3 of Theorem 2.3.8 are satisfied.*
2. *The zero solution of scalar NIFrDE (2.129) with $m = \infty$ is strongly practically stable w.r.t. the quadruplet $(a(\lambda), b(A), B, T)$ $\Big($uniformly strongly practically stable w.r.t. $(a(\lambda), b(A), B, T)\Big)$ where the positive constants $T, \lambda, A, B\ :\ B < \lambda < A,\ a(\lambda) < b(A)$ are given and the functions $a, b \in \mathcal{K}$.*
3. *Condition 3 of Theorem 2.3.9 is satisfied,*

Then the zero solution of the system of NIFrDE (2.48) is strongly practically stable w.r.t. (λ, A, B, T) $\Big($uniformly strongly practically stable w.r.t. $(\lambda, A, B, T)\Big)$.

Proof The proof of Theorem 2.3.10 is similar to the one in Theorem 2.3.8, so we omit it. □

Example 2.3.5.3 Let $t_k = 2k, s_k = 2k + 1$ for $k = 0, 1, 2, \dots$. Consider a function $m \in C^q(\cup_{k=0}^{\infty}(t_k, s_k], (0, \infty)) \cup C(\cup_{k=0}^{\infty}(s_k, t_{k+1}], \mathbb{R}_+)$, $\lim_{t\to s_k+0} m(t) = m(s_k + 0) < \infty$, $m(s_k) = \lim_{t\to s_k-0} m(t)$, $m(t) \leq m(s_k)$ for $t \in [s_k, t_{k+1}]$, $k = 0, 1, 2, \dots$ and there exist positive constants $K, M,\ K < M$ such that $K \leq m(t) \leq M$ for $t \geq 0$. Note, for example, the function $m(t) = \sin^2(\frac{\pi}{2}t) + 0.5$, $t \geq 0$ satisfies the above conditions with $K = 0.5,\ M = 1.5$.

Consider the initial value problem for the non-instantaneous impulsive fractional differential equation with a Caputo derivative for $0 < q < 1$

$$\begin{aligned} &{}_{t_k}^{c}D^q x(t) = -0.5\frac{x}{m(t)}\Big({}_{t_k}^{C}D^q m(t) + m(t_k)\frac{(t - t_k)^{-q}}{\Gamma(1 - q)}\Big) \\ &\qquad\qquad \text{for } t \in (t_k, s_k],\ k = 0, 1, 2, \dots \\ &x(t) = \phi_k(t, x(t_k - 0)) \quad \text{for } t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \dots, \\ &x(0) = x_0, \end{aligned} \tag{2.172}$$

where $x, x_0 \in \mathbb{R}$,

$$f(t,x) = -0.5\frac{x}{m(t)}\Big({}_{t_k}^{C}D^q m(t) + m(t_k)\frac{(t-t_k)^{-q}}{\Gamma(1-q)}\Big), \quad t \in [t_k, s_k],\ k = 0, 1, 2, \dots,\ x \in \mathbb{R},$$

and

$$\phi_k(t,x) = \begin{cases} x\sqrt{1+\frac{1}{t}}, & \text{for } k = 0,\ t \in [0,1],\ x \in \mathbb{R} \\ x\sqrt{1-\frac{1}{t}}, & \text{for } k = 1, 2, \dots,\ t \in [2k-1, 2k],\ x \in \mathbb{R}. \end{cases}$$

Let $V(t,x) = m(t)x^2$. Then $V(t,0) = 0$, $b(|x|) \le V(t,x) \le a(|x|)$ with $b(s) = Ks^2$, $a(s) = Ms^2$. For any $x_0 \in \mathbb{R}$ we obtain

$$\begin{aligned} &{}_{(2.172)}^{\ c}D_+^q V(t,x;t_k,x_0) \qquad (2.173) \\ &= 2xm(t)f(t,x) + x^2\, {}_{t_k}^{C}D^q m(t) + (x^2 - x_0^2)m(t_k)\frac{(t-t_k)^{-q}}{\Gamma(1-q)} \le 0 \end{aligned}$$

for $t \in (t_k, s_k)$, $x \in \mathbb{R}$.

Also, we obtain the inequalities

$$V(t, \phi_1(t,x)) \le m(t_1)\big(1 + \frac{1}{t}\big)x^2 = \psi_1(V(t_1,x)),\ t \in (t_1, s_1], x \in \mathbb{R}$$

and

$$V(t, \phi_k(t,x)) \le m(t_k)(1 - \frac{1}{t})x^2 = \Psi_k(V(t_k,x)),\ k = 0, 1, 2, \dots,\ t \in (t_k, s_k],\ x \in \mathbb{R}$$

where

$$\Psi_k(t,x) = \begin{cases} x(1+\frac{1}{t}), & \text{for } k = 0,\ t \in [0,1],\ x \in \mathbb{R} \\ x(1-\frac{1}{t}), & \text{for } k = 1, 2, \dots,\ t \in [2k-1, 2k],\ x \in \mathbb{R}. \end{cases}$$

Therefore all the conditions in Theorems 2.3.8, 2.3.9, and 2.3.10 are satisfied with $G(t,x) \equiv 0, t_0 = 0$. Consider the comparison scalar NIFrDE (2.129), $m = \infty$, with $G(t,x) \equiv 0, t_0 = 0$ where practical stability properties of the zero solution were discussed in Example 2.3.5.2. This allows us to discuss practical stability properties of the zero solution of the nonlinear NIFrDE (2.172). If $m(t) = \sin^2(\frac{\pi}{2}t) + 0.5$, $t \ge 0$, the following conclusions for the nonlinear NIFrDE (2.172) can be made:

– the zero solution of (2.172) is practically stable w.r.t. the couple $(\sqrt{\frac{0.5}{M}}, \sqrt{\frac{1}{K}})$, i.e., if $|x_0| < \sqrt{\frac{0.5}{M}}$, then $|x(t;0,x_0)| < \sqrt{\frac{1}{K}}$ for $t \ge 0$. It follows from Theorem 2.3.8 the practical stability of the zero solution of (2.129) with $m = \infty$ w.r.t. the couple $(0.5, 1)$;

– the zero solution of (2.172) is practically stable w.r.t. $(\sqrt{\frac{0.6}{M}}, \sqrt{\frac{1}{K}}, 5)$, i.e., if $|x_0| < \sqrt{\frac{0.6}{M}}$, then $|x(t; 0, x_0)| < \sqrt{\frac{1}{K}}$ for $t \geq 5$. It follows from Theorem 2.3.9 the practical quasi stability of the zero solution of (2.129) with $m = \infty$ w.r.t. $(0.6, 1, 5)$;
– the zero solution of (2.172) is strongly practically stable w.r.t. $(\sqrt{\frac{0.5}{M}}, \sqrt{\frac{1}{K}}, \sqrt{\frac{0.4}{K}}, 10)$, i.e., if $|x_0| < \sqrt{\frac{0.5}{M}}$, then $|x(t; 0, x_0)| < \sqrt{\frac{1}{K}}$ for $t \geq 0$ and $|x(t; 0, x_0)| < \sqrt{\frac{0.4}{K}}$ for $t \geq 10$. It follows from Theorem 2.3.10 the strong practical stability of the zero solution of (2.129) with $m = \infty$ w.r.t. $(0.5, 1, 0.4, 10)$. □

2.3.6 *Strict Stability of NIFrDE*

The usual stability concepts do not give any information concerning the rate of decay of the solutions, and hence are not strict concepts. As a result, strict stability was defined and criteria for such notions was discussed (see, for example, [75, 77, 111]).

We will define strict stability for fractional equations following the idea for ordinary differential equations (see, for example, [77]).

Definition 2.3.9 *The zero solution of the system NIFrDE (2.48) is said to be*

– strictly stable *if for given* $\epsilon_1 > 0$ *and any* $t_0 \in [0, s_0] \bigcup \cup_{k=1}^{\infty} [t_k, s_k)$ *there exists* $\delta_1 = \delta_1(t_0, \epsilon_1) > 0$ *such that for any initial point* $x_0 \in \mathbb{R}^n$ *the inequality* $||x_0|| < \delta_1$ *implies* $||x(t; t_0, x_0)|| < \epsilon_1$, $t \geq t_0$, *and for any* $\delta_2 = \delta_2(t_0, \epsilon_1)$, $\delta_2 \in (0, \delta_1]$ *there exists* $\epsilon_2 = \epsilon_2(t_0, \delta_2)$, $\epsilon_2 \in (0, \delta_2]$ *such that the inequality* $\delta_2 < ||x_0||$ *implies* $\epsilon_2 < ||x(t; t_0, x_0)||$ *for* $t \geq t_0$ *where* $x(t; t_0, x_0)$ *is a solution of the IVP for the NIFrDE (2.48);*
– uniformly strictly stable *if for any given* $\epsilon_1 > 0$ *there exists* $\delta_1 = \delta_1(\epsilon_1) > 0$ *such that for any initial time* $t_0 \in [0, s_0] \bigcup \cup_{k=1}^{\infty} [t_k, s_k)$ *and any initial point* $x_0 \in \mathbb{R}^n$ *the inequality* $||x_0|| < \delta_1$ *implies* $||x(t; t_0, x_0)|| < \epsilon_1$, $t \geq t_0$, *and for any* $\delta_2 \in (0, \delta_1]$ *there exists* $\epsilon_2 \in (0, \delta_2]$, $\epsilon_2 = \epsilon_2(\delta_2)$, *such that the inequality* $\delta_2 < ||x_0||$ *implies* $\epsilon_2 < ||x(t; t_0, x_0)||$ *for* $t \geq t_0$ *where* $x(t; t_0, x_0)$ *is a solution of the IVP for the NIFrDE (2.48).*

Example 2.3.6.1 (Strict Stability of NIFrDE) Consider the following scalar IVP for FrDE ${}_{t_0}^{c}D^q x = 0$, $t \geq t_0$, $x(t_0) = x_0$ with an arbitrary $t_0 \in \mathbb{R}_+$. Its solution $x(t) = x_0$ is uniformly strictly stable.

Now let $t_k = 2k$, $s_k = 2k + 1$, $k = 0, 1, 2, \dots$.

Consider the IVP for NIFrDE

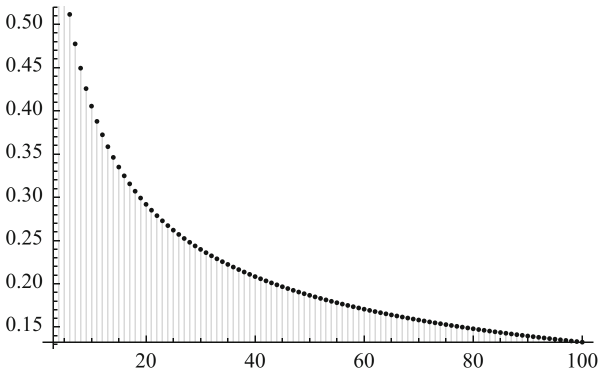

Fig. 2.14 Example 2.3.6.1. Graph of the impulsive functions $\prod_{i=1}^{n} \frac{2i}{2i+1}$.

$$\begin{aligned}
&{}_{t_k}^{c}D^q x = 0 \text{ for } t \in \cup_{k=0}^{\infty}(2k, 2k+1], \\
&x(t) = \Xi_k(t, x(2k-1-0)) \text{ for } \ t \in (2k-1, 2k],\ k = 1, 2, \dots, \qquad (2.174)\\
&x(0) = x_0,
\end{aligned}$$

where $x, x_0 \in \mathbb{R}$ and $\Xi_k(t, x) = a_k(t)x,\ a_k : [2k-1, 2k] \to \mathbb{R},\ k = 1, 2, \dots$. Then the solution of NIFrDE (2.174) is given by

$$x(t) = \begin{cases} x_0 & \text{for } \ t \in [0, 1] \\ x_0 a_{2k+1}(t) \prod_{j=p}^{k-1} a_{2j+1}(2j+2) & \text{for } t \in (2k+1, 2k+2],\ k = 1, 2, \dots \\ x_0 \prod_{j=p}^{k} a_{2j+1}(2j+2) & \text{for } t \in (2k, 2k+1],\ k = 1, 2, \dots. \end{cases}$$

The type of the non-instantaneous impulsive functions $\Xi_k(t, x)$, i.e., $a_k(t)$ has an influence on the behavior of the solution of NIFrDE.

Case 1. Let $a_k(t) = \frac{t}{t+1}$. Then $\lim_{n\to\infty} \prod_{i=1}^{n} \frac{2i}{2i+1} = 0$ (see the graph of $\prod_{i=1}^{n} \frac{2i}{2i+1}$ in Figure 2.14). Thus the zero solution of (2.174) is asymptotically stable (see Figure 2.15).

Case 2. Let $a_k(t) = 1 + \frac{(-1)^k}{t}$ for $k = 1, 2, 3, \dots$. Then $\lim_{n\to\infty} \prod_{i=1}^{n} \frac{(-1)^i}{2i} \neq 0$ (see Figure 2.16). Thus the solution of (2.174) is not asymptotically stable but it is strictly stable (see Figure 2.17). □

We will use the following set:

$$\mathcal{M} = \{a \in \mathcal{K} \text{ and } \lim_{s\to\infty} a(s) = \infty\},$$

We will use the following type of a couple of Caputo fractional differential equations with non-instantaneous impulses

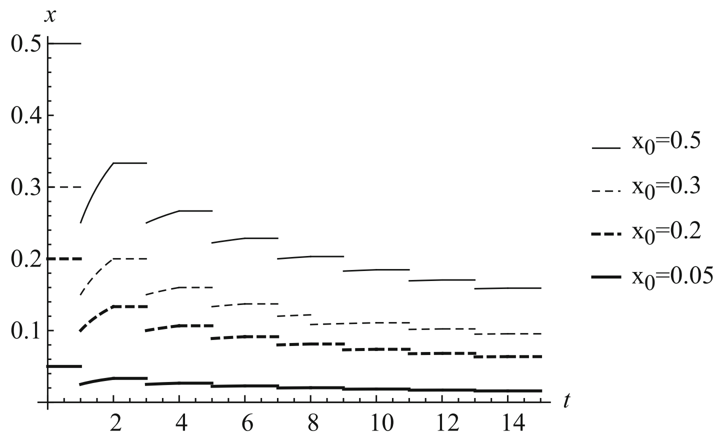

Fig. 2.15 Example 2.3.6.1. Graphs of solutions of (2.174) for $t_0 = 0$ and various x_0, *Case 1*.

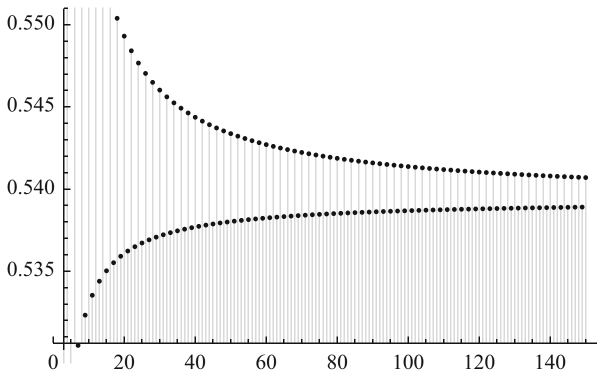

Fig. 2.16 Example 2.3.6.1. Graph of the impulsive functions $\prod_{i=1}^{n}(1 + \frac{(-1)^i}{2i})$.

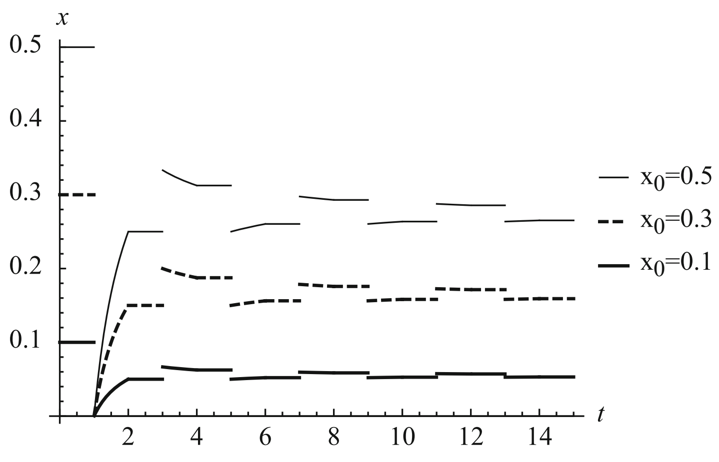

Fig. 2.17 Example 2.3.6.1. Graphs of solutions of (2.174) for $t_0 = 0$ and various x_0,*Case 2*.

$$
\begin{aligned}
&{}_{t_k}^{c}D^q u = g_1(t,u), \qquad {}_{t_k}^{c}D^q v = g_2(t,v), \\
&\qquad\qquad t \in [t_0,\infty)\bigcap\Big([0,s_0]\bigcup\cup_{k=1}^{\infty}(t_k,s_k]\Big), \\
&u(t) = \Phi_k(t,u(t_k-0)), \quad v(t) = \Psi_k(t,v(t_k-0)), \\
&\qquad\qquad t \in [t_0,\infty)\bigcap(s_k,t_{k+1}],\ k = 0,1,2,\dots \\
&u(t_0) = u_0, \qquad v(t_0) = v_0,
\end{aligned}
\tag{2.175}
$$

where $u, v \in \mathbb{R}$, $t_0 \in [0,s_0]\bigcup\cup_{k=1}^{\infty}[t_k,s_k)$, $g_1, g_2 : [0,s_0]\bigcup\cup_{k=1}^{\infty}(t_k,s_k]\times\mathbb{R}\to\mathbb{R}$, $g_j(t,0)\equiv 0, (j = 1,2)$, $\Phi_i, \Psi_i : [s_i,t_{i+1}]\times\mathbb{R}^n\to\mathbb{R}^n$, $\Phi_i(t,0) = 0,\ \Psi_i(t,0) = 0$, $(i = 0,1,2,3,\dots)$.

We will introduce the strict stability of the couple of Caputo fractional differential equations as follows:

Definition 2.3.10 *The zero solution of the couple of NIFrDE (2.175) is said to be*

- strictly stable in couple *if for given* $\epsilon_1 > 0$ *and* $t_0 \in [0,s_0]\bigcup\cup_{k=1}^{\infty}[t_k,s_k)$ *there exists* $\delta_1 = \delta_1(t_0,\epsilon_1) > 0$ *and for any* $\delta_2 = \delta_2(t_0,\epsilon_1)$, $\delta_2 \in (0,\delta_1]$ *there exists* $\epsilon_2 = \epsilon_2(t_0,\delta_2)$, $\epsilon_2 \in (0,\delta_2]$ *such that the inequalities* $|u_0| < \delta_1$ *and* $\delta_2 < |v_0|$ *imply* $|u(t;t_0,u_0)| < \epsilon_1$ *and* $\epsilon_2 < |v(t;t_0,v_0)|$ *for* $t \geq t_0$ *where the couple of functions* $\big(u(t;t_0,u_0), v(t;t_0,u_0)\big)$ *is a solution of the IVP for the couple of NIFrDE (2.175).*
- uniformly strictly stable in couple *if for any given* $\epsilon_1 > 0$ *there exists* $\delta_1 = \delta_1(\epsilon_1) > 0$ *and for any* $\delta_2 \in (0,\delta_1]$ *there exists* $\epsilon_2 \in (0,\delta_2]$, $\epsilon_2 = \epsilon_2(\delta_2)$, *such that for any initial time* $t_0 \in [0,s_0]\bigcup\cup_{k=1}^{\infty}[t_k,s_k)$ *the inequalities* $|u_0| < \delta_1$ *and* $\delta_2 < |v_0|$ *imply* $|u(t;t_0,u_0)| < \epsilon_1$ *and* $\epsilon_2 < |v(t;t_0,v_0)|$ *for* $t \geq t_0$ *where the couple of functions* $\big(u(t;t_0,u_0), v(t;t_0,u_0)\big)$ *is a solution of the IVP for the couple of NIFrDE (2.175).*

Remark 2.3.17 *Note if the zero solution of the couple of NIFrDE (2.175) is strictly stable,* $|v(t)| \leq |u(t)|, t \geq t_0$, *then according to Definition 2.3.10 the inequalities* $\delta_2 < |v_0| \leq |u_0| < \delta_1$ *provide* $\epsilon_2 < |v(t)| \leq |u(t)| < \epsilon_1$ *for* $t \geq t_0$, *i.e., the solutions remain in an appropriate tube.*

Remark 2.3.18 *If* $g_1(t,x) \equiv g_2(t,x)$, $\Psi_k(t,x) \equiv \Phi_k(t,x)$, $k = 1,2,\dots$ *in (2.175), then the strict stability (uniform strict stability) in a couple given by Definition 2.3.10 is reduced to a strict stability (uniform strict stability) of the zero solution of a scalar NIFrDE defined by Definition 2.3.9.*

Example 2.3.6.2 (Uniform Strict Stability in Couple) Let $t_k = 2k$, $s_k = 2k+1$ for $k = 0,1,2,\dots$. Consider the couple of Caputo fractional differential equations with non-instantaneous impulses

$$
\begin{aligned}
&{}_0^cD^q u = Au, \qquad\qquad {}_0^cD^q v = -Bv,\\
&\qquad\qquad t \in (2k, 2k+1],\ k = 0,1,2,\dots,\\
&u(t) = \frac{b_k}{E_q(A)} u(t_k - 0), \qquad v(t) = \frac{c_k}{E_q(-B)} v(t_k - 0)),\\
&\qquad\qquad t \in (2k+1, 2k+2],\ \ k = 0,1,2,\dots,\\
&u(0) = u_0, \qquad\qquad v(0) = v_0,
\end{aligned} \tag{2.176}
$$

where $u, v \in \mathbb{R}, A, B > 0, b_k :\ |b_k| \le 1, c_k :\ |c_k| \ge 1$ are given constants such that $\prod_{i=0}^{\infty} b_i \le M$ and $\prod_{i=0}^{\infty} c_i \ge N > 0$ with $N \le \frac{1}{E_q(-B)}$.

The solution of (2.176) is given by

$$
u(t) = \begin{cases} u_0\Big(\prod_{j=0}^{k-1} b_j\Big)E_q(A(t-2k)^q) & \text{for } t \in (2k, 2k+1],\ k = 0,1,2,\dots,\\ u_0\Big(\prod_{j=0}^{k} b_j\Big) & \text{for } t \in (2k+1, 2k+2],\ k = 0,1,2,\dots \end{cases}
$$

and

$$
v(t) = \begin{cases} v_0\Big(\prod_{j=0}^{k-1} c_j\Big)E_q(-B(t-2k)^q) & \text{for } t \in (2k, 2k+1],\ k = 0,1,2,\dots,\\ v_0\Big(\prod_{j=0}^{k} c_j\Big) & \text{for } t \in (2k+1, 2k+2],\ k = 0,1,2,\dots \end{cases}
$$

where the Mittag-Leffler function (with one parameter) is defined by $E_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+1)}$.

Let $\epsilon_1 > 0$ be arbitrary. Choose $\delta_1 = \frac{\epsilon_1}{ME_q(A)}$ and let $|u_0| < \delta_1$. Then from $1 \le E_q(A(t-2k)^q) \le E_q(A)$ for $t \in (2k, 2k+1]$ we obtain $|u(t)| \le |u_0|\Big(\prod_{j=1}^{k} b_j\Big)E_q(A) < \epsilon_1$. For any $\delta_2 \in (0, \delta_1]$ we choose $\epsilon_2 = \delta_2 E_q(-B)N \le \delta_2$. Then for $|v_0| > \delta_2$ using $1 \ge E_q(-B(t-2k)^q) \ge E_q(-B)$ we obtain $|v(t)| \ge |v_0|\Big(\prod_{j=1}^{k} c_j\Big)E_q(-B) > \epsilon_2$. Therefore, the zero solution of the couple of FrDE (2.176) is uniformly strictly stable in couple.

Note, for example, the above conclusion is true for $b_k = 1 - \frac{1}{2^k}, c_k = 1 + \frac{1}{2^k}, M = 0.288, N = 2.38$ (see Figure 2.18 for $q = 0.2$, $A = B = 1$, and various initial conditions).

The conclusion of Example 2.3.6.2 is true also for $A = B = 0$. □

We obtain sufficient conditions for strict stability of the system NIFrDE (2.48).

Theorem 2.3.11 (Strict Stability of NIFrDE) *Let the following conditions be satisfied:*

1. *Condition (H2.3.3) is satisfied*
2. *The functions* $g_1, g_2 \in [0, s_0] \bigcup \cup_{k=1}^{i} nfty(t_k, s_k]$ *satisfies the conditions (H2.3.2.2) and (H2.3.2.3) respectively and the functions* $\Phi_k, \Psi_k \in C([s_k, t_{k+1}] \times \mathbb{R}, \mathbb{R})$*,* $(k = 0, 1, 2, \dots)$*,* $\Phi_k(t, 0) = 0$*,* $\Psi_k(t, 0) = 0$ *satisfies the condition (H2.3.2.4).*

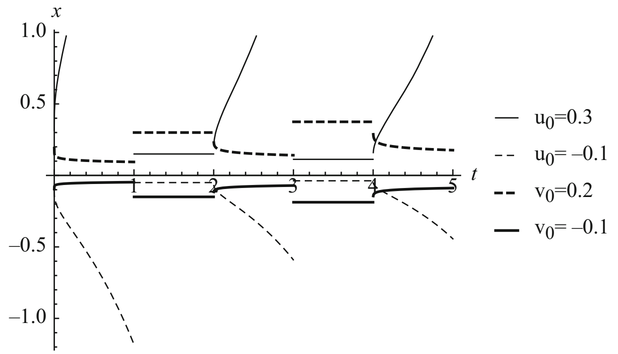

Fig. 2.18 Example 2.3.6.2. Graphs of solutions $(u(t), v(t))$ for various initial values.

3. *There exists a function* $V_1 \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ *such that* $V_1(t, 0) \equiv 0$ *for* $t \in \mathbb{R}_+$ *and*
   (i) *for any* $k = 0, 1, \ldots$ *and* $y_0 \in \mathbb{R}^n$ *the inequality*

$$ {}_{(2.48)}^{\;\;c}D_+^q V_1(t, x; t_k, y_0) \le g_1(t, V_1(t, x)) \quad \text{for } t \in (t_k, s_k),\ x \in \mathbb{R}^n $$

   holds;
   (ii) *for any* $k = 0, 1, \ldots$ *the inequality*

$$ V_1(t, \psi_k(t, x)) \le \Phi_k(t, V_1(s_k - 0, x)) \quad \text{for } t \in (s_k, t_{k+1}],\ x \in \mathbb{R}^n $$

   holds;
   (iii) $a(||x||) \le V_1(t, x)$ *for* $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, *where* $a \in \mathcal{K}$.
4. *There exists a function* $V_2 \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ *such that*
   (iv) *for any* $k = 0, 1, \ldots$ *and* $y_0 \in \mathbb{R}^n$ *the inequality*

$$ {}_{(2.48)}^{\;\;c}D_+^q V_2(t, x; t_k, y_0) \ge g_2(t, V_2(t, x)) \quad \text{for } t \in (t_k, s_k),\ x \in \mathbb{R}^n $$

   holds;
   (v) *for any* $k = 0, 1, \ldots$ *the inequality*

$$ V_2(t, \psi_k(t, x)) \ge \Psi_k(t, V_2(s_k - 0, x)) \quad \text{for } t \in (s_k, t_{k+1}],\ x \in \mathbb{R}^n $$

   hold;
   (vi) $c(||x||) \le V_2(t, x) \le b(||x||)$ *for* $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, *where* $b, c \in \mathcal{M}$.
5. *The zero solution of NIFrDE (2.175) is strictly stable in couple.*

Then the zero solution of the system NIFrDE (2.48) is strictly stable.

Proof Let $\epsilon_1 > 0$ and $t_0 \in [0, s_0] \bigcup \cup_{k=1}^{\infty} [t_k, s_k)$ be arbitrary. Without loss of generality we assume $t_0 \in [0, s_0]$.

From condition 5 there exists $\delta_1 = \delta_1(t_0, \epsilon_1) \geq 0$ and for any $\delta_2 \in (0, \delta_1]$ there exists $\epsilon_2 \in (0, \delta_2]$ such that $|u_0| < \delta_1$ and $|v_0| > \delta_2$ imply

$$|u(t; t_0, u_0)| < a(\epsilon_1) \quad \text{for } t \geq t_0, \tag{2.177}$$

$$|v(t; t_0, v_0)| > \epsilon_2 \quad \text{for } t \geq t_0, \tag{2.178}$$

where the couple $\big(u(t; t_0, u_0), v(t; t_0, v_0)\big)$ is a solution of (2.175).

Since $V_1(t_0, 0) = 0$ there exists $\delta_3 = \delta_3(t_0, \epsilon_1),\ \delta_3 \in (0, \delta_1)$ such that $V_1(t_0, x) < \delta_1$ for $||x|| < \delta_3$.

Choose $x_0 \in \mathbb{R}^n$ with $||x_0|| < \delta_3$ and let $x^*(t) = x(t; t_0, x_0)$ be a solution of the IVP for NIFrDE (2.48) for the initial data (t_0, x_0).

Let $u_0 = V_1(t_0, x_0)$ and $u^*(t) = u(t; t_0, u_0)$ be the maximal solution of the first equations in the IVP for NIFrDE (2.175) with initial value u_0. Note it exists according to condition 2 and Lemma 2.3.6 with $m = \infty$.

Then from the choice of δ_3 it follows $|u_0| = V_1(t_0, x_0) < \delta_1$ and therefore inequality (2.177) holds for $u^*(t)$.

From conditions 2, 3(i), 3(ii) of Theorem 2.3.11 and Lemma 2.3.8 (with $T = \infty$) applied to the solutions $x^*(t), u^*(t)$ the inequality

$$V_1(t, x^*(t)) \leq u^*(t),\ t \geq t_0 \tag{2.179}$$

holds. From condition 3(iii) and inequalities (2.177) for $u^*(t)$ and (2.179) we obtain

$$a(||x^*(t)||) \leq V(t, x^*(t)) \leq u^*(t) < a(\epsilon_1),\ t \geq t_0.$$

Now let $\delta_4 \in (0, \delta_3]$ be an arbitrary number. Then there exists $\delta_5 \in (0, \delta_4]$ such that $c(\delta_4) > \delta_5$. According to condition 5 for $\delta_5 \in (0, \delta_1]$ there exists $\epsilon_3 \in (0, \delta_5]$ such that $|v_0| > \delta_5$ implies

$$|v(t; t_0, v_0)| > \epsilon_3, \quad t \geq t_0. \tag{2.180}$$

Choose $\epsilon_4 > 0$ such that $\epsilon_4 < \min\{b^{-1}(\epsilon_3), \delta_4\}$.

Assume that the initial value x_0 additionally satisfies the inequality $||x_0|| > \delta_4$ and consider the minimal solution $v^*(t) = v(t; t_0, v_0)$ of the second equations of (2.175) with the initial value $v_0 = V_2(t_0, x_0)$ (it exists according to condition 2 and Lemma 2.3.7 with $m = \infty$). From the choice of x_0 and condition 4(iv) it follows that $|v_0| = V_2(t_0, x_0) \geq c(||x_0||) > c(\delta_4) > \delta_5$. Therefore, the function $v^*(t)$ satisfies the inequality (2.180). From condition 4(vi) we obtain $b(||x^*(t)||) \geq V(t, x^*(t)) \geq v^*(t) > \epsilon_3, t \geq t_0$. Therefore $||x^*(t)|| \geq b^{-1}(\epsilon_3) > \epsilon_4$ for $t \geq t_0$.

Since δ_4 is an arbitrary, from the above we have the strict stability of the zero solution of NIFrDE (2.48). □

Remark 2.3.19 *If all solutions of IVP for NIFrDE (2.48) satisfy $||x(t)|| \leq (\geq)||x_0||$, then the claim of Theorem 2.3.11 is true if the conditions 3, 4 are satisfied only for points $x, x_0 \in \mathbb{R}^n$ such that $||x|| \leq (\geq)||x_0||$.*

Theorem 2.3.12 (Uniform Strict Stability of NIFrDE) *Let the following conditions be satisfied:*

1. *Conditions 1 and 2 of Theorem 2.3.11 are satisfied.*
2. *There exists a function $V_1 \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ such that*
   (i) for any $k = 0, 1, 2, \ldots$ and $y_0 \in B(\lambda)$ the inequality
   $$ {}_{(2.48)}^{\;\;c}D_+^q V_1(t, x; t_k, y_0) \leq g_1(t, V_1(t, x)) \quad \text{for } t \in (t_k, s_k),\ x \in B(\lambda)$$
   holds where $\lambda > 0$ is a given number;
   (ii) for any $k = 0, 1, 2, \ldots$ the inequality
   $$V_1(t, \phi_k(t, x)) \leq \Phi_k(t, V_1(s_k - 0, x)) \ \text{for } t \in (s_k, t_{k+1}],\ x \in B(\lambda)$$
   holds;
   (iii) $a(||x||) \leq V_1(t, x) \leq b(||x||)$ for $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, where $a, b \in \mathcal{K}$.
3. *For each $\eta \in (0, \lambda)$ there exists a function $V_\eta \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ such that*
   (iv) for any $k = 0, 1, 2, \ldots$ and $y_0 \in B(\lambda)$ the inequality
   $$ {}_{(2.48)}^{\;\;c}D_+^q V_\eta(t, x; t_k, y_0) \ \geq g_2(t, V_\eta(t, x)) \quad \text{for } t \in [0, s_0] \bigcup \cup_{k=1}^{\infty}(t_k, s_k),$$
   $$x \in B(\lambda), ||x|| \geq \eta$$
   holds;
   (v) for any $k = 0, 1, 2, \ldots$ the inequality
   $$V_\eta(t, \phi_k(t, x)) \geq \Psi_k(t, V_\eta(s_k - 0, x)) \ \text{for } t \in (s_k, t_{k+1}],\ x \in B(\lambda), ||x|| \geq \eta$$
   holds;
   (vi) $c(||x||) \leq V_\eta(t, x) \leq d(||x||)$ for $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, where $d, c \in \mathcal{K}$
4. *The zero solution of the couple of NIFrDE (2.175) is uniformly strict stable in couple.*

Then the zero solution of the system NIFrDE (2.48) is uniformly strictly stable.

Proof Let $\epsilon_1 \in (0, \lambda]$ be an arbitrary number. From condition 4 there exists $\delta_1 = \delta_1(\epsilon_1) > 0$ and for any $\delta_2 \in (0, \delta_1]$ there exists $\epsilon_2 \in (0, \delta_2]$ such that for any $t_0 \in [0, s_0] \bigcup \cup_{k=0}^{\infty}[t_k, s_k)$ the inequalities $|u_0| < \delta_1$ and $\delta_2 < |v_0|$ imply

$$|u(t; t_0, u_0)| < a(\epsilon_1), \qquad t \geq t_0 \tag{2.181}$$

and

$$\epsilon_2 < |v(t; t_0, v_0)|, \qquad t \ge t_0 \tag{2.182}$$

where the couple of functions $\big(u(t; t_0, u_0), v(t; t_0, u_0)\big)$ is a solution of the IVP for NIFrDE (2.175).

Let $\delta_3 \in (0, \lambda)$ be such that $b(\delta_3) < \delta_1$. Choose $x_0 \in \mathbb{R}^n$ with $||x_0|| < \delta_3$ and $x^*(t) = x(t; t_0, x_0)$ be the solution of the IVP for NIFrDE (2.48) for the initial data (t_0, x_0).

Let $u_0 = V_1(t_0, x_0)$ and $u^*(t) = u(t; t_0, u_0)$ be the maximal solution of the first equations in the IVP for NIFrDE (2.175) with initial value u_0. Note it exists because of Lemma 2.3.6 with $m = \infty$. According to condition 2(iii) and the choice of x_0 we obtain $u_0 = V_1(t_0, x_0) \le b(||x_0||) < b(\delta_3) < \delta_1$. Therefore the function $u^*(t)$ satisfies (2.181).

Assume inequality

$$||x^*(t)|| < \epsilon_1 \ \text{ for } t \ge t_0 \tag{2.183}$$

is not true. There are three cases to consider.

Case 1. There exists a point $t^* > t_0,\ t^* \ne s_k, k = 0, 1, \ldots$ such that

$$||x^*(t)|| < \epsilon_1 \quad \text{for } t \in [t_0, t^*) \quad \text{and} \quad ||x^*(t^*)|| = \epsilon_1. \tag{2.184}$$

According to Lemma 2.3.8 for $T = t^*$ and $\Delta = B(\lambda)$ we obtain $V_1(t, x^*(t)) \le u^*(t)$ for $t \in [t_0, t^*]$. From condition 2(iii) and inequality (2.181) for $u^*(t)$ we get $a(\epsilon_1) = a(||x^*(t^*)||) \le V_1(t, x^*(t^*)) \le u^*(t^*) < a(\epsilon_1)$. We obtain a contradiction.

Case 2. There exists an integer $k \ge 0$ such that

$$||x^*(t)|| < \epsilon_1 \quad \text{for } t \in [t_0, s_k) \quad \text{and} \quad ||x^*(s_k - 0)|| = \epsilon_1. \tag{2.185}$$

As in the Case 1 with $t^* = s_k$ we obtain a contradiction.

Case 3. There exists an integer $k \ge 0$ such that

$$||x^*(t)|| < \epsilon_1 \quad \text{for } t \in [t_0, s_k] \quad \text{and} \quad ||x^*(s_k + 0)|| \ge \epsilon_1. \tag{2.186}$$

From Lemma 2.3.8 for $T = s_k$ and $\Delta = B(\lambda)$ we obtain $V_1(t, x^*(t)) \le u^*(t)$ for $t \in [t_0, s_k]$.

Then $x^*(s_k + 0) = \phi_k(s_k, x^*(s_k - 0))$ and according to conditions 2(ii) and 2(iii) we get

$$\begin{aligned} a(\epsilon_1) &\le a(||x^*(s_k + 0)||) = a(||\phi_k(s_k, x^*(s_k - 0)||) \\ &\le V_1(s_k, \phi_k(s_k, x^*(s_k - 0)) \ \le \Psi_k(s_k, V_1(s_k - 0, x^*(s_k - 0)) \\ &\le \Psi_k(s_k, u^*(s_k - 0)) = u^*(s_k + 0)) < a(\epsilon_1). \end{aligned} \tag{2.187}$$

The contradictions above prove inequality (2.183) is true.

Let $\delta_4 \in (0, \delta_3]$ be an arbitrary number. Then there exists $\delta_5 \in (0, \delta_4]$ such that $c(\delta_4) > \delta_5$. Let the initial value $x_0 \in \mathbb{R}^n$ additionally satisfy $||x_0|| > \delta_4$. From condition 3(*iv*) for $\eta = \delta_4$ there exists a function $V_\eta(t, x)$ and $V_\eta(t_0, x_0) \geq c(||x_0||) > c(\delta_4) > \delta_5$. Let $v_0 = V_\eta(t_0, x_0)$ and $v^*(t) = v(t; t_0, v_0)$ be the minimal solution of the second equations of the couple of NIFrDE (2.175) (it exists because of Lemma 2.3.7 with $m = \infty$). According to condition 4 there exists $\epsilon_2^* \in (0, \delta_5]$ such that $|v_0| > \delta_5$ implies the inequality (2.182) with $\epsilon_2 = \epsilon_2^*$.

Choose $\epsilon_3 \in (0, \delta_4]$ such that $\epsilon_3 < d^{-1}(\epsilon_2^*)$. Therefore, $\epsilon_3 < d^{-1}(\epsilon_2^*) \leq d^{-1}(\delta_5) \leq d^{-1}(c(\delta_4))$, and $c(\delta_4) < c(||x_0||) \leq V_\eta(t_0, x_0) \leq d(||x_0||)$, i.e., $||x_0|| > \epsilon_3$.

We will prove the inequality

$$||x^*(t)|| > \epsilon_3, \quad t \geq t_0. \tag{2.188}$$

Assume (2.188) is not true. There are three cases to consider.

Case 1. There exists a point $t^* > t_0,\ t^* \neq s_k, k = 0, 1, \ldots$ such that

$$||x^*(t)|| > \epsilon_3 \quad \text{for } t \in [t_0, t^*) \quad \text{and} \quad ||x^*(t^*)|| = \epsilon_3. \tag{2.189}$$

According to Lemma 2.3.9 for $T = t^*$, V_η, v^* and $\Delta = \{x : \ ||x|| \geq \epsilon_3\}$ we obtain $V_\eta(t, x^*(t)) \geq v^*(t)$ for $t \in [t_0, t^*]$. From condition 3(vi) and inequality (2.182) with $\epsilon_2 = \epsilon_2^*$ we get $d(\epsilon_3) = d(||x^*(t^*)||) \geq V_\eta(t, x^*(t^*)) \geq v^*(t^*) > \epsilon_2^* > d(\epsilon_3)$. We obtain a contradiction.

Case 2. There exists an integer $k \geq 0$ such that

$$||x^*(t)|| > \epsilon_3 \quad \text{for } t \in [t_0, s_k) \quad \text{and} \quad ||x^*(s_k - 0)|| = \epsilon_3. \tag{2.190}$$

As in the Case 1 with $t^* = s_k$ we obtain a contradiction.

Case 3. There exists an integer $k \geq 0$ such that

$$||x^*(t)|| > \epsilon_3 \quad \text{for } t \in [t_0, s_k] \quad \text{and} \quad ||x^*(s_k + 0)|| \leq \epsilon_3. \tag{2.191}$$

From Lemma 2.3.9 for $T = s_k$, V_η, v^* and $\Delta = \{x : \ ||x|| \geq \epsilon_3\}$ we obtain $V_\eta(t, x^*(t)) \geq v^*(t)$ for $t \in [t_0, s_k]$.

Then $x^*(s_k + 0) = \phi_k(s_k, x^*(s_k - 0))$ and according to conditions 3(v) and 3(vi) we get

$$\begin{aligned} d(\epsilon_3) &\geq d(||x^*(s_k + 0)||) = d(||\phi_k(s_k, x^*(s_k - 0)||) \\ &\geq V_\eta(s_k, \phi_k(s_k, x^*(s_k - 0)) \ \geq \Psi_k(s_k, V_\eta(s_k - 0, x^*(s_k - 0)) \\ &\geq \Psi_k(s_k, v^*(s_k - 0)) = v^*(s_k + 0)) > \epsilon_2^*. \end{aligned} \tag{2.192}$$

The contradictions above prove inequality (2.188) is true. □

Remark 2.3.20 *Note in the case of FrDE if conditions 2 and 3 of Theorem 2.3.12 are satisfied with $g_i(t,x) \equiv 0,\ i = 1,2$, then the zero solution of FrDE is uniformly strictly stable (see [7]).*

In the case of non-instantaneous impulses the condition $g_i(t,x) \equiv 0,\ i = 1,2$ is not enough for strict stability (see Example 2.3.6.1).

Sufficient conditions for strict stability could be obtained in the case of one Lyapunov function.

Theorem 2.3.13 *Let the following conditions be fulfilled:*

1. *The conditions 1, 2 of Theorem 2.3.11 are satisfied and the inequalities $g_2(t,u) \le g_1(t,u)$ for $t \in (t_0,\infty) \bigcap \cup_{k=0}^{\infty}(t_k,s_k), u \in \mathbb{R}$ and $\Psi_k(t,u) \le \Phi_k(t,u)$ for $t \in (s_k,t_{k+1}],\ u \in \mathbb{R},\ k = 1,2,\dots$ hold.*
2. *There exists a function $V \in \Lambda(\mathbb{R}_+,\mathbb{R}^n)$ such that*

   (i) for any $k = 0,1,2,\dots$ and $y_0 \in \mathbb{R}^n$ the inequality

$$g_2(t,V(t,x)) \le {}_{(2.48)}^{\ c}D_+^q V(t,x;t_k,y_0) \le g_1(t,V(t,x))$$
$$\text{for } t \in (t_k,s_k),\ x \in \mathbb{R}^n$$

   holds;

   (ii) for any $k = 0,1,2,\dots$ the inequality

$$\begin{aligned}\Psi_k(t,V(t_k-0,x)) &\le V(t,\phi_k(t,x))\\ &\le \Phi_k(t,V(t_k-0,x)) \text{ for } t \in (s_k,t_{k+1}],\ x \in \mathbb{R}^n\end{aligned}$$

   holds;

   (iii) $a(||x||) \le V(t,x) \le b(||x||)$ for $t \in \mathbb{R}_+,\ \ x \in \mathbb{R}^n$, where $a,b \in \mathcal{M}$.
3. *The zero solution of the couple of NIFrDE (2.175) is strictly stable (uniformly strictly stable) in couple.*

Then the zero solution of the system NIFrDE (2.48) is strictly stable (uniformly strictly stable).

The result of Theorem 2.3.13 is a special case of Theorem 2.3.11 and Theorem 2.3.12.

Example 2.3.6.3 (Uniform Strict Stability of NIFrDE) Let $s_0 = 0, s_k = 2k,\ t_k = 2k-1$ for $k \in \mathbb{Z}_+$. Consider the Caputo fractional differential equation with non-instantaneous impulses

$$\begin{aligned}&{}_0^cD^q x = 0, && t \in (2k,2k+1],\ k \in \mathbb{Z}_0\\ &x(t) = a_k(t_k-0), && t \in (2k+1,2k+2],\ \ k \in \mathbb{Z}_0\\ &x(0) = x_0, && v(0) = v_0,\end{aligned} \tag{2.193}$$

where $x \in \mathbb{R}$, $a_k = \sqrt{1-\frac{1}{2^k}}$ for k odd and $a_k = \sqrt{1+\frac{1}{2^k}}$ for k even.

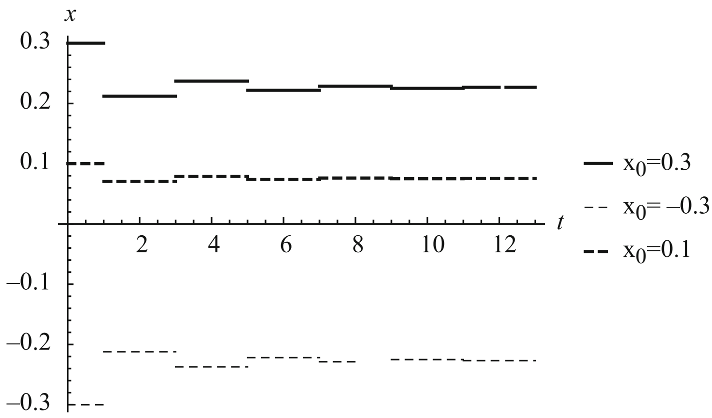

Fig. 2.19 Example 2.3.6.3. Graphs of solutions of (2.193) for various initial values x_0.

Note the IVP for NIFrDE (2.193) has a solution for which $|x(t)| \leq |x_0|$, $t \geq t_0$ (see Figure 2.19 for $q = 0.2$ and various initial values).

Let $V_1(t, x) = x^2$. Let $|x| \leq |x_0|$. From Example 2.3.1.5. and (2.98) with $m(t) \equiv 1$ we obtain

$$
{}^{c}_{(2.193)}D^q_+ V_1(t, x; t_k, x_0) = \Big(x^2 - y_0^2\Big)\frac{(t-t_k)^{-q}}{\Gamma(1-q)} \leq 0. \tag{2.194}
$$

Also, for k odd we get $(\sqrt{1 - \frac{1}{2^k}x})^2 \leq (1 + \frac{1}{2^k})x^2$.

For k even we get $(\sqrt{1 + \frac{1}{2^k}x})^2 = (1 + 2^k)V_1(t, x)$.

Therefore condition 3 of Theorem 2.3.11 is satisfied.

Let $V_2(t, x) = (2 - E_q(-(t-t_k)^q))x^2$ for $t \in (t_k, t_{k+1}]$. Let $|x| \leq |x_0|$. From Example 2.3.1.5, (2.98) with $m(t) = 2 - E_q(-(t-t_k)^q)$, $t \in (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots$ and ${}^{RL}_{t_k}D^q E_q(-(t-t_k)^q) = \frac{1}{t^q\Gamma(1-q)} - E_q(-(t-t_k)^q)$ we obtain

$$
\begin{aligned}
&{}^{c}_{(2.193)}D^q_+ V(t, x; 0, x_0) \\
&= x^2\, {}^{C}_{t_k}D^q\Big((2 - E_q(-(t-t_k)^q))\Big) + \Big(x^2 - y_0^2\Big)(2 - E_q(-(t_k - t_k)^q))\frac{(t-t_k)^{-q}}{\Gamma(1-q)} \\
&= x^2\Big(\frac{2}{(t-t_k)^q\Gamma(1-q)} - \frac{1}{(t-t_k)^q\Gamma(1-q)} + E_q(-(t-t_k)^q)\Big) \\
&\quad - (y_0)^2\frac{1}{(t-t_k)^q\Gamma(1-q)} \geq 0.
\end{aligned} \tag{2.195}
$$

Also, for k odd we get $(2 - E_q(-t^q))(\sqrt{1 - \frac{1}{2^k}x})^2 = (1 - \frac{1}{2^k})(2 - E_q(-t^q))x^2$.

For k even we get $(2 - E_q(-t^q))(\sqrt{1 + 2^k \frac{1}{2^k} x})^2 \geq (1 - \frac{1}{2^k}) V_2(t, x)$.

Therefore condition 4 of Theorem 2.3.11 is satisfied.

From Example 2.3.6.2 with $A = B = 0$ and Theorem 2.3.11 the zero solution of (2.193) is strictly stable (see Figure 2.19). □

2.4 Iterative Techniques for Caputo Fractional Differential Equations with Non-instantaneous Impulses

2.4.1 Monotone-Iterative Technique for Caputo Fractional Differential Equations with Non-instantaneous Impulses

The monotone iterative technique combined with the method of lower and upper solutions is applied to find the approximate solution of a scalar Caputo non-instantaneous impulsive fractional differential equation on a finite interval. A procedure for constructing two monotone functional sequences is given. The elements of these sequences are solutions of suitably chosen initial value problems for scalar linear non-instantaneous impulsive differential equations for which there is an explicit formula. Also, the elements of these sequences are lower/upper solutions of the problem. We prove that both sequences converge and their limits are minimal and maximal solutions of the problem.

Let two increasing finite sequences of points $\{t_i\}_{i=0}^{p+1}$ and $\{s_i\}_{i=0}^{p+1}$ are given such that $t_0 = 0 < s_i < t_{i+1} < s_{i+1}$, $i = 0, 1, 2, \ldots, p$, and $T = s_{p+1}$, p is a natural number.

Consider the initial value problem (IVP) for the nonlinear *non-instantaneous impulsive fractional differential equation* (NIFrDE)

$$\begin{aligned}
&{}_0^cD^q x(t) = f(t, x) \text{ for } t \in (t_k, s_k],\ k = 0, 1, \ldots, p, p+1, \\
&x(t) = \phi_k(t, x(t), x(s_k - 0)) \quad \text{for } t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \ldots, p, \\
&x(0) = x_0,
\end{aligned} \tag{2.196}$$

where $x, x_0 \in \mathbb{R}$, $f : \cup_{k=0}^{p+1}[t_k, s_k] \times \mathbb{R} \to \mathbb{R}$, $\phi_k : [s_k, t_{k+1}] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(k = 0, 1, 2, \ldots, p)$.

Consider the IVP for the linear FrDE

$${}_\tau^cD^q u(t) - \lambda u(t) = h(t) \quad \text{for } t \in\ [\tau, b] \quad \text{with } u(\tau) = a, \tag{2.197}$$

where $u \in \mathbb{R}$, $a, b, \lambda, \tau : \tau < b$ are given constants, $h \in C([\tau, b], \mathbb{R}$. According to Section 4.3.1 [68] the solution of (2.197) is given by

$$u(t) = aE_q(\lambda(t-\tau)^q) + \int_\tau^t (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q) h(s) ds, \quad t \in (\tau, b]. \tag{2.198}$$

We introduce the following classes of functions

$$NPC^1 = \Big\{u : [0,T] \to \mathbb{R} :\ u \in C^1(\cup_{k=0}^{p+1}[t_k, s_k], \mathbb{R}) :$$
$$u(s_k) = u(s_k - 0) = \lim_{t \uparrow s_k} u(t) < \infty,\ u'(s_k) = \lim_{t \uparrow s_k} u'(t) < \infty,\ k = 0, 1, 2, \dots, p,$$
$$u(s_k + 0) = \lim_{t \downarrow s_k} u(t) < \infty,\ k = 0, 1, 2, \dots, p\Big\},$$
$$PC^1([0,T]) = \Big\{u : [0,T] \to \mathbb{R} :\ u \in NPC^1,\ u \in C(\cup_{k=0}^{p}(s_k, t_{k+1}]\,\mathbb{R})\Big\}.$$

For any pair of functions $v, w \in PC^1$ such that $v(t) \le w(t)$ for $t \in [0,T]$, we define the sets

$$S(v,w) = \{u \in PC^1 :\ \ v(t) \le u(t) \le w(t), \quad t \in [0,T]\},$$
$$\Omega_k(t,v,w) = \{x \in \mathbb{R} :\ \ v(t) \le x \le w(t)\}\ \text{for}\ t \in [t_k, s_k]\}, \quad k = 0, 1, \dots, p+1,$$
$$\Lambda_k(t,v,w) = \{x \in \mathbb{R} :\ \ v(t) \le x \le w(t)\}\ \text{for}\ t \in (s_k, t_{k+1}]\}, k = 0, 1, \dots, p,$$
$$\Gamma_k(v,w) = \{x \in \mathbb{R} :\ \ v(s_k - 0) \le x \le w(s_k - 0)\},\ \ k = 0, 1, 2 \dots, p.$$

I. Lower and upper solutions of NIFrDE

Definition 2.4.1 *We say that the function $v(t) \in PC^1([0,T])$ is a minimal (maximal) solution of the IVP for NIFrDE (2.196) if it is a solution of (2.196) and for any solution $u(t) \in PC^1([0,T])$ of (2.196) the inequality $v(t) \le u(t)$ ($v(t) \ge u(t)$) holds on* $[0,T]$.

Applying Eq. (2.35) we will define lower and upper solutions of the IVP for NIFrDE (2.196).

Definition 2.4.2 *We say that the function $v(t) \in PC^1([0,T])$ is a lower (upper) solution of the IVP for NIFrDE (2.196) if*

$$\begin{aligned} &{}_0^cD^q v(t) \le (\ge) f(t,v)\ \text{for}\ t \in (t_k, s_k],\ k = 0, 1, 2, \dots, p+1, \\ &v(t) \le (\ge) \phi_k(t, v(t), v(s_k - 0)) \quad \text{for}\ t \in (s_k, t_{k+1}]\ \ k = 0, 1, 2, \dots, p, \\ &v(0) \le (\ge) x_0. \end{aligned} \tag{2.199}$$

or equivalently

$$v(t) \le (\ge) \begin{cases} x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, v(s))ds,\ t \in (0, s_0], \\ \phi_k(t, v(t), v(s_k - 0)), \qquad t \in (s_k, t_{k+1}], k = 0, 1, 2, \dots, p \\ \phi_{k-1}(t_k, v(t_k), v(s_{k-1} - 0)) \\ \quad + \frac{1}{\Gamma(q)} \int_{t_k}^t (t-s)^{q-1} f(s, v(s))ds,\ \ t \in (t_k, s_k],\ \ k = 1, 2, \dots, p+1. \end{cases} \tag{2.200}$$

In the main result we will need some results concerning existence and a formula for solutions of a scalar linear non-instantaneous impulsive fractional differential equation of the type

$$\begin{aligned}
&{}_0^cD^q u(t) + M_k u(t) = h_k(t), \quad \text{for } t \in (t_k, s_k],\ k = 0, 1, 2, \dots, p+1,\\
&u(t) = K_k(t)u(t) + L_k(t)u(s_k - 0) + \gamma_k(t), \quad \text{for } t \in (s_k, t_{k+1}],\\
&\qquad\qquad k = 0, 1, 2, \dots, p,\\
&u(0) = u_0,
\end{aligned} \tag{2.201}$$

where $u, u_0 \in \mathbb{R}$.

The formula for the solution of (2.201) is given in the following Lemma.

Lemma 2.4.1 *Let the functions $K_k \in C([s_k, t_{k+1}], \mathbb{R}/\{1\})$, $L_k, \gamma_k \in C([s_k, t_{k+1}], \mathbb{R})$, $k = 0, 1, 2, \dots, p$, the functions $h_k \in C([t_k, s_k], \mathbb{R})$ and the constants $M_k \in \mathbb{R}$, $k = 0, 1, 2, \dots, p+1$.*

Then the IVP for the scalar linear NIFrDE (2.201) has a unique solution $u \in PC^1([0, T]$ given by

$$u(t) = \begin{cases} x_0 E_q(-M_0 t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(-M_0(t-s)^q) h_0(s) ds \\ \qquad \textit{for } \ t \in [0, s_0], \\ \frac{L_k(t)u(s_k-0)+\gamma_k(t)}{1-K_k(t)} \quad \textit{for } \ t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \dots, p, \\ \frac{L_{k-1}(t_k)u(s_{k-1}-0)+\gamma_{k-1}(t_k)}{1-K_{k-1}(t_k)} E_q(-M_k(t-t_k)^q) \\ \quad + \int_{t_k}^t (t-s)^{q-1} E_{q,q}(-M_k(t-s)^q) h_k(s) ds \\ \qquad\qquad \textit{for } \ t \in [t_k, s_k],\ k = 1, 2, \dots, p+1. \end{cases} \tag{2.202}$$

Proof The proof is by induction.

For $t \in [0, s_0]$ formula (2.202) follows from (2.197) with $\tau = 0$, $\lambda = -M_0$, $b = s_0$, $a = x_0$, $h(t) = h_0(t)$.

Let $t \in (t_1, s_1]$. Then the function satisfies the fractional differential equation ${}_{t_1}^cD^q u(t) + M_1 u(t) = h_1(t)$. Apply formula (2.197) with $\tau = t_1$, $\lambda = -M_1$, $b = s_1$, $h(t) = h_1(t)$, $a = \frac{L_0(t_1)u(s_0-0)+\gamma_0(t_1)}{1-K_0(t_1)}$ and we obtain (2.1).

Following the above procedure we obtain (2.202). □

Lemma 2.4.2 (Lemma 2.1 [102]) *Let $m \in C^1([0, T], \mathbb{R})$. If there exists $t_1 \in [0, T]$ such that $m(t_1) = 0$ and $m(t) \le 0$ on $[0, t_1]$, then ${}_0^cD^q m(t_1) \ge 0$.*

Remark 2.4.1 *Note Lemma 2.4.2 is true if the interval is $[\tau, T]$, $\tau > 0$ and the lower limit of the fractional derivatives is τ.*

Lemma 2.4.3 *Let $v \in C^1([\tau, T], \mathbb{R})$ be such that*

$${}_\tau^cD^q v(t) \le -Mv(t) \ \textit{for } \ t \in (\tau, T], \quad v(\tau) \le 0 \tag{2.203}$$

where $M > 0$, $\tau \ge 0$.

Then $v(t) \leq 0$ *for* $t \in [\tau, T]$.

Proof The proof follows from Remark 2.4.1 and Lemma 2.4.2 applied to the interval $[\tau, T]$. Indeed, consider two cases.

Case 1. Let inequality (2.203) be strict and $v(\tau) < 0$. Then if assume the contrary, there exists $\tau_1 \in (\tau, T)$ such that $v(\tau_1) = 0$, $v(t) < 0$ for $t \in [\tau, \tau_1)$ and according to Lemma 2.4.2 we get ${}_{\tau}^{c}D^q v(\tau_1) \geq 0$ which contradicts inequality (2.203).

Case 2. Let at least one of inequality (2.203) and inequality $v(\tau) \leq 0$ be not strict. For an arbitrary number $\epsilon > 0$ we define $v_\epsilon(t) = v(t) - \epsilon$. Then $v_\epsilon(\tau) = v(\tau) - \epsilon \leq -\epsilon < 0$ and ${}_{\tau}^{c}D^q v_\epsilon(t) = {}_{\tau}^{c}D^q v(t) - {}_{\tau}^{c}D^q \epsilon = {}_{\tau}^{c}D^q v(t) \leq -Mv(t) < -M(v(t) - \epsilon) = -Mv_\epsilon(t)$. From Case 1 we have the result. □

Later we use the following result for scalar linear non-instantaneous impulsive fractional differential inequalities.

Lemma 2.4.4 *Let the following conditions be fulfilled:*

1. The functions $K_k \in C([s_k, t_{k+1}], [0, 1))$, $L_k \in C([s_k, t_{k+1}], \mathbb{R})$, $k = 0, 1, 2, \dots, p$, *and the constants* $M_k \in \mathbb{R}_+$, $k = 0, 1, 2, \dots, p + 1$.

2. The scalar function $m \in PC^1([0, T])$ *satisfies the inequalities*

$$\begin{aligned} &{}_{0}^{c}D^q m(t) \leq -M_k m(t) \;\; \text{for} \;\; t \in (t_k, s_k], \;\; k = 0, 1, 2, \dots, p + 1, \\ &m(t) \leq L_i(t) m(s_k - 0) + K_i(t) m(t) \;\; \text{for}\, t \in (s_k, t_{k+1}], \; k = 0, 1, 2, \dots, p, \\ &m(0) \leq 0. \end{aligned} \tag{2.204}$$

Then the inequality $m(t) \leq 0$ *holds for* $t \in [0, T]$.

Proof We will use induction to prove the result.

Let $t \in [0, s_0]$. Then the function $m(t) \in C^1([0, s_0], \mathbb{R})$ and satisfies the fractional differential inequality ${}_{0}^{c}D^q m(t) \leq -M_0 m(t)$ for $t \in [0, s_0]$, $m(0) \leq 0$. Apply Lemma 2.4.3 with $\tau = 0$, $M = M_0$ and obtain $m(t) \leq 0$ holds on $t \in [0, s_0]$.

Let $t \in (s_0, t_1]$. Then from (2.204), condition 1 of Lemma 2.4.4 and the above we get $m(t) \leq \frac{L_1(t)}{1 - K_1(t)} m(s_0 - 0) \leq 0$ on $(s_0, t_1]$.

Let $t \in (t_1, s_1]$. Consider the function $\overline{m}_1(t) = m(t)$ for $t \in (t_1, s_1]$ and $\overline{m}_1(t_1) = L_0(t_1) m(s_0 - 0) + K_0(t_1) m(t_1 - 0) \leq 0$. The function $\overline{m}_1(t) \in C^1([t_1, s_1], \mathbb{R})$ and satisfies the fractional differential inequality ${}_{t_1}^{c}D^q \overline{m}(t) \leq -M_1 \overline{m}(t)$ for $t \in (t_1, s_1]$. Apply Corollary 2.4.3 with $\tau = t_1$, $M = M_1$, $m(t) = \overline{m}(t)$ and obtain $\overline{m}(t) \leq 0$ holds on $t \in [t_1, s_1]$.

Continue this process and an induction argument proves the result. □

As a corollary of Lemma 2.4.4 with $p = \infty$ and $K_k = 0$ we obtain the following result which will be used in Chapter 3:

Lemma 2.4.5 *Let the following conditions be fulfilled:*

1. *Two increasing sequences of nonnegative points* $\{T_k\}_{k=0}^{\infty}$ *and* $\{s_k\}_{k=1}^{\infty}$ *are given with* $s_0 = T_0$, $T_{k-1} < s_k \leq T_k$, $k = 1, 2, \dots$ *and* $\lim_{k \to \infty} \{T_k\} = \infty$.

2. *The function* $m \in C(\cup_{k=0}^{\infty}(s_k, T_{k+1}), \mathbb{R})$, $m(T_k) = m(T_k - 0) = lim_{t\downarrow T_k} m(t)$, $m(T_k + 0) = lim_{t\uparrow T_k} m(t) < \infty$ *and for any* $t \in (s_k, T_{k+1})$ *the Caputo fractional Dini derivative of m exists at t and the inequalities*

$$\begin{aligned} &{}_{T_0}^{c}D_{+}^{q} m(t) \leq -am(t) \quad for\ t \in (s_k, T_{k+1}],\ k = 0, 1, 2, \ldots, \\ &m(t) \leq b_k m(T_k - 0) \quad for \quad t \in (T_k, s_k],\ k = 1, 2, \ldots, \\ &m(T_0) \leq 0, \end{aligned} \tag{2.205}$$

hold, where $a > 0$ *and* $b_k \in [0, 1), (k = 1, 2, \ldots)$ *are given constants.*

Then the function $m(t)$ *is nonpositive in* $[T_0, \infty)$.

In the case of continuously differentiable function on the corresponding subintervals the following result is true:

Lemma 2.4.6 *Let the following conditions be fulfilled:*

1. *Condition 1 of Lemma 2.4.5 is satisfied.*
2. *The function* $m \in C^1(\cup_{k=0}^{\infty}(s_k, T_{k+1}), \mathbb{R})$ *are such that* $m(T_k) = m(T_k - 0) = lim_{t\downarrow T_k} m(t)$, $m(T_k + 0) = lim_{t\uparrow T_k} m(t) < \infty$, $m'(T_k) = m(T_k - 0) = lim_{t\downarrow T_k} m'(t)$, $m'(T_k + 0) = lim_{t\uparrow T_k} m'(t) < \infty$ *and for any* $t \neq T_k$ *the Caputo fractional derivative of m exists at t and the inequalities*

$$\begin{aligned} &{}_{T_0}^{c}D^{q} m(t) \leq -am(t) \ \ for\ t \in (s_k, T_{k+1}),\ k = 0, 1, 2, \ldots, \\ &m(t) \leq b_k m(T_k - 0), \quad for \quad t \in (T_k, s_k],\ k = 1, 2, \ldots, \\ &m(T_0) \leq 0, \end{aligned} \tag{2.206}$$

hold, where $a > 0$ *and* $b_k \in [0, 1), (k = 1, 2, \ldots)$ *are given constants.*

Then the function $m(t)$ *is nonpositive in* $[T_0, \infty)$.

The proof of Lemma 2.4.6 is similar to the one of Lemma 2.4.4 where instead of Lemma 2.4.3 we apply Corollary 2.4 in [43].

Lemma 2.4.7 *Let the scalar function* $m \in PC^1([0, T])$ *satisfy the inequalities*

$$\begin{aligned} &{}_{0}^{c}D^{q} m(t) \leq 0 \ \ for \ \ t \in (t_k, s_k],\ \ k = 0, 1, 2, \ldots, p + 1, \\ &m(t) \leq 0 \ \ for\ t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \ldots, p, \\ &m(0) \leq 0. \end{aligned} \tag{2.207}$$

Then the inequality $m(t) \leq 0$ *holds for* $t \in [0, T]$.

The proof is similar to that in Lemma 2.4.4 so we omit it.

II. Monotone iterative technique to NIFrDE

We give an algorithm for constructing two monotonic sequences of successive approximations.

Theorem 2.4.1 *Let the following conditions be fulfilled:*

1. *The functions $v, w \in PC^1([0,T])$ are lower and upper solutions of the IVP for NIFrDE (2.196), respectively, and $v(t) \le w(t)$ for $t \in [0,T]$.*
2. *The function $f \in C(\cup_{k=0}^{p+1}[t_k, s_k], \mathbb{R})$ and there exist constants $M_k > 0$, $k = 0, 1, 2, \dots, p+1$ such that for any $x, y \in \Omega_k(t, v, w) : \; x \le y$ the inequality*

$$f(t,x) - f(t,y) \le -M_k(x-y), \quad t \in [t_k, s_k] \tag{2.208}$$

   holds.
3. *The functions $\phi_k \in C([s_k, t_{k+1}] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $k = 0, 1, 2, \dots, p$, and there exist functions $L_k \in C([s_k, t_{k+1}], [0, \infty))$ and $K_k \in C([s_k, t_{k+1}], [0, 1))$, $k = 0, 1, 2, \dots, p$ such that for any $t \in [s_k, t_{k+1}]$ and $x_1, x_2 \in \Lambda_k(t, v, w) : x_1 \le x_2$, $y_1, y_2 \in \Gamma_k(v, w) : y_1 \le y_2$ the inequality*

$$\phi_k(t, x_1, y_1) - \phi_k(t, x_2, y_2) \le K_k(t)(x_1 - x_2) + L_k(t)(y_1 - y_2) \tag{2.209}$$

   holds.

Then there exist two sequences of functions $\{v^{(n)}(t)\}_0^\infty$ and $\{w^{(n)}(t)\}_0^\infty$ such that:

a. *The sequences are increasing and decreasing correspondingly, i.e.*

$$v^{(n)}(t) \le v^{(n+1)}(t) \le w^{(n+1)}(t) \le w^{(n)}(t) \quad \textit{for } [0,T], \; n = 0, 1, 2, \dots;$$

b. *The functions $v^{(n)}, \; w^{(n)} \in PC^1([0,T], \mathbb{R}), \; n = 0, 1, 2, \dots,$ are lower and upper solutions of the IVP for NIFrDE (2.196) in $S(v, w)$ respectively;*
c. *Both sequences converge on $[0,T]$;*
d. *The limit's functions $V(t), \; W(t)$ of both sequences are the minimal and maximal solutions of IVP for NIFrDE (2.196) in $S(v, w)$, respectively.*
e. *If IVP for NIFrDE has a unique solution $u(t) \in S(v, w)$, then $V(t) \equiv u(t) \equiv W(t)$ on $[0,T]$.*

Remark 2.4.2 *Note if the function $f(t,x)$ is nondecreasing w.r.t. its second argument $x \in \Omega_k(t, v, w)$ for any fixed $t \in [t_k, s_k]$, $k = 0, 1, \dots, p+1$, then inequality (2.208) is satisfied.*

Remark 2.4.3 *If the function $\phi_k(t, x, y)$, $k = 0, 1, 2, \dots, p$, satisfies inequality (2.209), then it is nondecreasing in both arguments x and y.*

Proof For any arbitrary fixed function $\eta \in PC^1([0,T])$ we consider the IVP for the scalar linear NIFrDE

$$
\begin{aligned}
&{}_0^cD^q u(t) + M_k u(t) = \psi_k(t, \eta(t)), \quad \text{for } t \in (t_k, s_k], k = 0, 1, 2, \dots, p+1,\\
&u(t) = L_k(t)u(s_k - 0) + K_k(t)u(t) + \xi_k(t, \eta(t), \eta(s_k - 0)),\\
&\qquad\qquad \text{for } t \in (s_k, t_{k+1}], k = 0, 1, 2, \dots, p,\\
&u(0) = x_0,
\end{aligned} \tag{2.210}
$$

where $u \in \mathbb{R}$, the functions $\psi_k \in C([t_k, s_k] \times \mathbb{R}, \mathbb{R}),\ k = 0, 1, 2, \dots, p+1$ are defined by

$$\psi_k(t, x) = f(t, x) + M_k x, \quad t \in [t_k, s_k],\ x \in \mathbb{R}$$

and the functions $\xi_k \in C([s_k, t_{k+1}] \times \mathbb{R}^2, \mathbb{R})),\ \ k = 0, 1, 2, \dots, p$ are defined by

$$\xi_k(t, x, y) = \phi_k(t, x, y) - L_k(t)y - K_k(t)x, \quad t \in [s_k, t_{k+1}],\ \ x, y \in \mathbb{R}.$$

According to Lemma 2.4.1 for any fixed $\eta \in PC^1([0, T])$ the IVP for the linear NIFrDE (2.210) has a unique solution $x(t; \eta) \in PC^1([0, T])$ given by (2.1) with $h_k(t) = \psi_k(t, \eta(t))$ for $t \in [t_k, s_k], k = 0, 1, \dots, p+1$ and

$$\gamma_k(t) = \begin{cases} \xi_k(s_k + 0, \eta(s_k + 0), \eta(s_k - 0)), & \text{if } t = s_k\\ \xi_k(t, \eta(t), \eta(s_k - 0)), & \text{if } t \in (s_k, t_{k+1}]. \end{cases}$$

Define the operator $\Delta : PC^1([0, T]) \to PC^1([0, T])$ by $\Delta(\eta) = x(t)$, where $\eta \in PC^1([0, T])$ and $x(t) \in PC^1([0, T])$ is the unique solution of IVP for the linear NIFrDE (2.210) for the function η. Then $x(t)$ is given by

$$x(t) = \begin{cases} x_0 E_q(-M_0 t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(-M_0(t-s)^q)\big(f(s, \eta(s)) + M_0 \eta(s)\big) ds\\ \qquad \text{for } t \in [0, s_0],\\ \frac{L_k(t)x(s_k - 0) + \phi_k(t, \eta(t), \eta(s_k - 0)) - L_k(t)\eta(s_k - 0) - K_k(t)\eta(t)}{1 - K_k(t)}\\ \qquad \text{for } \ t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \dots, p,\\ \frac{L_k(t_k)x(s_k - 0) + \phi_k(t_k, \eta(t_k), \eta(s_k - 0)) - L_k(t_k)\eta(s_k - 0) - K_k(t_k)\eta(t_k)}{1 - K_k(t_k)} E_q(-M_k(t - t_k)^q)\\ \quad + \int_{t_k}^t (t-s)^{q-1} E_{q,q}(-M_k(t-s)^q)\big(f(s, \eta(s)) + M_k \eta(s)\big) ds,\\ \qquad \text{for } t \in [t_k, s_k],\ k = 1, 2, \dots, p+1. \end{cases}$$

The operator Δ has the following properties:

P1. If $\eta \in \Omega(v, w)$ is a lower (upper) solution of (2.196), then $\eta(t) \le (\ge) \mu(t),\ \ t \in [0, T]$ where $\mu = \Delta(\eta)$.

P2. $v \le \Delta(v)$ and $w \ge \Delta(w)$ in $[0, T]$.

P3. The operator Δ is nondecreasing in $S(v, w)$, i.e., for $\eta, \mu \in S(v, w) :\ \eta(t) \le \mu(t)$ for $t \in [0, T]$ the inequality $\Delta(\eta) \le \Delta(\mu)$ holds in $[0, T]$.

P4. If $\eta \in \Omega(v, w)$ is a lower (upper) solution of (2.196), then also $\mu = \Delta(\eta)$ is a lower (upper) solution of (2.196).
P5. If $\eta, \mu \in \Omega(v, w) : \ \eta \leq \mu$ are a lower solution and a upper solution of (2.196) respectively, then $\Delta(\eta) \leq \Delta(\mu)$.

We now prove property (P1). Let $\eta \in \Omega(v, w)$ be a lower solution of (2.196) and $\mu(t) = \Delta(\eta)$. Let $m(t) = \eta(t) - \mu(t),\ t \in [0, T]$. Then $m(0) \leq 0$.

For any $t \in (t_k, s_k]$, $k = 0, 1, \dots, p + 1$, the inequality

$$ {}_{t_k}^{c}D^q m(t) = {}_{t_k}^{c}D^q \eta(t) - {}_{t_k}^{c}D^q \mu(t) \leq f(t, \eta(t)) + M_k\mu(t) - f(t, \eta(t)) - M_k\eta(t) = -M_k m(t) $$

holds.

For any $t \in (s_k, t_{k+1}]$, $k = 0, 1, 2, \dots, p$, the inequality

$$ \begin{aligned} m(t) &\leq \phi_k(t, \eta(t), \eta(s_k - 0)) - L_k(t)\mu(s_k - 0) - K_k(t)\mu(t) - \phi_k(t, \eta(t), \eta(s_k - 0)) \\ &\qquad\qquad + L_k(t)\eta(s_k - 0) + K_k(t)\eta(t) \\ &= L_k(t)m(s_k - 0) + K_k(t)m(t) \end{aligned} $$

holds.

Therefore, the function $m(t)$ satisfies the inequalities (2.204). According to Lemma 2.4.4 the function $m(t)$ is nonpositive in $[0, T]$, i.e., $\eta \leq \Delta(\mu)$.

Analogously it can be proved that the inequality $w \geq \Delta(w)$ holds.

The property (P2) follows immediately from (P1).

We now prove property (P3). Let $\eta, \mu \in S(v, w)$ be arbitrary functions such that $\eta(t) \leq \mu(t)$ for $t \in [0, T]$. Let $x^{(1)} = \Delta(\eta)$ and $x^{(2)} = \Delta(\mu)$. Denote $g(t) = x^{(1)}(t) - x^{(2)}(t)$, $t \in [0, T]$.

For any $t \in (t_k, s_k]$, $k = 0, 1, \dots, p + 1$, applying condition 1 we obtain

$$ \begin{aligned} &{}_{t_k}^{c}D^q g(t) \\ &\quad = -M_k x^{(1)}(t) + f_k(t, \eta(t)) + M_k\eta(t) + M_k x^{(2)}(t) - f_k(t, \mu(t)) - M_k\mu(t) \\ &\quad = -M_k g(t) + f_k(t, \eta(t)) - f_k(t, \mu(t)) + M_k(\eta(t) - \mu(t)) \leq -M_k g(t). \end{aligned} \tag{2.211} $$

For any $t \in (s_k, t_{k+1}]$, $k = 0, 1, 2, \dots, p$, applying condition 2 we get the inequality

$$ \begin{aligned} g(t) &= x^{(1)}(t) - x^{(2)}(t) \\ &= L_k(t)g(s_k - 0) + K_k(t)g(t) \\ &\qquad + \phi_k(t, \eta(t), \eta(s_k - 0)) - \phi_k(t, \mu(t), \mu(s_k - 0)) \\ &\qquad - L_k(t)(\eta(s_k - 0) - \mu(s_k - 0)) - K_k(t)(\eta(t) - \mu(t)) \\ &\leq L_k(t)g(s_k - 0) + K_k(t)g(t). \end{aligned} \tag{2.212} $$

According to Lemma 2.4.4 the function $g(t)$ is nonpositive, i.e., $\Delta(\eta) \leq \Delta(\mu)$.

We now prove property (P4). Let $\eta \in \Omega(v, w)$ be a lower solution of (2.196). Consider the function $m = \Delta(\eta)$. According to (P1) the inequality $m(t) \geq \eta(t)$ holds on $[0, T]$. We will prove the function m is a lower solution of (2.196).

For any $t \in (t_k, s_k]$, $k = 0, 1, \ldots, p+1$, we obtain

$$\begin{aligned}
{}_{t_k}^{c}D^q m(t) &= -M_k m(t)(t) + f_k(t, \eta(t)) + M_k \eta(t) \\
&= f_k(t, m(t)) - M_k m(t) + f_k(t, \eta(t)) - f_k(t, m(t)) + M_k \eta(t) \\
&\leq f_k(t, m(t)).
\end{aligned} \tag{2.213}$$

For any $t \in (s_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, p$, we get the inequality

$$\begin{aligned}
m(t) &= L_k(t) m(s_k - 0) + K_k(t) m(t) + \phi_k(t, \eta(t), \eta(s_k - 0)) \\
&\quad - L_k(t) \eta(s_k - 0) - K_k(t) \eta(t) \\
&= \phi_k(t, m(t), m(s_k - 0)) + L_k(t)\big(m(s_k - 0) - \eta(s_k - 0)\big) \\
&\qquad + K_k(t)\big(m(t) - \eta(t)\big) \\
&\quad + \phi_k(t, \eta(t), \eta(s_k - 0)) - \phi_k(t, m(t), m(s_k - 0)) \\
&\leq \phi_k(t, m(t), m(s_k - 0)).
\end{aligned} \tag{2.214}$$

Inequalities (2.213) and (2.214) show the function $m(t)$ is a lower solution of IVP for NIFrDE (2.196).

Similarly, if $\eta \in \Omega(v, w)$ is an upper solution of NIFrDE (2.196), then the function $m = \Delta(\eta)$ is an upper solution of (2.196).

We now prove property (P5). Let $\eta, \mu \in \Omega(v, w)$ be a lower solution and a upper solution of (2.196) respectively. Denote $\xi = \Delta(\eta)$, $\varsigma = \Delta(\mu)$ and $m(t) = \xi(t) - \varsigma(t)$, $t \in [0, T]$. Then $m(0) \leq 0$.

For any $t \in (t_k, s_k]$, $k = 0, 1, \ldots, p+1$, applying condition 2 we obtain

$$\begin{aligned}
{}_{t_k}^{c}D^q m(t) &\leq -M_k \xi(t) + f(t, \eta(t)) + M_k \eta(t) - M_k \varsigma(t) - f(t, \mu(t)) - M_k \mu(t) \\
&= -M_k m(t) + f_k(t, \eta(t)) - f(t, \mu(t)) + M_k \eta(t) - M_k \mu(t) \\
&\leq -M_k m(t).
\end{aligned} \tag{2.215}$$

For any $t \in (s_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, p$, applying condition 3 we get the inequality

$$\begin{aligned}
m(t) \leq\ & L_k(t)\xi(s_k-0) + K_k(t)\xi(t) + \phi_k(t,\eta(t),\eta(s_k-0)) \\
& - L_k(t)\eta(s_k-0) - K_k(t)\eta(t) \\
& - L_k(t)\varsigma(s_k-0) - K_k(t)\varsigma(t) - \phi_k(t,\mu(t),\mu(s_k-0)) \\
& + L_k(t)\mu(s_k-0) + K_k(t)\mu(t) \\
=\ & L_k(t)m(s_k-0) + K_k(t)m(t) - L_k(t)\Big(\eta(s_k-0) - \mu(s_k-0)\Big) \\
& - K_k(t)\Big(\eta(t) - \mu(t)\Big) \\
& + \phi_k(t,\eta(t),\eta(s_k-0)) - \phi_k(t,m(t),m(s_k-0)) \\
\leq\ & L_k(t)m(s_k-0) + K_k(t)m(t).
\end{aligned} \tag{2.216}$$

According to Lemma 2.4.4 from inequalities (2.215) and (2.216) it follows that $m(t) \leq 0$ on $[0,T]$.

We define the sequences of functions $\{v^{(n)}(t)\}_0^\infty$ and $\{w^{(n)}(t)\}_0^\infty$ by the recurrence equalities

$$\begin{aligned}
& v^{(0)} = v, && w^{(0)} = w, \\
& v^{(n+1)} = \Delta(v^{(n)}), && w^{(n+1)} = \Delta(w^{(n)}), \quad n = 0,1,2,\dots.
\end{aligned}$$

Therefore, the functions $v^{(n)}(t)$ and $w^{(n)}(t)$ for any $n = 1,2,\dots$ satisfy the initial value problems

$$\begin{aligned}
& {}_{t_k}^{c}D^q(v^{(n)}(t)) + M_k v^{(n)}(t) = \psi_k(t, v^{(n-1)}(t)) \\
& \qquad \text{for } t \in (t_k, s_k],\ \ k = 0,1,2,\dots,p+1, \\
& v^{(n)}(t) = L_k v^{(n)}(s_k-0) + K_k v^{(n)}(t) + \xi_k(t, v^{(n-1)}(t), v^{(n-1)}(s_k-0)) \\
& \qquad \text{for } t \in (s_k, t_{k+1}],\ k = 0,1,2,\dots,p, \\
& v^{(n)}(0) = x_0,
\end{aligned} \tag{2.217}$$

and

$$\begin{aligned}
& {}_{t_k}^{c}D^q(w^{(n)}(t)) + M_k w^{(n)}(t) = \psi_k(t, w^{(n-1)}(t)) \\
& \qquad \text{for } t \in (t_k, s_k],\ \ i = 0,1,2,\dots,p+1, \\
& w^{(n)}(t) = L_k(t) w^{(n)}(s_k-0) + K_k(t) w^{(n)}(t) + \xi_k(t, w^{(n-1)}(t), w^{(n-1)}(s_k-0)) \\
& \qquad \text{for } t \in (s_k, t_{k+1}],\ i = 0,1,2,\dots,p, \\
& w^{(n)}(0) = x_0,
\end{aligned} \tag{2.218}$$

where

$$\psi_k(t,x) = f(t,x) + M_k x, \quad t \in [t_k, s_k], \ x \in \mathbb{R}$$

and

$$\xi_k(t,x,y) = \phi_k(t,x,y) - L_k(t)y - K_k(t)x, \quad t \in [s_k, t_{k+1}] \ x, y \in \mathbb{R}.$$

According to Lemma 2.4.1 the IVP for the linear NIFrDE (2.217) has a unique solution $v^{(n)}(t) \in PC^1([0,T])$ given by

$$v^{(n)}(t) = \begin{cases} x_0 E_q(-M_0 t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(-M_0(t-s)^q)\big(f(s, v^{(n-1)}(s)) + M_0 v^{(n-1)}(s)\big)ds \\ \qquad \text{for } t \in [0, s_0], \\ \frac{L_k(t)\left(v^{(n)}(s_k-0) - v^{(n-1)}(s_k-0)\right) - K_k(t)v^{(n-1)}(t) + \phi_k(t, v^{(n-1)}(t), v^{(n-1)}(s_k-0))}{1-K_k(t)} \\ \qquad \text{for } \ t \in (s_k, t_{k+1}], \ k = 0,1,2,\dots,p, \\ \frac{L_{k-1}(t_k)\left(v^{(n)}(s_{k-1}-0) - v^{(n-1)}(s_{k-1}-0)\right) - K_{k-1}(t_k)v^{(n-1)}(t_k) + \phi_{k-1}(t_k, v^{(n-1)}(t_k), v^{(n-1)}(s_{k-1}-0))}{1-K_{k-1}(t_k)} \\ \quad \times E_q(-M_k(t-t_k)^q) \\ \quad + \int_{t_k}^t (t-s)^{q-1} E_{q,q}(-M_k(t-s)^q)\big(f(s, v^{(n-1)}(s)) + M_k v^{(n-1)}(s)\big)ds, \\ \qquad \text{for } t \in [t_k, s_k], \ k = 1,2,\dots,p+1. \end{cases} \tag{2.219}$$

According to Lemma 2.4.1 the IVP for the linear NIFrDE (2.218) has a unique solution $w^{(n)}(t) \in PC^1([0,T])$ given by (2.219) where $v^{(n)}$ and $v^{(n-1)}$ are replaced by $w^{(n)}$ and $w^{(n-1)}$ respectively.

Also, the IVP for the linear NIFrDE (2.217) is equivalent to the following integral–algebraic equalities

$$v^{(n)}(t) = \begin{cases} x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \Big(f(s, v^{(n-1)}(s)) + M_0\big(v^{(n-1)}(s) - v^{(n)}(s)\big)\Big)ds \\ \qquad \text{for } t \in [0, s_0], \\ L_k(t)\big(v^{(n)}(s_k-0) - v^{(n-1)}(s_k-0)\big) + K_k(t)\big(v^{(n-1)}(t) - v^{(n)}(t)\big) \\ \quad + \phi_k(t, v^{(n-1)}(t), v^{(n-1)}(s_k-0)) \\ \quad \text{for } \ t \in (s_k, t_{k+1}], \ k = 0,1,2,\dots,p, \\ Ł_{k-1}(t)\big(v^{(n)}(s_{k-1}-0) - v^{(n-1)}(s_{k-1}-0)\big) \\ \quad + K_{k-1}(t_k)\big(v^{(n-1)}(t_k) - v^{(n)}(t_k)\big) \\ \quad + \phi_{k-1}(t_k, v^{(n-1)}(t_k), v^{(n-1)}(s_{k-1}-0)) \\ \quad + \frac{1}{\Gamma(q)} \int_{t_k}^t (t-s)^{q-1} \Big(f(s, v^{(n-1)}(s)) + M_k\big(v^{(n-1)}(s) - v^{(n)}(s)\big)\Big)ds, \\ \qquad \text{for } t \in [t_k, s_k], \ k = 1,2,\dots,p+1. \end{cases} \tag{2.220}$$

Similarly, the IVP for the linear NIFrDE (2.218) is equivalent to the integral–algebraic equalities (2.220) where $v^{(n)}$ and $v^{(n-1)}$ are replaced by $w^{(n)}$ and $w^{(n-1)}$ respectively.

According to (P4) the functions $v^{(n)}(t)$ and $w^{(n)}(t)$ are lower and upper solutions of IVP for NIFrDE (2.196) respectively and according to (P1), (P2), (P4), and (P5) the following inequalities

$$v^{(0)}(t) \leq v^{(1)}(t) \leq \cdots \leq v^{(n)}(t) \leq w^{(n)}(t) \leq \cdots \leq w^{(1)}(t) \leq w^{(0)}(t), \quad t \in [0, T] \tag{2.221}$$

hold.

We will prove the convergence of the sequence of functions $\{v^{(n)}(t)\}_0^\infty$ on $[0, T]$.

Let $t \in [0, s_0]$. Then any element $v^{(n)} \in C^1([0, s_0], \mathbb{R})$ and according to (2.220) we have

$$v^{(n)}(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \Big(f(s, v^{(n-1)}(s)) + M_0 \big(v^{(n-1)}(s) - v^{(n)}(s) \big) \Big) ds. \tag{2.222}$$

The sequence of functions $\{v^{(n)}(t)\}_0^\infty$ being monotonic and bounded is uniformly convergent on $[0, s_0]$. Let $V_1(t) = \lim_{n\to\infty} v^{(n)}(t),\ t \in [0, s_0]$. According to (2.221) the inequality

$$v(t) \leq V_1(t) \leq w(t),\ t \in [0, s_0] \tag{2.223}$$

holds. Take the limit in (2.222) and we obtain the Volterra fractional integral equation $V_1(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, V_1(s)) ds$. According to (2.12), (2.13), (2.14) with $\tau = 0, \tilde{x}_0 = x_0$, the limit's function $V_1(t)$ is a solution of the IVP for the FrDE

$$ {}_0^cD^q V_1(t) = f(t, V_1(t)),\ \ t \in [0, s_0], \qquad V_1(0) = x_0. \tag{2.224}$$

Let $t \in (s_0, t_1]$. Then any element $v^{(n)} \in C((s_0, t_1], \mathbb{R})$ and according to (2.220) we have

$$\begin{aligned} v^n(t) = {} & L_0(t) \big(v^{(n)}(s_0 - 0) - v^{(n-1)}(s_0 - 0) \big) + K_0(t) \big(v^{(n-1)}(t) - v^{(n)}(t) \big) \\ & + \phi_0(t, v^{(n-1)}(t), v^{(n-1)}(s_0 - 0)). \end{aligned} \tag{2.225}$$

From $v^{(n)}(t) \in PC^1([0, T])$, $\lim_{t\downarrow s_0} v^{(n)}(t) = v^{(n)}(s_0 + 0) < \infty$ exists. For any $n = 1, 2, \ldots$ we define the functions

$$\tilde{v}^{(n)}(t) = \begin{cases} v^{(n)}(s_0 + 0) & \text{for } t = s_0, \\ v^{(n)}(t) \text{ for } t \in (s_0, t_1]. \end{cases}$$

Then $\tilde{v}^{(n)} \in C([s_0, t_1], \mathbb{R})$. The sequence of functions $\{\tilde{v}^{(n)}(t)\}_0^\infty$ being monotonic and bounded is uniformly convergent on $[s_0, t_1]$. Let $V_2(t) = \lim_{n\to\infty} \tilde{v}^{(n)}(t),\ t \in [s_0, t_1]$. According to (2.221) the inequality

$$v(t) \leq V_2(t) \leq w(t),\ t \in (s_0, t_1] \tag{2.226}$$

holds. Take the limit in (2.225) and obtain for $t \in [t_1, s_1]$,

$$V_2(t) = \phi_0(t, V_2(t), V_1(s_0 - 0)), \quad t \in [s_0, t_1]. \tag{2.227}$$

Let $t \in [t_1, s_1]$. Then any element $v^{(n)} \in C^1([t_1, s_1], \mathbb{R})$ and according to (2.220) we have

$$\begin{aligned} v^{(n)}(t) &= L_0(t)\big(v^{(n)}(s_0 - 0) - v^{(n-1)}(s_0 - 0)\big) + K_0(t_1)\big(v^{(n-1)}(t_1) - v^{(n)}(t_1)\big) \\ &\quad + \phi_0(t_1, v^{(n-1)}(t_1), v^{(n-1)}(s_0 - 0)) \\ &\quad + \frac{1}{\Gamma(q)} \int_{t_1}^{t} (t-s)^{q-1} \Big(f(s, v^{(n-1)}(s)) + M_1\big(v^{(n-1)}(s) - v^{(n)}(s)\big)\Big) ds. \end{aligned} \tag{2.228}$$

The sequence of functions $\{v^{(n)}(t)\}_0^\infty$ being monotonic and bounded is uniformly convergent on $[t_1, s_1]$. Let $V_3(t) = \lim_{n\to\infty} v^{(n)}(t),\ t \in [t_1, s_1]$. According to (2.221) the inequality

$$v(t) \le V_3(t) \le w(t),\ t \in [t_1, s_1] \tag{2.229}$$

holds. Take the limit in (2.228) and obtain the fractional integral equation

$$V_3(t) = \phi_0(t_1, V_2(t_1), V_1(s_0 - 0)) + \frac{1}{\Gamma(q)} \int_{t_1}^{t} (t-s)^{q-1} f(s, V_3(s)) ds.$$

According to (2.12), (2.13), (2.14) with $\tau = t_1, \tilde{x}_0 = \phi_0(t_1, V_2(t_1), V_1(s_0 - 0))$, the limit's function $V_3(t)$ is a solution of the IVP for the FrDE

$${}_{t_1}^{c}D^q V_3(t) = f(t, V_3(t)),\ \ t \in [t_1, s_1], \qquad V_3(t_1) = \phi_0(t_1, V_2(t_1), V_1(s_0 - 0)). \tag{2.230}$$

By induction we can construct limit functions $V_{2k+2}(t) \in C([s_k, t_{k+1}], \mathbb{R})$, $k = 0, 1, 2, \ldots, p$, and $V_{2k+1}(t) \in C^1([t_k, s_k], \mathbb{R})$, $k = 0, 1, 2, \ldots, p+1$, which according to (2.224), (2.227), and (2.230) satisfy correspondingly the equations

$$V_{2k+2}(t) = \phi_k(t, V_{2k+2}(t), V_{2k+1}(s_k - 0)), \quad t \in [s_k, t_{k+1}],\ k = 0, 1, 2, \ldots, p, \tag{2.231}$$

and

$$\begin{aligned} &{}_{t_k}^{c}D^q\big(V_{2k+1}(t)\big) = f(t, V_{2k+1}(t)),\ \ t \in [t_k, s_k], \\ &V_{2k+1}(t_k) = \phi_{k-1}(t_k, V_{2k}(t_k), V_{2k-1}(s_{k-1} - 0)),\ \ k = 0, 1, 2, \ldots, p+1, \end{aligned} \tag{2.232}$$

where for $k = 0$ the initial value $\phi_0(0, V_0(0), V_{-1}(s_0 - 0))$ is replaced by x_0 in the initial condition of (2.232).

Define the function $V(t) \in PC^1([0, T], \mathbb{R})$ by

$$V(t) = \begin{cases} V_{2k+2}(t) & \text{for } t \in (s_k, t_{k+1}], k = 0, 1, 2, \ldots, p, \\ V_{2k+1}(t) & \text{for } t \in [t_k, s_k], k = 0, 1, 2, \ldots, p+1. \end{cases}$$

Similar to (2.223), (2.226), (2.229) it follows that $V(t) \in S(v,w)$. According to Definition 2.1.7 the function $V(t)$ is a solution of IVP for NIFrDE (2.196).

Similarly, using the sequence of successive approximations $\{w^{(n)}(t)\}_0^\infty$ we construct a function $W(t) \in S(v,w)$ which is a solution of IVP for NIFrDE (2.196).

We now prove that the functions $V(t)$ and $W(t)$ are minimal and maximal solutions of IVP for NIFrDE (2.196) in $S(v,w)$.

Let $u \in S(v,w)$ be a solution of IVP for NIFrDE (2.196). From inequalities (2.221) it follows that there exists a natural number N such that $v^{(N)}(t) \le u(t) \le w^{(N)}(t)$ for $t \in [0,T]$. Let $m(t) = v^{(N+1)}(t) - u(t)$, $t \in [0,T]$.

For any $t \in (t_k, s_k]$, $k = 0, 1, \dots, p+1$, we obtain

$$\begin{aligned}
{}_{t_k}^{c}D^q m(t) &= -M_k v^{(N+1)}(t) + f_k(t, v^{(N)}(t)) + M_k v^{(N)}(t) - f_k(t, u(t)) \\
&= -M_k m(t) + f_k(t, v^{(N)}(t)) - f_k(t, u(t)) \\
&\le -M_k m(t) + M_k\big(v^{(N)}(t) - u(t)\big) \\
&\le -M_k m(t).
\end{aligned}$$

For any $t \in (s_k, t_{k+1}]$, $k = 0, 1, 2, \dots, p$, we get the inequality

$$\begin{aligned}
m(t) &= L_k(t)v^{(N+1)}(s_k - 0) + K_k(t)v^{(N+1)}(t) + \phi_k(t, v^{(N)}(t), v^{(N)}(s_k - 0)) \\
&\quad - L_k(t)v^{(N)}(s_k - 0) - K_k(t)v^{(N)}(t) - \phi_k(t, u(t), u(s_k - 0)) \\
&= L_k(t)\big(v^{(N+1)}(s_k - 0) - v^{(N)}(s_k - 0)\big) + K_k(t)\big(v^{(N+1)}(t) - v^{(N)}(t)\big) \\
&\quad + \phi_k(t, v^{(N)}(t), v^{(N)}(s_k - 0)) - \phi_k(t, u(t), u(s_k - 0)) \\
&= L_k(t)m(s_k - 0) + K_k(t)m(t) + L_k(t)\big(u(s_k - 0) - v^{(N)}(s_k - 0)\big) \\
&\quad + K_k(t)\big(u(t) - v^{(N)}(t)\big) + \phi_k(t, v^{(N)}(t), v^{(N)}(s_k - 0)) - \phi_k(t, u(t), u(s_k - 0)) \\
&\le L_k(t)m(s_k - 0) + K_k(t)m(t) + L_k(t)\big(u(s_k - 0) - v^{(N)}(s_k - 0)\big) \\
&\quad + K_k(t)\big(u(t) - v^{(N)}(t)\big) + K_k\big(v^{(N)}(t) - u(t)\big) + L - k\big(v^{(N)}(s_k - 0) - u(s_k - 0)\big) \\
&= L_k(t)m(s_k - 0) + K_k(t)m(t).
\end{aligned}$$

According to Lemma 2.4.4 the function $m(t)$ is nonpositive, i.e., $v^{(N+1)}(t) \le u(t)$, $t \in [0,T]$.

Analogously the validity of inequality $w^{(N+1)}(t) \ge u(t)$ for $t \in [0,T]$ can be proved.

The inequalities $v^{(n)}(t) \le u(t) \le w^{(n)}(t)$ for $t \in [0,T]$ and $n = N, N+1, \dots$ prove $V_{2k+1}(t) \le u(t) \le W_{2k+1}(t)$ for $t \in [t_k, s_k], k = 0, 1, 2, \dots, p+1$, and $V_{2k+2}(t) \le u(t) \le W_{2k+2}(t)$ for $t \in [s_k, t_{k+1}], k = 0, 1, 2, \dots, p$.

Therefore, the functions $V(t)$ and $W(t)$ are minimal and maximal solutions of IVP for NIFrDE (2.196) in $S(v,w)$, respectively.

Suppose the IVP for NIFrDE (2.196) has a unique solution $u(t) \in S(v, w)$. Then from above it follows that $V(t) \equiv W(t) \equiv u(t)$ for $t \in [0, T]$.

$\square$

Remark 2.4.4 *The above procedure uses the approach established in [74] for ordinary differential equations where the iterates are solutions of the linear initial value problem. This poses a problem to compute the linear iterates since it involves Mittag-Leffler functions.*

2.4.2 Iterative Technique by Lower and Upper Solutions

In the monotone-iterative technique studied in the previous section the presence of the Mittag-Leffler functions with one and two parameters in the explicit formula for the successive approximations can cause some practical problems in applications. In the partial case of monotonic right side parts of the studied equation another iterative scheme for approximate solving is applied. This method is based also on the application of lower and upper solutions.

In the case when the right-hand sides of the NIFrDE are monotonic then we can apply another iterative technique which is easier to apply in practice.

Theorem 2.4.2 *Let the following conditions be fulfilled:*

1. *The functions $v, w \in PC^1([0, T])$ are lower and upper solutions of the IVP for NIFrDE(2.196), respectively, and $v(t) \leq w(t)$ for $t \in [0, T]$.*
2. *The functions $f \in C(\cup_{k=0}^{p+1}[t_k, s_k], \mathbb{R})$ and for any $x, y \in \Omega_k(t, v, w) : \ x \leq y$ and any fixed $t \in [t_k, s_k]$ the inequality $f(t, x) \leq f(t, y)$ holds.*
3. *The functions $\phi_k \in C([s_k, t_{k+1}] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $k = 0, 1, 2, \ldots, p$, and for any fixed $t \in [s_k, t_{k+1}]$ and $x_1, x_2 \in \Lambda_k(t, v, w) : x_1 \leq x_2$, $y_1, y_2 \in \Gamma_k(v, w) : y_1 \leq y_2$ the inequality $\phi_k(t, x_1, y_1) \leq \phi_k(t, x_2, y_2)$ holds.*

Then there exists a sequence of functions $\{v^{(n)}(t)\}_0^\infty$ such that:

a. *The sequences are defined by $v^{(0)}(t) = v(t)$ and for $n > 1$*

$$\begin{aligned}
&{}_0^cD^q\Big(v^{(n)}\Big)(t) = f(t, v^{(n-1)}(t)) \text{ for } t \in (t_k, s_k],\ k = 0, 1, \ldots, p+1, \\
&v^{(n)}(t) = \phi_k(t, v^{(n-1)}(t), v^{(n-1)}(s_k - 0)) \quad \text{for } t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \ldots, p, \\
&v^{(n)}(0) = x_0,
\end{aligned} \tag{2.233}$$

or by its equivalent form

$$v^{(n)}(t) = \begin{cases} x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, v^{(n-1)}(s))ds, \quad t \in [0, s_0], \\ \phi_k(t, v^{(n-1)}(t), v^{(n-1)}(s_k - 0)), \quad t \in (s_k, t_{k+1}], \ k = 0, 1, 2, \dots, p, \\ \phi_k(t_k, v^{(n-1)}(t_k), v^{(n-1)}(s_k - 0)) + \frac{1}{\Gamma(q)} \int_{t_k}^t (t-s)^{q-1} f(s, v^{(n-1)}(s))ds, \\ \qquad\qquad t \in [t_k, s_k], \ k = 1, 2, \dots, p+1. \end{cases}$$

b. *The sequence is increasing, i.e.,* $v^{(n)}(t) \le v^{(n+1)}(t) \le w(t)$ *for* $[0, T]$, $n = 0, 1, 2, \dots$;
c. *The sequence converges on* $[0, T]$*;*
d. *The limit's function* $V(t)$, *is the minimal solution of IVP for NIFrDE (2.196) in* $S(v, w)$.

Proof We will prove $v^{(0)}(t) \le v^{(1)}(t)$, $t \in [0, T]$. Let $m(t) = v^{(0)}(t) - v^{(1)}(t)$. Then the function $m(t)$ satisfies inequalities (2.207) and according to Lemma 2.4.7, $m(t) \le 0$, $t \in [0, T]$. By a similar argument we can show that $v^{(1)}(t) \le w(t)$, $t \in [0, T]$.

Assume

$$v^{(n-1)}(t) \le v^{(n)}(t) \le w(t), \quad t \in [0, T], \quad n > 1. \tag{2.234}$$

Let $m(t) = v^{(n)}(t) - v^{(n+1)}(t) \in PC^1([0, T])$. Then by the increasing nature of the functions f and ϕ_k it follows that

$$\begin{aligned} &{}_0^cD^q m(t) \\ &= {}_0^cD^q\big(v^{(n)}\big)(t) - {}_0^cD^q\big(v^{(n+1)}\big)(t) \\ &= f(t, v^{(n-1)}(t)) - f(t, v^{(n)}(t)) \\ &\le 0 \qquad \text{for } t \in (t_k, s_k], \ k = 0, 1, \dots, p+1, \end{aligned} \tag{2.235}$$

and

$$\begin{aligned} &m(t) \\ &= v^{(n)}(t) - v^{(n+1)}(t) \\ &= \phi(t, v^{(n-1)}(t), v^{(n-1)}(s_k - 0)) - \phi(t, v^{(n)}(t), v^{(n)}(s_k - 0)) \\ &\le 0 \qquad \text{for } t \in (s_k, t_{k+1}], \ k = 0, 1, 2, \dots, p. \end{aligned} \tag{2.236}$$

According to Lemma 2.4.7, $m(t) \le 0$, $t \in [0, T]$. By induction we have that

$$v^{(0)}(t) \le v^{(1)}(t) \le \dots \le v^{(n)}(t) \le \dots w(t.) \tag{2.237}$$

We will prove the convergence of the sequence of functions $\{v^{(n)}(t)\}_0^\infty$ on $[0, T]$. Any element $v^{(n)} \in C^1([0, s_0], \mathbb{R})$ and we have

$$v^{(n)}(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, v^{(n-1)}(s)) ds. \tag{2.238}$$

The sequence of functions $\{v^{(n)}(t)\}_0^\infty$ being monotonic and bounded is uniformly convergent on $[0, s_0]$. Let $V_1(t) = \lim_{n\to\infty} v^{(n)}(t),\ t \in [0, s_0]$. According to (2.237) the inequality

$$v(t) \leq V_1(t) \leq w(t),\ t \in [0, s_0] \tag{2.239}$$

holds. Take the limit in (2.238) and obtain the Volterra fractional integral equation $V_1(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, V_1(s)) ds$. According to (2.12), (2.13), (2.14) with $\tau = 0, \tilde{x}_0 = x_0$, the limit's function $V_1(t)$ is a solution of the IVP for the FrDE

$${}_0^cD^q V_1(t) = f(t, V_1(t)),\ \ t \in [0, s_0], \qquad V_1(0) = x_0. \tag{2.240}$$

Let $t \in (s_0, t_1]$. Then any element $v^{(n)} \in C((s_0, t_1], \mathbb{R})$ and we have

$$v^{(n)}(t) = \phi_k(t, v^{(n-1)}(t), v^{(n-1)}(s_k - 0)) \quad \text{for } t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \dots, p. \tag{2.241}$$

From $v^{(n)}(t) \in PC^1([0, T])$, $\lim_{t\downarrow s_0} v^{(n)}(t) = v^{(n)}(s_0 + 0) < \infty$ exists. For any $n = 1, 2, \dots$ we define the functions

$$\tilde{v}^{(n)}(t) = \begin{cases} v^{(n)}(s_0 + 0) & \text{for } \ t = s_0, \\ v^{(n)}(t) \text{ for } \ t \in (s_0, t_1]. \end{cases}$$

Then $\tilde{v}^{(n)} \in C([s_0, t_1], \mathbb{R})$. The sequence of functions $\{\tilde{v}^{(n)}(t)\}_0^\infty$ being monotonic and bounded is uniformly convergent on $[s_0, t_1]$. Let $V_2(t) = \lim_{n\to\infty} \tilde{v}^{(n)}(t),\ t \in [s_0, t_1]$. According to (2.237) the inequality

$$v(t) \leq V_2(t) \leq w(t),\ t \in (s_0, t_1] \tag{2.242}$$

holds. Take the limit in (2.225) and obtain for $t \in [t_1, s_1]$,

$$V_2(t) = \phi_1(t, V_2(t), V_1(s_0 - 0)), \quad t \in [s_0, t_1]. \tag{2.243}$$

The rest of the proof is similar to that in Theorem 2.4.1 and we omit it.

□

Theorem 2.4.3 *Let the following conditions be fulfilled:*

1. *The functions $v, w \in PC^1([0, T])$ are lower and upper solutions of the IVP for NIFrDE(2.196), respectively, and $v(t) \leq w(t)$ for $t \in [0, T]$.*

2. *The function* $f \in C(\cup_{k=0}^{p+1}[t_k, s_k], \mathbb{R})$ *and for any* $x, y \in \Omega_k(t, v, w) : \; x \leq y$ *and any* $t \in [t_k, s_k]$ *the inequality* $f(t, x) \leq f(t, y)$ *holds.*
3. *The functions* $\phi_k \in C([s_k, t_{k+1}] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $k = 0, 1, 2, \ldots, p$, *and for any* $t \in [s_k, t_{k+1}]$ *and* $x_1, x_2 \in \Lambda_k(t, v, w) : x_1 \leq x_2$, $y_1, y_2 \in \Gamma_k(v, w) : y_1 \leq y_2$ *the inequality* $\phi_k(t, x_1, y_1) \leq \phi_k(t, x_2, y_2)$ *holds.*

Then there exists a sequence of functions $\{w^{(n)}(t)\}_0^\infty$ *such that:*

a. *The sequences are defined by* $w^{(0)}(t) = w(t)$ *and for* $n > 1$

$$\begin{aligned}
&{}_0^cD^q\Big(w^{(n)}\Big)(t) = f(t, w^{(n-1)}(t)) \text{ for } t \in (t_k, s_k],\; k = 0, 1, \ldots, p+1,\\
&w^{(n)}(t) = \phi_k(t, w^{(n-1)}(t), w^{(n-1)}(s_k - 0)) \quad \text{for } t \in (s_k, t_{k+1}],\; k = 0, 1, 2, \ldots, p,\\
&w^{(n)}(0) = x_0,
\end{aligned} \tag{2.244}$$

or by its equivalent form

$$w^{(n)}(t) = \begin{cases} x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, w^{(n-1)}(s)) ds, \quad t \in [0, s_0],\\ \phi_k(t, w^{(n-1)}(t), w^{(n-1)}(s_k - 0)), \quad t \in (s_k, t_{k+1}],\; k = 0, 1, 2, \ldots, p,\\ \phi_k(t_k, w^{(n-1)}(t_k), w^{(n-1)}(s_k - 0)) + \frac{1}{\Gamma(q)} \int_{t_k}^t (t-s)^{q-1} f(s, w^{(n-1)}(s)) ds,\\ \qquad t \in [t_k, s_k],\; k = 1, 2, \ldots, p+1. \end{cases}$$

b. *The sequence is decreasing, i.e.,* $w^{(n)}(t) \leq w^{(n+1)}(t) \leq w(t)$ *for* $[0, T]$, $n = 0, 1, 2, \ldots$;
c. *The sequence converges on* $[0, T]$*;*
d. *The limit's function* $W(t)$ *is the maximal solution of IVP for NIFrDE (2.196) in* $S(v, w)$.

Proof The proof is similar to that in Theorem 2.4.2, so we omit it. □

Now with an example we will illustrate the algorithm for approximately solving a non-instantaneous impulsive differential equation.

Example 2.4.1 Let $0 = t_0 < s_0 = 0.5 < t_1 = 0.75 < s_1 = 1.3 = T$, $q = 0.3$. Consider the following nonlinear NIFrDE

$$\begin{aligned}
&{}_0^cD^{0.3}x(t) = f(t, x(t))) \text{ for } t \in (t_k, s_k],\; k = 0, 1,\\
&x(t) = e^{x(t)-2} + t(x(s_k - 0))^2 \quad \text{for } t \in (s_0, t_1],\\
&x(0) = 0,
\end{aligned} \tag{2.245}$$

where $x \in \mathbb{R}$ and $t \in [0, 0.5] \cup [0.75, 1.3]$, and the function $f(t, x)$ is defined by

$$f(t, x) = \begin{cases} x^2 + t(1-t), & t \in [0, 0.5],\\ x^2(t - 0.75)^{0.7}, & t \in [0.75, 1.3]. \end{cases}$$

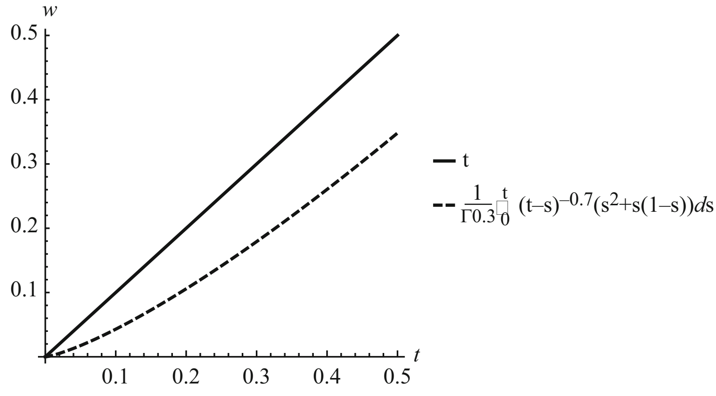

Fig. 2.20 Example 2.4.1. Graph of the upper solution on the interval $[0, 0.5]$.

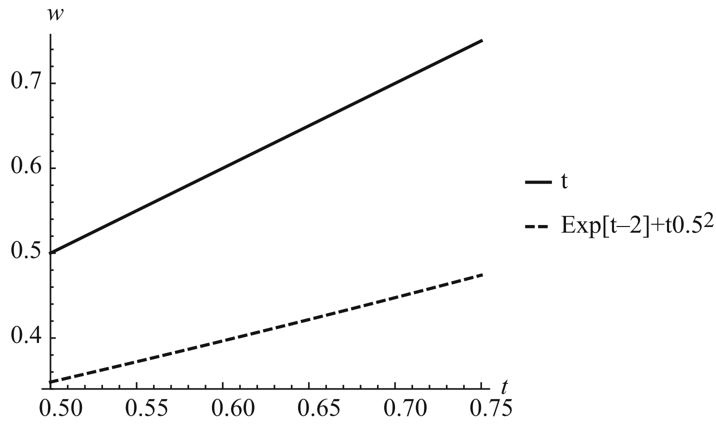

Fig. 2.21 Example 2.4.1. Graph of the upper solution on the interval of non-instantaneous impulse $[0.5, 0.75]$.

The function $v(t) = 0$, $t \in [0, T]$ is a lower solution of NIFrDE (see Definition 2.4.2 and inequalities (2.200)):

- the inequality $0 \leq 0 + \frac{1}{\Gamma(0.3)} \int_0^t (t-s)^{-0.7} s(1-s)ds$ holds in $[0, 0.5]$;
- the inequality $0 < e^{-2} + t\,0$ holds for $t \in [0.5, 0.75]$;
- the inequality $0 < e^{-2}$ holds in $[0.75, 1.3]$.

The function $w(t) = t$ is an upper solution (see Figure 2.23) because

- the inequality $t \geq 0 + \frac{1}{\Gamma(0.3)} \int_0^t (t-s)^{-0.7}(s^2 + s(1-s))ds$ holds in $[0, 0.5]$ (see Figure 2.20);
- the inequality $t > e^{t-2} + t\,0.5^2$ holds for $t \in [0.5, 0.75]$ (see Figure 2.21);
- the inequality $t \geq e^{0.75-2} + 0.75\;0.5^2 + \frac{1}{\Gamma(0.3)} \int_{0.75}^t (t-s)^{-0.7} s^2 (s-0.75)^{0.7} ds$ holds in $[0.75, 1.3]$ (see Figure 2.22).

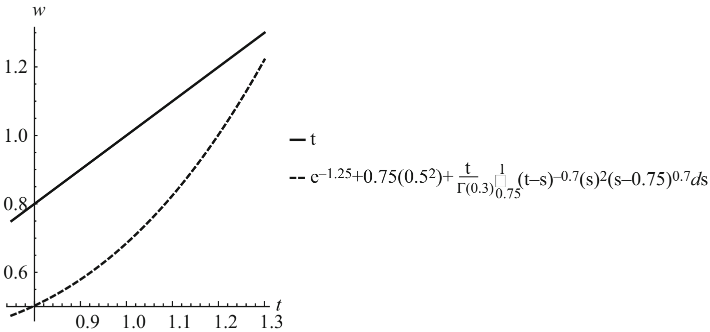

Fig. 2.22 Example 2.4.1. Graph of the upper solution on the interval [0.75, 1.3].

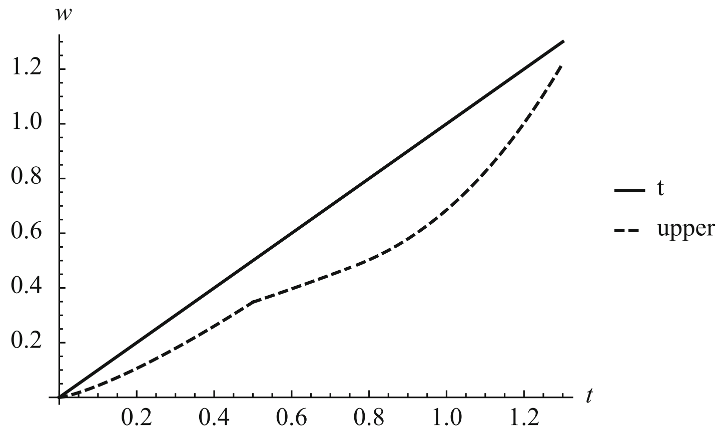

Fig. 2.23 Example 2.4.1. Graph of the upper solution on the interval [0, 1.3].

All conditions of Theorem 2.4.2 are satisfied. Therefore, there exists a minimal solution of the IVP for NIFrDE (2.245) which is a limit of the successive approximations $v^{(n)}(t), n = 1, 2, 3, \ldots$, given by

$$v^{(n)}(t) = \begin{cases} \frac{1}{\Gamma(0.3)} \int_0^t (t-s)^{-0.7}\Big(\big(v^{(n-1)}(s)\big)^2 + s(1-s)\Big)ds, \quad t \in [0, 0.5], \\ e^{v^{(n-1)}(t)-2} + t\Big(v^{(n-1)}(0.5-0)\Big)^2, \quad t \in (0.5, 0.75], \\ e^{v^{(n-1)}(0.75)-2} + 0.75\Big(v^{(n-1)}(0.5-0)\Big)^2 \\ \quad + \frac{1}{\Gamma(0.3)} \int_{0.75}^t (t-s)^{-0.7}\big(v^{(n-1)}(s)\big)^2 (s-0.75)^{0.7} ds, \quad t \in [0.75, 1.3], \end{cases}$$

i.e., $v^{(0)}(s) \equiv 0$,

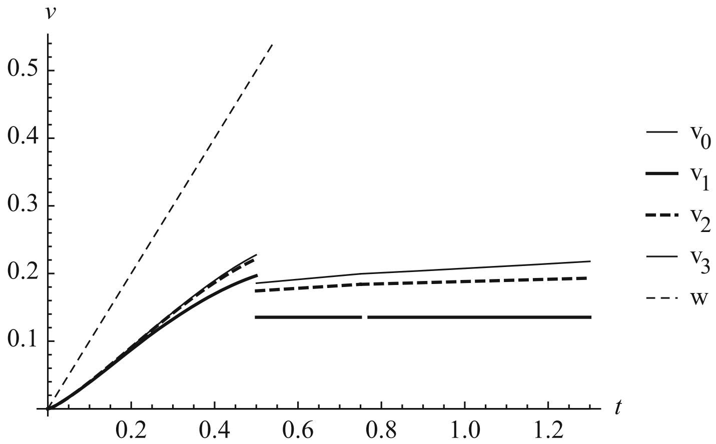

Fig. 2.24 Example 2.4.1. Graphs of lower approximations on the interval [0, 1.3].

$$v^{(1)}(t) = \begin{cases} \frac{1}{\Gamma(0.3)} \int_0^t (t-s)^{-0.7} s(1-s) ds, & t \in [0, 0.5], \\ e^{-2}, & t \in (0.5, 0.75], \\ e^{-2}, & t \in [0.75, 1.3]. \end{cases}$$

$$v^{(2)}(t) = \begin{cases} \frac{1}{\Gamma(0.3)} \int_0^t (t-s)^{-0.7} \Big(\big(v^{(1)}(s)\big)^2 + s(1-s) \Big) ds, & t \in [0, 0.5], \\ e^{e^{-2}-2} + t\big(v^{(1)}(0.5-0)\big)^2, & t \in (0.5, 0.75], \\ e^{e^{-2}-2} + 0.7\big(v^{(1)}(0.5-0)\big)^2 \\ \quad + \frac{1}{\Gamma(0.3)} \int_{0.75}^t (t-s)^{-0.7} e^{-4} (s-0.75)^{0.7} ds, & t \in [0.75, 1.3]. \end{cases}$$

$$v^{(3)}(t) = \begin{cases} \frac{1}{\Gamma(0.3)} \int_0^t (t-s)^{-0.7} \Big(\big(v^{(2)}(s)\big)^2 + s(1-s) \Big) ds, & t \in [0, 0.5], \\ e^{v^{(2)}(t)-2} + t\Big(\big(v^{(2)}(0.5-0)\big)^2 \Big)^2, & t \in (0.5, 0.75], \\ e^{e^{e^{-2}-2}+0.75e^{-4}-2} + 0.75\Big(\big(v^{(2)}(0.5-0)\big)^2 \Big)^2 \\ \quad + \frac{1}{\Gamma(0.3)} \int_{0.75}^t (t-s)^{-0.7} \big(v^{(2)}(s)\big)^2 (s-0.75)^{0.7} ds, & t \in [0.75, 1.3]. \end{cases}$$

The graphs of the lower approximations $v^{(n)}(t), n = 0, 1, 2, 3$ are given in Figure 2.24. We see the sequence of lower approximations is increasing and staying between the initial lower solution $v(t) \equiv 0$ and the initial upper solution $w(t) = t$.

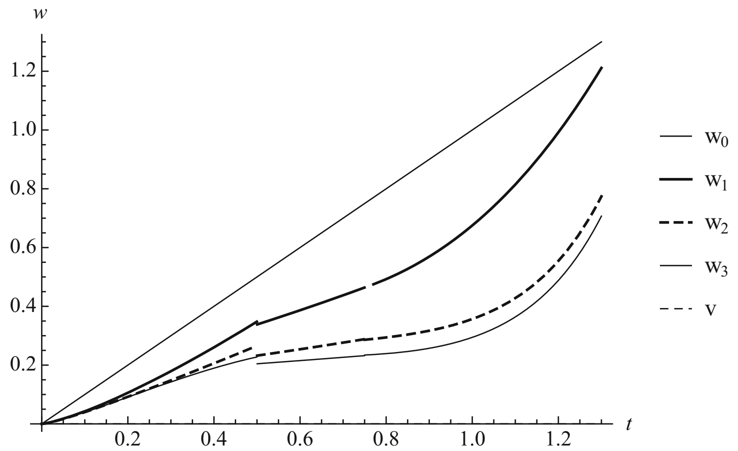

Fig. 2.25 Example 2.4.1. Graphs of upper approximations on the interval [0, 1.3].

Similarly, applying Theorem 2.4.3 we construct upper successive approximations by

$$w^{(0)}(t) = t,$$

$$w^{(1)}(t) = \begin{cases} \frac{1}{\Gamma(0.3)} \int_0^t (t-s)^{-0.7}(s^2 + s(1-s))ds, \quad t \in [0, 0.5], \\ e^{t-2} + t0.5^2, \quad t \in (0.5, 0.75], \\ e^{0.75-2} + 0.75\ 0.5^2 + \frac{1}{\Gamma(0.3)} \int_{0.75}^t (t-s)^{-0.7} s^2 (s-0.75)^{0.7} ds, \ t \in [0.75, 1.3]. \end{cases}$$

$$w^{(k+1)}(t) = \begin{cases} \frac{1}{\Gamma(0.3)} \int_0^t (t-s)^{-0.7}\Big(\big(w^{(k)}(s)\big)^2 + s(1-s)\Big)ds, \quad t \in [0, 0.5], \\ e^{w^{(k)}(t)-2} + t\big(w^{(k)}(0.5-0)\big)^2, \quad t \in (0.5, 0.75], \\ e^{w^{(k)}(0.75)-2} + 0.75\big(w^{(k)}(0.5-0)\big)^2 \\ \quad + \frac{1}{\Gamma(0.3)} \int_{0.75}^t (t-s)^{-0.7} (w^{(k)}(s))^2 (s-0.75)^{0.7} ds, \ t \in [0.75, 1.3], \ k = 1, 2, \dots. \end{cases}$$

The graphs of the upper approximations $w^{(n)}(t), n = 0, 1, 2, 3$ are given in Figure 2.25. We see the sequence of upper approximations is decreasing and staying between the initial lower solution $v(t) \equiv 0$ and the initial upper solution $w(t) = t$.

□

Springer
http://www.springer.com/978-3-319-66383-8

Non-Instantaneous Impulses in Differential Equations
Agarwal, R.P.; Hristova, S.; O' Regan, D.
2017, XI, 251 p. 49 illus., Hardcover
ISBN: 978-3-319-66383-8