

## Chapter 2

# Secular Diffusion

### 2.1 Introduction

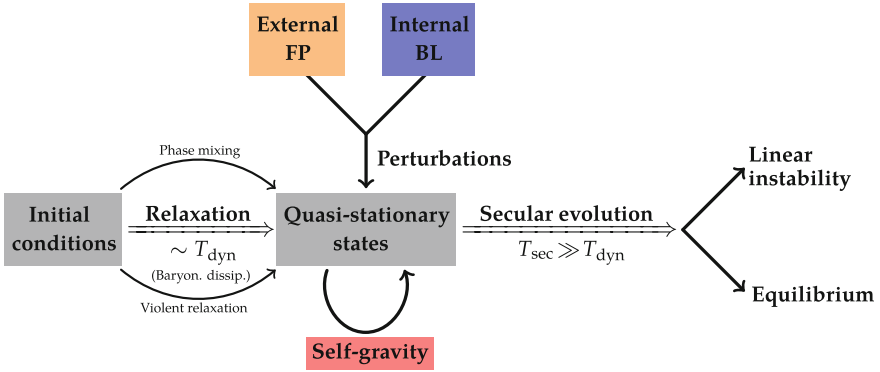
The previous chapter described the typical fate of self-gravitating systems, which is summed up in Fig. 2.1. As a result of both phase mixing (see Fig. 1.4) and violent relaxation (see Fig. 1.5), self-gravitating systems very efficiently reach quasi-stationary states for the collisionless mean field dynamics. The systems are virialised and the mean potentials do not strongly fluctuate anymore. Stars follow their orbit set up by the mean field potential and are typically uniformly distributed in phase along each of them. Yet, as gravity is a long-range interaction, self-gravitating systems have the ability to amplify and dress perturbations (see, e.g., Fig. 1.7). These collective effects have two main consequences. They may first lead to the spontaneous growth of dynamical instabilities if ever the system was dynamically unstable. Moreover, even for genuinely stable systems, these effects can also lead to polarisation, i.e. a dressing of perturbations and therefore a boost in the amplitude of the fluctuations in the system. This self-gravitating amplification is especially important for cold dynamical systems, i.e. within which most of the gravitational support comes from centrifugal forces and for which the velocity dispersion is low. This makes the system strongly responsive. This is for example important for stellar discs, where new stars, born on the cold orbits of the gas, are constantly being supplied to the system.

Once the system has reached a quasi-stationary state through these various mixing processes, the mean collisionless dynamics maintains stationarity and such a quiescent system can now only slowly evolve on long timescales.<sup>1</sup> This is the timescale for secular evolution, which will be our main interest here. At this stage, only additional fluctuations can drive the system's evolution. Such considerations fall within

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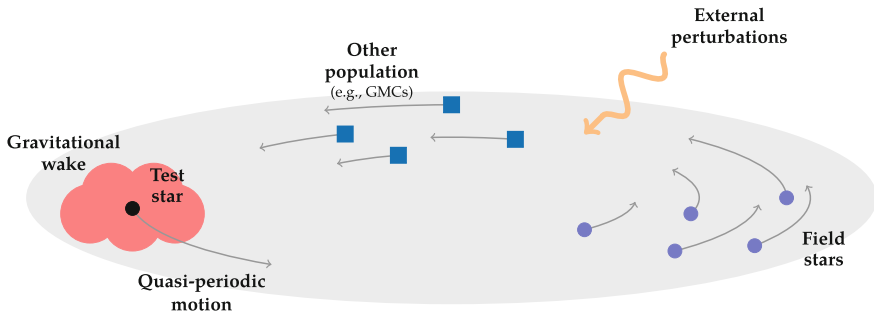
The work presented in this chapter is based on Fouvry et al. (2015a, b, 2016a, b).

<sup>1</sup>Another possibility allowing self-gravitating systems to reach more probable and hotter configurations is for them to spontaneously develop an instability, such as a bar (Hohl 1971), leading as well to an efficient rearrangement of the orbital structure. Such outcomes are not investigated in the present thesis.



**Fig. 2.1** Illustration of the typical fate of a self-gravitating system. On the first few dynamical times, the system’s evolution is governed by the Vlasov-Poisson system (see Eq. (1.9)), and as a result of violent relaxation and phase mixing, the system reaches a quasi-stationary state, generically difficult to predict. Following this relaxation, the mean system does not evolve anymore on short timescales. On longer timescales, the fluctuations present in the system, amplified by self-gravity, induce a long-term distortion of the system’s orbital structure: it is the regime of secular evolution. These perturbations may either be external (via the so-called dressed Fokker-Planck equation, see Sect. 2.2) or internal (via the inhomogeneous Balescu-Lenard equation, see Sect. 2.3). Finally, as a result of the secular evolution, the system may either become dynamically unstable or may reach an equilibrium state, if it exists

the general framework of the fluctuation-dissipation theorem, for which fluctuations dissipate and lead to a diffusion in the system. Let us now introduce an important dichotomy on which the two upcoming sections rely. There are two main channels to induce fluctuations in a system. Fluctuations of the first type are induced by external stochastic perturbations, whose contributions will be felt by the system and will lead therein to slow orbital distortions. As will be discussed in detail in the next section, the efficiency of such secular dynamics is dictated in particular by the match between the temporal frequencies of these perturbations and the system’s natural intrinsic frequencies. We call this framework the collisionless framework. Another source of fluctuations is also present in any system made of a finite number  $N$  of particles: these are finite- $N$  effects, also called Poisson shot noise. This graininess can not only be triggered by the finite number of constituents in the system, but can also originate from the variety of its components, e.g., the existence of a mass spectrum of components. As a direct consequence of the finite number of particles, the system’s self-induced potential is not perfectly smooth, and therefore fluctuates around its mean quasi-stationary value. These unavoidable and non-vanishing fluctuations may then act as the source of a secular irreversible evolution. We call this framework the collisional framework, in the sense that it relies on encounters between the finite number of particles. Let us finally note that whatever the source of the perturbations, these fluctuations are dressed by collective effects. A proper accounting of the importance of the gravitational polarisation is at the heart of the upcoming derivations. On the long-term, secular dynamics may lead the system towards a dynamical instability (see, e.g., Fig. 4.6) or towards an equilibrium state, if it exists.



**Fig. 2.2** Illustration of the different components involved in the secular dynamics of a test star in a stellar disc. To describe the complex quasiperiodic motion of the test star, one has to rely on angle-action coordinates. The disc being self-gravitating, any perturbation is dressed by collective effects and surrounded by a gravitational wake. The long-term orbital diffusion of the test star may then be induced either by external perturbations (e.g., cosmic environment) or internal perturbations (e.g., finite- $N$  effects)

The main elements involved in the secular evolution of a self-gravitating system can also be considered directly in physical space. This is illustrated in Fig. 2.2, where we consider the example of the diffusion of a test star in a stellar disc. Because a stellar disc is inhomogeneous, the trajectories of individual stars are intricate. Assuming their motions to be quasiperiodic, one can describe the associated orbits thanks to angle-action coordinates (see Eq. (1.6)). The disc being self-gravitating, any test star is surrounded by a “polarisation cloud” associated with collective effects (see Fig. 1.7). The test star may then diffuse as a result of two types of fluctuations. These perturbations may first be external, imposed for example by the cosmological environment. The long-term effects of these externally-induced perturbations can be described by the secular collisionless diffusion equation and will be considered in Sect. 2.2. There exist also intrinsic perturbations due to the finite number of field stars or due to the discreteness of another component of the disc (e.g., giant molecular clouds). Such an internally-induced evolution can be described by the inhomogeneous Balescu-Lenard equation and will be considered in Sect. 2.3.

This dichotomy is essential for all the upcoming sections. It allows us to distinguish secular evolution induced by the system’s environment from secular evolution induced by the system’s internal properties. It is therefore an useful tool to disentangle the respective contributions from nurture and nature in driving the evolution of a self-gravitating system. The aim of the present chapter is to detail the relevant formalisms allowing for the description of long-term evolutions induced by (internal or external) potential fluctuations. The following chapters will illustrate applications of these formalisms to various astrophysical systems. Let us first focus in Sect. 2.2 on the collisionless framework, where the dynamics is driven by external perturbations. Then, in Sect. 2.3, we will consider the collisional framework of diffusion, sourced by the discreteness of these self-gravitating systems.

## 2.2 Collisionless Dynamics

Let us first describe the collisionless diffusion that external potential fluctuations may induce. Such externally-driven secular evolution can be addressed via the so-called dressed secular collisionless diffusion equation, where the source of evolution is taken to be potential fluctuations from an external bath. It has already been a theme of active research, as we now briefly review. Binney and Lacey (1988) computed the first- and second-order diffusion coefficients in action space describing the orbital diffusion occurring in a system because of fluctuations in the gravitational potential. This first approach however did not account for collective effects, i.e. the ability of the system to dress and amplify perturbations. Weinberg (1993) emphasised the importance of self-gravity for the non-local and collective relaxation of stellar systems. Weinberg (2001a, b) considered similar secular evolutions while accounting for the self-gravitating amplification of perturbations, and studied the impacts of the properties of the noise processes. Ma and Bertschinger (2004) relied on a quasilinear approach to investigate the diffusion of dark matter induced by cosmological fluctuations. Pichon and Aubert (2006) sketched a time-decoupling approach to solve the collisionless Boltzmann equation in the presence of external perturbations and applied it to a statistical study of the effect of dynamical flows through dark matter haloes on secular timescales. The approach developed therein is close to the one presented in Fouvy et al. (2015b). Chavanis (2012a) considered the evolution of homogeneous collisionless systems when forced by an external perturbation, while Nardini et al. (2012) investigated similarly the effects of stochastic forces on the long-term evolution of long-range interacting systems.

In the upcoming section, let us follow Fouvy et al. (2015b) and present a derivation of the appropriate secular resonant collisionless dressed diffusion equation. This derivation is based on a quasilinear timescale decoupling of the collisionless Boltzmann equation. This yields two evolution equations, one for the fast dynamical evolution and amplification of perturbations within the system, and one for the secular evolution of the system's mean DF.

### 2.2.1 Evolution Equations

Let us consider a collisionless self-gravitating quasi-stationary system undergoing external stochastic perturbations. The mean system being quasi-stationary, we introduce its quasi-stationary Hamiltonian  $H_0$ , associated with the mean potential  $\psi_0$ . We assume that throughout its evolution, the system remains integrable, so that one can always define an angle-action mapping  $(\mathbf{x}, \mathbf{v}) \mapsto (\boldsymbol{\theta}, \mathbf{J})$  appropriate for the Hamiltonian  $H_0$ . Thanks to Jeans theorem (Jeans 1915), the mean DF of the system,  $F$ , depends only on the actions, so that  $F = F(\mathbf{J}, t)$ . We suppose that an external source is perturbing the system, and we expand the system's total DF and Hamiltonian as

$$\begin{cases} F^{\text{tot}}(\mathbf{J}, \boldsymbol{\theta}, t) = F(\mathbf{J}, t) + \delta F(\mathbf{J}, \boldsymbol{\theta}, t), \\ H^{\text{tot}}(\mathbf{J}, \boldsymbol{\theta}, t) = H_0(\mathbf{J}, t) + \delta\psi^e(\mathbf{J}, \boldsymbol{\theta}, t) + \delta\psi^s(\mathbf{J}, \boldsymbol{\theta}, t). \end{cases} \quad (2.1)$$

In the decompositions from Eq. (2.1), one should pay attention to the presence of two types of potential perturbations. Here,  $\delta\psi^e$  corresponds to an external stochastic perturbation, while  $\delta\psi^s$  corresponds to the self-response of the system induced by its self-gravity (Weinberg 2001a). This additional perturbation is crucial to capture the system's gravitational susceptibility, i.e. its ability to amplify perturbations. We place ourselves in the limit of small perturbations, so that  $\delta F \ll F$ , and  $\delta\psi^e, \delta\psi^s \ll \psi_0$ . Assuming that the system evolves in a collisionless fashion, its dynamics is fully described by the collisionless Boltzmann equation (1.9) reading

$$\frac{\partial F^{\text{tot}}}{\partial t} + [F^{\text{tot}}, H^{\text{tot}}] = 0, \quad (2.2)$$

where  $[\cdot, \cdot]$  stands for the Poisson bracket as defined in Eq. (1.2). Let us then inject the decomposition from Eq. (2.1) into Eq. (2.2) to get

$$\frac{\partial F}{\partial t} + \frac{\partial \delta F}{\partial t} + [F, H_0] + [F, \delta\psi^e + \delta\psi^s] + [\delta F, H_0] + [\delta F, \delta\psi^e + \delta\psi^s] = 0. \quad (2.3)$$

Because we assumed the mean DF to be quasi-stationary, i.e.  $F = F(\mathbf{J}, t)$ , one has  $[F, H_0] = 0$ , since  $H_0 = H_0(\mathbf{J}, t)$ . Let us now take an average of Eq. (2.3) w.r.t. the angles  $\boldsymbol{\theta}$ . In Eq. (2.3), all the terms linear in the perturbations vanish, and we get a secular evolution equation for the mean DF  $F$  as

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left[ \int \frac{d\boldsymbol{\theta}}{(2\pi)^d} \delta F \frac{\partial [\delta\psi^e + \delta\psi^s]}{\partial \boldsymbol{\theta}} \right], \quad (2.4)$$

where  $d$  is the dimension of the physical space, e.g.,  $d=2$  for a razor-thin disc. At this stage, let us note that  $\partial F / \partial t$  can be considered as a second-order term as it is the product of two fluctuations. Keeping only first-order terms in Eq. (2.3) (quasilinear approximation), one finally gets a second evolution equation of the form

$$\frac{\partial \delta F}{\partial t} + \boldsymbol{\Omega} \cdot \frac{\partial \delta F}{\partial \boldsymbol{\theta}} - \frac{\partial F}{\partial \mathbf{J}} \cdot \frac{\partial [\delta\psi^e + \delta\psi^s]}{\partial \boldsymbol{\theta}} = 0, \quad (2.5)$$

where we used the assumptions from Eq. (2.1) to rewrite the Poisson brackets. We also introduced the mean orbital frequencies  $\boldsymbol{\Omega} = \partial H_0 / \partial \mathbf{J}$ . The two evolution equations (2.4) and (2.5) are the two coupled evolution equations from which one can obtain the secular collisionless diffusion equation. Equation (2.5) describes the evolution of the perturbation  $\delta F$  on dynamical timescales, while Eq. (2.4) describes the long-term evolution of the quasi-stationary DF  $F$ . Let us now solve Eq. (2.5) to describe the dynamical amplification of perturbations. Its solution, when injected in Eq. (2.4), will then allow for the description of the secular evolution of the system's mean quasi-stationary DF.

As the angles  $\theta$  are  $2\pi$ -periodic, let us define the discrete Fourier transform w.r.t. these variables as

$$X(\theta, \mathbf{J}) = \sum_{\mathbf{m} \in \mathbb{Z}^d} X_{\mathbf{m}}(\mathbf{J}) e^{i\mathbf{m} \cdot \theta} \quad ; \quad X_{\mathbf{m}}(\mathbf{J}) = \int \frac{d\theta}{(2\pi)^d} X(\theta, \mathbf{J}) e^{-i\mathbf{m} \cdot \theta}, \quad (2.6)$$

so that Eq. (2.5) immediately becomes

$$\frac{\partial \delta F_{\mathbf{m}}}{\partial t} + i\mathbf{m} \cdot \boldsymbol{\Omega} \delta F_{\mathbf{m}} - i\mathbf{m} \cdot \frac{\partial F}{\partial \mathbf{J}} [\delta \psi_{\mathbf{m}}^e + \delta \psi_{\mathbf{m}}^s] = 0. \quad (2.7)$$

We now introduce the assumption of timescale decoupling, also coined Bogoliubov's ansatz. Indeed, let us assume that the fluctuations (i.e.  $\delta F$ ,  $\delta \psi^e$ , and  $\delta \psi^s$ ) evolve rapidly on dynamical timescales, while the mean orbit-averaged quantities (such as  $F$ ) only evolve on secular timescales, i.e. over many dynamical times. As a consequence, in Eq. (2.7), we may push the secular time to infinity, while assuming in the meantime that  $\partial F / \partial \mathbf{J} = \text{cst}$ . Forgetting transient terms and bringing the initial time to  $-\infty$  to consider only the forced regime of evolution, Eq. (2.7) can then be solved explicitly as

$$\delta F_{\mathbf{m}}(\mathbf{J}, t) = \int_{-\infty}^t d\tau e^{-i\mathbf{m} \cdot \boldsymbol{\Omega}(t-\tau)} i\mathbf{m} \cdot \frac{\partial F}{\partial \mathbf{J}} [\delta \psi_{\mathbf{m}}^e + \delta \psi_{\mathbf{m}}^s](\mathbf{J}, \tau). \quad (2.8)$$

We define the temporal Fourier transform with the convention

$$\widehat{f}(\omega) = \int_{-\infty}^{+\infty} dt f(t) e^{i\omega t} \quad ; \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \widehat{f}(\omega) e^{-i\omega t}. \quad (2.9)$$

Taking the temporal Fourier transform of Eq. (2.7), we immediately get

$$\delta \widehat{F}_{\mathbf{m}}(\mathbf{J}, \omega) = -\frac{\mathbf{m} \cdot \partial F / \partial \mathbf{J}}{\omega - \mathbf{m} \cdot \boldsymbol{\Omega}} \left[ \delta \widehat{\psi}_{\mathbf{m}}^e(\mathbf{J}, \omega) + \delta \widehat{\psi}_{\mathbf{m}}^s(\mathbf{J}, \omega) \right], \quad (2.10)$$

so that we expressed the DF's perturbations in terms of the potential fluctuations.

### 2.2.2 Matrix Method

The next step of the calculation is to account for the system's self-gravity, i.e. the fact that the perturbing DF  $\delta F$  should be consistent with the self-induced potential perturbation  $\delta \psi^s$  and its associated density  $\delta \rho^s$ . One has

$$\delta \rho^s(\mathbf{x}) = \int d\mathbf{v} \delta F(\mathbf{x}, \mathbf{v}). \quad (2.11)$$

In Eq. (2.11), the potential and density perturbations are connected through Poisson's equation  $\Delta\delta\psi^s = 4\pi G\delta\rho^s$ . The method to deal with this self-consistency constraint is to follow Kalnajs matrix method (Kalnajs 1976). Let us introduce a representative biorthogonal basis of potential and densities  $\psi^{(p)}$  and  $\rho^{(p)}$  satisfying

$$\Delta\psi^{(p)} = 4\pi G\rho^{(p)} \quad ; \quad \int d\mathbf{x} \psi^{(p)*}(\mathbf{x}) \rho^{(q)}(\mathbf{x}) = -\delta_p^q. \quad (2.12)$$

We will then use these basis elements to represent any potential and density disturbances in the system. The potential perturbations  $\delta\psi^s$  and  $\delta\psi^e$  may therefore be written as

$$\delta\psi^s(\mathbf{x}, t) = \sum_p a_p(t) \psi^{(p)}(\mathbf{x}) \quad ; \quad \delta\psi^e(\mathbf{x}, t) = \sum_p b_p(t) \psi^{(p)}(\mathbf{x}), \quad (2.13)$$

and we introduce as  $c_p = a_p + b_p$  the total potential perturbation. The linearity of Poisson's equation immediately ensures that one also has the decomposition  $\delta\rho^s(\mathbf{x}, t) = \sum_p a_p(t) \rho^{(p)}(\mathbf{x})$ . Multiplying Eq. (2.11) by  $\psi^{(p)*}(\mathbf{x})$  and integrating over  $d\mathbf{x}$ , we get

$$a_p(t) = - \sum_m \int d\mathbf{x} d\mathbf{v} \delta F_m(\mathbf{J}, t) e^{im\cdot\theta} \psi^{(p)*}(\mathbf{J}). \quad (2.14)$$

The transformation to angle-action coordinates  $(\mathbf{x}, \mathbf{v}) \mapsto (\boldsymbol{\theta}, \mathbf{J})$  is canonical so that it conserves infinitesimal volumes, i.e. one has  $d\mathbf{x}d\mathbf{v} = d\boldsymbol{\theta}d\mathbf{J}$ . Equation (2.14) can then be rewritten as

$$a_p(t) = -(2\pi)^d \sum_m \int d\mathbf{J} \delta F_m(\mathbf{J}, t) \psi_m^{(p)*}(\mathbf{J}), \quad (2.15)$$

where  $\psi_m^{(p)}(\mathbf{J})$  stands for the Fourier transformed basis elements in angles following Eq. (2.6). Thanks to Eq. (2.10) and taking a temporal Fourier transform, we finally obtain

$$\widehat{a}_p(\omega) = (2\pi)^d \sum_q \widehat{c}_q(\omega) \sum_m \int d\mathbf{J} \frac{\mathbf{m} \cdot \partial F / \partial \mathbf{J}}{\omega - \mathbf{m} \cdot \boldsymbol{\Omega}} \psi_m^{(p)*}(\mathbf{J}) \psi_m^{(q)}(\mathbf{J}). \quad (2.16)$$

Let us finally introduce the system's response matrix  $\widehat{\mathbf{M}}$  as

$$\widehat{\mathbf{M}}_{pq}(\omega) = (2\pi)^d \sum_m \int d\mathbf{J} \frac{\mathbf{m} \cdot \partial F / \partial \mathbf{J}}{\omega - \mathbf{m} \cdot \boldsymbol{\Omega}} \psi_m^{(p)*}(\mathbf{J}) \psi_m^{(q)}(\mathbf{J}), \quad (2.17)$$

so that Eq. (2.16) becomes

$$\widehat{\mathbf{a}}(\omega) = \widehat{\mathbf{M}}(\omega) \cdot \widehat{\mathbf{c}}(\omega). \quad (2.18)$$

One should note that the response matrix depends only on the mean state of the system, since  $\partial F/\partial \mathbf{J}$  only evolves on secular timescales, the perturbing and self-gravitating potentials are absent, and the basis elements  $\psi^{(p)}$  from Eq. (2.12) are chosen once for all. Assuming that the mean system is linearly stable, so that  $[\mathbf{I} - \widehat{\mathbf{M}}(\omega)]^{-1}$  admits no poles in the upper complex plane, one can invert Eq. (2.18) to get

$$\widehat{\mathbf{c}}(\omega) = [\mathbf{I} - \widehat{\mathbf{M}}(\omega)]^{-1} \cdot \widehat{\mathbf{b}}(\omega), \quad (2.19)$$

where  $\mathbf{I}$  stands for the identity matrix. Equation (2.19) is a crucial relation, which allows us to express the total perturbations as a function of the external perturbation only, whose statistical properties may be characterised. Equation (2.19) describes the short timescale (dynamical) response of the system and the associated self-gravitating amplification.

### 2.2.3 Diffusion Coefficients and Statistical Average

Let us now describe how these solutions may be used in Eq. (2.4) to describe the secular evolution of the system. The l.h.s. of Eq. (2.4) requires us to evaluate an expression of the form

$$\frac{1}{(2\pi)^d} \int d\boldsymbol{\theta} \delta F(\mathbf{J}, \boldsymbol{\theta}, t) \frac{\partial[\delta\psi^e + \delta\psi^s]}{\partial \boldsymbol{\theta}} = - \sum_m \delta F_m \, i\mathbf{m} [\delta\psi_m^{e*} + \delta\psi_m^{s*}], \quad (2.20)$$

where we used the fact that  $\delta\psi_{-m} = \delta\psi_m^*$ . Thanks to the resolution from Eq. (2.8), we may now rewrite Eq. (2.4) as

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left[ \sum_m \mathbf{m} D_m(\mathbf{J}, t) \mathbf{m} \cdot \frac{\partial F}{\partial \mathbf{J}} \right], \quad (2.21)$$

where the diffusion coefficients  $D_m(\mathbf{J}, t)$  are given by

$$D_m(\mathbf{J}, t) = \sum_{p,q} \psi_m^{(p)}(\mathbf{J}) \psi_m^{(q)*}(\mathbf{J}) c_q^*(t) \int_{-\infty}^t d\tau e^{-i\mathbf{m} \cdot \boldsymbol{\Omega}(t-\tau)} c_p(\tau). \quad (2.22)$$

The amplification relation from Eq. (2.19) allows us to rewrite Eq. (2.22) as a function of the external perturbation  $\widehat{\mathbf{b}}$  only, to get

$$\begin{aligned} D_m(\mathbf{J}, t) &= \frac{1}{(2\pi)^2} \sum_{p,q} \sum_{p_1, q_1} \psi_m^{(p)}(\mathbf{J}) \psi_m^{(q)*}(\mathbf{J}) \int d\omega e^{i\omega t} [\mathbf{I} - \widehat{\mathbf{M}}(\omega)]_{qq_1}^{-1*} \widehat{b}_{q_1}^*(\omega) \\ &\quad \times \int_{-\infty}^t d\tau e^{-i\mathbf{m} \cdot \boldsymbol{\Omega}(t-\tau)} \int d\omega' e^{-i\omega' \tau} [\mathbf{I} - \widehat{\mathbf{M}}(\omega')]_{pp_1}^{-1} \widehat{b}_{p_1}(\omega'). \end{aligned} \quad (2.23)$$



The final step of the derivation is to consider statistical averages over various realisations of the perturbations, i.e. to consider only the mean response of the system. Let us denote as  $\langle \cdot \rangle$  the ensemble average operation on such different realisations. When applying this average, we assume that the response matrix  $\widehat{\mathbf{M}}$ , as well as the DF  $F$  and its gradients  $\partial F / \partial \mathbf{J}$ , do not change significantly from one realisation to another. Thanks to these assumptions, Eq. (2.21) becomes

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left[ \sum_{\mathbf{m}} \mathbf{m} \langle D_{\mathbf{m}}(\mathbf{J}, t) \rangle \mathbf{m} \cdot \frac{\partial F}{\partial \mathbf{J}} \right]. \quad (2.24)$$

Let us now suppose that the external perturbations are stationary in time, so that one can introduce the corresponding temporal autocorrelation function  $\mathbf{C}$  as

$$\mathbf{C}_{k\ell}(t_1 - t_2) = \langle b_k(t_1) b_\ell^*(t_2) \rangle, \quad (2.25)$$

where it is assumed that the exterior perturbation is of zero mean. When Fourier transformed, Eq. (2.25) becomes

$$\langle \widehat{b}_k(\omega) \widehat{b}_\ell^*(\omega') \rangle = 2\pi \delta_D(\omega - \omega') \widehat{\mathbf{C}}_{k\ell}(\omega). \quad (2.26)$$

One can now immediately rewrite the averaged diffusion coefficients from Eq. (2.24) as

$$\langle D_{\mathbf{m}}(\mathbf{J}, t) \rangle = \frac{1}{2\pi} \sum_{p,q} \psi_{\mathbf{m}}^{(p)}(\mathbf{J}) \psi_{\mathbf{m}}^{(q)*}(\mathbf{J}) \int d\omega \int_{-\infty}^0 d\tau' e^{-i(\omega - \mathbf{m} \cdot \boldsymbol{\Omega})\tau'} \left[ [\mathbf{I} - \widehat{\mathbf{M}}]^{-1} \cdot \widehat{\mathbf{C}} \cdot [\mathbf{I} - \widehat{\mathbf{M}}]^{-1} \right]_{pq}(\omega), \quad (2.27)$$

where we relied on the hermiticity of the response matrix  $\widehat{\mathbf{M}}^* = \widehat{\mathbf{M}}^\dagger$ . One should note that after the ensemble average, the diffusion coefficients become (explicitly) independent of  $t$  (while they still depend on the secular timescale via the slow variations of  $F$ ). To shorten temporarily the notations, let us introduce the notation  $\widehat{\mathbf{L}} = [\mathbf{I} - \widehat{\mathbf{M}}]^{-1} \cdot \widehat{\mathbf{C}} \cdot [\mathbf{I} - \widehat{\mathbf{M}}]^{-1}$ . In Eq. (2.27), one must then evaluate a double integral of the form

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \widehat{\mathbf{L}}(\omega) \int_{-\infty}^0 d\tau' e^{-i(\omega - \mathbf{m} \cdot \boldsymbol{\Omega})\tau'} &= \frac{i}{2\pi} \int_{-\infty}^{+\infty} d\omega \frac{\widehat{\mathbf{L}}(\omega)}{\omega - \mathbf{m} \cdot \boldsymbol{\Omega}} \\ &= \frac{i}{2\pi} \mathcal{P} \int_{-\infty}^{+\infty} d\omega \frac{\widehat{\mathbf{L}}(\omega)}{\omega - \mathbf{m} \cdot \boldsymbol{\Omega}} + \frac{1}{2} \widehat{\mathbf{L}}(\mathbf{m} \cdot \boldsymbol{\Omega}), \end{aligned} \quad (2.28)$$

where to perform the integration over  $\tau'$ , we kept only the boundary term for  $\tau' = 0$ , by adding a small imaginary part to the frequency  $\omega$ , so that  $\omega = \omega + i0^+$ , which ensures the convergence for  $\tau \rightarrow -\infty$ . To evaluate the last integral over  $\omega$ , we also relied on Plemelj formula

$$\frac{1}{x \pm i0^+} = \mathcal{P}\left(\frac{1}{x}\right) \mp i\pi\delta_{\mathbf{D}}(x), \quad (2.29)$$

where  $\mathcal{P}$  stands for Cauchy principal value. The last step of the derivation is to note that the contributions associated with the principal value in Eq. (2.28) have no impact on the secular diffusion equation. Indeed, Eq. (2.22) gives us that the diffusion coefficients are such that  $D_{-m}(\mathbf{J}) = D_m^*(\mathbf{J})$ . Since we are summing on all vectors  $\mathbf{m} \in \mathbb{Z}^d$ , we may then rewrite Eq. (2.24) as

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left[ \sum_{\mathbf{m}} \mathbf{m} \langle \text{Re}[D_{\mathbf{m}}(\mathbf{J}, t)] \rangle \mathbf{m} \cdot \frac{\partial F}{\partial \mathbf{J}} \right]. \quad (2.30)$$

Equations (2.17) and (2.25) impose  $\widehat{\mathbf{M}}^* = \widehat{\mathbf{M}}^t$ , and  $\widehat{\mathbf{C}}^* = \widehat{\mathbf{C}}^t$ , so that the matrix  $\widehat{\mathbf{L}}$  defined above Eq. (2.28) is also hermitian. We finally recover the collisionless secular dressed secular diffusion equation (sometimes also called “dressed Fokker-Planck equation”) as

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left[ \sum_{\mathbf{m}} \mathbf{m} D_{\mathbf{m}}(\mathbf{J}) \mathbf{m} \cdot \frac{\partial F}{\partial \mathbf{J}} \right], \quad (2.31)$$

where the anisotropic diffusion coefficients are given by

$$D_{\mathbf{m}}(\mathbf{J}) = \frac{1}{2} \sum_{p,q} \psi_{\mathbf{m}}^{(p)}(\mathbf{J}) \psi_{\mathbf{m}}^{(q)*}(\mathbf{J}) \left[ [\mathbf{I} - \widehat{\mathbf{M}}]^{-1} \cdot \widehat{\mathbf{C}} \cdot [\mathbf{I} - \widehat{\mathbf{M}}]^{-1} \right]_{pq} \quad (\omega = \mathbf{m} \cdot \boldsymbol{\Omega}). \quad (2.32)$$

Let us finally introduce the total diffusion flux  $\mathcal{F}_{\text{tot}}$  as

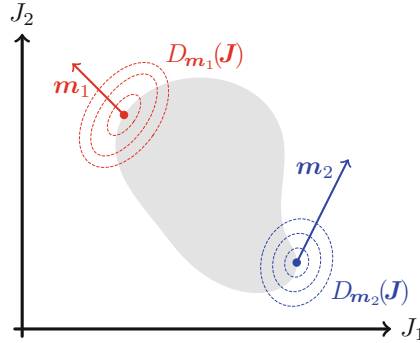
$$\mathcal{F}_{\text{tot}} = \sum_{\mathbf{m}} \mathbf{m} D_{\mathbf{m}}(\mathbf{J}) \mathbf{m} \cdot \frac{\partial F}{\partial \mathbf{J}}, \quad (2.33)$$

so that Eq. (2.31) becomes

$$\frac{\partial F}{\partial t} = \text{div}(\mathcal{F}_{\text{tot}}). \quad (2.34)$$

With this convention,  $-\mathcal{F}_{\text{tot}}$  corresponds to the direction along which individual particles diffuse. Equation (2.31) is the main result of this section.

Let us now briefly discuss the physical content of Eq. (2.31). First, because it is written as the divergence of a flux, the total number of stars is conserved during the diffusion. One can also note that the diffusion coefficients  $D_{\mathbf{m}}(\mathbf{J})$  from Eq. (2.32) capture the joint and coupled contributions from the external perturbations (via the autocorrelation matrix  $\widehat{\mathbf{C}}$ ) and from the self-gravitating susceptibility of the system (via the response matrix  $\widehat{\mathbf{M}}$ ). The total diffusion coefficients appear therefore as a collaboration between the strength of the external perturbations and the local strength of the system’s amplification. As Eq. (2.31) describes a resonant diffusion, the external perturbing power spectrum and the system’s susceptibility have to be evaluated



**Fig. 2.3** Illustration of the strong anisotropy of the diffusion in action space captured by Eq. (2.31). The background *grey* domain illustrates the region where the system's DF,  $F$ , is present. For a given resonance vector  $\mathbf{m}$ , one can compute the associated diffusion coefficients  $D_{\mathbf{m}}(\mathbf{J})$ , whose level contours are represented with *dotted* colored lines. In the region where  $D_{\mathbf{m}}(\mathbf{J})$  is maximum, following Eq. (2.31), one expects the associated flux to be aligned with the direction of  $\mathbf{m}$ . As a consequence, depending on which resonance vector locally dominates the diffusion, the DF's diffusion can occur along significantly different directions

at the local intrinsic frequency  $\omega = \mathbf{m} \cdot \boldsymbol{\Omega}$ . In this sense, this diffusion equation is appropriate to capture the nature of a collisionless system, via its natural frequencies and susceptibility, as well as its nurture, via the structure of the power spectrum of the external perturbations.

In addition, one can also note that the diffusion Eq. (2.31) takes the form of a strongly anisotropic diffusion equation in action space. It is anisotropic not only because the diffusion coefficients  $D_{\mathbf{m}}(\mathbf{J})$  depend on the position in action space, but also because the diffusion associated with one resonance vector  $\mathbf{m}$  corresponds to a diffusion in the preferential direction of the vector  $\mathbf{m}$ . For a given resonance  $\mathbf{m}$ , the diffusion is maximum along  $\mathbf{m}$  and vanishes in the orthogonal directions. A qualitative illustration of the properties of Eq. (2.31) is given in Fig. 2.3. Finally, note that Eq. (2.31) is indeed an illustration of the fluctuation-dissipation theorem. The autocorrelation of the fluctuating potential drives the diffusion of the system's orbital structure.

### 2.3 Self-induced Collisional Dynamics

In the previous section, we considered the collisionless case where a secular diffusion is induced by external perturbations. However, a given self-gravitating system, even when isolated, may also undergo a secular evolution as a result of its own intrinsic graininess. This is a collisional evolution sourced by finite- $N$  effects.

The dynamics and thermodynamics of systems with long-range interactions has recently been a subject of active research (Campa et al. 2009; Campa et al. 2014), which led to a much better understanding of the equilibrium properties of these systems, their specificities such as negative specific heats (Antonov 1962; Lynden-Bell and Wood 1968; Lynden-Bell 1999), as well as various kinds of phase transitions and ensemble inequivalences. However, the precise description of their dynamical evolution remains to be improved to offer explicit predictions. We refer the reader to Chavanis (2010, 2013a,b) for a historical account of the development of kinetic theories of plasmas, stellar systems, and other systems with long-range interactions, but let us briefly recall here the main milestones.

The first kinetic theory focusing on the statistical description of the evolution of a large number of particles was considered by Boltzmann in the case of dilute neutral gases (Boltzmann 1872). For such systems, particles do not interact except during strong local collisions. The gas is assumed to be spatially homogeneous and Boltzmann equation describes the evolution of the system's velocity distribution  $f(\mathbf{v}, t)$  as a result of strong collisions. This kinetic equation satisfies a H-theorem, associated with an increase of Boltzmann's entropy.

Boltzmann's approach was extended to charged gases (plasmas) by Landau (Landau 1936). For plasmas, particles interact via long-range Coulombian forces, but because of electroneutrality and Debye shielding (Debye and Hückel 1923a,b), these interactions are screened on a length scale of the order of the Debye length, and collisions become essentially local. Neutral plasmas are spatially homogeneous, so that the kinetic equation describes again the evolution of the velocity DF  $f(\mathbf{v}, t)$ , driven by close electrostatic encounters. Because the encounters are weak, one can expand the Boltzmann equation in the limit of small deflections and perform a linear trajectory approximation. In the weak coupling approximation, this leads to the so-called Landau equation. The Landau equation exhibits two formal divergences: one at small scales due to the neglect of strong collisions and one logarithmic divergence at large scales due to the neglect of collective effects, i.e. the dressing of particles by their polarisation cloud (a particle of a given charge has the tendency to be surrounded by a cloud of particles of opposite charges). Landau regularised these divergences by introducing a lower cut-off at the impact parameter producing a deflection of  $90^\circ$  (this is the Landau length) as well as an upper cut-off at the Debye length.

Collective effects were later rigorously taken into account in Balescu (1960) and Lenard (1960), leading to the Balescu-Lenard equation for plasmas. The Balescu-Lenard equation is similar to the Landau equation, except that it includes the square of the dielectric function in the denominator of the potential of interaction in Fourier space. This dielectric function first appeared as a probe of the dynamical stability of plasmas based on the linearised Vlasov equation (Vlasov 1938, 1945). In the Balescu-Lenard equation, the dielectric function accounts for Debye shielding and removes the large scale logarithmic divergence present in the Landau equation. The Landau equation is recovered from the Balescu-Lenard equation by replacing the dressed potential of interaction by its bare expression, i.e. by replacing the dielectric function by unity. In addition, the Balescu-Lenard equation, as given originally by Balescu and Lenard, exhibits a local resonance condition, encapsulated in a Dirac

$\delta_D$ —function. For such systems, resonant contributions are the drivers of the secular evolution. Integrating over this resonance condition leads to the original form of the kinetic equation given by Landau.

In parallel to the developments of kinetic equations for plasmas, the secular evolution of self-gravitating systems was also investigated. Self-gravitating systems are spatially inhomogeneous, but the first kinetic theories (Jeans 1929; Chandrasekhar 1942; Chandrasekhar 1943a, b) were all based on the assumption that collisions (i.e. close encounters) between stars can be treated with a local approximation, as if the system were infinite and homogeneous. Relying on the idea that a given star undergoes a large number of weak deflections, Chandrasekhar (1949) developed an analogy with Brownian motion. He started from a Fokker-Planck writing of the diffusion equation and computed the diffusion and friction coefficients relying on a binary collision theory. This led to a kinetic equation, often called Fokker-Planck equation in astrophysics, which is the gravitational equivalent of the Landau equation from plasmas. This equation exhibits similarly two divergences: one at small scales due to the mishandling of strong collisions, and one at large scales due to the local approximation, i.e. the assumption that the system is infinite and homogeneous. In the treatment of Chandrasekhar, strong collisions are taken into account without having to introduce a cut-off, so that the small scale divergence is regularised at the gravitational Landau length. The large scale divergence is usually regularised by introducing a cut-off at the Jeans length, which is the gravitational equivalent of the Debye length. This gravitational Landau equation is often considered to be relevant to describe the collisional dynamics of spherical systems such as globular clusters. Let us however note that the associated treatment based on the local approximation remains unsatisfactory, in particular because of the unavoidable appearance of a logarithmic divergence at large scales. In addition, within this framework, one cannot account for collective effects, i.e. the dressing of stars by their polarisation cloud, i.e. the fact that the gravitational force being attractive, a given star has the tendency to be surrounded by a cloud of stars. This increases its effective gravitational mass and reduces the collisional relaxation time.

In order to fully account for these properties, the kinetic theory of self-gravitating systems was recently generalised to fully inhomogeneous systems, either when collective effects are neglected (Polyachenko and Shukhman 1982; Chavanis 2010, 2013b) leading to the inhomogeneous Landau equation, or when they are accounted for leading to the inhomogeneous Balescu-Lenard equation (Heyvaerts 2010; Chavanis 2012b).<sup>2</sup> These kinetic equations, presented and discussed in detail in the upcoming section, are valid at order  $1/N$ , where  $N$  is the number of stars in the system. Having accounted for the finite extension of the system, these equations no longer present a divergence at large scales. In order to deal with the system's inhomogeneity, they are written in angle-action coordinates (see Sect. 1.3), which allow for the description of stars' intricate dynamics in spatially inhomogeneous

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<sup>2</sup>The inhomogeneous Balescu-Lenard equation with angle-action variables was previously derived in Luciani and Pellat (1987); Mynick (1988), where the self-gravitating dressing of perturbations is introduced at a formal level without an explicit representation.

and multi-periodic systems. These equations involve similarly a resonance condition encapsulated in a Dirac  $\delta_D$ -function (see Fig. 2.5), which generalises the one present in the homogeneous Balescu-Lenard equation. Finally, in order to capture collective effects, the inhomogeneous Balescu-Lenard equation also involves the system's response matrix (see Eq. (2.17)) expressed in angle-action variables. This generalises the dielectric function appearing in the homogeneous Balescu-Lenard equation for plasmas. This dressing accounts for anti-shielding, i.e. the fact that the gravitational mass of a star is enhanced by its polarisation, leading to a reduction of the relaxation time. The upcoming chapters will emphasise how these powerful and predictive kinetic equations may be used in the astrophysical context to probe complex secular regimes.

There are two standard methods to derive kinetic equations for a  $N$ -body system with long-range pairwise interactions. The first approach is based on Liouville's equation for the  $N$ -body distribution function of the system. One has to write the first two equations of the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy. The hierarchy is then closed by considering only contributions of order  $1/N$ . One may then solve the second equation of the BBGKY hierarchy to express the 2-body correlation function in terms of the system's 1-body DF. One finally substitutes this expression in the first equation of the BBGKY hierarchy to obtain the closed self-consistent kinetic equation satisfied by the 1-body DF. The same results can also be obtained thanks to projection operator techniques. The second method relies on the Klimontovich equation (Klimontovich 1967), which describes the dynamics of the system's DF written as a sum of  $\delta_D$  functions. This exact DF is then decomposed in two parts, a smooth component and fluctuations. One can then write two evolution equations, one for the smooth mean component, and one for the fluctuations. This coupled system is then closed by neglecting non-linear terms in the evolution of the fluctuations (quasilinear approximation). The final step in this approach is to solve the equation for the fluctuations to express their properties as a function of the underlying smooth component. Injecting this result in the first evolution equation for the smooth part, one obtains a self-consistent kinetic equation. These two methods are physically equivalent, while technically different. Finally, we recently presented in Fouvry et al. (2016a, b) a third approach based on a functional rewriting of the evolution equations. This approach starts from the first two equations of the BBGKY hierarchy truncated at order  $1/N$ . Introducing auxiliary fields, the evolution of the two coupled dynamical quantities, 1-body DF and 2-body autocorrelation, can then be rewritten as a traditional functional integral. By functionally integrating over the 2-body autocorrelation, one obtains a new constraint connecting the 1-body DF and the auxiliary fields. When inverted, this constraint finally allows for the derivation of the closed non-linear kinetic equation satisfied by the 1-body DF. Finally, in Heyvaerts et al. (2017), we recently showed how the Balescu-Lenard equation could be recovered via the direct computation of the first- and second-order diffusion coefficients, following a Fokker-Planck approach (Binney and Tremaine 2008).

In the upcoming sections, we will follow Chavanis (2012b) and present a derivation of the inhomogeneous Balescu-Lenard equation based on the resolution of the Klimontovich equation. We decided to present this derivation in the main text, in order to emphasise the various similarities it shares with the previous collisionless diffusion equation. In Appendix 2.A, we present the derivation of the BBGKY hierarchy. This allows us to revisit in Appendix 2.B the derivation of the inhomogeneous Balescu-Lenard equation first presented by Heyvaerts (2010) and based on the direct resolution of the BBGKY hierarchy. Finally, in Appendix 2.C, we consider the third approach to the derivation of kinetic equations based on a functional integral rewriting.

### 2.3.1 Evolution Equations

Let us consider an isolated system made of  $N$  particles of individual mass  $\mu = M_{\text{tot}}/N$ , where  $M_{\text{tot}}$  is the total active mass of the system, embedded in a physical space of dimension  $d$ . We note as  $(\mathbf{x}_i, \mathbf{v}_i)$  the position and velocity of particle  $i$  in an inertial frame. The individual dynamics of these particles is entirely described by Hamilton's equations which read

$$\mu \frac{d\mathbf{x}_i}{dt} = \frac{\partial H}{\partial \mathbf{v}_i} \quad ; \quad \mu \frac{d\mathbf{v}_i}{dt} = -\frac{\partial H}{\partial \mathbf{x}_i}, \quad (2.35)$$

where the Hamiltonian of the system contains all the binary interactions between particles as

$$H = \frac{\mu}{2} \sum_{i=1}^N \mathbf{v}_i^2 + \mu^2 \sum_{i < j}^N U(|\mathbf{x}_i - \mathbf{x}_j|). \quad (2.36)$$

In Eq. (2.36), we introduced the binary potential of interaction  $U(|\mathbf{x}_i - \mathbf{x}_j|)$ , given by  $U(|\mathbf{x}|) = -G/|\mathbf{x}|$  in the gravitational context. While capturing the exact dynamics of the system, one major drawback of Eq. (2.35) is that one has to deal with a set of  $N$  coupled differential equations. In Appendix 2.A, we show how these equations may be rewritten as an ordered hierarchy of evolution equations, the BBGKY hierarchy. Such a rewriting is at the heart of the derivation of the inhomogeneous Balescu-Lenard equation proposed in Heyvaerts (2010) and revisited in Appendix 2.B. Here, we intend to follow a different route, and rewrite Hamilton's equations (2.35) as a single evolution equation in phase space. To do so, let us introduce the discrete distribution function  $F_d(\mathbf{x}, \mathbf{v}, t)$  as

$$F_d(\mathbf{x}, \mathbf{v}, t) = \mu \sum_{i=1}^N \delta_D(\mathbf{x} - \mathbf{x}_i(t)) \delta_D(\mathbf{v} - \mathbf{v}_i(t)). \quad (2.37)$$

Let us also introduce the associated self-consistent potential  $\psi_d$  as

$$\psi_d(\mathbf{x}, \mathbf{v}, t) = \int d\mathbf{x}' d\mathbf{v}' U(|\mathbf{x} - \mathbf{x}'|) F_d(\mathbf{x}', \mathbf{v}', t). \quad (2.38)$$

One can show that  $F_d$  satisfies the Klimontovich equation (Klimontovich 1967), given by

$$\frac{\partial F_d}{\partial t} + [F_d, H_d] = 0, \quad (2.39)$$

where we introduced the Hamiltonian  $H_d$  as

$$H_d(\mathbf{x}, \mathbf{v}, t) = \frac{1}{2} \mathbf{v}^2 + \psi_d(\mathbf{x}, t). \quad (2.40)$$

At this stage, note that the Klimontovich equation (2.39) captures the exact same dynamics as Hamilton's equations (2.35), while being defined on a phase space of dimension  $2d$ . Let us assume that the system's DF and potential may be decomposed as the sum of a smooth component and a fluctuating one, so that

$$\begin{cases} F_d = F + \delta F, \\ \psi_d = \psi_0 + \delta\psi. \end{cases} \quad (2.41)$$

Let us emphasise how similar the decompositions from Eqs. (2.1) and (2.41) are. In addition to this decomposition, we assume that the smooth component  $F$  only evolves on secular timescales, while the fluctuating component  $\delta F$  evolves much faster on dynamical timescales. We also assume that the mean potential is integrable, so that there exists angle-action coordinates  $(\boldsymbol{\theta}, \mathbf{J})$  appropriate for the smooth quasi-stationary potential  $\psi_0$ . Thanks to Jeans theorem, the system's mean DF being quasi-stationary, it can be written as  $F(\mathbf{x}, \mathbf{v}, t) = F(\mathbf{J}, t)$ . Performing the same timescale decoupling and quasilinear approximation as in Eq. (2.3), one gets two evolution equations from Eq. (2.39). First a secular evolution equation for  $F$  as

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left[ \int \frac{d\boldsymbol{\theta}}{(2\pi)^d} \delta F \frac{\partial \delta\psi}{\partial \boldsymbol{\theta}} \right], \quad (2.42)$$

and an evolution equation for the perturbation  $\delta F$  as

$$\frac{\partial \delta F}{\partial t} + \boldsymbol{\Omega} \cdot \frac{\partial \delta F}{\partial \boldsymbol{\theta}} - \frac{\partial F}{\partial \mathbf{J}} \cdot \frac{\partial \delta\psi}{\partial \boldsymbol{\theta}} = 0. \quad (2.43)$$

These two evolution equations govern the evolution of the smooth DF  $F$  and the fluctuations  $\delta F$  at order  $1/N$ . They are the direct counterparts of Eqs. (2.4) and (2.5). Here, the potential fluctuations are not due to an external forcing, but to the intrinsic finite- $N$  Poisson shot noise. As was assumed in Eq. (2.7), we place ourselves within



the adiabatic approximation so that the time variations of  $F$  may be neglected on the timescales for which the fluctuations  $\delta F$  and  $\delta\psi$  evolve. In order to be valid, such an approximation requires to have  $N \gg 1$ . Finally, as in Eq. (2.19), we assume that the DF  $F$  remains Vlasov-stable throughout its evolution, so that its evolution is only governed by correlations and not by dynamical instabilities.

### 2.3.2 Fast Timescale Amplification

The first step of our calculation is to study the short timescale evolution equation (2.43), during which perturbations build up. As in Eq. (2.6), let us perform a Fourier transform w.r.t. to the angles  $\theta$ . Let us also define the Laplace transform of the fluctuations with the convention

$$\tilde{f}(\omega) = \int_0^{+\infty} dt f(t) e^{i\omega t} \quad ; \quad f(t) = \frac{1}{2\pi} \int_{\mathcal{B}} d\omega \tilde{f}(\omega) e^{-i\omega t}, \quad (2.44)$$

where the Bromwich contour  $\mathcal{B}$  in the complex  $\omega$ -plane should pass above all the poles of the integrand, i.e.  $\text{Im}[\omega]$  should be large enough. The Fourier-Laplace transform of the DF's fluctuations  $\delta F$  is therefore given by

$$\delta \tilde{F}_{m_1}(\mathbf{J}_1, \omega_1) = \int \frac{d\theta_1}{(2\pi)^d} \int_0^{+\infty} dt e^{-i(m_1 \cdot \theta_1 - \omega_1 t)} \delta F(\theta_1, \mathbf{J}_1, t). \quad (2.45)$$

One can perform a similar transformation for the potential fluctuations  $\delta\psi$ . Let us define the Fourier transform of the initial value of the DF as

$$\delta \hat{F}_{m_1}(\mathbf{J}_1, 0) = \int \frac{d\theta_1}{(2\pi)^d} e^{-i(m_1 \cdot \theta_1)} \delta F(\theta_1, \mathbf{J}_1, 0). \quad (2.46)$$

Relying on Bogoliubov's ansatz,  $F = \text{cst.}$ , we multiply Eq. (2.43) by  $\int d\theta_1 / (2\pi)^d \int dt e^{-i\mathbf{m}_1 \cdot \theta_1 - \omega_1 t}$  to get

$$\delta \tilde{F}_{m_1}(\mathbf{J}_1, \omega_1) = \frac{\mathbf{m}_1 \cdot \partial F / \partial \mathbf{J}_1}{\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - \omega_1} \delta \tilde{\psi}_{m_1}(\mathbf{J}_1, \omega_1) + \frac{\delta \hat{F}_{m_1}(\mathbf{J}_1, 0)}{i(\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - \omega_1)}. \quad (2.47)$$

Equation (2.47) relates the fluctuations in the potential  $\delta\psi$  to the induced response  $\delta F$  in the system's DF. One now has to account for the fact that these perturbations are self-consistently generated by the system itself, i.e.  $\delta\psi$  corresponds to potential fluctuations generated by the perturbing density  $\delta\rho$  associated with the DF  $\delta F$ . To do so, we follow the matrix method introduced in Sect. 2.2.2. Relying on basis elements  $(\psi^{(p)}, \rho^{(p)})$  as introduced in Eq. (2.12), we follow Eq. (2.13) and decompose the self-induced potential perturbations  $\delta\psi$  as

$$\delta\psi(\boldsymbol{\theta}_1, \mathbf{J}_1, t) = \sum_p a_p(t) \psi^{(p)}(\boldsymbol{\theta}_1, \mathbf{J}_1) \quad ; \quad \delta\tilde{\psi}_{\mathbf{m}_1}(\mathbf{J}_1, \omega_1) = \sum_p \tilde{a}_p(\omega_1) \psi_{\mathbf{m}_1}^{(p)}(\mathbf{J}_1), \quad (2.48)$$

where  $\widehat{a}_p(\omega)$  stands for the Laplace transform of the basis coefficients and  $\psi_{\mathbf{m}_1}^{(p)}(\mathbf{J}_1)$  for the Fourier transformed basis elements as introduced in Eq. (2.15). In order to capture this self-consistency, we follow the same method as presented in Eq. (2.16). We start from  $\delta\tilde{\rho} = \int d\mathbf{v} \delta\tilde{F}(\mathbf{x}, \mathbf{v})$ , multiply this relation by  $\psi^{(p)*}(\mathbf{x})$ , integrate it w.r.t.  $\mathbf{x}$ , and rely on the fact that  $d\mathbf{x}d\mathbf{v} = d\boldsymbol{\theta}d\mathbf{J}$  as the transformation  $(\mathbf{x}, \mathbf{v}) \mapsto (\boldsymbol{\theta}, \mathbf{J})$  is canonical. Equation (2.47) finally gives

$$\tilde{a}_p(\omega_1) = -(2\pi)^d \sum_q [\mathbf{I} - \widehat{\mathbf{M}}(\omega_1)]_{pq}^{-1} \sum_{\mathbf{m}_2} \int d\mathbf{J}_2 \frac{\delta\widehat{F}_{\mathbf{m}_2}(\mathbf{J}_2, 0)}{i(\mathbf{m}_2 \cdot \boldsymbol{\Omega}_2 - \omega_1)} \psi_{\mathbf{m}_2}^{(q)*}(\mathbf{J}_2). \quad (2.49)$$

In Eq. (2.49), we recover the role played by the system's susceptibility through the system's response matrix  $\widehat{\mathbf{M}}$  introduced in Eq. (2.17). Note as well that we assumed the system to be stable, so that one could indeed compute the matrix  $[\mathbf{I} - \widehat{\mathbf{M}}(\omega_1)]^{-1}$ . Let us now introduce the system's dressed susceptibility coefficients  $1/\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_2}$  as

$$\frac{1}{\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2, \omega)} = \sum_{p,q} \psi_{\mathbf{m}_1}^{(p)}(\mathbf{J}_1) [\mathbf{I} - \widehat{\mathbf{M}}(\omega)]_{pq}^{-1} \psi_{\mathbf{m}_2}^{(q)*}(\mathbf{J}_2), \quad (2.50)$$

so that Eq. (2.49) when multiplied by  $\psi_{\mathbf{m}_1}^{(p)}(\mathbf{J}_1)$  and summed over “ $p$ ” gives

$$\delta\tilde{\psi}_{\mathbf{m}_1}(\mathbf{J}_1, \omega_1) = -(2\pi)^d \sum_{\mathbf{m}_2} \int d\mathbf{J}_2 \frac{1}{\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2, \omega_1)} \frac{\delta\widehat{F}_{\mathbf{m}_2}(\mathbf{J}_2, 0)}{i(\mathbf{m}_2 \cdot \boldsymbol{\Omega}_2 - \omega_1)}. \quad (2.51)$$

Equation (2.51) gives the Laplace transform of the response potential as a function of the initial conditions in the DF's fluctuations. It describes the dynamical amplification of the perturbations occurring in the system.

### 2.3.3 Estimating the Collision Operator

Thanks to Eq. (2.51), one may now proceed to the evaluation of the collision operator in the r.h.s. of Eq. (2.42). As was argued in Eq. (2.24), let us emphasise that here we are interested in the system's mean evolution averaged over various realisations. We may then take the ensemble average of the evolution equation (2.42). When taking this average, we assume that the response matrix  $\widehat{\mathbf{M}}$  as well as the DF  $F$  and its gradients  $\partial F / \partial \mathbf{J}$  do not change significantly from one realisation to another. Equation (2.42) becomes

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial \mathbf{J}_1} \cdot [\mathcal{F}_{\text{tot}}(\mathbf{J}_1)], \quad (2.52)$$

where we introduced the total diffusion flux  $\mathcal{F}_{\text{tot}}$  as

$$\mathcal{F}_{\text{tot}}(\mathbf{J}) = \int \frac{d\boldsymbol{\theta}}{(2\pi)^d} \left\langle \delta F \frac{\partial \delta \psi}{\partial \boldsymbol{\theta}} \right\rangle, \quad (2.53)$$

where  $\langle \cdot \rangle$  stands for the ensemble average operation. Taking a Fourier transform w.r.t. the angles as well as an inverse Laplace transform, Eq.(2.53) gives

$$\mathcal{F}_{\text{tot}}(\mathbf{J}_1) = - \sum_{\mathbf{m}_1} \int_{\mathcal{B}_1} \frac{d\omega_1}{2\pi} \int_{\mathcal{B}_2} \frac{d\omega_2}{2\pi} i\mathbf{m}_1 e^{-i\omega_1 t} e^{-i\omega_2 t} \langle \delta \tilde{F}_{\mathbf{m}_1}(\mathbf{J}_1, \omega_1) \delta \tilde{\psi}_{-\mathbf{m}_1}(\mathbf{J}_1, \omega_2) \rangle, \quad (2.54)$$

where  $\mathcal{B}_1$  (resp.  $\mathcal{B}_2$ ) stands for the Bromwich contour associated with the inverse Laplace transform w.r.t.  $\omega_1$  (resp.  $\omega_2$ ). Let us now rely on Eqs.(2.47) and (2.51) to express Eq.(2.54) only as a function of the fluctuations in the initial conditions captured by  $\delta \hat{F}$ . Equation (2.54) can be written as

$$\mathcal{F}_{\text{tot}}(\mathbf{J}_1) = \mathcal{F}_{\text{tot}}^{(I)}(\mathbf{J}_1) + \mathcal{F}_{\text{tot}}^{(II)}(\mathbf{J}_1), \quad (2.55)$$

where the two components  $\mathcal{F}_{\text{tot}}^{(I)}(\mathbf{J}_1)$  and  $\mathcal{F}_{\text{tot}}^{(II)}(\mathbf{J}_1)$  are respectively given by

$$\begin{aligned} \mathcal{F}_{\text{tot}}^{(I)}(\mathbf{J}_1) &= i \sum_{\mathbf{m}_1} \mathbf{m}_1 \int_{\mathcal{B}_1} \frac{d\omega_1}{2\pi} e^{-i\omega_1 t} \int_{\mathcal{B}_2} \frac{d\omega_2}{2\pi} e^{-i\omega_2 t} (2\pi)^{2d} \frac{\mathbf{m}_1 \cdot \partial F / \partial \mathbf{J}_1}{\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - \omega_1} \sum_{\mathbf{m}_2, \mathbf{m}_3} \int d\mathbf{J}_2 d\mathbf{J}_3 \left\{ \frac{1}{\mathbf{m}_2 \cdot \boldsymbol{\Omega}_2 - \omega_1} \right. \\ &\quad \times \left. \frac{1}{\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2, \omega_1)} \frac{1}{\mathcal{D}_{-\mathbf{m}_1, \mathbf{m}_3}(\mathbf{J}_1, \mathbf{J}_3, \omega_2)} \frac{1}{\mathbf{m}_3 \cdot \boldsymbol{\Omega}_3 - \omega_2} \langle \delta \hat{F}_{\mathbf{m}_2}(\mathbf{J}_2, 0) \delta \hat{F}_{\mathbf{m}_3}(\mathbf{J}_3, 0) \rangle \right\}, \\ \mathcal{F}_{\text{tot}}^{(II)}(\mathbf{J}_1) &= -i \sum_{\mathbf{m}_1} \mathbf{m}_1 \int_{\mathcal{B}_1} \frac{d\omega_1}{2\pi} e^{-i\omega_1 t} \int_{\mathcal{B}_2} \frac{d\omega_2}{2\pi} e^{-i\omega_2 t} (2\pi)^d \frac{1}{\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - \omega_1} \\ &\quad \times \sum_{\mathbf{m}_2} \int d\mathbf{J}_2 \frac{1}{\mathcal{D}_{-\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2, \omega_2)} \frac{1}{\mathbf{m}_2 \cdot \boldsymbol{\Omega}_2 - \omega_2} \langle \delta \hat{F}_{\mathbf{m}_1}(\mathbf{J}_1, 0) \delta \hat{F}_{\mathbf{m}_2}(\mathbf{J}_2, 0) \rangle. \end{aligned} \quad (2.56)$$

In order to evaluate the two expressions from Eq.(2.56), one needs to compute the statistical expectation of the product  $\langle \delta \hat{F}_{\mathbf{m}_1}(\mathbf{J}_1, 0) \delta \hat{F}_{\mathbf{m}_2}(\mathbf{J}_2, 0) \rangle$  that we will now evaluate.

Let us recall here that the fluctuations  $\delta F$  introduced in Eq.(2.41) are given by  $\delta F = F_d - F$ , i.e. stand for the difference between the actual discrete DF  $F_d$  and the smooth mean field one  $F$ . Starting from the expression (2.37) of the discrete distribution function  $F_d$  and temporarily dropping the time dependence,  $t=0$ , to shorten the notations, one can write

$$\langle \delta F(\boldsymbol{\theta}_1, \mathbf{J}_1) \delta F(\boldsymbol{\theta}_2, \mathbf{J}_2) \rangle = \mu^2 \sum_{i,j}^N \langle \delta_D(\boldsymbol{\theta}_1 - \boldsymbol{\theta}_i) \delta_D(\mathbf{J}_1 - \mathbf{J}_i) \delta_D(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_j) \delta_D(\mathbf{J}_2 - \mathbf{J}_j) \rangle - F(\mathbf{J}_1) F(\mathbf{J}_2). \quad (2.57)$$

Here, we relied on the fact the fluctuations are of zero mean so that  $\langle \delta F \rangle = 0$ . Let us now evaluate the first term from Eq.(2.57) which reads

$$\begin{aligned}
& \mu^2 \sum_{i,j}^N \langle \delta_D(\boldsymbol{\theta}_1 - \boldsymbol{\theta}_i) \delta_D(\mathbf{J}_1 - \mathbf{J}_i) \delta_D(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_j) \delta_D(\mathbf{J}_2 - \mathbf{J}_j) \rangle \\
&= \mu^2 \sum_i^N \langle \delta_D(\boldsymbol{\theta}_1 - \boldsymbol{\theta}_i) \delta_D(\mathbf{J}_1 - \mathbf{J}_i) \delta_D(\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2) \delta_D(\mathbf{J}_1 - \mathbf{J}_2) \rangle \\
&+ \mu^2 \sum_{i \neq j}^N \langle \delta_D(\boldsymbol{\theta}_1 - \boldsymbol{\theta}_i) \delta_D(\mathbf{J}_1 - \mathbf{J}_i) \delta_D(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_j) \delta_D(\mathbf{J}_2 - \mathbf{J}_j) \rangle \\
&= \mu F(\mathbf{J}_1) \delta_D(\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2) \delta_D(\mathbf{J}_1 - \mathbf{J}_2) + F(\mathbf{J}_1) F(\mathbf{J}_2), \quad (2.58)
\end{aligned}$$

where, to get the last line, we assumed that the particles are initially uncorrelated and used the fact that  $\langle F_d \rangle = F$ . Injecting Eq. (2.58) into Eq. (2.57), we get the relation

$$\langle \delta F(\boldsymbol{\theta}_1, \mathbf{J}_1) F(\boldsymbol{\theta}_2, \mathbf{J}_2) \rangle = \mu F(\mathbf{J}_1) \delta_D(\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2) \delta_D(\mathbf{J}_1 - \mathbf{J}_2). \quad (2.59)$$

Finally, taking the Fourier transform of Eq. (2.59), one gets the needed correlations in the initial conditions as

$$\langle \delta \widehat{F}_{m_1}(\mathbf{J}_1, 0) \delta \widehat{F}_{m_2}(\mathbf{J}_2, 0) \rangle = \frac{\mu}{(2\pi)^d} \delta_{m_1}^{-m_2} \delta_D(\mathbf{J}_1 - \mathbf{J}_2) F(\mathbf{J}_1). \quad (2.60)$$

The two components of the diffusion flux from Eq. (2.56) then become

$$\begin{aligned}
\mathcal{F}_{\text{tot}}^{(I)}(\mathbf{J}_1) &= -i\mu(2\pi)^d \sum_{m_1} \mathbf{m}_1 \int_{\mathcal{B}_1} \frac{d\omega_1}{2\pi} e^{-i\omega_1 t} \int_{\mathcal{B}_2} \frac{d\omega_2}{2\pi} e^{-i\omega_2 t} \frac{\mathbf{m}_1 \cdot \partial F / \partial \mathbf{J}_1}{\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - \omega_1} \\
&\quad \times \sum_{m_2} \int d\mathbf{J}_2 \frac{1}{\mathbf{m}_2 \cdot \boldsymbol{\Omega}_2 - \omega_1} \frac{1}{\mathcal{D}_{m_1, m_2}(\mathbf{J}_1, \mathbf{J}_2, \omega_1)} \frac{1}{\mathcal{D}_{-m_1, -m_2}(\mathbf{J}_1, \mathbf{J}_2, \omega_2)} \frac{F(\mathbf{J}_2)}{\mathbf{m}_2 \cdot \boldsymbol{\Omega}_2 + \omega_2}, \\
\mathcal{F}_{\text{tot}}^{(II)}(\mathbf{J}_1) &= i\mu \sum_{m_1} \mathbf{m}_1 \int_{\mathcal{B}_1} \frac{d\omega_1}{2\pi} e^{-i\omega_1 t} \int_{\mathcal{B}_2} \frac{d\omega_2}{2\pi} e^{-i\omega_2 t} \frac{1}{\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - \omega_1} \\
&\quad \times \frac{1}{\mathcal{D}_{-m_1, -m_1}(\mathbf{J}_1, \mathbf{J}_1, \omega_2)} \frac{F(\mathbf{J}_1)}{\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 + \omega_2}. \quad (2.61)
\end{aligned}$$

Let us now proceed to the successive evaluations of both terms in Eq. (2.61). Let us first evaluate the term  $\mathcal{F}_{\text{tot}}^{(I)}(\mathbf{J}_1)$ , which corresponds to the diffusion component of the kinetic equation. Here, the difficulty is to deal with the resonant poles appearing in Eq. (2.61). We follow the argument presented in Eq. (51.17) of Pitaevskii and Lifshitz (2012) and note that considering only contributions that do not decay in time, one can perform the substitution

$$\frac{1}{\mathbf{m}_2 \cdot \boldsymbol{\Omega}_2 - \omega_1} \frac{1}{\mathbf{m}_2 \cdot \boldsymbol{\Omega}_2 + \omega_2} \longrightarrow (2\pi)^2 \delta_D(\omega_1 + \omega_2) \delta_D(\mathbf{m}_2 \cdot \boldsymbol{\Omega}_2 - \omega_1). \quad (2.62)$$

This substitution allows us to perform the integrations w.r.t.  $\omega_1$  and  $\omega_2$  in Eq. (2.61), so that  $\mathcal{F}_{\text{tot}}^{(I)}$  becomes

$$\mathcal{F}_{\text{tot}}^{(I)}(J_1) = i\mu(2\pi)^d \sum_{\mathbf{m}_1} \int_{\mathcal{B}_1} d\omega_1 \sum_{\mathbf{m}_2} \int dJ_2 \frac{\delta_D(\mathbf{m}_2 \cdot \boldsymbol{\Omega}_2 - \omega_1)}{\omega_1 - \mathbf{m}_1 \cdot \boldsymbol{\Omega}_1} \frac{\mathbf{m}_1 \cdot \partial F / \partial \mathbf{J}_1 F(J_2)}{|\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_2}(J_1, J_2, \mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - \omega_1)|^2}, \quad (2.63)$$

where we relied on the relation  $1/\mathcal{D}_{-\mathbf{m}_1, -\mathbf{m}_2}(J_1, J_2, -\omega) = 1/\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_2}^*(J_1, J_2, \omega)$  (see note [83] in Chavanis (2012b)). In Eq. (2.63), we may finally perform the integration w.r.t.  $\omega_1$  by lowering the contour  $\mathcal{B}_1$  to the lower axis and using the Landau prescription  $\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 \rightarrow \mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - i0^+$  associated with the fact that the contour  $\mathcal{B}_1$  has to pass above the pole. We finally rely on Plemelj formula from Eq. (2.29). Because  $\mathcal{F}_{\text{tot}}^{(I)}$  is a real quantity, only the Dirac delta remains, and Eq. (2.63) finally becomes

$$\mathcal{F}_{\text{tot}}^{(I)}(J_1) = \mu\pi(2\pi)^d \sum_{\mathbf{m}_1, \mathbf{m}_2} \int dJ_2 \frac{\delta_D(\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - \mathbf{m}_2 \cdot \boldsymbol{\Omega}_2)}{|\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_2}(J_1, J_2, \mathbf{m}_1 \cdot \boldsymbol{\Omega}_1)|^2} \mathbf{m}_1 \cdot \frac{\partial F}{\partial \mathbf{J}_1} F(J_2). \quad (2.64)$$

Coming back to Eq. (2.61), let us now evaluate the second flux component  $\mathcal{F}_{\text{tot}}^{(II)}(J_1)$ . This term is associated with the drift component of the kinetic equation. To perform the integrations over  $\omega_1$  and  $\omega_2$ , we distort once again the Bromwich contours  $\mathcal{B}_1$  and  $\mathcal{B}_2$  towards negative imaginary parts, while still remaining above all the singularities of the integrand. When deformed to large negative imaginary parts, the exponential terms  $e^{-i\omega_1 t}$  and  $e^{-i\omega_2 t}$  tend to 0, so that one should only account for the contributions from the poles. For  $\mathcal{B}_1$ , we note that there is only one pole in  $\omega_1 = \mathbf{m}_1 \cdot \boldsymbol{\Omega}_1$  and that this pole is located along the real axis. One should also pay a careful attention to the direction of integration, so that here one has  $\int d\omega_1 f(\omega_1)/(\omega_1 - \omega_0) = -2i\pi f(\omega_0)$ . For the integration w.r.t.  $\omega_2$ , one first notes an obvious pole along the real axis in  $\omega_2 = -\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1$ . In addition, because the system is assumed to be stable, all the singularities associated with the susceptibility coefficients  $\omega_2 \mapsto 1/\mathcal{D}(\omega_2)$  are located below the real axis. Such poles will then be multiplied by a decaying in time exponential. Considering only contributions which do not decay in time, we restrict ourselves only to the real pole in  $\omega_2 = -\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1$ , and pay as well a careful attention to the sign of the residues. Equation (2.61) gives

$$\begin{aligned} \mathcal{F}_{\text{tot}}^{(II)}(J_1) &= i\mu \sum_{\mathbf{m}_1} \mathbf{m}_1 \frac{1}{\mathcal{D}_{-\mathbf{m}_1, -\mathbf{m}_1}(J_1, J_1, -\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 + i0^+)} F(J_1) \\ &= \mu \sum_{\mathbf{m}_1} \mathbf{m}_1 \text{Im} \left[ \frac{1}{\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_1}(J_1, J_1, \mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 + i0^+)} \right] F(J_1), \end{aligned} \quad (2.65)$$

where one should pay attention to the small positive imaginary part  $i0^+$  which was added following Landau prescription  $\omega_2 \rightarrow \omega_2 + i0^+$ . This emphasises the fact that the contour  $\mathcal{B}_2$  has to pass above the pole. In Eq. (2.65), we also note that the two time dependences introduced by the two inverse Laplace transforms cancelled out, so that  $\mathcal{F}_{\text{tot}}^{(II)}$  does not explicitly depend on time. To get the second relation in Eq. (2.65), we performed the change  $\mathbf{m}_1 \rightarrow -\mathbf{m}_1$ , and relied on the fact that  $\mathcal{F}_{\text{tot}}^{(II)}$  is a real quantity, hence the imaginary part. The calculation of the imaginary part in Eq. (2.65) is

presented in Eq. (2.155). We refer to this calculation, so that we can finally rewrite Eq. (2.65) as

$$\mathcal{F}_{\text{tot}}^{(II)}(\mathbf{J}_1) = -\mu\pi(2\pi)^d \sum_{\mathbf{m}_1, \mathbf{m}_2} \mathbf{m}_1 \int d\mathbf{J}_2 \frac{\delta_D(\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - \mathbf{m}_2 \cdot \boldsymbol{\Omega}_2)}{|\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2, \mathbf{m}_1 \cdot \boldsymbol{\Omega}_1)|^2} F(\mathbf{J}_1) \mathbf{m}_2 \cdot \frac{\partial F}{\partial \mathbf{J}_2}. \quad (2.66)$$

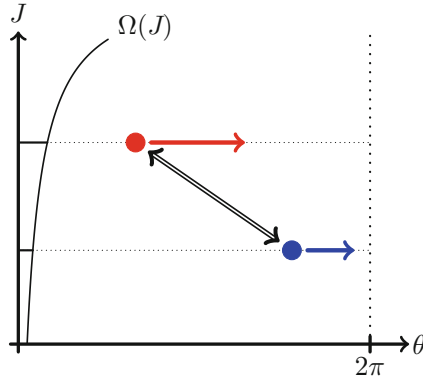
We have now evaluated the two components of the diffusion flux  $\mathcal{F}_{\text{tot}}$  from Eq. (2.52). We have therefore derived a closed kinetic equation, the inhomogeneous Balescu-Lenard equation, that will be presented in detail in the upcoming section.

### 2.3.4 The Balescu-Lenard Equation

Combining Eqs. (2.64) and (2.66), one can estimate the total diffusion flux  $\mathcal{F}_{\text{tot}}$ . Equation (2.52) immediately gives the associated closed diffusion equation. This is the inhomogeneous Balescu-Lenard equation which reads

$$\begin{aligned} \frac{\partial F}{\partial t} = \pi(2\pi)^d \mu \frac{\partial}{\partial \mathbf{J}_1} \cdot \left[ \sum_{\mathbf{m}_1, \mathbf{m}_2} \mathbf{m}_1 \int d\mathbf{J}_2 \frac{\delta_D(\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - \mathbf{m}_2 \cdot \boldsymbol{\Omega}_2)}{|\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2, \mathbf{m}_1 \cdot \boldsymbol{\Omega}_1)|^2} \right. \\ \left. \times \left( \mathbf{m}_1 \cdot \frac{\partial}{\partial \mathbf{J}_1} - \mathbf{m}_2 \cdot \frac{\partial}{\partial \mathbf{J}_2} \right) F(\mathbf{J}_1, t) F(\mathbf{J}_2, t) \right]. \end{aligned} \quad (2.67)$$

Let us now detail the physical content of this diffusion equation. Let us first note that Eq. (2.67) is written as the divergence of a flux, so that it conserves the total number of particles. The presence of the prefactor  $\mu = M_{\text{tot}}/N$  illustrates the fact that the Balescu-Lenard equation was obtained thanks to a kinetic development at order  $1/N$ . It captures first-order contributions associated with finite- $N$  effects. In Eq. (2.67), one should note in particular the presence of a resonance condition encapsulated by the Dirac delta  $\delta_D(\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - \mathbf{m}_2 \cdot \boldsymbol{\Omega}_2)$ , where  $\mathbf{m}_1, \mathbf{m}_2 \in \mathbb{Z}^d$  are resonance vectors. This is associated with an integration over the dummy variable  $\mathbf{J}_2$  scanning action space looking for locations where the resonance condition is satisfied. Figure 2.4 illustrates the gist of these resonant encounters, when the motion of the particles is directly considered in angle-action space. Similarly, Fig. 2.5 illustrates qualitatively such a non-local resonance condition in the case of a razor-thin disc. One should note that such resonant encounters are non-local in the sense that they do not require the resonating orbits to be close in position nor in action space. Equation (2.67) finally involves the dressed susceptibility coefficients  $1/\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2, \omega)$  introduced in Eq. (2.50). They encode the strength of the self-gravitating amplification within the system. Let us finally note that Eq. (2.67) scales like  $1/(ND^2)$ , so that increasing  $N$  or increasing the heat content of the system have the same effect by slowing down the diffusion.



**Fig. 2.4** Illustration in angle-action space of the resonance condition  $\delta_D(\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - \mathbf{m}_2 \cdot \boldsymbol{\Omega}_2)$  occurring in the Balescu-Lenard equation (2.67). In angle-action space, the trajectories of the particles are straight lines, with an intrinsic frequency  $\Omega(J)$ . This frequency depends only on the actions and is illustrated by the left curve. Here, the frequency associated with the *red* particle is twice the one of the *blue* particle: the particles are in resonance. These resonant encounters in angle-action space are the ones captured by the Balescu-Lenard equation (2.67)

Equation (2.67) can be rewritten as an anisotropic Fokker-Planck diffusion equation by introducing the relevant drift and diffusion coefficients. It becomes

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial J_1} \cdot \left[ \sum_{\mathbf{m}_1} \mathbf{m}_1 \left( A_{\mathbf{m}_1}(J_1) F(J_1) + D_{\mathbf{m}_1}(J_1) \mathbf{m}_1 \cdot \frac{\partial F}{\partial J_1} \right) \right], \quad (2.68)$$

where  $A_{\mathbf{m}_1}(J_1)$  and  $D_{\mathbf{m}_1}(J_1)$  are respectively the drift and diffusion coefficients associated with a given resonance vector  $\mathbf{m}_1$ . As the Balescu-Lenard equation describes the self-consistent evolution of the DF  $F$ , the drift and diffusion coefficients depend secularly on  $F$ . This dependence was not written out explicitly to simplify the notations. Following Eq. (2.67), the drift coefficients are given by

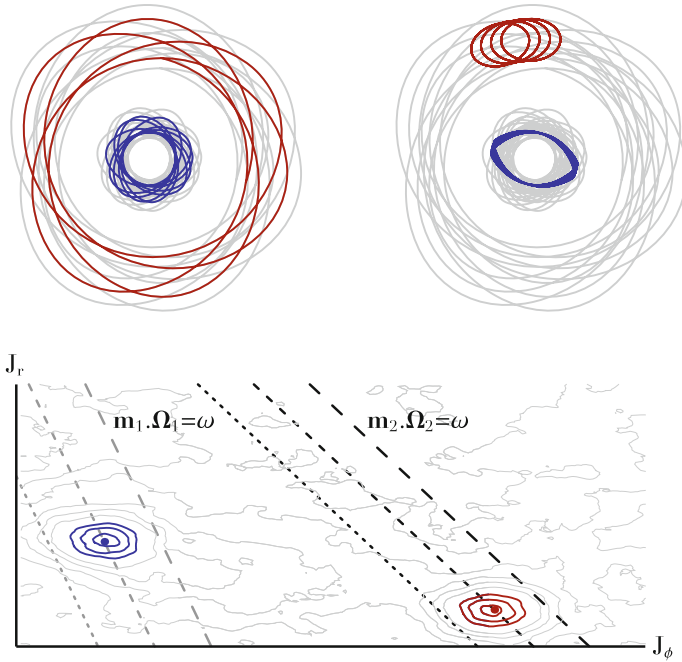
$$A_{\mathbf{m}_1}(J_1) = -\pi(2\pi)^d \mu \sum_{\mathbf{m}_2} \int dJ_2 \frac{\delta_D(\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - \mathbf{m}_2 \cdot \boldsymbol{\Omega}_2)}{|\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_2}(J_1, J_2, \mathbf{m}_1 \cdot \boldsymbol{\Omega}_1)|^2} \mathbf{m}_2 \cdot \frac{\partial F}{\partial J_2}, \quad (2.69)$$

while the diffusion coefficients are given by

$$D_{\mathbf{m}_1}(J_1) = \pi(2\pi)^d \mu \sum_{\mathbf{m}_2} \int dJ_2 \frac{\delta_D(\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - \mathbf{m}_2 \cdot \boldsymbol{\Omega}_2)}{|\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_2}(J_1, J_2, \mathbf{m}_1 \cdot \boldsymbol{\Omega}_1)|^2} F(J_2). \quad (2.70)$$

Finally, let us introduce the total diffusion flux  $\mathcal{F}_{\text{tot}}(J)$  as

$$\mathcal{F}_{\text{tot}}(J) = \sum_{\mathbf{m}} \mathbf{m} \left( A_{\mathbf{m}}(J) F(J) + D_{\mathbf{m}}(J) \mathbf{m} \cdot \frac{\partial F}{\partial J} \right), \quad (2.71)$$



**Fig. 2.5** Illustration of the resonance condition  $\delta_D(\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - \mathbf{m}_2 \cdot \boldsymbol{\Omega}_2)$  of the Balescu-Lenard equation (2.67) in the case of a razor-thin stellar disc. *Top-left panel* A set of two resonant orbits in the inertial frame. *Top-right panel* The same two orbits in the rotating frame in which they are in resonance—here through an ILR-COR coupling (see Fig. 3.4). *Bottom panel* Fluctuations in action space of the system’s DF sourced by finite- $N$  effects, exhibiting overdensities for the *blue* and *red* orbits. The *dashed* lines correspond to three contour levels of the intrinsic frequency respectively associated with the resonance vector  $\mathbf{m}_1$  (grey lines) and  $\mathbf{m}_2$  (black lines). The two sets of orbits satisfy the resonance condition  $\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - \mathbf{m}_2 \cdot \boldsymbol{\Omega}_2 = 0$ , and therefore lead to a secular diffusion of the system’s orbital structure according to the Balescu-Lenard equation (2.67). Let us emphasise that resonant orbits need not be caught in the same resonance ( $\mathbf{m}_1 \neq \mathbf{m}_2$ ), be close in position space nor in action space

so that the Balescu-Lenard equation becomes

$$\frac{\partial F}{\partial t} = \text{div}(\mathcal{F}_{\text{tot}}). \quad (2.72)$$

Here, with this convention,  $-\mathcal{F}_{\text{tot}}$  corresponds to the direction along which individual particles diffuse.



### 2.3.5 The Bare Case: The Landau Equation

When collective effects are neglected, the Balescu-Lenard equation (2.67) becomes the Landau equation (Polyachenko and Shukhman 1982; Chavanis 2007, 2010, 2013b) reading

$$\begin{aligned} \frac{\partial F}{\partial t} = \pi(2\pi)^d \mu \frac{\partial}{\partial \mathbf{J}_1} \cdot & \left[ \sum_{\mathbf{m}_1, \mathbf{m}_2} \mathbf{m}_1 \int d\mathbf{J}_2 \delta_D(\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - \mathbf{m}_2 \cdot \boldsymbol{\Omega}_2) |A_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2)|^2 \right. \\ & \left. \times \left( \mathbf{m}_1 \cdot \frac{\partial}{\partial \mathbf{J}_1} - \mathbf{m}_2 \cdot \frac{\partial}{\partial \mathbf{J}_2} \right) F(\mathbf{J}_1, t) F(\mathbf{J}_2, t) \right]. \end{aligned} \quad (2.73)$$

In Eq.(2.73), the dressed susceptibility coefficients  $1/|\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2, \omega)|^2$  from Eq.(2.50) are replaced by the bare ones  $|A_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2)|^2$ . These are defined as the Fourier transform of the interaction potential  $U$  (Lynden-Bell 1994; Pichon 1994; Chavanis 2013b), so that

$$A_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2) = \frac{1}{(2\pi)^{2d}} \int d\boldsymbol{\theta}_1 d\boldsymbol{\theta}_2 U(\mathbf{x}(|\boldsymbol{\theta}_1, \mathbf{J}_1) - \mathbf{x}(\boldsymbol{\theta}_2, \mathbf{J}_2)|) e^{-i(\mathbf{m}_1 \cdot \boldsymbol{\theta}_1 - \mathbf{m}_2 \cdot \boldsymbol{\theta}_2)}. \quad (2.74)$$

In addition, these coefficients satisfy the symmetry relations

$$A_{\mathbf{m}_2, \mathbf{m}_1}(\mathbf{J}_2, \mathbf{J}_1) = A_{-\mathbf{m}_1, -\mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2) = A_{\mathbf{m}_1, \mathbf{m}_2}^*(\mathbf{J}_1, \mathbf{J}_2). \quad (2.75)$$

Note that the kinetic equations (2.67) and (2.73) share the same overall structure.

### 2.3.6 The Multi-Component Case

A crucial strength of the Balescu-Lenard formalism, already emphasised in Heyvaerts (2010) and Chavanis (2013b), is that this formalism also allows for a self-consistent description of the simultaneous evolution of multiple populations of various masses. Let us now detail the structure of such a multi-component diffusion equation. (See Appendix 6.B for an illustration of how the multi-component Balescu-Lenard equation may be derived in the specific context of quasi-Keplerian systems.) Here, we consider a system made of multiple components, indexed by the letters “a” and “b”. The particles of the component “a” have an individual mass  $\mu_a$  and follow the DF  $F^a$ . Each DF  $F^a$  is normalised such that  $\int d\mathbf{x} d\mathbf{v} F^a = M_{\text{tot}}^a$ , where  $M_{\text{tot}}^a$  is the total active mass of component “a”. The evolution of each DF is then given by

$$\begin{aligned} \frac{\partial F^a}{\partial t} = \pi(2\pi)^d \frac{\partial}{\partial \mathbf{J}_1} \cdot \left[ \sum_{\mathbf{m}_1, \mathbf{m}_2} \mathbf{m}_1 \int d\mathbf{J}_2 \frac{\delta_D(\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - \mathbf{m}_2 \cdot \boldsymbol{\Omega}_2)}{|\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2, \mathbf{m}_1 \cdot \boldsymbol{\Omega}_1)|^2} \right. \\ \left. \times \sum_b \left\{ \mu_b F^b(\mathbf{J}_2) \mathbf{m}_1 \cdot \frac{\partial F^a}{\partial \mathbf{J}_1} - \mu_a F^a(\mathbf{J}_1) \mathbf{m}_2 \cdot \frac{\partial F^b}{\partial \mathbf{J}_2} \right\} \right]. \end{aligned} \quad (2.76)$$

In the multi-component case, the susceptibility coefficients are still given by Eq. (2.50). However, the response matrix now encompasses all the active components of the system, so that

$$\hat{\mathbf{M}}_{pq}(\omega) = (2\pi)^d \sum_{\mathbf{m}} \int d\mathbf{J} \frac{\mathbf{m} \cdot \partial(\sum_b F^b)/\partial \mathbf{J}}{\omega - \mathbf{m} \cdot \boldsymbol{\Omega}} \psi_{\mathbf{m}}^{(p)*}(\mathbf{J}) \psi_{\mathbf{m}}^{(q)}(\mathbf{J}). \quad (2.77)$$

Similarly to Eq. (2.68), the multi-component Balescu-Lenard equation may also be written as an anisotropic diffusion equation, so that

$$\frac{\partial F^a}{\partial t} = \frac{\partial}{\partial \mathbf{J}_1} \cdot \left[ \sum_{\mathbf{m}_1} \mathbf{m}_1 \sum_b \left\{ \mu_a A_{\mathbf{m}_1}^b(\mathbf{J}_1) F^a(\mathbf{J}_1) + \mu_b D_{\mathbf{m}_1}^b(\mathbf{J}_1) \mathbf{m}_1 \cdot \frac{\partial F^a}{\partial \mathbf{J}_1} \right\} \right]. \quad (2.78)$$

In Eq. (2.78), we introduced the multi-component drift and diffusion coefficients  $A_{\mathbf{m}_1}^b(\mathbf{J}_1)$  and  $D_{\mathbf{m}_1}^b(\mathbf{J}_1)$ . They depend on the location  $\mathbf{J}_1$  in action space, the considered resonance  $\mathbf{m}_1$ , and the component “b”, whose DF is the underlying DF used to estimate them. In analogy with Eq. (2.69), the multi-component drift coefficients are given by

$$A_{\mathbf{m}_1}^b(\mathbf{J}_1) = -\pi(2\pi)^d \sum_{\mathbf{m}_2} \int d\mathbf{J}_2 \frac{\delta_D(\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - \mathbf{m}_2 \cdot \boldsymbol{\Omega}_2)}{|\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2, \mathbf{m}_1 \cdot \boldsymbol{\Omega}_1)|^2} \mathbf{m}_2 \cdot \frac{\partial F^b}{\partial \mathbf{J}_2}, \quad (2.79)$$

while the diffusion ones, similarly to Eq. (2.70), read

$$D_{\mathbf{m}_1}^b(\mathbf{J}_1) = \pi(2\pi)^d \sum_{\mathbf{m}_2} \int d\mathbf{J}_2 \frac{\delta_D(\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - \mathbf{m}_2 \cdot \boldsymbol{\Omega}_2)}{|\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2, \mathbf{m}_1 \cdot \boldsymbol{\Omega}_1)|^2} F^b(\mathbf{J}_2). \quad (2.80)$$

One should pay attention to the fact that the multi-component drift and diffusion coefficients from Eqs. (2.79) and (2.80) do not have the same dimension as the single component ones. In order to emphasise the process of mass segregation, let us finally rewrite Eq. (2.78) as

$$\frac{\partial F^a}{\partial t} = \frac{\partial}{\partial \mathbf{J}_1} \cdot \left[ \sum_{\mathbf{m}_1} \mathbf{m}_1 \left\{ \mu_a A_{\mathbf{m}_1}^{\text{tot}}(\mathbf{J}_1) F^a(\mathbf{J}_1) + D_{\mathbf{m}_1}^{\text{tot}}(\mathbf{J}_1) \mathbf{m}_1 \cdot \frac{\partial F^a}{\partial \mathbf{J}_1} \right\} \right], \quad (2.81)$$

where we introduced the total drift and diffusion coefficients  $A_{m_1}^{\text{tot}}(\mathbf{J}_1)$  and  $D_{m_1}^{\text{tot}}(\mathbf{J}_1)$  as

$$A_{m_1}^{\text{tot}}(\mathbf{J}_1) = \sum_b A_{m_1}^b(\mathbf{J}_1) \quad ; \quad D_{m_1}^{\text{tot}}(\mathbf{J}_1) = \sum_b \mu_b D_{m_1}^b(\mathbf{J}_1). \quad (2.82)$$

In Eq. (2.81), let us note that the only differences between the different components is the presence of the mass prefactor  $\mu_a$  in front of the total drift coefficient. This leads to the process of mass segregation, when a spectrum of mass is involved.

### 2.3.7 *H-Theorem*

Following closely Heyvaerts (2010), let us define the system's entropy  $S(t)$  as

$$S(t) = - \int d\mathbf{J}_1 s(F(\mathbf{J}_1, t)), \quad \text{with} \quad s(x) = x \log(x), \quad (2.83)$$

where  $s(x)$  corresponds to Boltzmann's entropy function. Differentiating Eq. (2.83) once w.r.t. to time yields

$$\frac{dS}{dt} = - \int d\mathbf{J}_1 s'(F(\mathbf{J}_1)) \frac{\partial F}{\partial t}. \quad (2.84)$$

Let us follow the definition of the total diffusion flux  $\mathcal{F}_{\text{tot}}(\mathbf{J}_1)$  from Eq. (2.71) to rewrite  $\mathcal{F}_{\text{tot}}(\mathbf{J}_1)$  as

$$\mathcal{F}_{\text{tot}}(\mathbf{J}_1) = \sum_{m_1, m_2} m_1 \int d\mathbf{J}_2 \alpha_{m_1, m_2}(\mathbf{J}_1, \mathbf{J}_2) \left[ m_1 \cdot \frac{\partial}{\partial \mathbf{J}_1} - m_2 \cdot \frac{\partial}{\partial \mathbf{J}_2} \right] F(\mathbf{J}_1) F(\mathbf{J}_2), \quad (2.85)$$

with  $\alpha_{m_1, m_2}(\mathbf{J}_1, \mathbf{J}_2)$  given by

$$\alpha_{m_1, m_2}(\mathbf{J}_1, \mathbf{J}_2) = \pi (2\pi)^d \mu \frac{\delta_D(m_1 \cdot \boldsymbol{\Omega}_1 - m_2 \cdot \boldsymbol{\Omega}_2)}{|\mathcal{D}_{m_1, m_2}(\mathbf{J}_1, \mathbf{J}_2, m_1 \cdot \boldsymbol{\Omega}_1)|^2} \geq 0. \quad (2.86)$$

Integrating Eq. (2.84) by parts and ignoring boundary terms, one gets

$$\frac{dS}{dt} = \int d\mathbf{J}_1 s''(F(\mathbf{J}_1)) \frac{\partial F}{\partial \mathbf{J}_1} \cdot \mathcal{F}_{\text{tot}}(\mathbf{J}_1). \quad (2.87)$$

Thanks to the rewriting from Eq. (2.85), one can express Eq. (2.87) as

$$\frac{dS}{dt} = \sum_{m_1, m_2} \int d\mathbf{J}_1 d\mathbf{J}_2 \alpha_{m_1, m_2}(\mathbf{J}_1, \mathbf{J}_2) s_1''(m_1 \cdot \mathbf{F}'_1) \left[ F_2(m_1 \cdot \mathbf{F}'_1) - F_1(m_2 \cdot \mathbf{F}'_2) \right], \quad (2.88)$$

where we used the shortened notations  $s_i'' = s''(F(\mathbf{J}_i))$ ,  $F_i = F(\mathbf{J}_i)$ , and  $\mathbf{F}_i' = \partial F / \partial \mathbf{J}_i$ . Equation (2.88) can be symmetrised via the substitutions  $\mathbf{m}_1 \leftrightarrow \mathbf{m}_2$  and  $\mathbf{J}_1 \leftrightarrow \mathbf{J}_2$ . As  $\alpha_{\mathbf{m}_2, \mathbf{m}_1}(\mathbf{J}_2, \mathbf{J}_1) = \alpha_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2)$ , Eq. (2.88) finally becomes

$$\frac{dS}{dt} = \frac{1}{2} \sum_{\mathbf{m}_1, \mathbf{m}_2} \int d\mathbf{J}_1 d\mathbf{J}_2 \alpha_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2) \left[ F_2 s_1'' (\mathbf{m}_1 \cdot \mathbf{F}_1')^2 - (\mathbf{m}_1 \cdot \mathbf{F}_1') (\mathbf{m}_2 \cdot \mathbf{F}_2') (F_1 s_1'' + F_2 s_2'') + F_1 s_2'' (\mathbf{m}_2 \cdot \mathbf{F}_2')^2 \right]. \quad (2.89)$$

As Boltzmann's entropy function<sup>3</sup> satisfies  $s''(x) = 1/x$ , the square bracket in Eq. (2.89) may immediately be factored as

$$\frac{1}{F_1 F_2} \left[ F_2 (\mathbf{m}_1 \cdot \mathbf{F}_1') - F_1 (\mathbf{m}_2 \cdot \mathbf{F}_2') \right]^2 \geq 0, \quad (2.90)$$

so that one finally gets  $dS/d\tau \geq 0$ . The Balescu-Lenard equation (2.67) therefore satisfies Boltzmann's H-theorem. This entropy increase corresponds to heat generation as the orbital structure of the system secularly rearranges itself driven by self-induced collisional effects. The previous demonstration naturally extends both to the Landau equation (2.73), but also more interestingly to the multi-component Balescu-Lenard equation (2.76). Indeed, defining the system's total entropy  $S_{\text{tot}}$  as

$$S_{\text{tot}}(t) = - \int d\mathbf{J}_1 \sum_a \frac{1}{\mu_a} s(F^a(\mathbf{J}_1, t)), \quad (2.91)$$

and following the same approach, one can show that for  $s''(x) = 1/x$ , one has  $dS_{\text{tot}}/dt \geq 0$ . Let us finally note that this does not necessarily imply that the entropy of each individual component increases.

## 2.4 Conclusion

In this chapter, we presented two important sources of diffusion to induce secular evolution in self-gravitating systems. The first source, presented in Sect. 2.2, considers the case of a collisionless system undergoing external perturbations. The second source, presented in Sect. 2.3, is captured by the Balescu-Lenard equation, which describes the long-term effects of finite- $N$  fluctuations on isolated discrete self-gravitating systems. In our two derivations, we emphasised the strong similarities existing between the two approaches, as can be seen in particular in their similar decoupled evolution equations. Let us finally underline that both Eqs. (2.31) and (2.67) share the properties that they describe strongly anisotropic diffusion in action space (see Fig. 2.3), account for the system's internal susceptibility (via the response matrix from Eq. (2.17) and the associated gravitational polarisation, see

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<sup>3</sup>Any double primitive of  $1/x$  would work as well.

Fig. 1.6). Because they are sourced by different fluctuations, either external or internal, these two orbital diffusion processes provide the ideal frameworks in which to study the secular evolution of self-gravitating systems.

The rest of the thesis is focused on illustrating for various astrophysical systems how these formalisms allow for a detailed description of their secular dynamics. In Chap. 3, we will consider the case of razor-thin stellar discs. In order to obtain simple quadratures for the diffusion fluxes, we will develop a razor-thin WKB formalism (i.e. restriction to radially tightly wound perturbations) providing a straightforward understanding of the regions of maximum amplification within the disc. We will illustrate how the functional form of the diffusion coefficients explains the self-induced formation of resonant ridges in the disc’s DF, as observed in numerical simulations. In Chap. 4, we will resort to the same razor-thin stellar discs, but will devote our efforts to correctly account for the disc’s self-gravity and the associated strong amplification. This will be shown to significantly hasten the diffusion in the disc. In addition, in Appendix 4.D, we will illustrate how the same method may also be applied to study the long-term dynamics of 3D spherical systems such as dark matter haloes. This framework provides a promising way to investigate the secular transformation of dark matter haloes’ cusps into cores. In Chap. 5, we will extend our WKB approximation to apply it to thickened stellar discs. We will investigate various possible mechanisms of thickening such as the disc’s internal Poisson shot noise, a series of central decaying bars, or the joint evolution of giant molecular clouds within the disc. Finally, in Chap. 6, we will consider the case of quasi-Keplerian systems, such as galactic centres, for which the presence of a dominating central body imposes a degenerate Keplerian dynamics. Once tailored for such systems, we will detail in particular how the Balescu-Lenard formalism recovers the process of “resonant relaxation” specific to these systems.

### 2.4.1 *Future Works*

The previous formalisms could be generalised in various ways.

In Appendix 2.C, we presented a new method based on a functional approach to derive the inhomogeneous Landau equation. Because of the simplicity of the required calculations, this throws new light on the complex dynamical processes at play. One could hope to generalise this calculation to account for collective effects and recover the inhomogeneous Balescu-Lenard equation. Such a calculation is expected to be more demanding, as it will involve a Fredholm type equation, such as Eq. (2.125). Similarly, we showed in Fouvry et al. (2016b) how the same functional approach could also be transposed to the kinetic theory of two-dimensional point vortices (Chavanis 2012c, d). One should investigate other physical systems for which this approach could also be successful. Finally, it would be of particular interest to apply this method to derive a closed kinetic equation when higher order correlation terms are accounted for. This could for example allow us to describe the dynamics of 1D homogeneous systems, for which the  $1/N$  Balescu-Lenard colli-

sion term vanishes by symmetry (Eldridge and Feix 1963; Kadomtsev and Pogutse 1970). This is also the case for the Hamiltonian Mean Field model (HMF) in the homogeneous limit (Chavanis et al. 2005; Bouchet and Dauxois 2005; Rocha Filho et al. 2014).

Inspired by Pichon and Aubert (2006), the previous approaches could also be extended and developed for open systems, by accounting for possible sources and sinks of particles. Similarly, it could also prove interesting to investigate the Balescu-Lenard equation in a context where the system's number of particles gets to evolve during the secular evolution, to describe for example the progressive dissolution of overdensities, etc. Similarly, as can be seen in the proposed derivations, all these formalisms rely on the fundamental assumption of integrability, i.e. on the existence of angle-action coordinates. It would be of interest to investigate as well how such approaches could be tailored to deal with chaotic behaviours and their associated diffusions. Finally, one could also investigate within these frameworks the role that gas may play on the dynamical properties of the system. Indeed, one crucial property of gas is that it cannot shell-cross so that it shocks. This typically means that the gas component is dynamically much colder than its stellar counterpart, which alters the system's dynamical susceptibility.

## Appendix

### A Derivation of the BBGKY Hierarchy

In this Appendix, let us briefly recover the fundamental equations of the BBGKY hierarchy. This decomposition is at the heart of the derivation of the Balescu-Lenard equation presented in Heyvaerts (2010). Such a derivation with similar notations is presented in Fouvry et al. (2016a). Let us consider an isolated system made of  $N$  identical particles of individual mass  $\mu = M_{\text{tot}}/N$ , where  $M_{\text{tot}}$  is the total mass of the system. The individual dynamics of these particles is exactly described by Hamilton's equation which read

$$\mu \frac{d\mathbf{x}_i}{dt} = \frac{\partial H}{\partial \mathbf{v}_i} \quad ; \quad \mu \frac{d\mathbf{v}_i}{dt} = -\frac{\partial H}{\partial \mathbf{x}_i}, \quad (2.92)$$

where  $(\mathbf{x}_i, \mathbf{v}_i)$  corresponds to the position and velocity of particle  $i$ . The total Hamiltonian  $H$  appearing in Eq.(2.92) encompasses all the binary interactions between particles, so that

$$H = \frac{\mu}{2} \sum_{i=1}^N \mathbf{v}_i^2 + \mu^2 \sum_{i < j}^N U(|\mathbf{x}_i - \mathbf{x}_j|), \quad (2.93)$$

where  $U(|\mathbf{x}|)$  stands for the interaction potential, e.g.,  $U(|\mathbf{x}|) = -G/|\mathbf{x}|$  in the gravitational context. As will be underlined in Chap. 6 when considering quasi-Keplerian

systems, one can easily add an external potential to this Hamiltonian, and the associated hierarchy equations are straightforward to deduce. While Eq. (2.92) captures the individual dynamics of the system's components, we are interested in a statistical description of our system. As a consequence, let us introduce the system's  $N$ -body probability distribution function (PDF)  $P_N(\mathbf{x}_1, \mathbf{v}_1, \dots, \mathbf{x}_N, \mathbf{v}_N, t)$  which gives the probability of finding at time  $t$  particle 1 at position  $\mathbf{x}_1$  with velocity  $\mathbf{v}_1$ , particle 2 at position  $\mathbf{x}_2$  with velocity  $\mathbf{v}_2$ , etc. We choose the convention that

$$\int d\Gamma_1 d\Gamma_2 \dots d\Gamma_N P_N(\Gamma_1, \dots, \Gamma_N, t) = 1, \quad (2.94)$$

where we introduced the phase space coordinates  $\Gamma_i = (\mathbf{x}_i, \mathbf{v}_i)$ , so that  $d\Gamma_i = d\mathbf{x}_i d\mathbf{v}_i$ . The evolution of  $P_N$  is given by Liouville's equation (see Eq. (1.9)) which reads

$$\frac{\partial P_N}{\partial t} + \sum_{i=1}^N \left[ \mathbf{v}_i \cdot \frac{\partial P_N}{\partial \mathbf{x}_i} + \mu \mathcal{F}_i^{\text{tot}} \cdot \frac{\partial P_N}{\partial \mathbf{v}_i} \right] = 0, \quad (2.95)$$

where we introduced the total force,  $\mathcal{F}_i^{\text{tot}}$ , exerted on particle  $i$  as

$$\mathcal{F}_i^{\text{tot}} = \sum_{j \neq i}^N \mathcal{F}_{ij} = - \sum_{j \neq i}^N \frac{\partial U_{ij}}{\partial \mathbf{x}_i}. \quad (2.96)$$

In Eq. (2.96),  $\mathcal{F}_{ij}$  stands for the force exerted by particle  $j$  on particle  $i$ . It satisfies  $\mathcal{F}_{ij} = -\partial U_{ij} / \partial \mathbf{x}_i$ , where we wrote the interaction potential as  $U_{ij} = U(|\mathbf{x}_i - \mathbf{x}_j|)$ . At this stage, let us insist on the fact that Liouville's equation (2.95) is an exact equation, which encompasses the same information as Hamilton's equation (2.92). It is however defined on the (large) space of configurations  $(\Gamma_1, \dots, \Gamma_N)$ . In order to reduce the dimension of the space where the evolution equations are defined, let us introduce the reduced PDFs  $P_n$  for  $1 \leq n \leq N$  as

$$P_n(\Gamma_1, \dots, \Gamma_n, t) = \int d\Gamma_{n+1} \dots d\Gamma_N P_N(\Gamma_1, \dots, \Gamma_N, t). \quad (2.97)$$

Relying on the symmetry of  $P_N$  w.r.t. permutations of its arguments, we may integrate Eq. (2.95) w.r.t.  $d\Gamma_{n+1} \dots d\Gamma_N$  to obtain the evolution equation satisfied by  $P_n$ . This gives the general equation of the BBGKY hierarchy which reads

$$\frac{\partial P_n}{\partial t} + \sum_{i=1}^n \mathbf{v}_i \cdot \frac{\partial P_n}{\partial \mathbf{x}_i} + \sum_{i=1}^n \sum_{k=1, k \neq i}^n \mu \mathcal{F}_{ik} \cdot \frac{\partial P_n}{\partial \mathbf{v}_i} + (N-n) \sum_{i=1}^n \int d\Gamma_{n+1} \dots d\Gamma_N \mu \mathcal{F}_{i,n+1} \cdot \frac{\partial P_{n+1}}{\partial \mathbf{v}_i} = 0. \quad (2.98)$$

One can note that Eq. (2.98) is defined on the smaller space  $(\Gamma_1, \dots, \Gamma_n)$ . The three first terms only involve the first  $n$  particles, while the last collision term involves the reduced  $P_{n+1}$  of higher order, i.e. the BBGKY hierarchy is not closed. In order to

simplify the prefactors present in Eq. (2.98), let us introduce the reduced distribution functions  $f_n$  as

$$f_n(\Gamma_1, \dots, \Gamma_n, t) = \mu^n \frac{N!}{(N-n)!} P_n(\Gamma_1, \dots, \Gamma_n, t). \quad (2.99)$$

The hierarchy from Eq. (2.98) immediately becomes

$$\frac{\partial f_n}{\partial t} + \sum_{i=1}^n \mathbf{v}_i \cdot \frac{\partial f_n}{\partial \mathbf{x}_i} + \sum_{i=1}^n \sum_{k=1, k \neq i}^n \mu \mathcal{F}_{ik} \cdot \frac{\partial f_n}{\partial \mathbf{v}_i} + \sum_{i=1}^n \int d\Gamma_{n+1} \mathcal{F}_{i,n+1} \cdot \frac{\partial f_{n+1}}{\partial \mathbf{v}_i} = 0. \quad (2.100)$$

Equation (2.100) corresponds to the traditional writing of the BBGKY hierarchy. In order to emphasise the importance of the contributions arising from correlations between particles, let us introduce the cluster representation of the reduced distribution functions. We therefore define the 2-body correlation  $g_2$  as

$$f_2(\Gamma_1, \Gamma_2) = f_1(\Gamma_1) f_1(\Gamma_2) + g_2(\Gamma_1, \Gamma_2), \quad (2.101)$$

where the dependences w.r.t.  $t$  were not written out explicitly to simplify the notations. Similarly, we introduce the 3-body autocorrelation  $g_3$ , so that  $f_3$  reads

$$f_3(\Gamma_1, \Gamma_2, \Gamma_3) = f_1(\Gamma_1) f_1(\Gamma_2) f_1(\Gamma_3) + f_1(\Gamma_1) g_2(\Gamma_2, \Gamma_3) + f_1(\Gamma_2) g_2(\Gamma_1, \Gamma_3) + f_1(\Gamma_3) g_2(\Gamma_1, \Gamma_2) + g_3(\Gamma_1, \Gamma_2, \Gamma_3). \quad (2.102)$$

Thanks to the convention from Eq. (2.94), it is straightforward to check that one has the normalisations

$$\int d\Gamma_1 f_1(\Gamma_1) = \mu N; \quad \int d\Gamma_1 d\Gamma_2 g_2(\Gamma_1, \Gamma_2) = -\mu^2 N; \quad \int d\Gamma_1 d\Gamma_2 d\Gamma_3 g_3(\Gamma_1, \Gamma_2, \Gamma_3) = 2\mu^3 N. \quad (2.103)$$

As the mass of the individual particles is given by  $\mu = M_{\text{tot}}/N$ , one immediately gets the scalings w.r.t. the number of particles as  $|f_1| \sim 1$ ,  $|g_2| \sim 1/N$ , and  $|g_3| \sim 1/N^2$ . Thanks to these decompositions, the two first equations of the BBGKY hierarchy from Eq. (2.100) respectively become

$$\frac{\partial f_1}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f_1}{\partial \mathbf{x}_1} + \left[ \int d\Gamma_2 \mathcal{F}_{12} f_1(\Gamma_2) \right] \cdot \frac{\partial f_1}{\partial \mathbf{v}_1} + \int d\Gamma_2 \mathcal{F}_{12} \cdot \frac{\partial g_2(\Gamma_1, \Gamma_2)}{\partial \mathbf{v}_1} = 0, \quad (2.104)$$

and

$$\begin{aligned} & \frac{1}{2} \frac{\partial g_2}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial g_1}{\partial \mathbf{x}_1} + \left[ \int d\Gamma_3 \mathcal{F}_{13} f_1(\Gamma_3) \right] \cdot \frac{\partial g_2}{\partial \mathbf{v}_1} + \mu \mathcal{F}_{12} \cdot \frac{\partial f_1}{\partial \mathbf{v}_1} f_1(\Gamma_2) + \left[ \int d\Gamma_3 \mathcal{F}_{13} g_2(\Gamma_2, \Gamma_3) \right] \cdot \frac{\partial f_1}{\partial \mathbf{v}_1} \\ & + \mu \mathcal{F}_{12} \cdot \frac{\partial g_2}{\partial \mathbf{v}_1} + \left[ \int d\Gamma_3 \mathcal{F}_{13} \frac{\partial g_3(\Gamma_1, \Gamma_2, \Gamma_3)}{\partial \mathbf{v}_1} \right] + (1 \leftrightarrow 2) = 0, \end{aligned} \quad (2.105)$$



where  $(1 \leftrightarrow 2)$  stands for the permutation of indices 1 and 2, and applies to all preceding terms. When considering the long-term evolution induced by discreteness effects, one may perform a truncation at order  $1/N$  of the two equations (2.104) and (2.105). This requires to rely on the scalings from Eq. (2.103), as well as on the fact that  $\mu \sim 1/N$  and  $\mathcal{F}_{ij} \sim 1$ . In Eq. (2.104), all the terms are at least of order  $1/N$ , so that they should all be conserved. In Eq. (2.105), all the terms on the first line are of order  $1/N$  and have to be conserved, while all the terms on the second line are of order  $1/N^2$  and may therefore be neglected.<sup>4</sup> In addition to these truncations, and in order to consider quantities of order 1, let us introduce the system's 1-body DF  $F$  and the 2-body correlation function  $\mathcal{C}$  as

$$F = f_1 \quad ; \quad \mathcal{C} = \frac{g_2}{\mu}. \quad (2.106)$$

It is straightforward to note that  $|F| \sim 1$  and  $|\mathcal{C}| \sim 1$ . When truncated at order  $1/N$ , Eqs. (2.104) and (2.105) finally take the form

$$\frac{\partial F}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial F}{\partial \mathbf{x}_1} + \left[ \int d\Gamma_2 \mathcal{F}_{12} F(\Gamma_2) \right] \cdot \frac{\partial F}{\partial \mathbf{v}_1} + \mu \int d\Gamma_2 \mathcal{F}_{12} \cdot \frac{\partial \mathcal{C}(\Gamma_1, \Gamma_2)}{\partial \mathbf{v}_1} = 0, \quad (2.107)$$

and

$$\begin{aligned} & \frac{1}{2} \frac{\partial \mathcal{C}}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial \mathcal{C}}{\partial \mathbf{x}_1} + \left[ \int d\Gamma_3 \mathcal{F}_{13} F(\Gamma_3) \right] \cdot \frac{\partial \mathcal{C}}{\partial \mathbf{v}_1} + \mathcal{F}_{12} \cdot \frac{\partial F}{\partial \mathbf{v}_1} F(\Gamma_2) \\ & + \left[ \int d\Gamma_3 \mathcal{F}_{13} \mathcal{C}(\Gamma_2, \Gamma_3) \right] \cdot \frac{\partial F}{\partial \mathbf{v}_1} + (1 \leftrightarrow 2) = 0. \end{aligned} \quad (2.108)$$

These two coupled evolution equations (2.107) and (2.108) only involve the system's 1-body DF  $F$  and its 2-body autocorrelation function  $\mathcal{C}$ . They are the two central equations, from which one may derive the inhomogeneous Balescu-Lenard equation, as proposed in Heyvaerts (2010) and revisited in Appendix 2.B. The second and third terms in Eq. (2.107) correspond to the Vlasov advection term associated with the mean self-consistent potential generated by the stars. The fourth term is a collisional correction, as it scales like  $\mu \propto 1/N$ , and captures the effects of 2-body correlations on the dynamics of the 1-body DF. Similarly, in Eq. (2.108), the second and third terms are associated with the mean Vlasov advection due to the system's mean potential. The fourth term is a source term, which depends only on the 1-body DF and sources the dynamics of the 2-body correlation. Finally, the last term in Eq. (2.108) corresponds to the collective effects and is associated with the fact that the system can amplify and dress perturbations.

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<sup>4</sup>There is a subtlety with the first term on the second line of Eq. (2.105). While being of order  $1/N^2$ , it can become arbitrarily large when particle 2 approaches particle 1, due to the divergence of the interaction force at small separation. This term describes strong collisions and is not accounted for in the present formalism of resonance-driven diffusion.

## B Derivation of the Balescu-Lenard Equation via the BBGKY Hierarchy

In this Appendix, we revisit the derivation of the Balescu-Lenard equation (2.67) following the method presented in Heyvaerts (2010). This method, based on the direct resolution of the BBGKY hierarchy, is complementary to the second approach subsequently proposed by Chavanis (2012b), based on the Klimontovich equation and already presented in Sect. 2.3.

As already shown in Appendix 2.A, at order  $1/N$ , the dynamics of a self-gravitating system made of  $N$  identical particles is fully characterised by its 1-body DF  $F$  and 2-body autocorrelation  $\mathcal{C}$ . These two dynamical quantities are coupled by the two first truncated equations of the BBGKY hierarchy, namely Eqs. (2.107) and (2.108). They can be rewritten as

$$\frac{\partial F}{\partial t} + \left[ \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{x}_1} + \left[ \int d\Gamma_2 \mathcal{F}_{12} F(\Gamma_2) \right] \cdot \frac{\partial}{\partial \mathbf{v}_1} \right] F = -\mu \int d\Gamma_2 \mathcal{F}_{12} \cdot \frac{\partial \mathcal{C}(\Gamma_1, \Gamma_2)}{\partial \mathbf{v}_1}, \quad (2.109)$$

and

$$\begin{aligned} \frac{\partial \mathcal{C}}{\partial t} + \left[ \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{x}_1} + \left[ \int d\Gamma_3 \mathcal{F}_{13} F(\Gamma_3) \right] \cdot \frac{\partial}{\partial \mathbf{v}_1} \right] \mathcal{C} + \left[ \mathbf{v}_2 \cdot \frac{\partial}{\partial \mathbf{x}_2} + \left[ \int d\Gamma_3 \mathcal{F}_{23} F(\Gamma_3) \right] \cdot \frac{\partial}{\partial \mathbf{v}_2} \right] \mathcal{C} \\ + \left[ \int d\Gamma_3 \mathcal{F}_{13} \mathcal{C}(\Gamma_2, \Gamma_3) \right] \cdot \frac{\partial F}{\partial \mathbf{v}_1} + \left[ \int d\Gamma_3 \mathcal{F}_{23} \mathcal{C}(\Gamma_1, \Gamma_3) \right] \cdot \frac{\partial F}{\partial \mathbf{v}_2} = -\mathcal{F}_{12} \cdot \left[ \frac{\partial}{\partial \mathbf{v}_1} - \frac{\partial}{\partial \mathbf{v}_2} \right] F(\Gamma_1) F(\Gamma_2). \end{aligned} \quad (2.110)$$

### B.1 Solving for the Autocorrelation

The first step of the derivation is to solve Eq. (2.110) to obtain  $\mathcal{C} = \mathcal{C}[F]$ . Injecting this solution in the r.h.s. of Eq. (2.109), one finally obtains a closed kinetic equation involving  $F$  only.

First, one can note that Eq. (2.110) is linear in the correlation  $\mathcal{C}$ , symmetric in 1 and 2, and its r.h.s. is a source term  $S_2(\Gamma_1, \Gamma_2, t)$  depending only on the 1-body DF  $F$ . It reads

$$S_2(\Gamma_1, \Gamma_2, t) = -\mathcal{F}_{12} \cdot \left[ \frac{\partial}{\partial \mathbf{v}_1} - \frac{\partial}{\partial \mathbf{v}_2} \right] F(\Gamma_1) F(\Gamma_2). \quad (2.111)$$

Equation (2.110) can be solved for  $\mathcal{C}(\Gamma_1, \Gamma_2, t)$  by working out the Green's function  $\mathcal{G}^{(2)}(\Gamma_1, \Gamma_2, \Gamma'_1, \Gamma'_2, \tau)$  of the linear differential operator in its l.h.s. Indeed, the solution for  $\mathcal{C}(\Gamma_1, \Gamma_2, t)$  can be written as

$$\mathcal{C}(\Gamma_1, \Gamma_2, t) = \int_0^{+\infty} d\tau \int d\Gamma'_1 d\Gamma'_2 \mathcal{G}^{(2)}(\Gamma_1, \Gamma_2, \Gamma'_1, \Gamma'_2, \tau) S_2(\Gamma'_1, \Gamma'_2, t - \tau). \quad (2.112)$$

Injecting Eq. (2.112) into Eq. (2.110), one gets the propagation equation satisfied by  $\mathcal{G}^{(2)}$ . It reads

$$\begin{aligned} \frac{\partial \mathcal{G}^{(2)}}{\partial \tau} + \left[ \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{x}_1} + \left[ \int d\Gamma_3 \mathcal{F}_{13} F(\Gamma_3) \right] \cdot \frac{\partial}{\partial \mathbf{v}_1} \right] \mathcal{G}^{(2)} + \left[ \mathbf{v}_2 \cdot \frac{\partial}{\partial \mathbf{x}_2} + \left[ \int d\Gamma_3 \mathcal{F}_{23} F(\Gamma_3) \right] \cdot \frac{\partial}{\partial \mathbf{v}_2} \right] \mathcal{G}^{(2)} \\ + \left[ \int d\Gamma_3 \mathcal{F}_{13} \mathcal{G}^{(2)}(\Gamma_3, \Gamma_2, \Gamma'_1, \Gamma'_2, \tau) \right] \cdot \frac{\partial F}{\partial \mathbf{v}_1} + \left[ \int d\Gamma_3 \mathcal{F}_{23} \mathcal{G}^{(2)}(\Gamma_1, \Gamma_3, \Gamma'_1, \Gamma'_2, \tau) \right] \cdot \frac{\partial F}{\partial \mathbf{v}_2} = 0, \end{aligned} \quad (2.113)$$

where we assumed that the source term  $S_2(t)$  was effectively turned on only for  $t \geq 0$ , so that  $S_2(t < 0) = 0$ . Moreover, the Green's function initially satisfies  $\mathcal{G}^{(2)}(\Gamma_1, \Gamma_2, \Gamma'_1, \Gamma'_2, 0) = \delta_D(\Gamma_1 - \Gamma'_1) \delta_D(\Gamma_2 - \Gamma'_2)$ . Once the autocorrelation has been expressed as a function of  $F$ , i.e.  $\mathcal{C} = \mathcal{C}[F]$ , one may finally proceed to the evaluation of the collision operator  $\mathcal{C}[F]$  appearing in the r.h.s. of Eq. (2.109), which reads

$$\mathcal{C}[F] = -\mu \int d\Gamma_2 \mathcal{F}_{12} \cdot \frac{\partial \mathcal{C}[F](\Gamma_1, \Gamma_2)}{\partial \mathbf{v}_1}. \quad (2.114)$$

When considering Eq. (2.113), it is crucial to note that this propagation equation acts separately on the variables  $(\Gamma_1, \Gamma'_1)$  and  $(\Gamma_2, \Gamma'_2)$ . The initial condition of  $\mathcal{G}^{(2)}$  is also separable. Let us then solve Eq. (2.113) by factoring the 2-body Green's function  $\mathcal{G}^{(2)}$  as the product of two 1-body Green's function  $\mathcal{G}^{(1)}$ , so that

$$\mathcal{G}^{(2)}(\Gamma_1, \Gamma_2, \Gamma'_1, \Gamma'_2, \tau) = \mathcal{G}^{(1)}(\Gamma_1, \Gamma'_1, \tau) \mathcal{G}^{(1)}(\Gamma_2, \Gamma'_2, \tau), \quad (2.115)$$

where the 1-body Green's function  $\mathcal{G}^{(1)}$  satisfies the linearised 1-body Vlasov equation, namely

$$\begin{aligned} \frac{\partial \mathcal{G}^{(1)}(\Gamma_1, \Gamma'_1, \tau)}{\partial \tau} + \left[ \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{x}_1} + \left[ \int d\Gamma_2 \mathcal{F}_{12} F(\Gamma_2) \right] \cdot \frac{\partial}{\partial \mathbf{v}_1} \right] \mathcal{G}^{(1)}(\Gamma_1, \Gamma'_1, \tau) \\ + \left[ \int d\Gamma_2 \mathcal{G}^{(1)}(\Gamma_2, \Gamma'_1, \tau) \mathcal{F}_{12} \right] \cdot \frac{\partial F}{\partial \mathbf{v}_1} = 0, \end{aligned} \quad (2.116)$$

with the initial condition  $\mathcal{G}^{(1)}(\Gamma_1, \Gamma'_1, 0) = \delta_D(\Gamma_1 - \Gamma'_1)$ .<sup>5</sup> Because of the causality requirement, one needs to solve Eq. (2.116) only for  $\tau \geq 0$ . To do so, we rely on Bogoliubov's ansatz, which assumes that the system's 1-body DF  $F$  only evolves on a slow secular timescale, while the fluctuations and correlations evolve on a fast dynamical timescale. As a consequence, in Eq. (2.116), which describes the evolution of fluctuations, one may assume  $F$  to be frozen. Because of this decoupling, the correlations at a given time  $t$  can be seen as functionals of  $F$  evaluated at the very same time. To solve Eq. (2.116), let us perform a Laplace transform following the conventions from Eq. (2.44). One gets

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<sup>5</sup>Heyvaerts (2010) interestingly notes that if one was to account for contributions associated with strong collisions, such as in the first term of the second line of Eq. (2.105), the previous property of separability would not hold anymore.

$$\begin{aligned}
& -i\omega \tilde{\mathcal{G}}^{(1)}(\Gamma_1, \Gamma'_1, \omega) + \left[ \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{x}_1} + \left[ \int d\Gamma_2 \mathcal{F}_{12} F(\Gamma_2) \right] \cdot \frac{\partial}{\partial \mathbf{v}_1} \right] \tilde{\mathcal{G}}^{(1)}(\Gamma_1, \Gamma'_1, \omega) \\
& + \left[ \int d\Gamma_2 \tilde{\mathcal{G}}^{(1)}(\Gamma_2, \Gamma'_1, \omega) \mathcal{F}_{12} \right] \cdot \frac{\partial F}{\partial \mathbf{v}_1} = \delta_D(\Gamma_1 - \Gamma'_1). \tag{2.117}
\end{aligned}$$

## B.2 Application to Inhomogeneous Systems

Let us now assume that the system's mean potential is integrable, so that the physical phase space coordinates  $(\mathbf{x}, \mathbf{v})$  may be remapped to angle-action ones  $(\boldsymbol{\theta}, \mathbf{J})$ . Such a mapping allows for a simple description of the intricate trajectories of individual particles. This change of coordinates is canonical and the infinitesimal volumes are conserved, i.e.  $d\Gamma = d\mathbf{x}d\mathbf{v} = d\boldsymbol{\theta}d\mathbf{J}$ . Thanks to the adiabatic approximation (Heyvaerts 2010; Chavanis 2012b, 2013b), let us also assume that the system's 1-body DF is a quasi-stationary solution of the collisionless dynamics, so that  $F(\boldsymbol{\theta}, \mathbf{J}, t) = F(\mathbf{J}, t)$ . The angle-action coordinates satisfy two important additional properties. First, the derivatives along the mean motion take the simple form

$$\mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{x}_1} + \left[ \int d\Gamma_2 \mathcal{F}_{12} F(\Gamma_2) \right] \cdot \frac{\partial}{\partial \mathbf{v}_1} = \boldsymbol{\Omega}_1 \cdot \frac{\partial}{\partial \boldsymbol{\theta}_1}, \tag{2.118}$$

where  $\boldsymbol{\Omega}_1$  are the intrinsic frequencies of motion associated with the mean potential. Secondly, the Poisson brackets are invariant under the change of coordinates  $(\mathbf{x}, \mathbf{v}) \mapsto (\boldsymbol{\theta}, \mathbf{J})$ , so that for any functions  $L_1(\mathbf{x}, \mathbf{v})$  and  $L_2(\mathbf{x}, \mathbf{v})$ , one has

$$\frac{\partial L_1}{\partial \mathbf{x}} \cdot \frac{\partial L_2}{\partial \mathbf{v}} - \frac{\partial L_1}{\partial \mathbf{v}} \cdot \frac{\partial L_2}{\partial \mathbf{x}} = \frac{\partial L_1}{\partial \boldsymbol{\theta}} \cdot \frac{\partial L_2}{\partial \mathbf{J}} - \frac{\partial L_1}{\partial \mathbf{J}} \cdot \frac{\partial L_2}{\partial \boldsymbol{\theta}}. \tag{2.119}$$

With these transformations, Eq. (2.117) becomes

$$-i\omega \tilde{\mathcal{G}}^{(1)}(\Gamma_1, \Gamma'_1, \omega) + \boldsymbol{\Omega}_1 \cdot \frac{\partial \tilde{\mathcal{G}}^{(1)}(\Gamma_1, \Gamma'_1, \omega)}{\partial \boldsymbol{\theta}_1} - \int d\Gamma_2 \tilde{\mathcal{G}}^{(1)}(\Gamma_2, \Gamma'_1, \omega) \frac{\partial U_{12}}{\partial \boldsymbol{\theta}_1} \cdot \frac{\partial F}{\partial \mathbf{J}_1} = \delta_D(\Gamma_1 - \Gamma'_1). \tag{2.120}$$

Following the convention from Eq. (2.6), let us multiply Eq. (2.117) by  $1/(2\pi)^d e^{-i\mathbf{m}_1 \cdot \boldsymbol{\theta}_1}$  and integrate it w.r.t.  $\boldsymbol{\theta}_1$  to get

$$\begin{aligned}
& -i\omega \tilde{\mathcal{G}}_{\mathbf{m}_1}^{(1)}(\mathbf{J}_1, \Gamma'_1, \omega) + i\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 \tilde{\mathcal{G}}_{\mathbf{m}_1}^{(1)}(\mathbf{J}_1, \Gamma'_1, \omega) \\
& - (2\pi)^d i\mathbf{m}_1 \cdot \frac{\partial F}{\partial \mathbf{J}_1} \sum_{\mathbf{m}_2} \int d\mathbf{J}_2 \tilde{\mathcal{G}}_{\mathbf{m}_2}^{(1)}(\mathbf{J}_2, \Gamma'_1, \omega) A_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2) = \frac{e^{-i\mathbf{m}_1 \cdot \boldsymbol{\theta}'_1}}{(2\pi)^d} \delta_D(\mathbf{J}_1 - \mathbf{J}'_1), \tag{2.121}
\end{aligned}$$

where the bare susceptibility coefficients  $A_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2)$  were introduced in Eq. (2.74). Equation (2.121) can easily be rewritten as

$$\begin{aligned}
& \tilde{\mathcal{G}}_{m_1}^{(1)}(J_1, \Gamma'_1, \omega) + (2\pi)^d \frac{\mathbf{m}_1 \cdot \partial F / \partial \mathbf{J}_1}{\omega - \mathbf{m}_1 \cdot \boldsymbol{\Omega}_1} \sum_{m_2} \int dJ_2 \tilde{\mathcal{G}}_{m_2}^{(1)}(J_2, \Gamma'_1, \omega) A_{m_1, m_2}(J_1, J_2) \\
& = \frac{i}{(2\pi)^d} \frac{e^{-i\mathbf{m}_1 \cdot \boldsymbol{\theta}'_1}}{\omega - \mathbf{m}_1 \cdot \boldsymbol{\Omega}_1} \delta_D(J_1 - J'_1).
\end{aligned} \tag{2.122}$$

At this stage, let us note that Eq. (2.122) takes the form of a Fredholm equation, as the Green's function appears twice on the l.h.s, in particular once as an integral term. The method to solve such an equation is to rely on Kalnajs matrix method (Kalnajs 1976). Let us therefore introduce a basis of potential and densities  $(\psi^{(p)}, \rho^{(p)})$  as in Eq. (2.12), thanks to which the potential perturbations may be decomposed. Let us first develop the interaction potential  $U$  on these elements. We consider the function  $\mathbf{x}_1 \mapsto U(|\mathbf{x}_1 - \mathbf{x}_2|)$  and decompose it on the basis elements  $\psi^{(p)}(\mathbf{x}_1)$ . This takes the form  $U(|\mathbf{x}_1 - \mathbf{x}_2|) = \sum_p u_p(\mathbf{x}_2) \psi^{(p)}(\mathbf{x}_1)$ , where the coefficients  $u_p(\mathbf{x}_2)$  are given by

$$u_p(\mathbf{x}_2) = - \int d\mathbf{x}_1 U(|\mathbf{x}_1 - \mathbf{x}_2|) \rho^{(p)*}(\mathbf{x}_1) = -\psi^{(p)*}(\mathbf{x}_2). \tag{2.123}$$

Because they were defined as the Fourier transform in angles of the interaction potential, the bare susceptibility coefficients from Eq. (2.74) can immediately be rewritten as

$$A_{m_1, m_2}(J_1, J_2) = - \sum_p \psi_{m_1}^{(p)}(J_1) \psi_{m_2}^{(p)*}(J_2). \tag{2.124}$$

In order to invert the l.h.s. of Eq. (2.122), let us perform on  $\tilde{\mathcal{G}}_{m_1}^{(1)}$  the same operations than the ones operating on  $\tilde{\mathcal{G}}_{m_2}^{(1)}$ . This amounts to multiplying Eq. (2.122) by  $(2\pi)^d \sum_{m_1} \int dJ_1 \psi_{m_1}^{(q)*}(J_1)$ , so that it becomes

$$\begin{aligned}
& \left[ (2\pi)^d \sum_{m_1} \int dJ_1 \psi_{m_1}^{(q)*}(J_1) \tilde{\mathcal{G}}_{m_1}^{(1)}(J_1, \Gamma'_1, \omega) \right] \\
& - \sum_p \left[ (2\pi)^d \sum_{m_1} \int dJ_1 \frac{\mathbf{m}_1 \cdot \partial F / \partial \mathbf{J}_1}{\omega - \mathbf{m}_1 \cdot \boldsymbol{\Omega}_1} \psi_{m_1}^{(p)}(J_1) \psi_{m_1}^{(q)*}(J_1) \right] \left[ (2\pi)^d \sum_{m_2} \int dJ_2 \tilde{\mathcal{G}}_{m_2}^{(1)}(J_2, \Gamma'_1, \omega) \psi_{m_2}^{(p)*}(J_2) \right] \\
& = \sum_{m_1} \frac{i e^{-i\mathbf{m}_1 \cdot \boldsymbol{\theta}'_1}}{\omega - \mathbf{m}_1 \cdot \boldsymbol{\Omega}_1} \psi_{m_1}^{(q)*}(J'_1).
\end{aligned} \tag{2.125}$$

In order to clarify Eq. (2.125), let us introduce the notations

$$K_p(\Gamma'_1, \omega) = (2\pi)^d \sum_m \int dJ \tilde{\mathcal{G}}_m^{(1)}(J, \Gamma'_1, \omega) \psi_m^{(p)*}(J) \quad ; \quad L_p(\Gamma'_1, \omega) = \sum_m \frac{i e^{-i\mathbf{m} \cdot \boldsymbol{\theta}'_1}}{\omega - \mathbf{m} \cdot \boldsymbol{\Omega}_1} \psi_m^{(p)*}(J'_1). \tag{2.126}$$

Recalling also the expression of the response matrix from Eq. (2.17), we may finally rewrite Eq. (2.125) under the shortened form

$$K_p(\Gamma'_1, \omega) - \sum_q \widehat{\mathbf{M}}_{pq}(\omega) K_q(\Gamma'_1, \omega) = L_p(\Gamma'_1, \omega). \quad (2.127)$$

Assuming that the system considered is dynamically stable, so that  $[\mathbf{I} - \widehat{\mathbf{M}}(\omega)]$  can be inverted, Eq. (2.127) finally leads to

$$K_p(\Gamma'_1, \omega) = \sum_q [\mathbf{I} - \widehat{\mathbf{M}}(\omega)]_{pq}^{-1} L_q(\Gamma'_1, \omega). \quad (2.128)$$

Thanks to Eq. (2.128), one can finally rewrite Eq. (2.122) as

$$\tilde{\mathcal{G}}_{m_1}^{(1)}(\mathbf{J}_1, \Gamma'_1, \omega) = \frac{1}{(2\pi)^d} \frac{i e^{-i\mathbf{m}_1 \cdot \boldsymbol{\theta}'_1}}{\omega - \mathbf{m}_1 \cdot \boldsymbol{\Omega}_1} \delta_D(\mathbf{J}_1 - \mathbf{J}'_1) + \frac{\mathbf{m}_1 \cdot \partial F / \partial \mathbf{J}_1}{\omega - \mathbf{m}_1 \cdot \boldsymbol{\Omega}_1} \sum_{m'_1} \frac{1}{\mathcal{D}_{m_1, m'_1}(\mathbf{J}_1, \mathbf{J}'_1, \omega)} \frac{i e^{-i\mathbf{m}'_1 \cdot \boldsymbol{\theta}'_1}}{\omega - \mathbf{m}'_1 \cdot \boldsymbol{\Omega}'_1}, \quad (2.129)$$

where the dressed susceptibility coefficients,  $1/\mathcal{D}_{m_1, m'_1}$ , have been introduced in Eq. (2.50). Thanks to the inverse Fourier transform from Eq. (2.6), one can finally obtain the expression of  $\tilde{\mathcal{G}}^{(1)}(\Gamma_1, \Gamma'_1, \omega)$  as

$$\begin{aligned} \tilde{\mathcal{G}}^{(1)}(\Gamma_1, \Gamma'_1, \omega) &= \sum_{m_1, m'_1} \frac{i e^{i(\mathbf{m}_1 \cdot \boldsymbol{\theta}_1 - \mathbf{m}'_1 \cdot \boldsymbol{\theta}'_1)}}{\omega - \mathbf{m}_1 \cdot \boldsymbol{\Omega}_1} \left[ \frac{\delta_{m_1}^{m'_1}}{(2\pi)^d} \delta_D(\mathbf{J}_1 - \mathbf{J}'_1) + \frac{\mathbf{m}_1 \cdot \partial F / \partial \mathbf{J}_1}{(\omega - \mathbf{m}'_1 \cdot \boldsymbol{\Omega}'_1) \mathcal{D}_{m_1, m'_1}(\mathbf{J}_1, \mathbf{J}'_1, \omega)} \right] \\ &= \sum_{m_1, m'_1} \tilde{\mathcal{G}}_{m_1, m'_1}^{(1)}(\mathbf{J}_1, \mathbf{J}'_1, \omega) e^{i(\mathbf{m}_1 \cdot \boldsymbol{\theta}_1 - \mathbf{m}'_1 \cdot \boldsymbol{\theta}'_1)}. \end{aligned} \quad (2.130)$$

### B.3 Rewriting the Collision Operator

Thanks to the explicit expression of the 1-body Green's function from Eq. (2.130), one may now proceed to the evaluation of the collision operator from Eq. (2.114). Let us first rely on Bogoliubov's ansatz, so that in Eq. (2.112) we may do the replacement  $S_2(\Gamma'_1, \Gamma'_2, t - \tau) \rightarrow S_2(\Gamma'_1, \Gamma'_2, t)$ . Relying on the factorisation of the Green's function from Eq. (2.115) and the inverse Laplace transform from Eq. (2.44), the collision operator takes the form

$$\begin{aligned} \mathcal{C}[F] &= \int_0^{+\infty} d\tau \int d\Gamma_2 d\Gamma'_1 d\Gamma'_2 \int_{\mathcal{B}} \frac{d\omega}{2\pi} \int_{\mathcal{B}'} \frac{d\omega'}{2\pi} e^{-i(\omega + \omega')\tau} \\ &\quad \times \mu \mathcal{F}_{12} \cdot \frac{\partial}{\partial \mathbf{v}_1} \left\{ \tilde{\mathcal{G}}^{(1)}(\Gamma_1, \Gamma'_1, \omega) \tilde{\mathcal{G}}^{(1)}(\Gamma_2, \Gamma'_2, \omega') \left[ \mathcal{F}_{1'2'} \cdot \left[ \frac{\partial}{\partial \mathbf{v}'_1} - \frac{\partial}{\partial \mathbf{v}'_2} \right] F(\Gamma'_1) F(\Gamma'_2) \right] \right\}, \end{aligned} \quad (2.131)$$

where the Laplace transformed 1-body Green's function was obtained in Eq. (2.130). The rest of the calculations is now to rewrite Eq. (2.131) under a simpler form. Let us first rely on the properties from Eq. (2.119) to rewrite the various terms appearing in Eq. (2.131) in angle-action space. One has

$$\begin{aligned}\mathcal{F}_{12} \cdot \frac{\partial \tilde{\mathcal{G}}^{(1)}(\Gamma_1)}{\partial \mathbf{v}_1} &= -\frac{\partial U_{12}}{\partial \boldsymbol{\theta}_1} \cdot \frac{\partial \tilde{\mathcal{G}}^{(1)}(\Gamma_1)}{\partial \mathbf{J}_1} + \frac{\partial U_{12}}{\partial \mathbf{J}_1} \cdot \frac{\partial \tilde{\mathcal{G}}^{(1)}(\Gamma_1)}{\partial \boldsymbol{\theta}_1} \\ &= -\frac{\partial}{\partial \mathbf{J}_1} \cdot \left[ \int \frac{d\boldsymbol{\theta}_1}{(2\pi)^d} \frac{\partial U_{12}}{\partial \boldsymbol{\theta}_1} \tilde{\mathcal{G}}^{(1)}(\Gamma_1) \right],\end{aligned}\quad (2.132)$$

where we used the shortened notation  $\tilde{\mathcal{G}}^{(1)}(\Gamma_1) = \tilde{\mathcal{G}}^{(1)}(\Gamma_1, \Gamma'_1, \omega)$ . To obtain the second line of Eq. (2.132), we relied on Schwarz's theorem. We also relied on the fact that during the secular diffusion, the 1-body DF is of the form  $F = F(\mathbf{J}_1, t)$ , allowing us to perform an angle average w.r.t.  $\boldsymbol{\theta}_1$ . Similarly, one can write

$$\mathcal{F}_{1'2'} \cdot \left[ \frac{\partial}{\mathbf{v}'_1} - \frac{\partial}{\mathbf{v}'_2} \right] F(\Gamma'_1) F(\Gamma'_2) = - \left[ \frac{\partial U_{1'2'}}{\partial \boldsymbol{\theta}'_1} \cdot \frac{\partial F}{\partial \mathbf{J}'_1} F(\mathbf{J}'_2) + \frac{\partial U_{2'1'}}{\partial \boldsymbol{\theta}'_2} \cdot \frac{\partial F}{\partial \mathbf{J}'_2} F(\mathbf{J}'_1) \right]. \quad (2.133)$$

Let us now use the two previous rewritings, as well as Eq. (2.130), to rewrite the collision operator from Eq. (2.131) in angle-action space. After integrating w.r.t.  $\boldsymbol{\theta}_1$ ,  $\boldsymbol{\theta}_2$ ,  $\boldsymbol{\theta}'_1$ , and  $\boldsymbol{\theta}'_2$ , it reads

$$\begin{aligned}\mathcal{C}[F] &= \int_0^{+\infty} d\tau \int d\mathbf{J}_2 d\mathbf{J}'_1 d\mathbf{J}'_2 \int_{\mathcal{B}} \frac{d\omega}{2\pi} \int_{\mathcal{B}'} \frac{d\omega'}{2\pi} e^{-i(\omega+\omega')\tau} \mu(2\pi)^{3d} \\ &\quad \times \frac{\partial}{\partial \mathbf{J}_1} \cdot \left[ \sum_{\mathbf{m}_1, \mathbf{m}_2} \sum_{\mathbf{m}'_1, \mathbf{m}'_2} \tilde{\mathcal{G}}_{\mathbf{m}_1, \mathbf{m}'_1}^{(1)}(\omega) \tilde{\mathcal{G}}_{\mathbf{m}_2, \mathbf{m}'_2}^{(1)}(\omega') \mathbf{m}_1 A_{-\mathbf{m}_1, \mathbf{m}_2} \right. \\ &\quad \left. \times \left[ A_{\mathbf{m}'_1, -\mathbf{m}'_2} \mathbf{m}'_1 \cdot \frac{\partial F}{\partial \mathbf{J}'_1} F(\mathbf{J}'_2) + A_{\mathbf{m}'_2, -\mathbf{m}'_1} \mathbf{m}'_2 \cdot \frac{\partial F}{\partial \mathbf{J}'_2} F(\mathbf{J}'_1) \right] \right],\end{aligned}\quad (2.134)$$

where we used the shortened notations  $\tilde{\mathcal{G}}_{\mathbf{m}_1, \mathbf{m}'_1}^{(1)}(\omega) = \tilde{\mathcal{G}}_{\mathbf{m}_1, \mathbf{m}'_1}^{(1)}(\mathbf{J}_1, \mathbf{J}'_1, \omega)$  and  $A_{\mathbf{m}_1, \mathbf{m}_2} = A_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2)$ . Let us now use the explicit expression of the Fourier coefficients of the 1-body Green's function from Eq. (2.130). Equation (2.134) becomes

$$\begin{aligned}\mathcal{C}[F] &= - \int_0^{+\infty} d\tau \int d\mathbf{J}_2 d\mathbf{J}'_1 d\mathbf{J}'_2 \int_{\mathcal{B}} \frac{d\omega}{2\pi} \int_{\mathcal{B}'} \frac{d\omega'}{2\pi} e^{-i(\omega+\omega')\tau} \mu(2\pi)^d \\ &\quad \times \frac{\partial}{\partial \mathbf{J}_1} \cdot \left[ \sum_{\mathbf{m}_1, \mathbf{m}_2} \sum_{\mathbf{m}'_1, \mathbf{m}'_2} \frac{1}{\omega - \omega_1} \frac{1}{\omega' - \omega_2} \mathbf{m}_1 A_{-\mathbf{m}_1, \mathbf{m}_2} \right. \\ &\quad \times \left[ \delta_{\mathbf{m}'_1}^{m'_1} \delta_D(\mathbf{J}_1 - \mathbf{J}'_1) + (2\pi)^d \frac{\mathbf{m}_1 \cdot \partial F / \partial \mathbf{J}_1}{(\omega - \omega'_1) \mathcal{D}_{\mathbf{m}_1, \mathbf{m}'_1}(\omega)} \right] \left[ \delta_{\mathbf{m}'_2}^{m'_2} \delta_D(\mathbf{J}_2 - \mathbf{J}'_2) + (2\pi)^d \frac{\mathbf{m}_2 \cdot \partial F / \partial \mathbf{J}_2}{(\omega' - \omega'_2) \mathcal{D}_{\mathbf{m}_2, \mathbf{m}'_2}(\omega')} \right] \\ &\quad \left. \times \left[ A_{\mathbf{m}'_1, -\mathbf{m}'_2} \mathbf{m}'_1 \cdot \frac{\partial F}{\partial \mathbf{J}'_1} F(\mathbf{J}'_2) + A_{\mathbf{m}'_2, -\mathbf{m}'_1} \mathbf{m}'_2 \cdot \frac{\partial F}{\partial \mathbf{J}'_2} F(\mathbf{J}'_1) \right] \right],\end{aligned}\quad (2.135)$$

where we used the shortened notations  $1/\mathcal{D}_{\mathbf{m}_1, \mathbf{m}'_1}(\omega) = 1/\mathcal{D}_{\mathbf{m}_1, \mathbf{m}'_1}(\mathbf{J}_1, \mathbf{J}'_1, \omega)$ , as well as  $\omega_{1/2} = \mathbf{m}_{1/2} \cdot \boldsymbol{\Omega}_{1/2}$  and  $\omega'_{1/2} = \mathbf{m}'_{1/2} \cdot \boldsymbol{\Omega}'_{1/2}$ . The next step of the calculation is to deal with the integration and sum w.r.t.  $\mathbf{J}_2$  and  $\mathbf{m}_2$ . One can write

$$\sum_{m_2'} \int d\mathbf{J}_2 \frac{A_{-m_1, m_2}}{\omega' - \omega_2} \left[ \delta_{m_2'}^{m_2'} \delta_D(\mathbf{J}_2 - \mathbf{J}_2') + (2\pi)^d \frac{\mathbf{m}_2 \cdot \partial F / \partial \mathbf{J}_2}{(\omega' - \omega_2') \mathcal{D}_{m_2, m_2'}(\omega')} \right] = -\frac{1}{\omega' - \omega_2'} \frac{1}{\mathcal{D}_{-m_1, m_2'}(\omega')}, \quad (2.136)$$

where we relied on the intrinsic definition of the dressed susceptibility coefficients  $1/\mathcal{D}_{m_1, m_2}$  given by

$$\frac{1}{\mathcal{D}_{m_1, m_2}(\mathbf{J}_1, \mathbf{J}_2, \omega)} = -A_{m_1, m_2}(\mathbf{J}_1, \mathbf{J}_2) - (2\pi)^d \sum_{m_3} \int d\mathbf{J}_3 \frac{\mathbf{m}_3 \cdot \partial F / \partial \mathbf{J}_3}{\omega - \mathbf{m}_3 \cdot \boldsymbol{\Omega}_3} \frac{A_{m_1, m_3}(\mathbf{J}_1, \mathbf{J}_3)}{\mathcal{D}_{m_3, m_2}(\mathbf{J}_3, \mathbf{J}_1, \omega)}. \quad (2.137)$$

Equation (2.137) is straightforward to obtain thanks to the basis decompositions of the susceptibility coefficients from Eqs. (2.50) and (2.124), and the definition of the response matrix from Eq. (2.17). Equation (2.135) becomes

$$\begin{aligned} \mathcal{C}[F] &= \int_0^{+\infty} d\tau \int d\mathbf{J}_1' d\mathbf{J}_2' \int_B \frac{d\omega}{2\pi} \int_{B'} \frac{d\omega'}{2\pi} e^{-i(\omega + \omega')\tau} \mu (2\pi)^d \\ &\times \frac{\partial}{\partial \mathbf{J}_1} \cdot \left[ \sum_{m_1} \sum_{m_1', m_2'} \frac{1}{\omega - \omega_1} \frac{1}{\omega' - \omega_2'} \mathbf{m}_1 \frac{1}{\mathcal{D}_{-m_1, m_2'}(\omega')} \right. \\ &\times \left[ \delta_{m_1'}^{m_1'} \delta_D(\mathbf{J}_1 - \mathbf{J}_1') + (2\pi)^d \frac{\mathbf{m}_1 \cdot \partial F / \partial \mathbf{J}_1}{(\omega - \omega_1') \mathcal{D}_{m_1, m_1'}(\omega)} \right] \\ &\times \left. \left[ A_{m_1', -m_2'} \mathbf{m}_1' \cdot \frac{\partial F}{\partial \mathbf{J}_1'} F(\mathbf{J}_2') + A_{m_2', -m_1'} \mathbf{m}_2' \cdot \frac{\partial F}{\partial \mathbf{J}_2'} F(\mathbf{J}_1') \right] \right]. \quad (2.138) \end{aligned}$$

The next step of the calculation is to perform the integration and the sum w.r.t.  $\mathbf{J}_1'$  and  $\mathbf{m}_1'$ . These only act on the two last lines of Eq. (2.138). As previously, to perform this calculation, we rely on the intrinsic definition of the dressed susceptibility coefficients from Eq. (2.137). One has to deal with two distinct contributions: the first one  $\mathcal{C}_1[F]$  associated with the term in  $\mathbf{m}_1' \cdot \partial F / \partial \mathbf{J}_1' F(\mathbf{J}_2')$  and the second one  $\mathcal{C}_2[F]$  associated with the term in  $\mathbf{m}_2' \cdot \partial F / \partial \mathbf{J}_2' F(\mathbf{J}_1')$ . The first contribution  $\mathcal{C}_1[F]$  takes the form

$$\begin{aligned} \mathcal{C}_1[F] &= \sum_{m_1'} \int d\mathbf{J}_1' \left[ \delta_{m_1'}^{m_1'} \delta_D(\mathbf{J}_1 - \mathbf{J}_1') + (2\pi)^d \frac{\mathbf{m}_1 \cdot \partial F / \partial \mathbf{J}_1}{(\omega - \omega_1') \mathcal{D}_{m_1, m_1'}(\omega)} \right] A_{m_1', -m_2'} \mathbf{m}_1' \cdot \frac{\partial F}{\partial \mathbf{J}_1'} F(\mathbf{J}_2') \\ &= -\frac{1}{\mathcal{D}_{m_1, -m_2'}(\omega)} \mathbf{m}_1 \cdot \frac{\partial F}{\partial \mathbf{J}_1} F(\mathbf{J}_2'). \quad (2.139) \end{aligned}$$

The second contribution  $\mathcal{C}_2[F]$  takes the form

$$\begin{aligned} \mathcal{C}_2[F] &= \sum_{m_1'} \int d\mathbf{J}_1' \left[ \delta_{m_1'}^{m_1'} \delta_D(\mathbf{J}_1 - \mathbf{J}_1') + (2\pi)^d \frac{\mathbf{m}_1 \cdot \partial F / \partial \mathbf{J}_1}{(\omega - \omega_1') \mathcal{D}_{m_1, m_1'}(\omega)} \right] A_{m_2', -m_1'} \mathbf{m}_2' \cdot \frac{\partial F}{\partial \mathbf{J}_2'} F(\mathbf{J}_1') \\ &= A_{m_2', -m_1'} \mathbf{m}_2' \cdot \frac{\partial F}{\partial \mathbf{J}_2'} F(\mathbf{J}_1) + \mathbf{m}_1 \cdot \frac{\partial F}{\partial \mathbf{J}_1} \mathbf{m}_2' \cdot \frac{\partial F}{\partial \mathbf{J}_2'} (2\pi)^d \sum_{m_1'} \int d\mathbf{J}_1' \frac{F(\mathbf{J}_1') A_{m_2', -m_1'}}{(\omega - \omega_1') \mathcal{D}_{m_1, m_1'}(\omega)}. \quad (2.140) \end{aligned}$$



Let us now rewrite Eq. (2.138) by relying on the matrix method, i.e. by using the basis elements  $\psi^{(p)}$ . The bare and dressed susceptibility coefficients take the form

$$A_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2) = -\psi_{\mathbf{m}_1}^{(\alpha)}(\mathbf{J}_1) \psi_{\mathbf{m}_2}^{(\alpha)*}(\mathbf{J}_2) ; \quad \frac{1}{\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2, \omega)} = \psi_{\mathbf{m}_1}^{(\alpha)}(\mathbf{J}_1) \varepsilon_{\alpha\beta}^{-1}(\omega) \psi_{\mathbf{m}_2}^{(\beta)*}(\mathbf{J}_2), \quad (2.141)$$

where we introduced the matrix  $\varepsilon(\omega) = \mathbf{I} - \widehat{\mathbf{M}}(\omega)$ , with  $\widehat{\mathbf{M}}(\omega)$  the response matrix from Eq. (2.17) and  $\mathbf{I}$  the identity matrix. In Eq. (2.141) and the following, all the sums over the greek indices are implied. Let us finally define the matrix  $\mathbf{H}(\omega)$  as

$$H_{\alpha\beta}(\omega) = (2\pi)^d \sum_{\mathbf{m}} \int d\mathbf{J} \frac{F(\mathbf{J})}{\omega - \mathbf{m} \cdot \boldsymbol{\Omega}} \psi_{\mathbf{m}}^{(\alpha)*}(\mathbf{J}) \psi_{-\mathbf{m}}^{(\beta)*}(\mathbf{J}). \quad (2.142)$$

Gathering the two contributions from Eqs. (2.139) and (2.140), and after some straightforward calculations, one can rewrite Eq. (2.138) as

$$\begin{aligned} \mathcal{C}[F] = & -\int_0^{+\infty} d\tau \int_{\mathcal{B}} \frac{d\omega}{2\pi} \int_{\mathcal{B}'} \frac{d\omega'}{2\pi} e^{-i(\omega+\omega')\tau} \mu \frac{\partial}{\partial \mathbf{J}_1} \cdot \left[ \sum_{\mathbf{m}_1} \frac{1}{\omega - \omega_1} \mathbf{m}_1 \right. \\ & \times \left\{ \psi_{-\mathbf{m}_1}^{(\alpha)}(\mathbf{J}_1) \varepsilon_{\alpha\beta}^{-1}(\omega') H_{\beta\delta}(\omega') \varepsilon_{\gamma\delta}^{-1}(\omega) \psi_{\mathbf{m}_1}^{(\gamma)}(\mathbf{J}_1) \mathbf{m}_1 \cdot \frac{\partial F}{\partial \mathbf{J}_1} \right. \\ & + \psi_{-\mathbf{m}_1}^{(\alpha)}(\mathbf{J}_1) [\varepsilon_{\alpha\gamma}^{-1}(\omega') - \delta_{\alpha\gamma}] \psi_{-\mathbf{m}_1}^{(\gamma)*}(\mathbf{J}_1) F(\mathbf{J}_1) \\ & + \psi_{-\mathbf{m}_1}^{(\alpha)}(\mathbf{J}_1) \varepsilon_{\alpha\gamma}^{-1}(\omega') \varepsilon_{\delta\lambda}^{-1}(\omega) H_{\lambda\gamma}(\omega) \psi_{\mathbf{m}_1}^{(\delta)}(\mathbf{J}_1) \mathbf{m}_1 \cdot \frac{\partial F}{\partial \mathbf{J}_1} \\ & \left. \left. - \psi_{-\mathbf{m}_1}^{(\alpha)}(\mathbf{J}_1) \varepsilon_{\delta\lambda}^{-1}(\omega) H_{\lambda\alpha}(\omega) \psi_{\mathbf{m}_1}^{(\delta)}(\mathbf{J}_1) \mathbf{m}_1 \cdot \frac{\partial F}{\partial \mathbf{J}_1} \right\} \right]. \quad (2.143) \end{aligned}$$

At this stage, let us perform the integration w.r.t.  $\tau$  and  $\omega'$  in Eq. (2.143). It generically takes the form

$$\int_0^{+\infty} d\tau \int_{\mathcal{B}'} \frac{d\omega'}{2\pi} e^{-i(\omega+\omega')\tau} g(\omega, \omega'). \quad (2.144)$$

The integration over  $\tau$  is straightforward provided that  $\omega + \omega'$  has a negative imaginary part. Introducing  $p > 0$ , we perform the substitution  $\omega + \omega' \rightarrow \omega + \omega' - ip$ , and evaluate the integration over  $\tau$  as

$$(2.144) = \lim_{p \rightarrow 0} \int_{\mathcal{B}'} \frac{d\omega'}{2\pi} \frac{-i}{\omega + \omega' - ip} g(\omega, \omega'). \quad (2.145)$$

As the system is supposed to be stable, the poles of the function  $\omega' \mapsto g(\omega, \omega')$  are all in the lower-half complex plane, and the Bromwich contour  $\mathcal{B}'$  has to pass above all these singularities. The only pole in  $\omega'$  which remains is then  $\omega' = -\omega + ip$ , which is in the upper plane. The integration on  $\omega'$  is then carried out thanks to the residue theorem by closing the contour  $\mathcal{B}'$  in the upper half complex plane – this is possible

because the integrand decreases sufficiently fast at infinity like  $1/|\omega'|^2$ . One gets

$$(2.144) = \lim_{p \rightarrow 0} g(\omega, -\omega + ip). \quad (2.146)$$

Let us now consider the integration w.r.t.  $\omega$  in Eq. (2.143). First, one can note that the fourth term of Eq. (2.143) vanishes when integrated upon  $\omega$ . Indeed, by construction the Bromwich contour  $\mathcal{B}$  has to pass above all the singularities of the functions of  $+\omega$ . The contour  $\mathcal{B}$  can then be closed in the upper half complex plane and, because it surrounds no singularities, gives a vanishing result for this term. Equation (2.143), when rearranged, becomes

$$\begin{aligned} \mathcal{C}[F] = & \lim_{p \rightarrow 0} - \int_{\mathcal{B}} \frac{d\omega}{2\pi} \mu \frac{\partial}{\partial \mathbf{J}_1} \cdot \left[ \sum_{\mathbf{m}_1} \frac{1}{\omega - \omega_1} \mathbf{m}_1 \right. \\ & \times \left\{ \psi_{-\mathbf{m}_1}^{(\alpha)}(\mathbf{J}_1) [\varepsilon_{\alpha\gamma}^{-1}(-\omega + ip) - \delta_{\alpha\gamma}] \psi_{-\mathbf{m}_1}^{(\gamma)*}(\mathbf{J}_1) F(\mathbf{J}_1) \right. \\ & \left. \left. + \psi_{-\mathbf{m}_1}^{(\alpha)}(\mathbf{J}_1) \varepsilon_{\alpha\beta}^{-1}(-\omega + ip) \varepsilon_{\gamma\delta}^{-1}(\omega) \psi_{\mathbf{m}_1}^{(\gamma)}(\mathbf{J}_1) [H_{\beta\delta}(-\omega + ip) + H_{\delta\beta}(\omega)] \mathbf{m}_1 \cdot \frac{\partial F}{\partial \mathbf{J}_1} \right\} \right]. \end{aligned} \quad (2.147)$$

Let us now evaluate the term within brackets in the second term of Eq. (2.147). It reads

$$[H_{\beta\delta}(-\omega + ip) + H_{\delta\beta}(\omega)] = (2\pi)^d \sum_{\mathbf{m}_2} \int d\mathbf{J}_2 \psi_{\mathbf{m}_2}^{(\delta)*}(\mathbf{J}_2) \psi_{-\mathbf{m}_2}^{(\beta)*}(\mathbf{J}_2) F(\mathbf{J}_2) \left[ \frac{1}{\omega - \omega_2} - \frac{1}{\omega - (\omega_2 + ip)} \right], \quad (2.148)$$

where we used the notation  $\omega_2 = \mathbf{m}_2 \cdot \boldsymbol{\Omega}(\mathbf{J}_2)$ . As one takes the limit  $p \rightarrow 0$ , a naive reading of Eq. (2.148) would indicate that Eq. (2.148) vanishes. However, one should be careful with the two poles  $\omega = \omega_2$  and  $\omega = \omega_2 + ip$ , as these two poles are on opposite sides of the prescribed integration contour  $\mathcal{B}$ . Indeed, when lowering the integration  $\mathcal{B}$  to the real axis, the pole  $\omega = \omega_2$  remains below the contour, while the one in  $\omega = \omega_2 + ip$  is above it. Relying on Plemelj formula from Eq. (2.29), one can rewrite Eq. (2.148) as

$$\begin{aligned} [H_{\beta\delta}(-\omega + ip) + H_{\delta\beta}(\omega)] &= (2\pi)^d \sum_{\mathbf{m}_2} \int d\mathbf{J}_2 \psi_{\mathbf{m}_2}^{(\delta)*}(\mathbf{J}_2) \psi_{-\mathbf{m}_2}^{(\beta)*}(\mathbf{J}_2) F(\mathbf{J}_2) \left[ \frac{1}{\omega - \omega_2 + i0} - \frac{1}{\omega - \omega_2 - i0} \right] \\ &= -2\pi i (2\pi)^d \sum_{\mathbf{m}_2} \int d\mathbf{J}_2 \psi_{\mathbf{m}_2}^{(\delta)*}(\mathbf{J}_2) \psi_{-\mathbf{m}_2}^{(\beta)*}(\mathbf{J}_2) F(\mathbf{J}_2) \delta_{\mathbf{D}}(\omega - \omega_2). \end{aligned} \quad (2.149)$$

When lowering the contour  $\mathcal{B}$  to the real axis, one can also compute the integration w.r.t.  $\omega$  for the first term in Eq. (2.147). Because the system is stable, the poles of  $\varepsilon_{\alpha\gamma}^{-1}(-\omega + ip)$  are all located in the upper half plane and there remains only one pole on the real axis in  $\omega = \omega_1$ . The Bromwich contour  $\mathcal{B}$  is then closed in the lower half plane and only encloses this second pole. Paying attention to the direction of integration, the residue gives a factor  $-2i\pi$ , and Eq. (2.147) becomes

$$\begin{aligned}
\mathcal{C}[F] = i\mu \frac{\partial}{\partial J_1} \cdot & \left[ \sum_{\mathbf{m}_1} \mathbf{m}_1 \psi_{-\mathbf{m}_1}^{(\alpha)}(\mathbf{J}_1) [\varepsilon_{\alpha\gamma}^{-1}(-\omega_1 + i0) - \delta_{\alpha\gamma}] \psi_{-\mathbf{m}_1}^{(\gamma)*}(\mathbf{J}_1) F(\mathbf{J}_1) \right. \\
& + (2\pi)^d \sum_{\mathbf{m}_1, \mathbf{m}_2} \mathbf{m}_1 \int d\mathbf{J}_2 [\psi_{-\mathbf{m}_1}^{(\alpha)}(\mathbf{J}_1) \varepsilon_{\alpha\beta}^{-1}(-\omega_2) \psi_{-\mathbf{m}_2}^{(\beta)*}(\mathbf{J}_2)] \\
& \times [\psi_{\mathbf{m}_1}^{(\gamma)}(\mathbf{J}_1) \varepsilon_{\gamma\delta}^{-1}(\omega_2) \psi_{\mathbf{m}_2}^{(\delta)*}(\mathbf{J}_2)] \frac{\mathbf{m}_1 \cdot \partial F / \partial \mathbf{J}_1 F(\mathbf{J}_2)}{\omega_2 - \omega_1 + i0} \Big],
\end{aligned} \tag{2.150}$$

where one should pay attention to the small positive imaginary part in the pole  $1/(\omega_2 - \omega_1 + i0)$  associated with the fact that the contour  $\mathcal{B}$  passes above the pole  $\omega = \omega_1$ . Relying on the expression of the susceptibility coefficients from Eq. (2.141), one can rewrite Eq. (2.150) as

$$\begin{aligned}
\mathcal{C}[F] = i\mu \frac{\partial}{\partial J_1} \cdot & \left[ - \sum_{\mathbf{m}_1} \mathbf{m}_1 \left( \frac{1}{\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_1}(\mathbf{J}_1, \mathbf{J}_1, \omega_1 + i0)} + A_{\mathbf{m}_1, \mathbf{m}_1}(\mathbf{J}_1, \mathbf{J}_1) \right) F(\mathbf{J}_1) \right. \\
& + (2\pi)^d \sum_{\mathbf{m}_1, \mathbf{m}_2} \mathbf{m}_1 \int d\mathbf{J}_2 \frac{1}{\mathcal{D}_{-\mathbf{m}_1, -\mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2, -\omega_2)} \frac{1}{\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2, \omega_2)} \frac{\mathbf{m}_1 \cdot \partial F / \partial \mathbf{J}_1 F(\mathbf{J}_2)}{\omega_2 - \omega_1 + i0} \Big],
\end{aligned} \tag{2.151}$$

where we performed the change  $\mathbf{m}_1 \rightarrow -\mathbf{m}_1$  in the first term. Note that  $A_{\mathbf{m}_1, \mathbf{m}_1}(\mathbf{J}_1, \mathbf{J}_1)$  is real, thanks to Eq. (2.141). Let us now rely on the fact that the collision term  $\mathcal{C}[F]$  is real. As a consequence, because of the prefactor “ $i$ ” in Eq. (2.150), we may restrict ourselves only to the imaginary part of the terms within brackets. The first term requires us to study

$$\text{Im} \left[ \frac{1}{\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_1}(\mathbf{J}_1, \mathbf{J}_1, \omega_1 + i0)} \right] = \frac{1}{2i} \psi_{\mathbf{m}_1}^{(\alpha)}(\mathbf{J}_1) [\varepsilon_{\alpha\beta}^{-1}(\omega_1 + i0) - \varepsilon_{\beta\alpha}^{-1*}(\omega_1 + i0)] \psi_{\mathbf{m}_1}^{(\beta)*}(\mathbf{J}_1). \tag{2.152}$$

In order to compute the term within brackets, we rely on the identity (Heyvaerts 2010)

$$\varepsilon^{-1} - (\varepsilon^{-1})^\dagger = \varepsilon^{-1} (\varepsilon^\dagger - \varepsilon) (\varepsilon^\dagger)^{-1}. \tag{2.153}$$

The term within parenthesis in Eq. (2.153) can be evaluated and reads

$$\begin{aligned}
[\varepsilon^\dagger - \varepsilon]_{\gamma\delta}(\omega_1 + i0) &= -(2\pi)^d \sum_{\mathbf{m}_2} \int d\mathbf{J}_2 \mathbf{m}_2 \cdot \frac{\partial F}{\partial \mathbf{J}_2} \psi_{\mathbf{m}_2}^{(\gamma)*}(\mathbf{J}_2) \psi_{\mathbf{m}_2}^{(\delta)}(\mathbf{J}_2) \left[ \left( \frac{1}{\omega_1 - \omega_2 + i0} \right)^* - \frac{1}{\omega_1 - \omega_2 + i0} \right] \\
&= -2\pi i (2\pi)^d \sum_{\mathbf{m}_2} \int d\mathbf{J}_2 \delta_D(\omega_1 - \omega_2) \mathbf{m}_2 \cdot \frac{\partial F}{\partial \mathbf{J}_2} \psi_{\mathbf{m}_2}^{(\gamma)*}(\mathbf{J}_2) \psi_{\mathbf{m}_2}^{(\delta)}(\mathbf{J}_2).
\end{aligned} \tag{2.154}$$

Combining Eqs. (2.152) and (2.154), one finally gets the relation

$$\text{Im} \left[ \frac{1}{\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_1}(\mathbf{J}_1, \mathbf{J}_1, \omega_1 + i0)} \right] = -\pi (2\pi)^d \sum_{\mathbf{m}_2} \int d\mathbf{J}_2 \frac{\delta_D(\omega_1 - \omega_2)}{|\mathcal{D}_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2, \omega_1)|^2} \mathbf{m}_2 \cdot \frac{\partial F}{\partial \mathbf{J}_2}. \tag{2.155}$$

This contribution corresponds to the drift term in the Balescu-Lenard equation. To evaluate the second term in Eq. (2.151), we rely on the relation  $1/\mathcal{D}_{-m_1, -m_2}(\mathbf{J}_1, \mathbf{J}_2, -\omega) = 1/\mathcal{D}_{m_1, m_2}^*(\mathbf{J}_1, \mathbf{J}_2, \omega)$  (see note [83] in Chavanis (2012b)). Thanks to Plemelj formula, it immediately gives the contribution

$$\begin{aligned} & \text{Im} \left[ (2\pi)^d \sum_{m_1, m_2} m_1 \int d\mathbf{J}_2 \frac{1}{\mathcal{D}_{-m_1, -m_2}(\mathbf{J}_1, \mathbf{J}_2, -\omega_2)} \frac{1}{\mathcal{D}_{m_1, m_2}(\mathbf{J}_1, \mathbf{J}_2, \omega_2)} \frac{m_1 \cdot \partial F / \partial \mathbf{J}_1 F(\mathbf{J}_2)}{\omega_2 - \omega_1 + i0} \right] \\ &= -\pi(2\pi)^d \sum_{m_1, m_2} m_1 \int d\mathbf{J}_2 \frac{\delta_D(\omega_1 - \omega_2)}{|\mathcal{D}_{m_1, m_2}(\mathbf{J}_1, \mathbf{J}_2, \omega_1)|^2} m_1 \cdot \frac{\partial F}{\partial \mathbf{J}_1} F(\mathbf{J}_2). \end{aligned} \quad (2.156)$$

This contribution corresponds to the diffusion term in the Balescu-Lenard equation. Gathering the two contributions from Eqs. (2.155) and (2.156), and paying a careful to the signs of the various terms, one gets the final expression of the collision term  $\mathcal{C}[F]$  as

$$\mathcal{C}[F] = \pi(2\pi)^d \mu \frac{\partial}{\partial \mathbf{J}_1} \cdot \left[ \sum_{m_1, m_2} m_1 \frac{\delta_D(m_1 \cdot \boldsymbol{\Omega}_1 - m_2 \cdot \boldsymbol{\Omega}_2)}{|\mathcal{D}_{m_1, m_2}(\mathbf{J}_1, \mathbf{J}_2, m_1 \cdot \boldsymbol{\Omega}_1)|^2} \left( m_1 \cdot \frac{\partial}{\partial \mathbf{J}_1} - m_2 \cdot \frac{\partial}{\partial \mathbf{J}_2} \right) F(\mathbf{J}_1) F(\mathbf{J}_2) \right]. \quad (2.157)$$

This allows us to recover the inhomogeneous Balescu-Lenard equation (2.67).

## C Functional Approach to the Landau Equation

The previous sections presented two complementary derivations of the Balescu-Lenard equation, respectively based on the Klimontovich equation and the BBGKY hierarchy. In this Appendix, let us present an alternative approach based on a functional integral rewriting of the dynamics. In a little-known seven-page paper, Jolicoeur and Le Guillou (1989) presented how the general functional integral framework (Faddeev and Slavnov 1993) was suited for the study of classical kinetic theory. Using this formalism and starting from Liouville's equation, they recovered the BBGKY hierarchy. More importantly, they illustrated how this approach allows for a simple derivation of the homogeneous Balescu-Lenard equation (Balescu 1960; Lenard 1960) of plasma physics. In the context of inhomogeneous systems, we presented in Fouvry et al. (2016a) how this same functional approach may be used to recover the inhomogeneous Landau equation (2.73). Relying on the analogy between self-gravitating systems and 2D systems of point vortices (Chavanis 2002), a similar derivation in the context of 2D hydrodynamics was also presented in Fouvry et al. (2016b). In order to offer some new insights on the content of the collisional kinetic equations (2.67) and (2.73), we will now present this alternative derivation in the context of inhomogeneous systems.

### C.1 Functional Integral Formalism

As previously, let us consider a system made of  $N$  identical particles. At order  $1/N$ , its dynamics is fully described by the two first truncated equations of the BBGKY hierarchy (2.107) and (2.108), which involve the system's 1-body DF  $F$  and the 2-body autocorrelation  $\mathcal{C}$ . The first step of the present derivation is to rewrite these two coupled evolution equations under a functional form. As an illustration of this method, let us consider a dynamical quantity  $f$  depending on time  $t$  and defined on a phase space  $\Gamma$ . We assume that this quantity follows an evolution equation of the form  $[\partial_t + L]f = 0$ , where  $L$  is a differential operator. Let us now introduce an auxiliary field  $\lambda$  defined on the same space than  $f$  to rewrite the evolution constraint of  $f$  as a functional integral of the form (see Jolicœur and Le Guillou (1989); Fouvry et al. (2016a) for more details)

$$1 = \int \mathcal{D}f \mathcal{D}\lambda \exp \left[ i \int dt d\Gamma \lambda [\partial_t + L]f \right]. \quad (2.158)$$

In Eq. (2.158), we define the action  $S[F, \lambda] = i \int dt d\Gamma \lambda [\partial_t + L]f$  as the argument of the exponential.<sup>6</sup> It is important to note that the evolution equation satisfied by  $f$  corresponds to the quantity by which the auxiliary field  $\lambda$  is multiplied in the action.

When considering the two coupled evolution equations (2.107) and (2.108), one may proceed to a similar transformation. Let us define the phase space coordinates as  $\Gamma = (\mathbf{x}, \mathbf{v})$ . By introducing two auxiliary fields  $\lambda_1(t, \Gamma_1)$  and  $\lambda_2(t, \Gamma_1, \Gamma_2)$ , respectively associated with  $F$  and  $\mathcal{C}$ , Eqs. (2.107) and (2.108) can be rewritten under the compact functional form

$$1 = \int \mathcal{D}F \mathcal{D}\mathcal{C} \mathcal{D}\lambda_1 \mathcal{D}\lambda_2 \exp \left\{ i \int dt d\Gamma_1 \lambda_1 (A_1 F + B_1 \mathcal{C}) + \frac{i}{2} \int dt d\Gamma_1 d\Gamma_2 \lambda_2 (A_2 \mathcal{C} + D_2 \mathcal{C} + S_2) \right\}. \quad (2.159)$$

In Eq. (2.159), we introduced the operators  $A_1$ ,  $B_1$ ,  $A_2$ ,  $D_2$ , and  $S_2$  as

$$\begin{aligned} A_1 F &= \left[ \frac{\partial}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{x}_1} + \left[ \int d\Gamma_2 \mathcal{F}_{12} F(\Gamma_2) \right] \cdot \frac{\partial}{\partial \mathbf{v}_1} \right], \\ B_1 \mathcal{C} &= \mu \int d\Gamma_2 \mathcal{F}_{12} \cdot \frac{\partial \mathcal{C}(\Gamma_1, \Gamma_2)}{\partial \mathbf{v}_1}, \\ A_2 \mathcal{C} &= \left[ \frac{\partial}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{x}_1} + \mathbf{v}_2 \cdot \frac{\partial}{\partial \mathbf{x}_2} + \int d\Gamma_3 F(\Gamma_3) \left[ \mathcal{F}_{13} \cdot \frac{\partial}{\partial \mathbf{v}_1} + \mathcal{F}_{23} \cdot \frac{\partial}{\partial \mathbf{v}_2} \right] \right] \mathcal{C}(\Gamma_1, \Gamma_2), \\ D_2 \mathcal{C} &= \left[ \int d\Gamma_3 \mathcal{F}_{13} \mathcal{C}(\Gamma_2, \Gamma_3) \right] \cdot \frac{\partial F}{\partial \mathbf{v}_1} + (1 \leftrightarrow 2), \\ S_2 &= F(\Gamma_2) \mathcal{F}_{12} \cdot \frac{\partial F}{\partial \mathbf{v}_1} + (1 \leftrightarrow 2). \end{aligned} \quad (2.160)$$

<sup>6</sup>This should not be mixed up with the angle-action coordinates  $(\boldsymbol{\theta}, \mathbf{J})$  from inhomogeneous dynamics.

In Eq. (2.159), we did not write explicitly the dependence w.r.t.  $t$  to simplify the notations. In the expression of  $B_1\mathcal{C}$ , let us emphasise the presence of the small factor  $\mu = M_{\text{tot}}/N$ , which illustrates the fact that we relied on a kinetic development at order  $1/N$ . Finally, the prefactor  $1/2$  in Eq. (2.159) was only added for later convenience and does not play any role for the final expression of the evolution equation, since it was added as a global prefactor. Let us recall here the physical content of the various terms appearing in Eq. (2.159). Here,  $A_1 F$  corresponds to the 1–body Vlasov advection term, and  $B_1\mathcal{C}$  to the  $1/N$  sourcing of the evolution of the 1–body DF under the effect of the 2–body autocorrelation. Similarly,  $A_2\mathcal{C}$  encompasses the usual 2–body Vlasov advection term,  $D_2\mathcal{C}$  corresponds to the dressing of particles by collective effects, and  $S_2$  is a source term depending only on  $F$ , which sources the dynamics of  $\mathcal{C}$ .

Relying on basic manipulation, one can rewrite Eq. (2.159) as

$$1 = \int \mathcal{D}F \mathcal{D}\mathcal{C} \mathcal{D}\lambda_1 \mathcal{D}\lambda_2 \exp \left\{ i \int dt d\Gamma_1 \lambda_1(\Gamma_1) A_1 F(\Gamma_1) + \frac{i}{2} \int dt d\Gamma_1 d\Gamma_2 \lambda_2(\Gamma_1, \Gamma_2) G(\Gamma_1, \Gamma_2) - \frac{i}{2} \int dt d\Gamma_1 d\Gamma_2 \mathcal{C}(\Gamma_1, \Gamma_2) E(\Gamma_1, \Gamma_2) \right\}, \quad (2.161)$$

where it is crucial to note that all the dependences w.r.t.  $\mathcal{C}$  were gathered in the prefactor of the second line. In Eq. (2.161), we introduced the quantity  $G(\Gamma_1, \Gamma_2)$  as

$$G(\Gamma_1, \Gamma_2) = \mathcal{F}_{12} \cdot \left[ F(\Gamma_2) \frac{\partial F}{\partial \mathbf{v}_1} - F(\Gamma_1) \frac{\partial F}{\partial \mathbf{v}_2} \right], \quad (2.162)$$

for which we used the relation  $\mathcal{F}_{21} = -\mathcal{F}_{12}$ . In Eq. (2.161), we also introduced the quantity  $E(\Gamma_1, \Gamma_2)$  as

$$E(\Gamma_1, \Gamma_2) = A_2 \lambda_2(\Gamma_1, \Gamma_2) + \int d\Gamma_3 \left[ \mathcal{F}_{13} \lambda_2(\Gamma_2, \Gamma_3) + \mathcal{F}_{23} \lambda_2(\Gamma_1, \Gamma_3) \right] \cdot \frac{\partial F}{\partial \mathbf{v}_3} + \mu \mathcal{F}_{12} \cdot \left[ \frac{\partial \lambda_1}{\partial \mathbf{v}_1} - \frac{\partial \lambda_1}{\partial \mathbf{v}_2} \right]. \quad (2.163)$$

Equation (2.163) was obtained thanks to an integration by parts. In order to invert the time derivative  $\partial \mathcal{C} / \partial t$  present in the term  $\lambda_2 A_2 \mathcal{C}$  from Eq. (2.159), we assumed  $t \in [0; T]$ , where  $T$  is an arbitrary upper temporal bound, along with the boundary conditions  $\mathcal{C}(t=0)=0$  (the system is supposed to be initially uncorrelated) and  $\lambda_2(T)=0$  (we are free to impose a condition on  $\lambda_2$ ). As presented in Fouvry et al. (2016a), let us now neglect collective effects. This amounts to neglecting the contributions associated with the term  $D_2\mathcal{C}$  in Eq. (2.159), so that Eq. (2.163) becomes

$$E(\Gamma_1, \Gamma_2) = A_2 \lambda_2(\Gamma_1, \Gamma_2) + \mu \mathcal{F}_{12} \cdot \left[ \frac{\partial \lambda_1}{\partial \mathbf{v}_1} - \frac{\partial \lambda_1}{\partial \mathbf{v}_2} \right]. \quad (2.164)$$

In order to obtain a closed kinetic equation involving  $F$  only, the traditional approach would be to start from Eq. (2.159) and proceed in the following way. By functionally integrating Eq. (2.159) w.r.t.  $\lambda_2$ , one gets a constraint of the form  $A_2\mathcal{C} + D_2\mathcal{C} + S_2 = 0$ ,

which couples  $F$  and  $\mathcal{C}$ . This constraint must then be inverted to give  $\mathcal{C}=\mathcal{C}[F]$ . One then uses this substitution in Eq. (2.159), and functionally integrates this equation w.r.t.  $\lambda_1$ , to obtain a kinetic equation involving  $F$  only. This gives the Balescu-Lenard equation (or the Landau equation when collective effects are not accounted for). This approach is identical to the direct resolution of the BBGKY hierarchy presented in Appendix 2.B.

However, based on the rewriting from Eq. (2.161), Jolicoeur and Le Guillou (1989) suggested a different strategy. One may indeed first integrate functionally Eq. (2.161) w.r.t.  $\mathcal{C}$ , to obtain a constraint of the form  $E[F, \lambda_1, \lambda_2]=0$ . Once inverted, this offers a relation of the form  $\lambda_2=\lambda_2[F, \lambda_1]$ . One may then substitute this relation in Eq. (2.161), to obtain a functional equation which only involves  $F$  and  $\lambda_1$ . The final step is then to functionally integrate this equation w.r.t.  $\lambda_1$ , to obtain a closed kinetic equation involving  $F$  only. Let us now show how this alternative approach allows for the derivation of the inhomogeneous Landau equation.

## C.2 Application to Inhomogeneous Systems

As in Sect. 2.3, let us assume that the system's mean potential is integrable, so that one may always remap the physical phase space coordinates  $(\mathbf{x}, \mathbf{v})$  to the angle-action ones  $(\boldsymbol{\theta}, \mathbf{J})$ . Relying on the adiabatic approximation (Heyvaerts 2010; Chavanis 2012b, 2013b), we assume that the 1-body DF is a quasi-stationary solution of the Vlasov equation, so that  $F(\boldsymbol{\theta}, \mathbf{J})=F(\mathbf{J})$ , where the dependence w.r.t.  $t$  has not been written out to shorten the notations. Since  $\lambda_1$  is the auxiliary field associated with  $F$ , one also has  $\lambda_1(\boldsymbol{\theta}, \mathbf{J})=\lambda_1(\mathbf{J})$ , while the second auxiliary field  $\lambda_2(\boldsymbol{\theta}_1, \mathbf{J}_1, \boldsymbol{\theta}_2, \mathbf{J}_2)$  still fully depends on all angle-action coordinates. Let us note that for homogeneous systems, the system's invariance by translation would impose  $\lambda_2(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2)=\lambda_2(\mathbf{x}_1-\mathbf{x}_2, \mathbf{v}_1, \mathbf{v}_2)$ . Relying on the angle-action properties from Eqs. (2.118) and (2.119), one may now rewrite the various operators appearing in Eq. (2.161). Equation (2.160) gives

$$A_1 F = \frac{\partial F}{\partial t}. \quad (2.165)$$

Similarly, Eq. (2.162) can be rewritten as

$$G(\Gamma_1, \Gamma_2) = - \left[ F(\mathbf{J}_2) \frac{\partial U_{12}}{\partial \boldsymbol{\theta}_1} \cdot \frac{\partial F}{\partial \mathbf{J}_1} + F(\mathbf{J}_1) \frac{\partial U_{21}}{\partial \boldsymbol{\theta}_2} \cdot \frac{\partial F}{\partial \mathbf{J}_2} \right]. \quad (2.166)$$

Finally, the constraint  $E(\Gamma_1, \Gamma_2)=0$  from Eq. (2.164) takes the form

$$\frac{\partial \lambda_2}{\partial t} + \boldsymbol{\Omega}_1 \cdot \frac{\partial \lambda_2}{\partial \boldsymbol{\theta}_1} + \boldsymbol{\Omega}_2 \cdot \frac{\partial \lambda_2}{\partial \boldsymbol{\theta}_2} - \mu \left[ \frac{\partial U_{12}}{\partial \boldsymbol{\theta}_1} \cdot \frac{\partial \lambda_1}{\partial \mathbf{J}_1} + \frac{\partial U_{21}}{\partial \boldsymbol{\theta}_2} \cdot \frac{\partial \lambda_1}{\partial \mathbf{J}_2} \right] = 0. \quad (2.167)$$

### C.3 Inverting the Constraint

In order to invert Eq. (2.167), we once again rely on Bogoliubov's ansatz by assuming that the fluctuations (such as  $\mathcal{C}$  and  $\lambda_2$ ) evolve much faster than the mean dynamical orbit-averaged quantities (such as  $F$  and  $\lambda_1$ ). As a consequence, on the timescale on which  $\lambda_2$  evolves, one can assume  $F$  and  $\lambda_1$  to be frozen, while on the timescale of secular evolution, one can assume  $\lambda_2$  to be equal to the asymptotic value associated with the current value of  $F$  and  $\lambda_1$ . As defined in Eq. (2.6), let us perform a Fourier transform w.r.t. the angles  $\theta$ . We decompose the interaction potential  $U_{12}$  as

$$U_{12} = U(\mathbf{x}(\theta_1, \mathbf{J}_1) - \mathbf{x}(\theta_2, \mathbf{J}_2)) = \sum_{\mathbf{m}_1, \mathbf{m}_2} A_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2) e^{i(\mathbf{m}_1 \cdot \theta_1 - \mathbf{m}_2 \cdot \theta_2)}, \quad (2.168)$$

where the bare susceptibility coefficients  $A_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2)$  were already introduced in Eq. (2.74). Multiplying Eq. (2.167) by  $1/(2\pi)^{2d} e^{i(\mathbf{m}_1 \cdot \theta_1 - \mathbf{m}_2 \cdot \theta_2)}$  and integrating it w.r.t.  $\theta_1$  and  $\theta_2$ , we obtain

$$\frac{\partial \lambda_{-\mathbf{m}_1, \mathbf{m}_2}}{\partial t} - i\Delta\omega \lambda_{-\mathbf{m}_1, \mathbf{m}_2} = -i\mu A_{\mathbf{m}_1, \mathbf{m}_2}^* \left[ \mathbf{m}_1 \cdot \frac{\partial \lambda_1}{\partial \mathbf{J}_1} - \mathbf{m}_2 \cdot \frac{\partial \lambda_1}{\partial \mathbf{J}_2} \right], \quad (2.169)$$

where we used the shortening notations  $\lambda_{-\mathbf{m}_1, \mathbf{m}_2} = \lambda_{-\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2)$ ,  $A_{\mathbf{m}_1, \mathbf{m}_2} = A_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2)$ , and  $\Delta\omega = \mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - \mathbf{m}_2 \cdot \boldsymbol{\Omega}_2$ . Thanks to the boundary condition  $\lambda_2(T) = 0$  introduced in Eq. (2.163), and relying on the adiabatic approximation that  $\lambda_1$  is frozen, one can straightforwardly solve the differential equation (2.169) as

$$\lambda_{-\mathbf{m}_1, \mathbf{m}_2}(t) = \mu A_{\mathbf{m}_1, \mathbf{m}_2}^* \left[ \mathbf{m}_1 \cdot \frac{\partial \lambda_1}{\partial \mathbf{J}_1} - \mathbf{m}_2 \cdot \frac{\partial \lambda_1}{\partial \mathbf{J}_2} \right] \frac{1 - e^{i\Delta\omega(t-T)}}{\Delta\omega}. \quad (2.170)$$

In order to consider only the forced regime of evolution, let us now assume that the arbitrary temporal bound  $T$  is large compared to the considered time  $t$ . Therefore, we place ourselves in the limit  $T \rightarrow +\infty$ . Let us finally recall the formula

$$\lim_{T \rightarrow +\infty} \frac{e^{iT\Delta\omega} - 1}{\Delta\omega} = i\pi \delta_D(\Delta\omega), \quad (2.171)$$

so that Eq. (2.170) immediately gives

$$\lim_{T \rightarrow +\infty} \lambda_{-\mathbf{m}_1, \mathbf{m}_2}(t) = i\pi \mu A_{\mathbf{m}_1, \mathbf{m}_2}^* \left[ \mathbf{m}_1 \cdot \frac{\partial \lambda_1}{\partial \mathbf{J}_1} - \mathbf{m}_2 \cdot \frac{\partial \lambda_1}{\partial \mathbf{J}_2} \right] \delta_D(\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - \mathbf{m}_2 \cdot \boldsymbol{\Omega}_2). \quad (2.172)$$

Thanks to Bogoliubov's ansatz, we therefore inverted the constraint  $E[F, \lambda_1, \lambda_2] = 0$  from Eq. (2.167), to obtain  $\lambda_2 = \lambda_2[F, \lambda_1]$ .



### C.4 Recovering the Landau Collision Operator

Let us now substitute the inverted expression from Eq. (2.172) into the functional integral from Eq. (2.161), which then only involves  $F$  and  $\lambda_1$ . The remaining action term  $S[F, \lambda_1]$  reads

$$S[F, \lambda_1] = i \int dt d\Gamma_1 \lambda_1 A_1 F + \frac{i}{2} \int dt d\Gamma_1 d\Gamma_2 \lambda_2 [F, \lambda_1] G(\Gamma_1, \Gamma_2). \quad (2.173)$$

Thanks to the expressions of  $A_1$  and  $G$  from Eqs. (2.165) and (2.166), and using a Fourier transform in angles as in Eq. (2.6), one can rewrite Eq. (2.173) as

$$\begin{aligned} S[F, \lambda_1] = & i \int dt d\Gamma_1 \lambda_1(\Gamma_1) \frac{\partial F}{\partial t} \\ & + \frac{i}{2} \int dt d\Gamma_1 d\Gamma_2 \sum_{m_1, m_2} \text{Im} \left[ A_{m_1, m_2} \lambda_{-m_1, m_2} \right] \left[ m_1 \cdot \frac{\partial F}{\partial \mathbf{J}_1} F(\mathbf{J}_2) - m_2 \cdot \frac{\partial F}{\partial \mathbf{J}_2} F(\mathbf{J}_1) \right]. \end{aligned} \quad (2.174)$$

Thanks to the inversion from Eq. (2.172), one immediately has

$$\text{Im} \left[ A_{m_1, m_2} \lambda_{-m_1, m_2} \right] = \pi \mu \delta_D(m_1 \cdot \mathbf{\Omega}_1 - m_2 \cdot \mathbf{\Omega}_2) |A_{m_1, m_2}|^2 \left[ m_1 \cdot \frac{\partial \lambda_1}{\partial \mathbf{J}_1} - m_2 \cdot \frac{\partial \lambda_1}{\partial \mathbf{J}_2} \right]. \quad (2.175)$$

Injecting this result in Eq. (2.174), one gets

$$\begin{aligned} S[F, \lambda_1] = & i \int dt d\Gamma_1 \lambda_1 \frac{\partial F}{\partial t} + \frac{i}{2} \int dt d\Gamma_1 d\Gamma_2 \sum_{m_1, m_2} \pi \mu \delta_D(m_1 \cdot \mathbf{\Omega}_1 - m_2 \cdot \mathbf{\Omega}_2) |A_{m_1, m_2}|^2 \\ & \times \left[ m_1 \cdot \frac{\partial \lambda_1}{\partial \mathbf{J}_1} - m_2 \cdot \frac{\partial \lambda_1}{\partial \mathbf{J}_2} \right] \left[ m_1 \cdot \frac{\partial F}{\partial \mathbf{J}_1} F(\mathbf{J}_2) - m_2 \cdot \frac{\partial F}{\partial \mathbf{J}_2} F(\mathbf{J}_1) \right]. \end{aligned} \quad (2.176)$$

The final step of the calculation is to rewrite the second term of Eq. (2.176) under the form  $\int dt d\Gamma_1 \lambda_1(\Gamma_1) \dots$ . This is a straightforward calculation, which requires to use an integration by parts and to permute accordingly the indices  $1 \leftrightarrow 2$ . Equation (2.176) can finally be rewritten as

$$\begin{aligned} S[F, \lambda_1] = & i \int dt d\Gamma_1 \lambda_1(\Gamma_1) \left\{ \frac{\partial F}{\partial t} - \pi (2\pi)^d \mu \frac{\partial}{\partial \mathbf{J}_1} \cdot \left[ \sum_{m_1, m_2} m_1 \int d\mathbf{J}_2 \delta_D(m_1 \cdot \mathbf{\Omega}_1 - m_2 \cdot \mathbf{\Omega}_2) |A_{m_1, m_2}|^2 \right. \right. \\ & \left. \left. \times \left[ m_1 \cdot \frac{\partial F}{\partial \mathbf{J}_1} F(\mathbf{J}_2) - m_2 \cdot \frac{\partial F}{\partial \mathbf{J}_2} F(\mathbf{J}_1) \right] \right] \right\}, \end{aligned} \quad (2.177)$$

where the additional prefactor  $(2\pi)^d$  comes from the transformation  $\int d\Gamma_2 f(\mathbf{J}_2) = (2\pi)^d \int d\mathbf{J}_2 f(\mathbf{J}_2)$ . Integrating functionally Eq. (2.177) w.r.t.  $\lambda_1$ , one finally obtains a closed form expression for the kinetic equation as

$$\frac{\partial F}{\partial t} = \pi(2\pi)^d \mu \frac{\partial}{\partial \mathbf{J}_1} \cdot \left[ \sum_{\mathbf{m}_1, \mathbf{m}_2} \mathbf{m}_1 \int d\mathbf{J}_2 \delta_D(\mathbf{m}_1 \cdot \boldsymbol{\Omega}_1 - \mathbf{m}_2 \cdot \boldsymbol{\Omega}_2) |A_{\mathbf{m}_1, \mathbf{m}_2}(\mathbf{J}_1, \mathbf{J}_2)|^2 \right. \\ \left. \times \left( \mathbf{m}_1 \cdot \frac{\partial}{\partial \mathbf{J}_1} - \mathbf{m}_2 \cdot \frac{\partial}{\partial \mathbf{J}_2} \right) F(\mathbf{J}_1, t) F(\mathbf{J}_2, t) \right]. \quad (2.178)$$

As a conclusion, relying on a functional integral formalism, we were able to exactly recover the inhomogeneous Landau equation (2.73). Such a new calculation provides additional insights on the origin of these diffusion equations. A natural next step would be to show how it may be used to account for collective effects and recover the inhomogeneous Balescu-Lenard equation (2.67). Such a derivation is expected to be more involved, as one will have to deal with a self-consistent Fredholm equation associated with the polarisation dressing of the potential fluctuations (similar to the one obtained in Eq. (2.122)). As illustrated in the two derivations from Sect. 2.3 and Appendix 2.B, this requires to rely on Kalnajs matrix method (Kalnajs 1976) and to introduce potential-density basis elements. Jolicoeur and Le Guillou (1989) managed to develop such a self-consistent calculation in the homogeneous context of plasma physics, where both the resonance condition and the Fredholm equation are simpler. The generalisation of this method to inhomogeneous systems will be the subject of a future work. Finally, because of its alternative point of view, this approach may also turn out fruitful to tackle the question of obtaining closed kinetic equations when higher order correlation terms are taken into account.

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