Chapter 2
Duality

In this text, we use the dual characterization of the ergodic maximizing value to expose a natural bridge between maximizing probabilities and sub-actions. Since the earliest works in ergodic optimization theory, it became clear however that this constant can be presented in various equivalent ways. In this chapter, some alternative expressions that could be considered to define the ergodic maximizing value are brought to the attention of the reader. Furthermore, a proof of the dual formula is provided.

2.1 A Multifaceted Constant

There are several interesting descriptions of the ergodic maximizing value (1.1). For instance, as a topologically mixing subshift of finite type is an example of dynamics satisfying the specification property, from a classical result of Sigmund [97], the *periodic probabilities* (i.e., any σ-invariant Borel probability measure whose support is a periodic orbit) are a weak* dense subset of $\mathcal{M}_\sigma$. Actually, for symbolic systems, this fact is known at least since the work of Parthasarathy [91]. Therefore, one has

$$\beta_A = \sup\left\{ \int A\, d\mu : \mu \in \mathcal{M}_\sigma,\ \mu \text{ periodic probability} \right\}. \tag{2.1}$$

Naturally related to this formula, the rate at which the ergodic maximizing value is approached by integrals of the potential against periodic probabilities of at most a given period was fully studied in [27].

Birkhoff's ergodic theorem immediately suggests that the ergodic maximizing value (1.1) may be also expressed in terms of averages of Birkhoff sums, placing a major emphasis on orbits. Indeed, well-known characterizations of this constant are the following ones.

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Proposition 2.1 *For a continuous potential A,*

$$\begin{aligned}\beta_A &= \lim_{n\to\infty} \frac{1}{n} \max_{\mathbf{x}\in\Sigma} S_nA(\mathbf{x}) \\ &= \max_{\mathbf{x}\in\Sigma} \limsup_{n\to\infty} \frac{1}{n} S_nA(\mathbf{x}) \\ &= \max_{\mathbf{x}\in Reg(A)} \lim_{n\to\infty} \frac{1}{n} S_nA(\mathbf{x}),\end{aligned}$$

where $Reg(A)$ denotes the set of the points $\mathbf{x} \in \Sigma$ for which there exists the limit of $\frac{1}{n}S_nA(\mathbf{x})$ as n tends to infinite.

A proof of such equalities may be found, for instance, in [64]. For the sake of completeness, we present a similar argument.

Proof First note that $\{\max S_nA\}_{n\geq 1}$ is subadditive and thus the limit of $\frac{1}{n}\max S_nA$ exists (and is equal to $\inf_{n\geq 1} \frac{1}{n} \max S_nA$). Moreover, it is easy to see that

$$\sup_{\mathbf{x}\in \mathrm{Reg}(A)} \lim_{n\to\infty} \frac{1}{n} S_nA(\mathbf{x}) \leq \sup_{\mathbf{x}\in\Sigma} \limsup_{n\to\infty} \frac{1}{n} S_nA(\mathbf{x}) \leq \lim_{n\to\infty} \frac{1}{n} \max_{\mathbf{x}\in\Sigma} S_nA(\mathbf{x}).$$

On the one hand, recalling that Borel probabilities form a weak* compact set, it is not hard to check that, for any given sequence $\{\mathbf{x}_n\} \subset \Sigma$, the measures defined by $\mu_n := \frac{1}{n}\sum_{k=0}^{n-1} \delta_{\sigma^k(\mathbf{x}_n)}$ always accumulate on σ-invariant probabilities. Hence, choosing $\mathbf{x}_n$ such that $S_nA(x_n) = \max S_nA$, we conclude that $\lim \frac{1}{n} \max S_nA \leq \beta_A$.

On the other hand, Peres' lemma (see [92]) states that, for any $\mu \in \mathcal{M}_\sigma$, there exists $\mathbf{x} \in \Sigma$ such that

$$\int A\, d\mu \leq \frac{1}{n} S_nA(\mathbf{x}) \qquad \forall\, n \geq 1.$$

Considering then a maximizing probability $\mu \in m_A$, we have

$$\beta_A \leq \liminf_{n\to\infty} \frac{1}{n} S_nA(\mathbf{x}) \leq \limsup_{n\to\infty} \frac{1}{n} S_nA(\mathbf{x}) \leq \lim_{n\to\infty} \frac{1}{n} \max S_nA \leq \beta_A,$$

which shows that $\mathbf{x} \in \mathrm{Reg}(A)$ and therefore all equalities hold. □

The dual formula (1.2), which will be addressed in a moment, corresponds in Lagrangian Aubry-Mather theory to a similar description of Mañé's critical value (see [38] for details). We provide in the next result a representation of the ergodic maximizing value using Mañé's viewpoint (compare (2.2)–(5.9) in Chap. 5). As usual, we denote by $\mathrm{orb}(\mathbf{x})$ the orbit by σ of a point $\mathbf{x}$ of Σ.

Proposition 2.2 *Let $A : \Sigma \to \mathbb{R}$ be a continuous potential. Then*

$$\beta_A = \min\Big\{c \in \mathbb{R} : \sum_{\mathbf{y}\in orb(\mathbf{x})} (A - c)(\mathbf{y}) \leq 0, \quad \forall\, \mathbf{x} \in \Sigma \textit{ periodic point}\Big\}. \tag{2.2}$$

Proof Obviously $\int (A - \beta_A)\, d\mu \leq 0$ for all μ σ-invariant probability measure and, in particular, for all μ periodic probability. Moreover, if $\int (A - c)\, d\mu \leq 0$ for any μ periodic probability, thanks to (2.1), we deduce that $\beta_A \leq c$, which concludes the proof. □

2.2 The Dual Formula

In ergodic optimization, the dual expression was established in two unpublished papers [41, 95] by different techniques. For convenience of the reader, we present a complete proof, which may be extended to other situations, like the one involving the study of minimizing configurations in almost-periodic environments [52]. A comparable argument was used in [13] to obtain the dual description of Mañé's critical value.

Theorem 2.3 *For a continuous potential $A : \Sigma \to \mathbb{R}$,*

$$\beta_A = \inf_{f \in C(\Sigma)} \max_{\mathbf{x} \in \Sigma} \left[A(\mathbf{x}) + f \circ \sigma(\mathbf{x}) - f(\mathbf{x})\right].$$

Proof Given any $f \in C(\Sigma)$, notice that, for all $\mu \in \mathcal{M}_\sigma$,

$$\int A\, d\mu = \int (A + f \circ \sigma - f)\, d\mu \leq \max(A + f \circ \sigma - f),$$

so we get $\beta_A \leq \inf_f \max_{\mathbf{x}} (A + f \circ \sigma - f)(\mathbf{x})$.

In order to obtain the opposite inequality, by conciseness we denote

$$\kappa := -\inf_f \max_{\mathbf{x}} (A + f \circ \sigma - f)(\mathbf{x}).$$

Notice that it is enough to prove the following claim.

Claim There is a measure $\mu \in \mathcal{M}_\sigma$ such that $\int g\, d\mu - \max(A + g) \leq \kappa$, for all $g \in C(\Sigma)$.

In fact, if such a claim holds, for $g = -A$, we have $-\kappa \leq \int A\, d\mu \leq \beta_A$, which is the inequality we are looking for. To demonstrate the claim, recall that a function $g \in C(\Sigma)$ is a (topological) coboundary when $g = f \circ \sigma - f$ for some $f \in C(\Sigma)$. Consider then the sets

$$C_1 := \{(g_1, t_1) \in C(\Sigma) \times \mathbb{R} : t_1 < 0,\ g_1 \text{ is a coboundary}\},$$
$$C_2 := \{(g_2, t_2) \in C(\Sigma) \times \mathbb{R} : \kappa + \max(A + g_2) < t_2\}.$$

Notice that C_1 and C_2 are nonempty convex sets. Besides, they are disjoint by the definition of κ and C_2 is open. Therefore, by the geometric version of Hahn-Banach theorem, there exists a non-null continuous linear form (ν, α) on $C(\Sigma) \times \mathbb{R}$ which separates C_1 and C_2:

$$\sup_{(g_1,t_1)\in C_1}\Big(\int g_1 d\nu + \alpha t_1\Big) \le \inf_{(g_2,t_2)\in C_2}\Big(\int g_2 d\nu + \alpha t_2\Big). \tag{2.3}$$

We must have $\alpha > 0$. Indeed, $\alpha = 0$ would imply $\int (f\circ\sigma - f)\,d\nu \le \int (f\circ\sigma - f \pm h)\,d\nu$ for all $f, h \in C(\Sigma)$, so that $\int h\,d\nu = 0$ would contradict $(\nu,\alpha) \neq (0,0)$. Besides, for $\alpha < 0$, by taking $-t_1$ and t_2 large enough, one would easily contradict the inequality (2.3).

Define then $\mu := -\nu/\alpha$. Hence, from (2.3) we obtain $\int g_1\,d\mu - t_1 \ge \int g_2\,d\mu - t_2$ for all $(g_i, t_i) \in C_i$, $i = 1, 2$. As t_1 tends to 0 and t_2 tends to $\kappa + \max(A + g_2)$, we get that

$$\int (g_2 - g_1) d\mu - \max(A + g_2) \le \kappa \tag{2.4}$$

for any coboundary g_1 and for all continuous function g_2. Therefore, the claim (and the theorem) will be proved when we show that μ is actually a σ-invariant probability.

Notice first that replacing g_2 by cg_2 in inequality (2.4), dividing by $c > 0$, and finally letting c tend to infinity yield $\int g_2 d\mu - \max g_2 \le 0$. Applying this last inequality to both g_2 and $-g_2$ now leads to

$$\min g_2 \le \int g_2\,d\mu \le \max g_2.$$

This shows that μ is a probability measure.

In order to prove that μ is σ-invariant, replace g_1 by cg_1 in (2.4), divide by $c > 0$, and then let c tend to infinity to obtain $\int g_1 d\mu \le 0$. Since $-g_1$ is also a coboundary, we conclude that $\int g_1 d\mu = 0$ for every coboundary g_1. $\square$

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