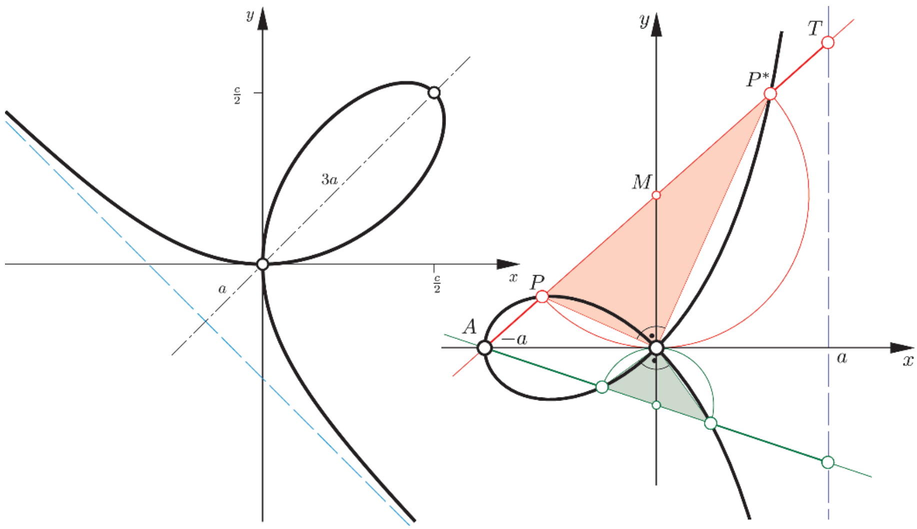
2 Equations, systems of equations



In this chapter, we examine elementary knowledge of mathematical relations from a higher point of view. Having understood the elementary relations thoroughly, we will conclude that many deceptively complex problems are grounded in simpler principles.

We will repeat the most elementary rules of algebra, such as the laws governing power functions, and the rules related to solving linear and quadratic equations.

Linear systems of equations will be particularly relevant to us, and may also be of interest to advanced readers, as they are not always trivial and often exhibit intricate interrelations with other branches of science, such as geography, physics, chemistry, photography, applied arts, and music. We will perform many calculations using real physical units. This will enable us to stay relevant to the solution of practical problems.

Due to the high computational complexity involved, we will only deal with higher-order algebraic equations conceptually. However, we will still find practical applications for their use.

At the end of the chapter, we will discuss further applications that cross over into fields of knowledge that are not exclusively concerned with mathematics.

G. Glaeser, *Math Tools*, https://doi.org/10.1007/978-3-319-66960-1_2

2.1 The fundamentals of numbers and equations

Calculations with floating point numbers and precision

In the following section, we will perform calculations almost exclusively with real numbers. In general, our pocket calculator shows such numbers as decimals. Most calculators perform their task with an accuracy of ≈13–15 digits. We have to keep in mind that the numbers we enter into our calculator might be subject to imprecision (due to estimation, imprecise measurement, or rounding). A definitely *correct digit* (except zero, if placed in order to fix the decimal point) will be called a *valid digit*. Thus, the number 0.000123 possesses *at most* three valid digits. We now have to consider the following theorem:

> After a multiplication or a division, the number of valid digits is equal to the minimum number of valid digits in all operands. The result of an addition or subtraction has no valid digits beyond the last decimal place where *both* operands possess valid digits. One might say that the chain is only as strong as its weakest link!

▸▸▸ **Application**: *average velocity*

Consider a simple equation in physics: $v = s/t$ (velocity = distance over time). A car drives a distance of 88 km in 1 h 6 min. What is the average velocity, and how accurately can it be calculated?

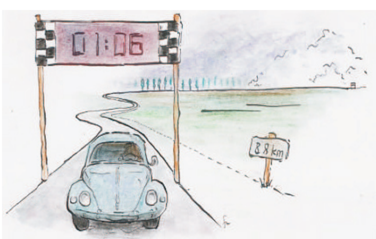


Fig. 2.1 average velocity

Solution:
1 h 6 min is equivalent to $1\,\mathrm{h} + \frac{6}{60}\,\mathrm{h} = 1.1\,\mathrm{h}$. The above equation yields

$$v = \frac{s}{t} = \frac{88\,\mathrm{km}}{1.1\,\mathrm{h}} = 80\frac{\mathrm{km}}{\mathrm{h}}.$$

The distance of the route travelled was taken from a street map, which would indicate 88 km, even if the actual distance is 87.500 km or 88.499 km. In a similar manner, most people might be satisfied with a time measurement that is exact with minute precision. Thus, the actual car might have taken 1 h 5 min 30 s or 1 h 6 min 29 s, and we still would have entered 1 h 6 min into our calculation. However, the time measurement fluctuates between

$$1\,\mathrm{h} + \frac{5}{60}\,\mathrm{h} + \frac{30}{3\,600}\,\mathrm{h} = 1.0917\mathrm{h} \text{ and } 1\,\mathrm{h} + \frac{6}{60}\,\mathrm{h} + \frac{29}{3\,600}\,\mathrm{h} = 1.108\mathrm{h}.$$

We can, therefore, conclude that the "accurate result" might be found anywhere between

$$v = \frac{88.499}{1.0917} = 81.065\,\text{km/h} \quad \text{and} \quad v = \frac{87.500}{1.108} = 78.971\,\text{km/h}.$$

Therefore, it is correct to say that many decimal places of the calculated average velocity are baseless and may lead to misleading and unjustified impressions of accuracy!

⊕ *Remark*: Physicists tend to perform calculations in units of measurement. The velocity – including that of a car – is usually given in m/s, and time in seconds. Since $x\frac{1\,\text{km}}{1\,\text{h}} = x\frac{1000\text{m}}{3600\,\text{s}}$, the following frequently employed relation is true:

$$x\,\frac{\text{km}}{\text{h}} = \frac{x}{3.6}\,\frac{\text{m}}{\text{s}} \quad \text{and} \quad y\,\frac{\text{m}}{\text{s}} = 3.6y\,\frac{\text{km}}{\text{h}}.$$

The following example demonstrates the risk of using units of measurement carelessly.
Consider the following – incorrect – chain of equations

$$1€ = 100\,\text{Cent} = 10\,\text{Cent} \cdot 10\,\text{Cent} = 0.1€ \cdot 0.1€ = 0.01€.$$

Can you find the mistake hidden therein? ⊕ ◂◂◂

▸▸▸ **Application**: ***moderate but constant***

Every driver ought to know that it is unwise to make up for lost time by "speeding". A moderate but constant velocity not only consumes less fuel (while also being less stressful), but is also more time-efficient than one might imagine. This provides an answer to the question "when to mount rain tires" in Formula 1 racing: Is it worth it to "put on" rain tires, which allow the vehicle to drive faster on wet surfaces, at the cost of a pit stop?
Let us consider a simple and very idealized example (Fig. 2.2): A race course of $s = 12\,\text{km}$ (where 6 km run across easy and 6 km across difficult terrain) is to be traversed by a runner (at a constant velocity of $12\,\frac{\text{km}}{\text{h}}$) and a mountain biker ($4\,\frac{\text{km}}{\text{h}}$ faster than the runner in simple terrain, and $4\,\frac{\text{km}}{\text{h}}$ slower than the runner in difficult terrain). Which participant takes less time to cross the finish line? How long must the trajectory across simple terrain be for both participants to finish at the same time?
Solution:
Time for the runner $t_1 = \frac{12\,\text{km}}{12\,\frac{\text{km}}{\text{h}}} = 1\,\text{h}$.
Time for the mountain biker $t_2 = \frac{6\,\text{km}}{(12{+}4)\,\frac{\text{km}}{\text{h}}} + \frac{6\,\text{km}}{(12{-}4)\,\frac{\text{km}}{\text{h}}} = \frac{9}{8}\,\text{h}$, or $7\frac{1}{2}$ minutes slower!
If the mountain biker is to cross the finish line at the same time as the runner, then the sum of the partial times for the different terrains must be 1 h:

$$\frac{x\,\text{km}}{16\frac{\text{km}}{\text{h}}} + \frac{(12-x)\,\text{km}}{8\frac{\text{km}}{\text{h}}} = 1\text{h} \Rightarrow x = 8\,\text{km}.$$

Fig. 2.2 the great race

It follows that the easier terrain should cover *two thirds* of the entire course.

⊕ *Remark*: The "accordion effect" is an interesting phenomenon to note at this point: If a line of vehicles travels on the fast lane at a speed of 160 km/h, it may only take a single car, "breaking out" at 130 km/h from behind a truck, to bring the faster vehicles to a full stop: The driver at the front has a certain reaction time, in which the velocity of 160 km/h may still be maintained. To avert a collision with the slower vehicle, the velocity needs to be reduced, until it is significantly lower than that of the obstructing vehicle. However, the same problem arises for the subsequent car in the fast lane. Sufficient distances between the cars help to avoid traffic jam scenarios of this kind. ⊕ ◂◂◂

▸▸▸ **Application**: ***overtaking calculations***

A car (speed v_1) drives past another slower moving car (speed v_2), travelling in the same direction. How much time does it take for the faster car to pass the slower one and how much distance must the faster car cover?

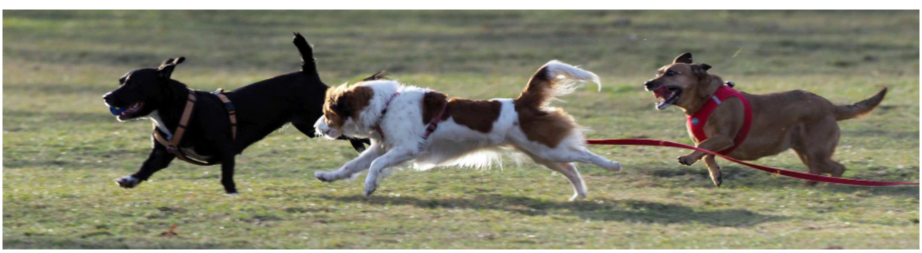


Fig. 2.3 overtaking with dog leash as tape measure

Solution:

The speed difference is $v_1 - v_2$. If we assume the length of the slower vehicle to be A meters and we should be B meters behind the vehicle before overtaking and cut in C meters after the vehicle, we have to pace across $\Delta s = A + B + C$ meters with a speed of $\Delta v = v_1 - v_2$. The time needed for that is $t = \Delta s / \Delta v$. Clearly, the process depends in an indirectly proportional way on the speed difference. From the beginning of the overtaking process to the end, the faster car must cover a distance of $s_{\text{total}} = v_1 \cdot t$.

Numerical example:

$v_1 = 108$ km/h (= 30 m/s), $v_2 = 90$ km/h (= 25 m/s), length of slower vehicle $A = 10$ m.

Good numbers for A and C might be 15 m. Then, we have $\Delta s = 40$ m, $t = 8$ s and $s_{\text{total}} = 30\text{m/s} \cdot 8\text{s} = 240$ m. ◂◂◂

▸▸▸ **Application**: ***making up ground***

Two joggers A and B run together at a speed of 3.5 m/s. Then, A decides to run back, whereas B intends to continue for another 300 m, and then, turn around. He, therefore, immediately speeds up to 4 m/s, and separates with the words "I will join you soon". When is "soon"?

Solution:

Jogger B gains 0.5 m every second. Thus, it will take him 1,200 seconds (20 minutes!) to join A again. ◂◂◂

Calculating with powers of ten

Fig. 2.4 orders of magnitude in the animal Kingdom I: The *lengths* of the animals are 4,000 mm, 400 mm, 40 mm, 4 mm.

The careful treatment of powers of ten is important whenever one uses mathematics, such as when estimating results or when correctly interpreting computer calculations.

Fig. 2.5 orders of magnitude in the animal Kingdom II: The *masses* of the animals are 10^9 mg, 10^6 mg, 10^3 mg, 1 mg.

If $n = 1, 2, \ldots$ is any natural number, then the number 10^n represents a 1 followed by n zeroes. When applying the rules for general powers that hold for any base, we obtain

$$\boxed{\begin{aligned} 10^0 &= 1, \quad & 10^1 &= 10, \\ 10^n \cdot 10^m &= 10^{n+m}, \quad & \frac{10^n}{10^m} &= 10^{n-m}, \\ 10^{-n} &= \frac{1}{10^n}, \quad & (10^n)^m &= 10^{nm}, \\ \sqrt[m]{10^n} &= 10^{\frac{n}{m}}. \end{aligned}}$$

The numbers n and m do not need to be natural and can each take on any real value.

▸▸▸ **Application**: ***car tire wear per kilometer*** (Fig. 2.6)
How strong is the wear of a car tire's outer surface if a brand new tire loses 1 cm in depth after being driven for about 50,000 kilometers?

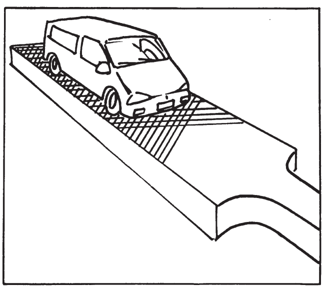


Fig. 2.6 tire wear ...

Fig. 2.7 ... with forward acting turbines

Solution:
If the outer surface wears down about $1\,\text{cm} = 10^{-2}\,\text{m}$ after $50{,}000\,\text{km} = 5 \cdot 10^4\,\text{km}$, then its wear per kilometer is

$$\frac{10^{-2}\text{m}}{5 \cdot 10^4} = \frac{10 \cdot 10^{-3}\text{m}}{5 \cdot 10^4} = 2 \cdot 10^{-7}\text{m} = 0.2 \cdot 10^{-6}\,\text{m} = 0.2\mu\text{m (micrometer)}.$$

⊕ *Remark*: A car tire has a diameter of approximately 65 cm (26 inches). Tread wear reduces the tire's diameter by a centimeter, and therefore, the circumference by approximately $2 \cdot 1.5\%$. The reduction of speed necessary to prevent being caught by speed cameras at $100\,\frac{\text{km}}{\text{h}}$ is at least $3\,\frac{\text{km}}{\text{h}}$. Tires for aircrafts wear out much faster and have to be replaced more often than those for other types of vehicles. Fig. 2.7 shows a landing on one of the shortest runways in the world (Madeira) and — due to the need to reverse thrust — the shimmering hot air emitted by the turbines. ⊕

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▸▸▸ **Application**: ***hard disk space***
Suppose a hard disk has 100 GB (gigabytes) of storage. How many typed

pages (75 characters per line, 40 lines per page), or alternatively how many color photos (at a resolution of $1,000 \times 1,000$ pixels) can be stored in an uncompressed form?

⊕ *Remark*: Note: 1 typewritten ASCII character takes up 1 byte of space. A color pixel occupies 3 bytes (red, green, and blue components at 1 byte each). 1 gigabyte is equivalent to 1,024 megabytes, 1 megabyte to 1,024 kilobytes, and 1 kilobyte to 1,024 bytes. Nevertheless, we can estimate 1 GB as being equivalent to roughly 10^9 bytes. Images are rarely stored in uncompressed bitmap format. Modern digital cameras allow taking photos at resolutions of $3,000 \times 2,000$ pixels or more. A 6 megapixel image would require 18 megabytes of space on the medium of storage. Instead, images tend to be stored in compressed form (JPG format) and can thus be fitted into about 2 megabytes – depending on the degree of compression and the type of the image. ⊕

Solution:
1 page $= 75 \cdot 40 = 3,000$ characters,
$\frac{100 \cdot 10^9}{3 \cdot 10^3} \approx 30 \cdot 10^6 = 30$ millions of pages,
$1,000 \times 1,000$ occupy $3 \cdot 10^6$ bytes $\Rightarrow \frac{100 \cdot 10^9}{3 \cdot 10^6} \approx 30 \cdot 10^3 = 30,000$ images. ◂◂◂

▸▸▸ **Application**: ***measuring distances through GPS***

Using GPS (Global Positioning System), it is possible to determine one's position at any point on Earth up to a few meters. We will discuss in Application p. 363 how these remarkably precise positions are calculated. In summary, it is necessary to take accurate measurements of one's distance to three satellites. These satellites continuously transmit characteristic signals that propagate at the speed of light. In order to determine the distance of the satellite, one must measure the time that it takes for the signal to reach the receiver, and multiply it by the speed of light. How accurately must such time measurements be taken so that the accuracy of position would fall within 1 m?

Solution:
The speed of light $300,000\,\frac{\mathrm{km}}{\mathrm{s}} = 3 \cdot 10^5\,\frac{\mathrm{km}}{\mathrm{s}} = 3 \cdot 10^8\,\frac{\mathrm{m}}{\mathrm{s}}$ implies that the time for the signal to cross a meter is $\frac{1}{3} \cdot 10^{-8}\,\mathrm{s} \approx 3.3 \cdot 10^{-9}\,\mathrm{s}$.
One should be able to measure time to an accuracy of a nanosecond. Even in a microsecond, the GPS signal will already have travelled a distance of 300 m.

⊕ *Remark*: Even atomic clocks (which are used on satellites) can "only" measure microseconds reliably (the magnitude of error of such a clock equates to about 1 second in 6 million years). To overcome this difficulty, the GPS employs a trick: The characteristic signal encodes a description within the wave itself: Since it has a period of 300 m in length, the exact position within the oscillation at the time the wave is received can be observed. Thus, it is possible to say how many nanoseconds have passed since the last microsecond. ⊕ ◂◂◂

▸▸▸ **Application**: ***of billions and trillions***
Comparing the English language with other languages, the big numbers have names that are "false friends". This leads quite often to misunderstandings and wrong translations. 10^9 is usually a "billion" in English, and a "milliard" in most other languages. 10^{12} is called a "trillion" instead of a "billion" elsewhere (`https://en.wikipedia.org/wiki/Names_of_large_numbers`). So, how far is a light year, how much is the gross product of Germany, when you find a German website with the content "Deutschland/Bruttoinlandsprodukt 3,73 Billionen USD (2013)", etc.

Solution:
In the case of the light year ($9 \cdot 10^{12}$km): "six trillion miles".
In the case of Germany's gross product: 3.73 billion USD (2013) or 3, 73 Milliarden USD, or, much better: $3.73 \cdot 10^9$ USD (2013). ◂◂◂

Manipulations of equations

Applied mathematics is constantly working with formulas, which consist of relations between given values (which may be variables or constants) that output new values (variables). Usually, there exists a given "explicit solution" for a preferred variable. If one wants to calculate another variable within such a formula, one must transform the whole equation. We will now briefly remind ourselves of the main rules which may be used during such transformations.

Elementary operations

Equivalence transformations are elementary (addition and subtraction as well as multiplication and division of equal terms or quantities on both sides of the equation, except division by zero). So, we find

$$a = b \Leftrightarrow a + c = b + c \quad \text{or} \quad a - c = b - c,$$

$$a = b \Leftrightarrow a\,c = b\,c \quad \text{or} \quad \frac{a}{c} = \frac{b}{c} \ (c \neq 0).$$

The product of a sum quantity and a sum is the sum of two products (distributive law):

$$a\,b + a\,c = a(b + c).$$

One often needs to employ

$$(a \pm b)^2 = a^2 \pm 2ab + b^2 \quad \text{and} \quad (a + b)(a - b) = a^2 - b^2.$$

Computing with powers (of the same base):

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ mal}} \Rightarrow a^n \cdot a^m = a^{n+m}, \text{ and } (a^n)^m = a^{n\,m}.$$

When raising values to a given power, parentheses are important, since

$$(a^n)^m \neq a^{(n^m)}. \tag{2.1}$$

⊕ *Remark*: For example, $(10^{10})^{10} = 10^{100}$ is a number with 100 zeros, and $10^{(10^{10})} = 10^{10000000000}$ is a number with 10 billion zeros. Even $4^{(4^4)} = 4^{256} \approx 10^{154}$ has 154 digits. The largest number that can be written using three digits is obviously $9^{(9^9)}$. This number cannot be represented on any computer (depending on the processor being used, the largest values used by computers fall within the range of 10^{300}), but can only be processed with software for algebraic computation, such as *Derive*. ⊕

Representing fractions

"Equating fractions":

$$\frac{a}{b} = \frac{c}{d} \Leftrightarrow a\,d = b\,c.$$

It follows from $\frac{1}{a} = \frac{1}{b}$ that $a = b$ and vice versa. Resolving double fractions means:

$$\frac{\frac{a}{b}}{c} = \frac{a}{b\,c}, \quad \frac{a}{\frac{b}{c}} = \frac{a\,c}{b}, \quad \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a\,d}{b\,c}.$$

"Common denominator":

$$\frac{a}{b\,c} + \frac{d}{b\,e} = \frac{a\,e + d\,c}{b\,c\,e}.$$

Proportions, intercept theorem

Let us now address *axioms*, which are simple and apparently intuitive statements that are, nevertheless, unprovable, but that must be assumed in order to deduct logical conclusions from them. An example of this is the *intercept theorem*:

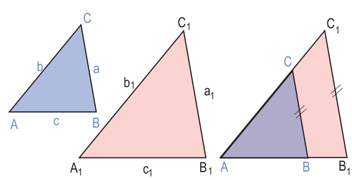

Fig. 2.8 similar triangles, intercept theorem

Consider a triangle ABC, whose side lengths are represented by a, b, c multiplied by a constant factor k (the similarity factor). Thus, it is possible to describe a similar triangle $A_1B_1C_1$ (Fig. 2.8) with side lengths

$$a_1 = k\,a, \;\; b_1 = k\,b, \;\; c_1 = k\,c.$$

These relations of similarity are not limited to the specific triangles ABC and $A_1B_1C_1$. In fact, they can be applied to any triangle with identical distance proportions.

Similar triangles have equal angles.
Furthermore, we have:

$$\overline{AB} : \overline{A_1B_1} = \overline{AC} : \overline{A_1C_1} \quad \text{and} \quad \overline{AB} : \overline{A_1B_1} = \overline{BC} : \overline{B_1C_1}. \tag{2.2}$$

If we move the triangles inside each other so that the legs of an angle join the legs of a corresponding angle of the other triangle, then the remaining two sides are parallel. The two ray sets are then described by formulas (2.2). In other words:

> If you intersect two neighboring sides of a triangle with a pair of parallel lines, then the ratio of the segments on one side equals the ratio of the segments on the other side, and it also equals the ratio of the segments on the parallels.

▸▸▸ **Application**: ***lens equation*** (Fig. 2.9, see also Application p. 54)
Consider a spherical, symmetrical, and convex lens such as a magnifying glass. A "thin lens" (where the spherical radius is much larger than the lens thickness) is a good approximation for light rays near the optical axis adhering to the following three laws (using the notation of Fig. 2.9a):

1. Any ray directed towards the lens center Z, "principal ray" PP^*, is not refracted.

2. Rays parallel to the optical axis $F\overline{F}$ are refracted so that their refractions are concurrent in $\overline{F}$ on the other side of the lens.

3. The outgoing rays of an object point P are refracted so that their refractions concur in an image point P^*.

Let the distance of the object point P to the lens center Z be denoted by g and $\overline{1Z} =: b$ (see Fig. 2.9).

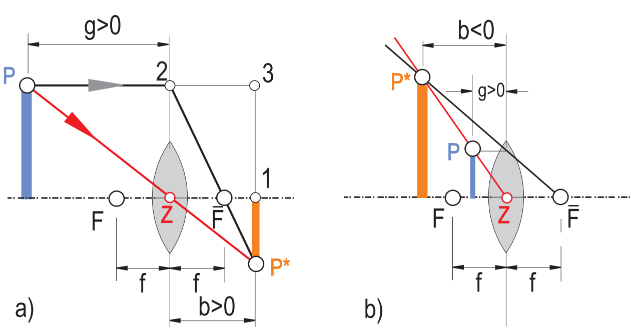

Fig. 2.9 lens equation

Deduce the *lens equation*, in other words the relation between g and b.

Solution:
The triangles $P2P^*$ and $Z\overline{F}P^*$ satisfy the conditions of the intercept theorem, which proves our claim, since

$$\overline{P^*P} : \overline{P^*Z} = \overline{P2} : \overline{Z\overline{F}} = g : f.$$

Analogously, for the triangles $P3P^*$ and $Z1P$:

$$\overline{P^*P} : \overline{P^*Z} = \overline{P3} : \overline{Z1} = (g+b) : b.$$

This yields

$$\frac{g}{f} = \frac{g+b}{b} \Rightarrow \frac{1}{f} = \frac{g+b}{bg} = \frac{g}{bg} + \frac{b}{bg}.$$

The simplified version is the easy-to-remember lens equation:

$$\boxed{\frac{1}{f} = \frac{1}{b} + \frac{1}{g}} \tag{2.3}$$

In practice, we often know f and g. The distance b of the image can be calculated by Formula (2.3) where it is multiplied by the common denominator fgb. This causes the other elements of the equation to move to one side of the equation. After that, b "cancels out":

$$b\,g = f\,g + b\,f \Rightarrow b\,g - b\,f = f\,g \Rightarrow b(g-f) = f\,g \Rightarrow b = \frac{f\,g}{g-f}.$$

If we introduce f as a "scalar" and set $g = k\,f$, we have

$$b = \frac{f\,k\,f}{k\,f - f} = \frac{k}{k-1}f.$$

Fig. 2.10 convex lens (magnifying glass) with fuel and a magnifying effect

In Fig. 2.9a, the image of the (real) object is reduced and upside-down. It is only magnified if we move closer to the object so that $g < f$. In that case, the image is upright but virtual (see Fig. 2.9b, Fig. 2.10):

$$0 < g < f \Rightarrow 0 < k < 1 \Rightarrow \frac{k}{k-1} < 0.$$

Fig. 2.11 concave lens (concave mirror) with fuel and a magnifying effect

The value $k = 1$ ($f = g$) is obviously a critical value (division by zero!): Thus, the image is "infinitely large" and is also "infinitely far away". For a very large k ($k \to \infty$), the factor $\frac{k}{k-1}$ converges towards 1, and the value of b converges to f accordingly.
Rays parallel to the optical axis are bundled in the opposite focal point.

⊕ *Remark*: A similar property of "burning mirrors" was allegedly utilized by the Greek mathematician *Archimedes* (298 – 212 BC) to sink Roman ships (Fig. 2.11, left). However, the accuracy of this legend is doubted, because such an undertaking in practice would be quite difficult. After all, cases are known where dried grass ignited due to the internal effect of dewdrops. Anyway, one can still count Archimedes among the greatest mathematicians. He lived in Syracuse and in Egypt, where he discovered the respective laws for the lever and the lift. He also gave the first good approximation of the mathematical constant π, and even of integral calculus (!). On a side note, he was also the inventor of the Archimedean screw. ⊕ ◂◂◂

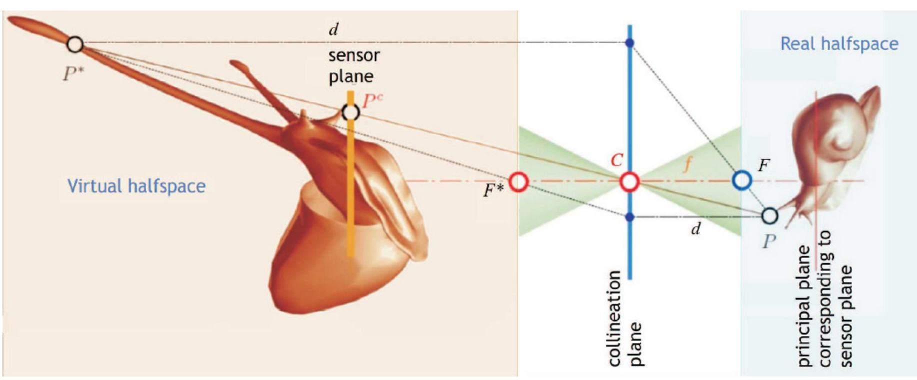

Fig. 2.12 To each object in space, the lens system creates a collinear virtual object. The sensor plane corresponds to a plane perpendicular to the optical axis. Perfectly sharp points in a photo lie in that plane.

▸▸▸ **Application**: ***spatial images produced by lens systems*** (Fig. 2.12)
The illustrated ideal lens system generates for each point P a virtual point

P^*. The pixel on the chip is the – theoretical – section of the ray PZ with the sensor plane.
In reality, there is no "pixel": It is rather a circular "image spot" on the sensor (chip), which is formed by all the light rays emanating from P, going through the circular aperture, and forming an oblique circular cone.[1] This circle is called the *blurry circle* in photography. ◂◂◂

▸▸▸ **Application**: ***parallel resistors*** (Fig. 2.13)
The lens equation also occurs in an interesting manner with resistors connected in parallel. Denote by R_1 and R_2 the resistance of two resistors connected in parallel (so-called partial resistors), then the total resistance R of the system is given by: $1/R = 1/R_1 + 1/R_2$.
The total resistance is always less than each partial resistance. Assuming that the total resistance $R = 50\,\Omega$ and one partial resistance $R_1 = 75\,\Omega$ are given, then we can find the other partial resistance R_2:

$$R_2 = \frac{R \cdot R_1}{R_1 - R} = \frac{50\,\Omega \cdot 75\,\Omega}{75\,\Omega - 50\,\Omega} = 150\,\Omega.$$ ◂◂◂

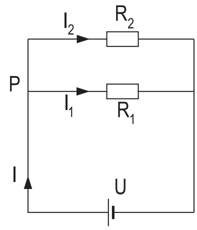

Fig. 2.13 parallel resistors

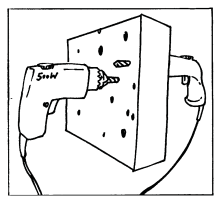
Fig. 2.14 performing task

▸▸▸ **Application**: ***speeding up a process*** (Fig. 2.14)
A machine completes a task in 8 hours. How much time will it take a second machine to complete the same task, so that both machines can do the job in 4.8 hours?
Solution:
Let W denote the total work. From physics we know that *Power* = *Work / Time*.
The performance of the first machine is $P_1 = \frac{W}{8}$, and the performance of the second machine is $P_2 = \frac{W}{x}$. The performance of both machines equals $P = \frac{W}{4.8}$. Since $P = P_1 + P_2$ the equation

$$\frac{W}{8} + \frac{W}{x} = \frac{W}{4.8} \Rightarrow \frac{1}{8} + \frac{1}{x} = \frac{1}{4.8}$$

is a "lens equation" from which $x = 12$ can be calculated as in Application p. 16. ◂◂◂

[1]For further information on this topic, see *G. Glaeser*: Geometry and its Applications. Springer Verlag, New York, 2012.

Equations involving a single root

▸▸▸ **Application**: ***compound interest*** (Fig. 2.15)
The formula for the return of an output capital K_0 over n years, with annual capital (interest rate p), reads

$$K = K_0 \left(1 + \frac{p}{100}\right)^n .$$

Let the initial deposit be $K_0 = 10,000.00$ €. After five years (with fixed interest rates) the resulting capital $K = 11,000.00$ € (already reduced by the capital gains tax) will be paid out. How much is the net interest rate (without taxes)?

Fig. 2.15 compound interest

Solution:
Firstly, we expect that, in general,

$$\frac{K}{K_0} = \left(1 + \frac{p}{100}\right)^n \Rightarrow 1 + \frac{p}{100} = \sqrt[n]{\frac{K}{K_0}}$$
$$\Rightarrow p = 100 \left(\sqrt[n]{\frac{K}{K_0}} - 1\right)$$

and calculate p

$$p = 100 \left(\sqrt[5]{\frac{11,000}{10,000}} - 1\right) = 1.92.$$

⊕ *Remark*: The recommended strategy is to reshape a formula so that it becomes generalized in such a way that one can only use numerical values. Firstly, calculations may become more accurate, and secondly, this reshaped formula can be used with arbitrary numerical values. ⊕ ◂◂◂

▸▸▸ **Application**: ***mathematical pendulum***
To determine the gravitational acceleration g at a particular place on Earth, one must measure the period of oscillation T of a simple pendulum of length L. What is the value obtained for g (in m/s^2) if we apply the following formula:

$$T = 2\pi\sqrt{\frac{L}{g}}. \tag{2.4}$$

Solution:

$$T = 2\pi\sqrt{\frac{L}{g}} \Rightarrow T^2 = 4\pi^2\frac{L}{g} \Rightarrow gT^2 = 4\pi^2 L \Rightarrow g = 4\pi^2\frac{L}{T^2}.$$

Numerical example:
$L = 0.5\,\mathrm{m}$, $T = 1.420\,\mathrm{s} \Rightarrow g = 9.789\mathrm{m/s}^2$ (Beware of the dimensions!)

⊕ *Remark*: Later, we will derive Formula (2.4) with the aid of integral calculus (Application p. 420). Note, incidentally, that the period of oscillation can be measured very accurately since it depends on the angle of deflection and, provided the pendulum is built so that the angle of deflection is sufficiently small, the period is independent of the oscillation amplitude. The pendulum is not, per se, a "perpetuum mobile"; so, energy is to be added constantly – in this case by means of a towing weight. ⊕ ◂◂◂

▸▸▸ **Application**: ***maximum speed in freefall***

Following *Newton*'s Formula (2.5), the air resistance depends on various factors, including the square of the speed v. Determine from this the maximum speed in free fall. ◂◂◂

Solution:

According to *Newton*, we have

$$F_W = c_W\, A\, \varrho \frac{v^2}{2}. \tag{2.5}$$

Here, ϱ is the density of the flowing fluid (gas or liquid) and v is the velocity of the body relative to the fluid. A is the cross-sectional area of the body and c_W is the drag coefficient specific to the body-type indicating how streamlined a barrier is. It is noteworthy that the force F_W depends only on the mass m (to a certain degree and via the cross-sectional area).

For the freefall we have:

$$m\,a = m\,g - F_W.$$

The current acceleration equals a $(0 \le a \le g)$. The weight $m\,a$ (usually $m\,g$) decreases with increasing resistance. If a reaches the value zero, then the maximum speed is reached. Then, we have

$$m\,g = c_W\, A\, \varrho \frac{v_{\max}^2}{2} \quad \Rightarrow \quad v_{\max} = \sqrt{\frac{2mg}{c_W\, A\, \varrho}}.$$

Numerical example: A human body with $A = 0.8\,\mathrm{m}^2$, a drag coefficient $c_W = 0.8$, and a mass of 70 kg achieves at an air density of $1.3\,\mathrm{kg/m}^3$ a terminal velocity of

$$v_{\max} = \sqrt{\frac{2 \cdot 70\mathrm{kg} \cdot 10\frac{\mathrm{m}}{\mathrm{s}^2}}{0.8 \cdot 0.8\mathrm{m}^2 \cdot 1.3\frac{\mathrm{kg}}{\mathrm{m}^3}}} \approx 41\,\mathrm{m/s}$$

(approximately 150 km/h). An ant that is 1 cm long with a mass of 40 mg, with the same c_W, and $A = 0.4\,\mathrm{cm}^2$ only achieves

$$v_{\max} = \sqrt{\frac{2 \cdot 40 \cdot 10^{-6}\mathrm{kg} \cdot 10\frac{\mathrm{m}}{\mathrm{s}^2}}{0.8 \cdot 0.4 \cdot 10^{-4}\mathrm{m}^2 \cdot 1.3\frac{\mathrm{kg}}{\mathrm{m}^3}}} \approx 4.5\,\mathrm{m/s}$$

(see Application p. 111).

▸▸▸ **Application**: ***plunge into water*** (Fig. 2.17)

What is the average deceleration experienced by a diver when the diver jumps from an altitude of H m into the water, and he reaches a depth of T m? The following formulas come into play:

Immersion speed is $v_0 = \sqrt{2g\,H}$ (deduction of the formula in Application p. 59, Formula (2.29)), uniform deceleration $a = \frac{v_0^2}{2T}$ (derivation of the formula in Application p. 59, Formula (2.27)).

Fig. 2.16 different styles of diving ...

Solution:
We substitute $v_0 = \sqrt{2g\,H}$ into $a = \dfrac{v_0^2}{2T}$ and obtain

$$a = \frac{(\sqrt{2g\,H})^2}{2T} = \frac{2g\,H}{2T} = \frac{g\,H}{T} = \frac{H}{T}\,g.$$

The average deceleration is directly proportional to the height H from which the diver jumps and "indirectly" proportional to the depth T of immersion.

Numerical example: With a dive from a five meter diving board ($H = 5\,\mathrm{m}$) and an immersion depth of $T = 4\,\mathrm{m}$, a negative acceleration of about a quarter of the gravitational acceleration is reached.

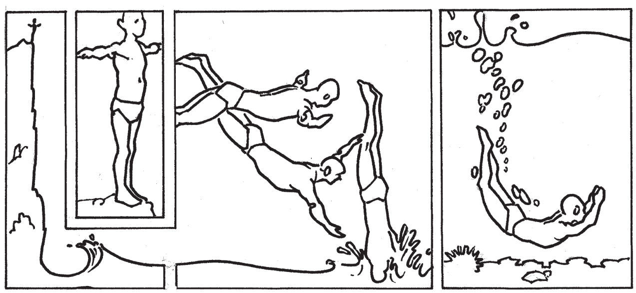

Fig. 2.17 going loco down in Acapulco

Much more extreme, of course, is the practice of the famous "death jumpers" of Acapulco/Mexico. They jump from a height of 40 m and have to roll quickly when submersed in water, because the water at the immersion point is only 3.6 m deep. The deceleration is then $11\,g$ on average.

⊕ *Remark*: The formulas are also valid when applied to the frontal impact of a vehicle crashing against a wall with an airbag of great quality. The deformation of the hood of a car and the relatively small breaking distance by the airbag together produce about $T \approx 0.8\,\mathrm{m}$. Upon impact at $72\,\mathrm{km/h} = 20\mathrm{m/s}$, there is an average deceleration of about $25\,g$. This is almost at the limit ($30\,g$) of what the human organism can survive in the short term. ⊕ ◂◂◂

2.2 Linear equations

General information about algebraic equations

Mathematicians distinguish between algebraic and transcendental equations. For algebraic equations, clear statements about their solutions (number, multiplicity) can be made. One often has to compute the zeros of a function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0. \tag{2.6}$$

Mathematicians often use the "sum notation"

$$f(x) = \sum_{k=0}^{n} a_k x^k.$$

The natural number n is called the *degree of the equation*. The algebraic equations of degree one are called linear equations; those of degree two are called quadratic equations.
A value x is called a *solution* or a *root* of an equation if $f(x) = 0$ is satisfied.

The simplest special case of an algebraic equation

For $n = 1$, Formula (2.6) has the form

$$f(x) = a_1 x^1 + a_0$$

which is usually written as

$$y = k\,x + d.$$

If $k \neq 0$, the equation has exactly one solution

$$y = 0 \Rightarrow k\,x + d = 0 \Rightarrow x = -\frac{d}{k}.$$

▸▸▸ **Application**: ***linear profit function ("naive approach")***
A product is to be produced and sold to a wholesaler. The necessary initial investment amount is K_0 MU (monetary units). The cost of generating each piece is E MU. Up to what number of pieces is the business "in the red", if one can arrive at a selling price of V MU per piece?
Solution:
The *profit function* for x units sold is linear in this simplified case

$$y = V\,x - (K_0 + E\,x) = (V - E)\,x - K_0.$$

The zero is obviously the critical number of pieces. Later on, one starts to make a profit. This is the case for

$$(V - E)\,x - K_0 = 0 \Rightarrow x = \frac{K_0}{V - E}.$$ ◂◂◂

▸▸▸ Application: ***conversion between temperature scales***

The unit for measuring temperature in physics is *Kelvin*, in Europe it is *Celsius*, and in the US it is *Fahrenheit*. The conversion from *Celsius* to *Kelvin* is easy: $K = C + 273.15°$. A conversion that is particularly simple is the one from *Celsius* to the *Réaumur* scale (Fig. 2.18): $R = \frac{4}{5}C$. In order to convert from *Celsius* to *Fahrenheit*, observe the following: $0° C$ corresponds to $32° F$ and $100° C$ corresponds to $212° F$. Determine conversion formulas between the two scales.

Fig. 2.18 Rarely seen: On Réaumur's scale, water also freezes at 0°, but it starts boiling at 80°.

Solution:

We first describe the temperature in *Celsius* as c and that in *Fahrenheit* as f. Then, we can apply the linear ansatz

$$f = k\,c + d.$$

We insert the two nodes, and thus, we obtain two linear equations

$$32 = 0\,k + d \quad \text{and} \quad 212 = 100\,k + d$$

which immediately yield $d = 32$, and thus, $k = \frac{9}{5} = 1.8$. So, we arrive at

$$f = \tfrac{9}{5}\,c + 32 \quad \text{or equivalently} \quad c = \tfrac{5}{9}\,(f - 32).$$

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▸▸▸ Application: ***expansion of a body when heated***

If the temperature of a body with volume V_1 is raised by Δt, its volume is increased to V_2 according to the following formula

$$V_2 = V_1\,(1 + \gamma\,\Delta t).$$

Determine the material constant γ if a given body's volume increases by 3% at $\Delta t = 50°$.

Solution:

Reshaping the given formula yields

$$\gamma = \frac{1}{\Delta t}\left(\frac{V_2}{V_1} - 1\right) = \frac{1}{50}(1.03 - 1) = \frac{0.03}{50} = 0.0006 = 6 \cdot 10^{-4}.$$

⊕ *Remark*: The material constant γ and Δt conform to the same principle regardless of whether we measure in *Celsius* or in *Kelvin* – as is common in physics (see Application p. 24). ⊕ ◀◀◀

Imperial vs. Metric system

▸▸▸ **Application**: ***conversion bar ↔ psi***

On many pressure gauges (e.g. on diving tanks), one can find values either in bar or in psi (pounds per square inch). A diver knows: A full tank is either $\approx$ 200 bar or $\approx 3,000$ psi. Calculate the exact conversion numbers of this linear proportional relation.

Solution:
By definition, we have 1 bar $= 10^5$kg/(ms^2) = 10^5N/m^2.
With 1 N $\approx$ 0.22481 pounds and 1 meter $\approx$ 39.37 inches (1”=2.54 cm), we then have 1 bar $= 22,481/39.37^2$ psi $=$ 14.5 psi or 1 psi $=$ 0.069 bar.

⊕ *Remark*: Note that this was not trivial, since we must not confuse masses and weights (forces). A pound is, physically speaking, a weight.
One bar equals approximately the ambient pressure at sea level. Every 10 meters under water, the pressure increases by one bar. ⊕ ◀◀◀

▸▸▸ **Application**: ***conversion miles per gallon ↔ liters per 100 km***

There are some things in daily life that are easier to understand with the use of some simple calculations. US-Americans are more familiar with “miles per gallon”, and Europeans only think in “liters per 100 km”. What is the formula for this *indirectly proportional* relation?

Solution:
1 fluid gallon equals $\approx$ 3.785 liters, 1 US mile $\approx$ 1.61 kilometers (therefore, e.g., $\approx$ 62 miles equal 100 km).
Suppose, we are driving such that this would be equivalent to using x liters per 100 kilometers and result in a distance of y miles per gallon.
Let us assume that we continue driving at this rate until we have used up a gallon. This is the case after $100 \cdot 3.785/x$ km or $100 \cdot 3.785/1.61/x$ miles ($\approx 235/x$ miles). Thus, we have

$$y \approx 235/x.$$

⊕ *Remark*: For example, five liters per 100 km would be equivalent to 235/5=47 miles per gallon. Note that you do not need a computer with Internet access to obtain this result; it may be quickly estimated by means of a mental calculation. By the way, this simple formula was not easy to find anywhere. ⊕ ◀◀◀

The following two questions were part of a written exam taken by the author in order to acquire an American driver’s licence. Due to unfamiliar measuring units, together with questions about “turn on red” and special regulations concerning school buses (unusual for Europeans), even seemingly simple tests can become critical ...

▸▸▸ **Application**: ***How long is the braking distance at 55 mph?***

The proposed answers were a) 143 ft, b) 243 ft, c) 443 ft. What to choose?

Solution:

Firstly, how many meters per second is 55 mph? We have $v = 55\text{mph} = 55 \cdot 1.61\text{km/h} = 88.5\text{km/h} \approx 25\text{m/s}$.
Secondly, a deceleration of $a \approx 0.6g \approx 6\text{m/s}^2$ might be a good value.
Then we get the time for the braking process: $v = at \Rightarrow t = 25/6\text{s} \approx 4\text{s}$.
This, finally, leads to the braking distance $s = a/2 \cdot t^2 \approx 3 \cdot 4^2 \approx 50\text{m} \approx 165$ feet.
Therefore, the author chose answer a) ...

⊕ *Remark*: The total stopping distance is the sum of the perception distance, the reaction distance, and the actual braking distance (once the brakes are put on). ⊕

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▸▸▸ **Application**: ***How many ounces of 86-proof liquor have the same amount of alcohol as a six-pack of beer?***

This question is less hard to answer once you know what 86-proof liquor means ....

Solution:

A six-pack has $6 \cdot 12 = 72$ fluid ounces. The alcohol content by volume of regular beer may be 5 percent, that of 86-proof liquor is 43 percent, that is, almost 9 times as much. Thus, the estimated answer would be $72/9 = 8$ fluid ounces. ◂◂◂

2.3 Systems of linear equations

Linear equations in two variables

The solution of a linear equation is so "trivial" that it needs no further explanation. It is scarcely any more difficult to solve a *system of linear equations*. Let us consider some introductory examples:

▸▸▸ **Application**: ***mixing task***

How many liters of hot water ($95°\,C$) does one need to raise the temperature of n liters of cold tap water ($15°\,C$) to that of bathwater ($35°\,C$)? What is the ratio of cold and hot water?

Solution:

Let x and y be the amount of cold water and hot water respectively.

1. Mass comparison: $x + y = n$
2. Energy comparison: $15\,x + 95\,y = 35 \cdot n$

Multiply the first equation by 15 and subtract it from the second equation. One will thus need $\frac{n}{4}$ liters of hot water. The ratio of *cold : hot* is, therefore,

$$x : y = \tfrac{3n}{4} : \tfrac{n}{4} = 3 : 1.$$

⊕ *Remark*: One could also have solved the above example as follows: The temperature difference between bathwater and cold water is $-20°$, and $60°$ for hot water. In this case, we use the energy comparison equation $-20\,x + 60\,y = 0$, from which it also follows that $x : y = 3 : 1$. ⊕ ◂◂◂

▸▸▸ **Application**: ***total resistance and partial resistance***

Two resistors R_1 and R_2 connected in (Fig. 2.13) behave like $1 : n$ and have a total resistance of R. What are the individual resistances?

Solution:

For resistors connected in parallel, it holds that (see Application p. 19)

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{R}.$$

By assumption, we have

$$R_1 : R_2 = 1 : n \Rightarrow \frac{1}{R_1} : \frac{1}{R_2} = n : 1.$$

We now set $x = \dfrac{1}{R_1}$ and $y = \dfrac{1}{R_2}$. Then, we have two linear equations

$$x + y = \frac{1}{R}, \quad x : y = n.$$

Following the second equation $x = n\,y$, and thus, from the first equation,

$$n\,y + y = \frac{1}{R} \Rightarrow y = \frac{1}{n+1}\frac{1}{R} \quad \text{respectively } x = \frac{n}{n+1}\frac{1}{R}.$$

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Since we had two unknowns in the above examples, we needed two equations. By skilfully adding, subtracting or using “substitution”, we were able to solve such a “$(2,2)$-System” quickly.
We now derive formulas for the solution (x/y) of a general system of linear equations in two variables, so we do not always have to worry about using such “tricks”.
A $(2,2)$-system has the general form

$$\begin{aligned} \text{(I)} \quad & a_{11}\,x + a_{12}\,y = b_1, \\ \text{(II)} \quad & a_{21}\,x + a_{22}\,y = b_2. \end{aligned} \tag{2.7}$$

Both of the equations are linear, for we have $y = -\frac{a_{i1}}{a_{i2}}x + \frac{b_i}{a_{i2}}$ $(i = 1, 2)$. We want to find a pair of values (x/y) that satisfies both equations. In order to “eliminate” one of the unknowns, we multiply (I) by a_{21} and (II) by a_{11} and get:

$$\begin{aligned} \text{(I)} \quad & a_{21}\,a_{11}\,x + a_{21}a_{12}\;y = a_{21}\,b_1, \\ \text{(II)} \quad & a_{11}\,a_{21}\,x + a_{11}\,a_{22}\,y = a_{11}\,b_2. \end{aligned} \tag{2.8}$$

Now, we subtract the upper from the lower equation, and thus, we get

$$\text{(II)} - \text{(I)} \quad 0 \cdot x + (a_{11}\,a_{22} - a_{21}a_{12})\,y = a_{11}\,b_2 - a_{21}\,b_1 \tag{2.9}$$

or

$$y = \frac{a_{11}\,b_2 - a_{21}\,b_1}{a_{11}\,a_{22} - a_{21}\,a_{12}} = \frac{D_y}{D}.$$

Now, x can be calculated from (I) or (II). The result is

$$x = \frac{a_{22}\,b_1 - a_{12}\,b_2}{a_{11}\,a_{22} - a_{21}\,a_{12}} = \frac{D_x}{D}.$$

The result in this form is ideally is verified by computers; it is easily programmable. It must be mentioned that there exist more efficient methods for solving linear equations. This applies, in particular, to systems of linear equations with three or more variables, but they require – at least programmatically – much greater effort, because different cases have to be considered. We gave formulas for x and y that always work as long as the denominator D is not 0. If the denominator “disappears”, then the system has no solution. The only exception being: The equations are (scalar) multiples of each other.

Then, of course all (infinitely many) pairs (x/y) satisfying equation (I) are solutions of the system.
It is difficult to learn these formulas by heart, because they involve confusing indices of the coefficients. Here we are aided by *Cramer*'s Rule (Gabriel *Cramer*, 1704–1752, a Swedish mathematician and philosopher) – a service that will subsequently be important. It is based on the common "matrix notation", in which system (2.7) – omitting variables x and y for short – is written as follows:

$$\left(\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right)$$

The right column can be described as a "spare column". From this matrix, one builds the three "determinants" (square number schemes) as follows:

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad D_x = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}, \quad D_y = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}. \tag{2.10}$$

The sub-determinants D_x and D_y arising from the "major determinant" D are formed by respectively replacing the x or y column by the spare column. The "value" of such a determinant can now be defined as the difference between the product of elements in the *principal diagonal* (from top left to bottom right) and the product of the members in the *anti-diagonal*, for example,

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\,a_{22} - a_{21}\,a_{12}. \tag{2.11}$$

▸▸▸ **Application**: ***intersecting two lines***
Where do the lines $2\,x + 4\,y = 5$ and $3\,x - 5\,y = 0$ meet?
Solution:
The intersection of two lines in the plane leads to a $(2,2)$-system. The determinants that are sought in this case are

$$D = \begin{vmatrix} 2 & 4 \\ 3 & -5 \end{vmatrix} = -22\ (\neq 0),\ \ D_x = \begin{vmatrix} 5 & 4 \\ 0 & -5 \end{vmatrix} = -25,\ \ D_y = \begin{vmatrix} 2 & 5 \\ 3 & 0 \end{vmatrix} = -15.$$

According to *Cramer*'s Rule, we have the intersection $S\left(\frac{25}{22}\Big/\frac{15}{22}\right)$. See also Application p. 57.

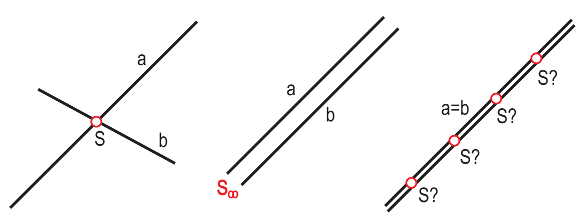

Fig. 2.19 three cases when intersecting two lines

⊕ *Remark*: If the principal determinant vanishes, the lines are either parallel – so that the "point at infinity" S_∞ is the solution – or they are identical – so that every point is a solution (Fig. 2.19, the situations depicted in the center and right).
If one uses the "substitution method", i.e. expressing either variable by the other and substituting them into one equation, one recognizes the exceptional case of $D = 0$ as follows: When the lines are identical, one obtains a "true statement" without further information about x and y; if they are different, then it is a "false statement".

When calculating with computers, it may be quite advantageous to work with "points at infinity" so that, instead of using $D = 0$, we use a very small value that is about $D = 10^{-10}$. This saves us from addressing a large number of different cases since the result will be "true" for values of precise decimals. ⊕ ◂◂◂

Linear equations in three or more variables

The advantage of *Cramer*'s Rule becomes even more apparent when we solve three linear equations containing three variables, that is, a $(3,3)$-system. We can then follow the exact same rules that apply for a $(2,2)$-system (though this can be a bit tedious to prove by recalculation). We will just need to know how to calculate a "three-row" determinant: Perform this calculation by calculating three of the "two-row" sub-determinants; this is called "expanding by complementary minors". This technique will be explained by means of several examples (Application p. 30, Application p. 32). Expanding by the minors complementary to the first column gives:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}. \tag{2.12}$$

We can obviously associate a_{ik} to an element in the sub-determinant created by deleting the i-th row and k-th column. It should be noted that each summand is given a sign according to the following scheme

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}. \tag{2.13}$$

▸▸▸ **Application**: ***intersection of three planes*** (Fig. 2.20)
We would like to find the common point S of the three planes

$$2x + y = 3, \quad y + 2z = 0, \quad y - z = 1$$

(see also the chapter on vector calculus).
Solution:
We write the three equations in matrix form. It must not be forgotten that zeros have to be placed if a variable does not show up:

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right).$$

The principal determinant is now

$$D = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} - 0\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = -6.$$

With the same sub-determinants, D_x assumes the following value

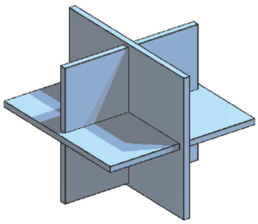

Fig. 2.20 the intersection of three planes . . .

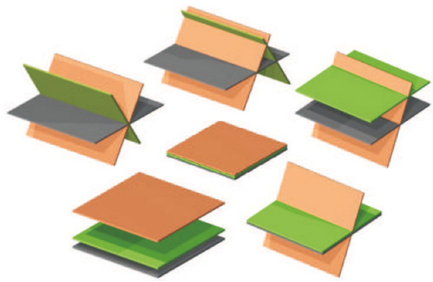

Fig. 2.21 . . . including special cases.

$$D_x = \begin{vmatrix} 3 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{vmatrix} = 3\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} - 0\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = -7.$$

Finally, we have

$$D_y = \begin{vmatrix} 2 & 3 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{vmatrix} = 2\begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} = -4, \quad D_z = \begin{vmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 2.$$

According to *Cramer*'s Rule, now $x = \dfrac{D_x}{D}$, $y = \dfrac{D_y}{D}$, $z = \dfrac{D_z}{D}$, and the intersection point has the coordinates

$$S\left(\frac{7}{6}\Big/\frac{2}{3}\Big/-\frac{1}{3}\right).$$

⊕ *Remark*: If the principal determinant vanishes, the lines of intersection of any pair of planes are either parallel or identical (Fig. 2.21). If the planes are identical, then each point of the plane(s) is a solution. The planes can also form a "pencil" when they have a common line. All of the points on this line are then a solution. This common line can also be the common "line at infinity" of three parallel planes. The planes can intersect each other even along three parallel lines (one of which may also be a line at infinity). Then, the common point at infinity of this line is a solution. In computer calculations – particularly in animations or simulations where such special cases can sometimes occur, but not always – one can continue in the cases with a zero determinant by setting the determinant to a very small value (e.g., $D = 10^{-10}$) in order to circumvent nasty case distinctions (see Application p. 29). ⊕

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▸▸▸ **Application**: ***Δ-connection, Y-connection*** (Fig. 2.22)
When transforming a Δ-connection into a Y-connection, the following equations occur (see Application p. 27):

$$r_k + r_i = \frac{R_j(R_k + R_i)}{R_1 + R_2 + R_3} = A_i.$$

The indices i, j, k run "cyclically", starting with $i = 1$, $j = 2$, and $k = 3$. The values r_i are to be expressed in terms of the R_i.

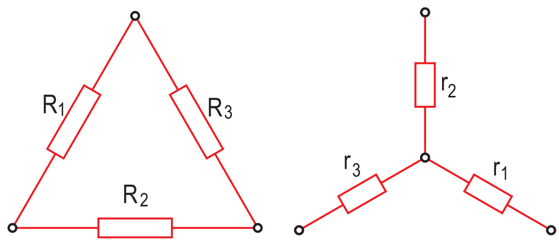

Fig. 2.22 Δ-connection, Y-connection

Solution:
Once again, we write down the three equations explicitly:

$$\begin{aligned} r_3 + r_1 &= s\,R_2(R_3 + R_1) = A_1, \\ r_1 + r_2 &= s\,R_3(R_1 + R_2) = A_2, \\ r_2 + r_3 &= s\,R_1(R_2 + R_3) = A_3 \end{aligned}$$

(with $s = 1/(R_1 + R_2 + R_3)$). In matrix form, the scheme is as follows:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & A_1 \\ 1 & 1 & 0 & A_2 \\ 0 & 1 & 1 & A_3 \end{array}\right).$$

The principal determinant is now

$$D = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 2.$$

Furthermore, we have

$$D_1 = \begin{vmatrix} A_1 & 0 & 1 \\ A_2 & 1 & 0 \\ A_3 & 1 & 1 \end{vmatrix} = A_1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - A_2\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + A_3\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = A_1 + A_2 - A_3.$$

We expand the remaining determinants for calculating r_2 and r_3 by expanding along the middle and right columns respectively:

$$D_2 = \begin{vmatrix} 1 & A_1 & 1 \\ 1 & A_2 & 0 \\ 0 & A_3 & 1 \end{vmatrix} = -A_1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + A_2\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - A_3\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = A_2 + A_3 - A_1,$$

$$D_3 = \begin{vmatrix} 1 & 0 & A_1 \\ 1 & 1 & A_2 \\ 0 & 1 & A_3 \end{vmatrix} = A_1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - A_2 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + A_3 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = A_3 + A_1 - A_2 .$$

The new resistors are then

$$r_i = \frac{D_i}{D} = \frac{A_i + A_j - A_k}{2} .$$

The indices i, j, k again run in a cyclical order, starting with $i = 1$, $j = 2$, and $k = 3$.

⊕ *Remark*: One remembers formulas better when they display regularities. Often such regularities in output are an indication that one has calculated correctly. *Einstein* once said that a formula is only correct if it is "beautiful". ⊕ ◂◂◂

▸▸▸ **Application**: ***parabola on three points*** (Fig. 2.23)
There are very common techniques for the approximation of complicated curves by simpler ones such as parabolas. A famous example of such a technique is the *Kepler*'s Fass-Rule (Fass = German word for barrel, see Section 8, page 413). For this rule, one derives the coefficients a, b, c of the equation of a parabola $y = ax^2 + bx + c$ passing through three points $P(u_i/v_i)$.

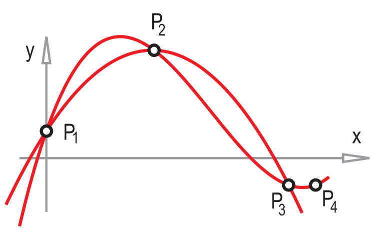

Fig. 2.23 quadratic parabola (3 points) vs. cubic parabola (4 points)

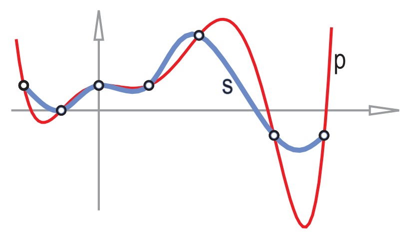

Fig. 2.24 parabola of degree 6 (7 points) vs. cubic spline

Solution:
The coordinates of the three points have to "fulfil" the equation of the parabola, so we immediately get three equations with three unknowns a, b, and c:

$$\begin{aligned} u_1^2\, a + u_1\, b + c &= v_1 , \\ u_2^2\, a + u_2\, b + c &= v_2 , \\ u_3^2\, a + u_3\, b + c &= v_3 . \end{aligned}$$

We solve this by means of *Cramer*'s Rule. The principal determinant is:

$$D = \begin{vmatrix} u_1^2 & u_1 & 1 \\ u_2^2 & u_2 & 1 \\ u_3^2 & u_3 & 1 \end{vmatrix} .$$

⊕ *Remark*: Generally, one can interpolate $n+1$ points by a "parabola of degree n" (Fig. 2.23 and Fig. 2.24). However, complicated curves in computer graphics are rarely approximated by parabolas of degrees higher than 3 – such curves tend to oscillate (especially at their end points, Fig. 2.24). Instead of choosing a higher degree curve, it is better to use a sequence of "cubic parabolas" which are joined as smoothly as possible. The overall approximating curve is a cubic spline. Such splines also require the solution of systems of linear equations. For more details, read the chapter on differential calculus (Application p. 290). ⊕ ◂◂◂

Systems of linear equations having higher degrees are not uncommon in applied mathematics. Often one can "linearize" complex problems by introducing additional variables. This allows us to obtain a simple system of linear equations with many unknowns, rather than a complicated system with a few unknowns.
Such (n,n)-systems can also be solved using an adjusted version of *Cramer*'s Rule. The determinants of degree n develop gradually by a summation of determinants of degree $n-1$ until they reach a degree of three. The higher the level, the more complex *Cramer*'s Rule becomes, and one often uses other methods with computers leading to a more efficient way to obtain solutions. Here, we have a simple example of using a $(4,4)$-system, this can be solved by skilfully inserting determinants (the "substitution method"):

▸▸▸ **Application**: ***combustion of alcohol in carbon dioxide and water***
Consider the "chemical reaction formula" when one has sufficient oxygen to burn alcohols – such as the ethanol found in sugarcane liquor, which decomposes C_2H_5OH – to CO_2 and H_2O.
Solution:
The following holds true

$$n_1 \cdot C_2H_5OH + n_2 \cdot O_2 = n_3 \cdot CO_2 + n_4 \cdot H_2O.$$

The four unknowns n_i can be calculated incrementally if we compare the quantity of the atoms:

$$\begin{array}{ll} \text{Carbon } (C) & 2\,n_1 = n_3, \\ \text{Hydrogen } (H) & (5+1)\,n_1 = 2\,n_4, \\ \text{Oxygen } (O) & n_1 + 2\,n_2 = 2\,n_3 + n_4. \end{array}$$

The first striking observation: We have three equations but four unknowns. However, there is an additional condition: The n_i must be integers. First, we substitute $n_1 = 1$ and hope to obtain integer solutions for the remaining n_iS (If we do not obtain such solutions: Chemists also reckon with fractions!). Now we only need to solve a $(3,3)$-system:

$$2 = n_3, \quad 6 = 2\,n_4, \quad 1 + 2\,n_2 = 2\,n_3 + n_4.$$

The solutions can immediately be seen: The first equation evaluates to $n_3 = 2$, the second one evaluates to $n_4 = 3$, and thus, the third evaluates to $n_2 = 3$. In fact, all solutions are integers (otherwise you will have fractions or will have to try $n_1 = 2, \ldots$). The chemical reaction formula is thus

$$C_2H_5OH + 3 \cdot O_2 = 2 \cdot CO_2 + 3 \cdot H_2O.$$

◂◂◂

▸▸▸ **Application**: ***explosive combustion of gasoline vapor***

Gasoline, especially, contains alkanes that have 6 to 9 carbon atoms in their molecules such as C_8H_{18}. This is used as a fuel due to the fact that the mixture of gasoline vapor and air combusts explosively into water and carbon dioxide given a certain composition. Calculate the coefficients of the chemical reaction equation

$$n_1 \cdot C_8H_{18} + n_2 \cdot O_2 = n_3 \cdot CO_2 + n_4 \cdot H_2O.$$

Solution:

We calculate the four unknowns n_i gradually by comparing the quantities of the atoms:

Carbon (C)	$8\,n_1 = n_3$
Hydrogen (H)	$18\,n_1 = 2\,n_4$
Oxygen (O)	$2\,n_2 = 2\,n_3 + n_4$

We have three equations but four unknowns. The n_i must be integers. First let us put again $n_1 = 1$ and hope to obtain integer solutions for the remaining n_i.

The $(3,3)$-system is now

$$8 = n_3, \quad 18 = 2\,n_4, \quad 2\,n_2 = 2\,n_3 + n_4.$$

The first equation gives $n_3 = 8$, the second gives $n_4 = 9$, and thus, the third gives $n_2 = (16 + 9)/2 = 12.5$, which is not an integer. Thus, we try $n_1 = 2$. Therefore, the appropriate $(3,3)$-system is

$$16 = n_3, \quad 36 = 2\,n_4, \quad 2\,n_2 = 2\,n_3 + n_4,$$

which gives the chemical reaction formula

$$2 \cdot C_8H_{18} + 25 \cdot O_2 = 16 \cdot CO_2 + 18 \cdot H_2O.$$

◂◂◂

2.4 Quadratic equations

The pure quadratic equation

The equation

$$x^2 = D$$

has two real solutions if $D > 0$:

$$x_1 = \sqrt{D}, \; x_2 = -\sqrt{D}.$$

The solutions for $D = 0$ are: $x_1 = x_2 = 0$. The solutions for $D < 0$ are *complex conjugates*. More on this in Section B.

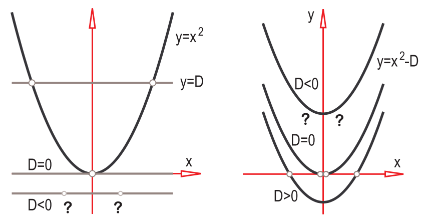

Fig. 2.25 graphical solution of $x^2 = D$

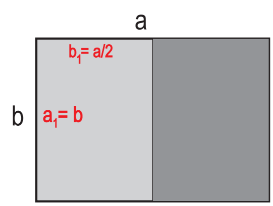

Fig. 2.26 convenient paper format

▸▸▸ **Application**: ***convenient paper format*** (Fig. 2.26)
Let the side lengths of a rectangular sheet of paper be a and b with a ratio $a : b$ such that two similar rectangles arise when folding the sheet. How should one read the aspect ratio? How long are the sides of the paper if the surface of the sheet is $1\,\mathrm{m}^2$?
Solution:

$$\frac{a_1}{b_1} = \frac{b}{\frac{a}{2}} = \frac{a}{b} \;\Rightarrow\; b^2 = \frac{a^2}{2} \;\Rightarrow\; a = b\sqrt{2}$$

(the solution $a = -b\sqrt{2}$ is not relevant).
If the area of the rectangle ($1\mathrm{m}^2$) is the only information given, b can be calculated, and thus, a as well:

$$a\,b = b\sqrt{2}\,b = b^2\sqrt{2} = 1$$

$$\Rightarrow\; b = \sqrt{\frac{1}{\sqrt{2}}} = 0.841\mathrm{m} \;\Rightarrow\; a = 1.189\mathrm{m}.$$

This format is called $A0$. Repeatedly folding the sheet gives the formats $A1 = 0.841 \times 0.595,\; A2 = 0.595 \times 0.420,\; A3 = 0.420 \times 0.297,\; A4 = 0.297 \times 0.210$ etc. (with areas $\frac{1}{2}\mathrm{m}^2,\; \frac{1}{4}\mathrm{m}^2,\; \frac{1}{8}\mathrm{m}^2,\; \frac{1}{16}\mathrm{m}^2,\; \ldots$). ◂◂◂

The general quadratic equation

In the general form of the quadratic equation,

$$Ax^2 + Bx + C = 0,$$

the sign of the so-called *discriminant*

$$D = B^2 - 4AC$$

determines the number of real solutions: For $D > 0$, it gives two real solutions; for $D = 0$, it gives one real solution; and for $D < 0$, it gives two complex conjugate solutions.
The solutions can be given directly

$$x_{1,2} = \frac{-B \pm \sqrt{D}}{2A}. \tag{2.14}$$

Proof: We first divide equation $Ax^2 + Bx + C = 0$ by A, and we get

$$x^2 + \frac{B}{A}x + \frac{C}{A} = 0.$$

Now, we complete to a full square

$$\left(x + \frac{B}{2A}\right)^2 + \frac{C}{A} - \left(\frac{B}{2A}\right)^2 = 0.$$

Hence, we have a purely quadratic equation:

$$\left(x + \frac{B}{2A}\right)^2 = \left(\frac{B}{2A}\right)^2 - \frac{C}{A} = \frac{B^2 - 4AC}{4A^2}.$$

The solutions are given by Formula (2.14). ⊙

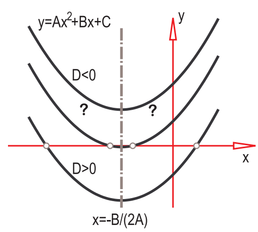

Fig. 2.27 graphical solution

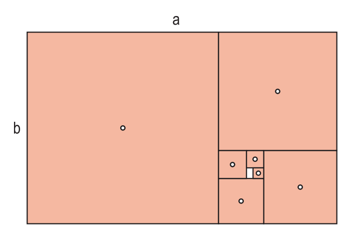

Fig. 2.28 harmonic rectangle

▸▸▸ **Application**: ***Golden Ratio, harmonic rectangles*** (Fig. 2.28)
The side lengths a, b of a rectangle are chosen such that $b < a$ where

$$b : a = a : (a + b) \Rightarrow a^2 = ab + b^2. \tag{2.15}$$

We then call this rectangle *harmonic*. Calculate the ratio of the side lengths (the *Golden Number*).
In addition, prove the following: One rectangle divides into a square and a smaller rectangle, and this smaller rectangle is again considered to be harmonic (this ratio is called the *Golden Section*).

Solution:
We first set $b = 1$ and then,

$$\frac{1}{a} = \frac{a}{a+1} \;\Rightarrow\; a^2 - a - 1 = 0 \;\Rightarrow$$
$$a_{1,2} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

We do not consider the negative solution (however, see Application p. 528). Thus

$$\boxed{a : b = \frac{1+\sqrt{5}}{2} : 1 \approx 1.62 : 1.} \tag{2.16}$$

Assuming that Formula (2.15) applies, we also get

$$a^2 = ab + b^2 \Rightarrow a^2 - ab = b^2 \Rightarrow a(a-b) = b^2 \Rightarrow (a-b) : b = b : a.$$

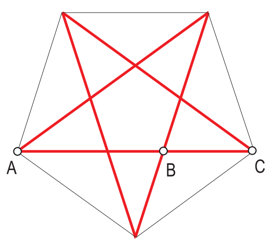

Fig. 2.29 pentagram

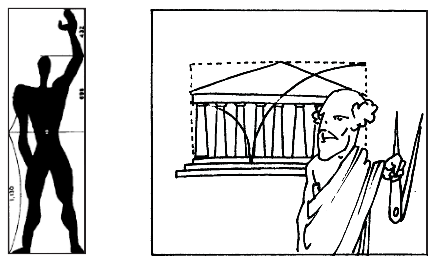

Fig. 2.30 Modulor, Parthenon

⊕ *Remark*: Harmonic rectangles or the golden ratio often occur in art.[2] Examples include the "Modulor" by *Corbusier* (Fig. 2.30, left),[3] the "magic pentagon" ("pentagram", Fig. 2.29: $\overline{AC} : \overline{AB} = \frac{1+\sqrt{5}}{2} : 1$), and the Parthenon (Greek temple, Fig. 2.30). In nature, the remarkable number $\frac{1+\sqrt{5}}{2}$ plays an important role. More on this can be found in Appendix B. ⊕ ◂◂◂

▸▸▸ **Application**: ***upward vertical throw*** (Fig. 2.31)
Let v_0 be the initial velocity and let g be the gravitational acceleration. The air resistance in this example is negligible.
Then, for an elapsed time of t seconds, the height h is given by

$$h = G(t) - B(t) = v_0 t - \frac{g}{2} t^2. \tag{2.17}$$

$G(t)$ is the component of the upwards uniform motion; $B(t)$ is the component of the downwards uniformly accelerated motion.
Compute t at given values of h and v_0.

---

[2]See `http://www.uni-hildesheim.de/~stegmann/goldschn.pdf` for an interesting article on the historical development of the Golden Section.

[3]Le Corbusier (Charles Edouard Jeanneret): *The Modulor: A Harmonious Measure to the Human Scale Universally Applicable to Architecture and Mechanics and Modulor 2 (Let the User Speak Next)*. 2 Volumes. Birkhäuser, Basel, 2000.

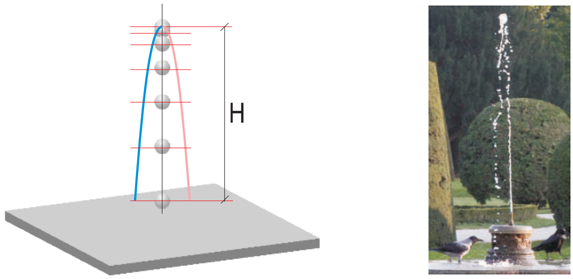

Fig. 2.31 upward vertical throw

Solution:
We collect together powers of t as follows: $\frac{g}{2}\,t^2 - v_0\,t + h = 0$ and find

$$A = \frac{g}{2},\; B = -v_0,\; C = h \Rightarrow D = (-v_0)^2 - 4\frac{g}{2}h = v_0^2 - 2gh.$$

This results in

$$t_{1,2} = \frac{v_0 \pm \sqrt{v_0^2 - 2gh}}{g}.$$

The values t_1 and t_2 are the times at which the thrown object reaches height h, the difference $t_2 - t_1$ being the time it takes to rise and fall back down to this height. For the highest point (the current height H), there must be a double solution:

$$D = 0 \Rightarrow v_0^2 - 2gH = 0 \Rightarrow H = \frac{v_0^2}{2g}. \tag{2.18}$$

Conversely, measuring from the current height H, the initial velocity v_0 can be determined as:

$$v_0 = \sqrt{2gH} \tag{2.19}$$

⊕ *Remark*: This formula unexpectedly provides a good service in Application p. 56. ⊕

⊕ *Remark*: If we *obliquely throw upwards* (with an angle tilted at α to the horizontal), the value of $v_0 \sin\alpha$ is used instead of v_0 (Application p. 142). ⊕ ◂◂◂

▸▸▸ **Application**: ***balance of the attraction forces*** (Fig. 2.32)
Let m_1 and m_2 be two masses with centers of gravity separated by a distance d. The center of gravity of a body P is located at a distance x from the center of gravity of m_1, on the line joining the centers of gravity of m_1 and m_2. For which x will the attractions between m_1 and m_2 compensate?
Solution:
According to *Newton*, the attraction is proportional to the mass, but it is inversely proportional to the square of the distance. For the body to be in equilibrium, one must have:

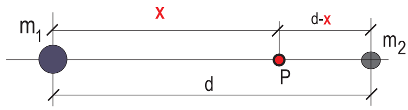

Fig. 2.32 Where does P have to lie in order to be attracted equally by m_1 and m_2?

$$\frac{m_1}{x^2} = \frac{m_2}{(d-x)^2} \qquad (0 < x < d)$$

$$\Rightarrow\ m_1(d^2 - 2dx + x^2) = m_2x^2 \Rightarrow\ \underbrace{(m_1 - m_2)}_{A}\, x^2 \underbrace{-2m_1 d}_{B}\, x + \underbrace{m_1 d^2}_{C} = 0.$$

Now we turn to Formula (2.14):

$$x_{1,2} = \frac{2m_1 d \pm \sqrt{4m_1^2 d^2 - 4(m_1 - m_2)m_1 d^2}}{2(m_1 - m_2)} = \frac{m_1 \pm \sqrt{m_1 m_2}}{m_1 - m_2}\, d.$$

Only one of the two solutions (namely the one associated with the "–") is relevant (since the forces must have opposite signs). This implies $x < d$ and

$$x = \frac{m_1 - \sqrt{m_1 m_2}}{m_1 - m_2}\, d. \tag{2.20}$$

⊕ *Remark*: It often proves to be convenient to consider only the *ratio* of the two masses. This could provide a better understanding of the relations. We can rewrite Formula (2.20) to show this by dividing the right term numerator and denominator by m_1

$$\Rightarrow\ x = \frac{1 - \sqrt{\frac{m_2}{m_1}}}{1 - \frac{m_2}{m_1}}\, d. \tag{2.21}$$

Now we employ a small trick: we have

$$\frac{a-b}{a^2 - b^2} = \frac{a-b}{(a+b)(a-b)} = \frac{1}{a+b}.$$

In our case, let $a = 1$ and $b = \sqrt{\frac{m_2}{m_1}}$. Thus, we can simplify Formula (2.21) to get

$$x = \frac{1}{1 + \sqrt{\frac{m_2}{m_1}}}\, d.$$

Numerical example: For a spacecraft between the Earth and the Moon (which have a mass ratio of $1:81$, $d \approx 384{,}000$km), this means

$$x = \frac{d}{1 + 1/9} = \frac{9}{10}\, d.$$

So, the spacecraft must be about 346,000 km from Earth, or rather 38,000km from Earth (measured from the centers of both the Earth and the Moon). ⊕

⊕ *Remark*: While the attraction of the Sun is quite strong, it is completely offset by the centrifugal force (the Earth, the Moon and the spacecraft rotate around the Sun at about the same speed). ⊕ ◀◀◀

2.5 Algebraic equations of higher degree

In applied mathematics, algebraic equations of higher degree(s) such as

$$f(x) = \sum_{k=0}^{n} a_k x^k = 0$$

are not uncommon. The solutions of such equations are called the "roots" or "zeros" of the equation. Equations of degree three and four can be solved by means of rather complicated formulas without too much effort and can even be solved exactly with a detour via the *complex numbers*.
Without going into too much detail or even using any "complex numbers" (see B), we only want to collect the most important facts on algebraic equations. The theory of these equations kept mathematicians busy over many centuries. The most important theorem was given and proved by C.F. *Gauß* in his PhD thesis:

The Fundamental Theorem of Algebra: An algebraic equation of degree n has n solutions when so-called "multiple solutions" and also when so-called "complex solutions" are counted.

⊕ *Remark*: The German mathematician Carl Friedrich *Gauß* (1777 – 1855) is one of the most important mathematicians of all time. He was involved in many fields of mathematics and its applications, especially in astronomy. Even up to the present, many of his elegant methods have remained unsurpassed. ⊕
For us, an important consequence of the fundamental theorem of algebra (see Section 2, page 41) is the following: *The number of real solutions is at most n, but it may reduce by multiples of 2. Multiplicities also count.*

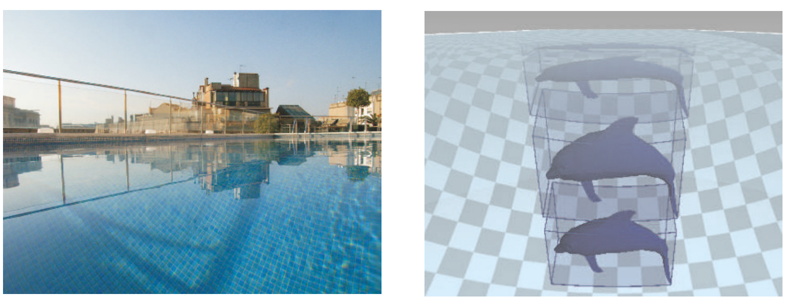

Fig. 2.33 images formed by refraction

▸▸▸ **Application**: ***refraction at a plane*** (Fig. 2.33)
To determine the "break point" of a light ray at the planar interface between two "media" (a water surface, a surface of a thick glass plate, etc.), we must

solve an algebraic equation of degree 4, like $f(x) = a_4\,x^4 + a_3\,x^3 + a_2\,x^2 + a_1\,x + a_0 = 0$. Two or four solutions are real. However, only one of these is relevant. It can, above all, be calculated accurately and efficiently by a computer.

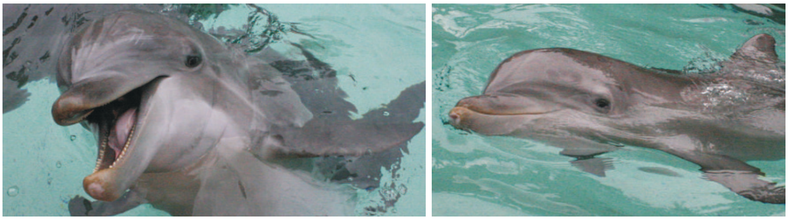

Fig. 2.34 right: heavily skewed beneath the surface

⊕ *Remark*: Consider the "elevation" of the pool bottom in Fig. 2.33. Both the photo on the left and the computer calculation on the right have a constant value as their depth! The right image shows four (!) dolphins that swim right above each other. The top dolphin is barely recognizable in the image. Also if we look at Fig. 2.34, the animals are "undistorted" above, but heavily skewed beneath the surface. ⊕ ◂◂◂

▸▸▸ **Application**: ***ray tracing***

Realistic computer images, to which we are now accustomed, are often generated by ray tracing programs. The "scene" is composed of "primitive algebraic building blocks", such as flat polygons (triangles), spheres, cones, cylinders, or annular surfaces (tori). Lines of sight are considered by the individual *pixels* of the screen ("picture elements"). They either disappear "into nothing", or else they meet such blocks. The identification of all possible intersections of the line of sight is done by means of the rigorous and efficient method of solving algebraic equations up to degree 4. If several intersections exist, only the foremost – thus seen as the visible – is considered. ◂◂◂

In general, when dealing with algebraic equations of a higher degree, we will have to make do with approximate solutions, as will be the case with non-algebraic expressions of the form $f(x) = 0$. We deal with this issue in the differential calculus section (*Newton*'s method, Chapter 6).
Sometimes the degree of an equation decreases. This occurs especially when we can use a substitution of the form $u = x^2$, $u = x^3$, etc.

▸▸▸ **Application**: ***reducible equation of degree six*** (Fig. 2.35)

Find all real solutions of the equation

$$x^6 - 2\,x^3 + 1 = 0.$$

Solution:
By means of the substitution $u = x^3$, we obtain the easily solvable quadratic equation:

$$u^2 - 2\,u + 1 = 0 \Rightarrow u_{1,2} = 1 \text{ (double solution!)}$$

Since $u = x^3 = 1$, a cubic equation arises. The latter has one real solution. Overall, our equation of degree six has, therefore, a – doubly counted – real zero. This solution might not be found by a program that works numerically. It will only estimate zeros (up to a certain precision) within intervals where the function changes sign (or vanishes). Thus, in this special case, it *may* find the only zero, but it will do so only by chance. ◀◀◀

Sometimes trivial solutions of a higher degree equation split off so that the degree of the equation can be reduced. Under these circumstances, the remaining solutions can be calculated exactly.

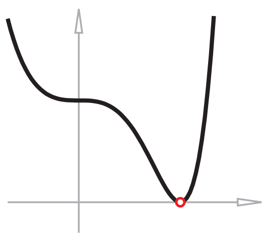

Fig. 2.35 $f(x) = x^6 - 2x^3 + 1$

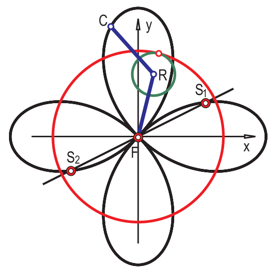

Fig. 2.36 Cartesian quadrifolium

▸▸▸ **Application**: ***algebraic curve of degree six (quadrifolium)*** (Fig. 2.36)
Descartes's *quadrifolium* is given by the equation

$$(x^2 + y^2)^3 = 27\,x^2 y^2. \tag{2.22}$$

(An equation in x, y where neither variable is isolated like $y = f(x)$ is called an *implicit equation.*) Find the up to six points of intersection with a straight line through the origin.

Solution:
In the general linear equation $y = kx + d$, we set $d = 0$ since the line passes through the origin (0/0). Then we insert into this Formula (2.22) and get

$$[x^2(1 + k^2)]^3 = 27\,k^2\,x^4 \Rightarrow (1 + k^2)^3\,x^6 = 27\,k^2\,x^4.$$

Now, we can cut out x^4 if we assume $x \neq 0$. Then, we have only a purely quadratic equation whose solutions can be directly written as

$$(1 + k^2)^3\,x^2 = 27\,k^2 \Rightarrow x_{1,2} = \pm\sqrt{\frac{27\,k^2}{(1 + k^2)^3}} = \pm\frac{3k\sqrt{3}}{\sqrt{(1 + k^2)^3}}.$$

The value $x = 0$ counts in the algebraic sense as a *four-fold solution.* In fact, one can see in Fig. 2.36 that the curve meets a general line through the origin at two points S_1 and S_2 (with the x-coordinates x_1 and x_2) and it additionally intersects at the origin four times.

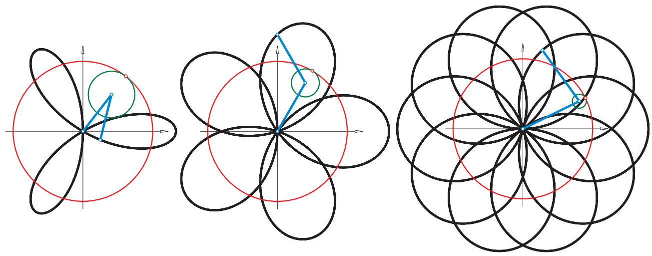

Fig. 2.37 generalization of the quadrifolium by varying the kinematic generation

⊕ *Remark*: The quadrifolium can be generated *kinematically* by a so-called *planetary gear* in which a rod FR uniformly rotates about a fixed origin F, while a second rod RC – of the same length – (Fig. 2.36) rotates about R with triple angular velocity. The same motion is generated by rolling a circle R on a fixed circle F. The ratio of the radii is $1:4$. This generation allows a generalization by varying the ratio of the radii. This produces a trefoil, quintifolium, etc. The trefoil is a curve of degree four. The degrees of the other curves are significantly higher. See also page 542. ⊕

◂◂◂

▸▸▸ **Application**: ***cubic curves*** (Fig. 2.38)

The folium of René Descartes (1596–1650) is best described in the coordinate system that bears his name (Cartesian). Its cubic equation is $x^3 + y^3 = c\,xy$. The intersection with rays $y = kx$ through the origin lead to simple linear equations: $x^3(1 + k^3) = c\,kx^2 \Rightarrow x = \frac{c\,k}{1+k^3}$.

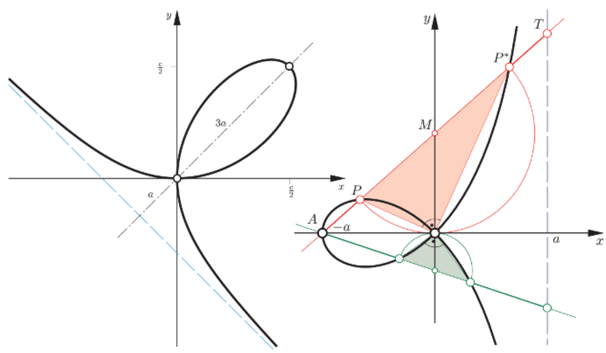

Fig. 2.38 Two “classics” among cubic curves: The folium of Descartes and the right strophoid. Both curves are described by algebraic equations of degree 3.

The strophoid is easily constructed: On each ray through the point $A(-a/0)$, one draws those points which are at distance d from the ray's intersection $M = (0/d)$ with the y-axis. This yields two points P and P^* on the curve. The curve has a double point at the origin, and its tangents there are the two angle bisectors of the coordinate axes. The algebraic equation of the curve is $(a-y)y^2 = x^2(a+x)$. Again, points on rays can be found via a linear equation.

◂◂◂

▸▸▸ **Application**: ***the Delian cube duplication problem***[4] (Fig. 2.39)
Legend has it that the Delphic oracle once suggested that the volume of the cube shaped altar in the temple to Apollo should be doubled. It was, therefore, necessary to determine the side lengths of two cubes whose volumes formed the ratio $1:2$. The solution is, of course, the cubic root of 2:

$$x^3 = 2a^3 \Rightarrow x = a \cdot \sqrt[3]{2}.$$

A remarkable approach is attributed to *Menaechmus*, who presented the problem to Plato's school. In his approach to the problem, he used curves that would later be classified by *Apollonius of Perga* (262–190 BC) as parabolas. We consider four cubes altogether with side lengths a, x, y, and $2a$. Let the volume of the second cube be twice that of the first, the volume of the third be twice that of the second, and that of the fourth be twice the volume of the third.

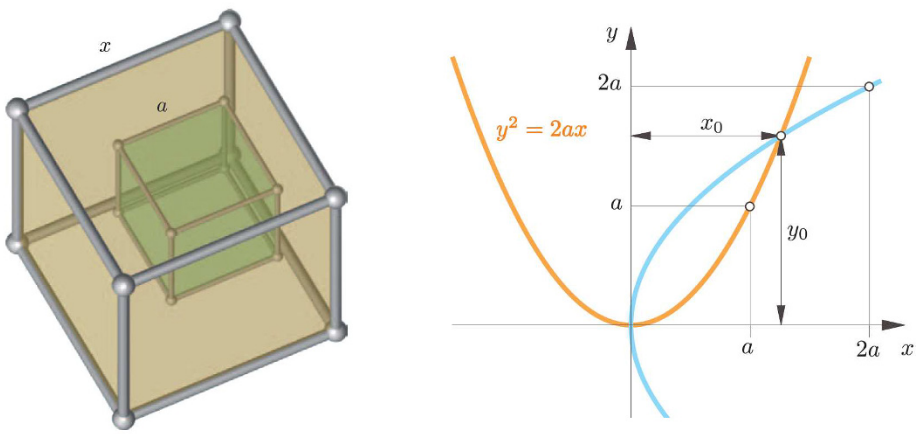

Fig. 2.39 left: *Menaechmus*'s method, right: intersection of quadratic parabolas

Then, from $a : x = x : y = y : 2a$, we obtain the system of equations

$$x^2 = ay, \quad y^2 = 2ax.$$

These two equations can be represented graphically as parabolas through the origin whose remaining point of intersection yields the side length of the desired "intermediate cube." By manipulating the equations, the system can be interpreted as the intersection of a circle with an equilateral hyperbola:

$$x^2 + y^2 = 2ax + ay, \quad x^2 - y^2 = ay - 2ax.$$

In each case, it amounts to a solution that the classical Greeks might have deemed inelegant. ◂◂◂

[4]G. Glaeser, K. Polthier: *Bilder der Mathematik*, Springer Spektrum, Heidelberg, 3^{rd} edition 2014.

2.6 Further applications

This section contains exercises related to the previous sections. They are mostly independent of one another and can be partially skipped without affecting one's comprehension of the concepts. The examples are somewhat more complex than the previous ones. The reader is encouraged to consider at least the comments on the results in this section.

▸▸▸ **Application**: ***How often do the hands of a clock overlap?***

Solution:
This problem can be solved quickly (otherwise you have to give a lavish argument): We start at exactly 12 noon, where both hands overlap. After a full turn of the minute hand, the hour hand has performed 1/12 (30°) of a full rotation. When the minute hand has rotated 12/11 of the full angle of rotation, then the hour hand has rotated 1/11 (almost 33°) of the full revolution, so that there is coverage again. Then, over a period of 12 hours, the hands will have overlapped 12/(12/11) times, that is, exactly eleven times.

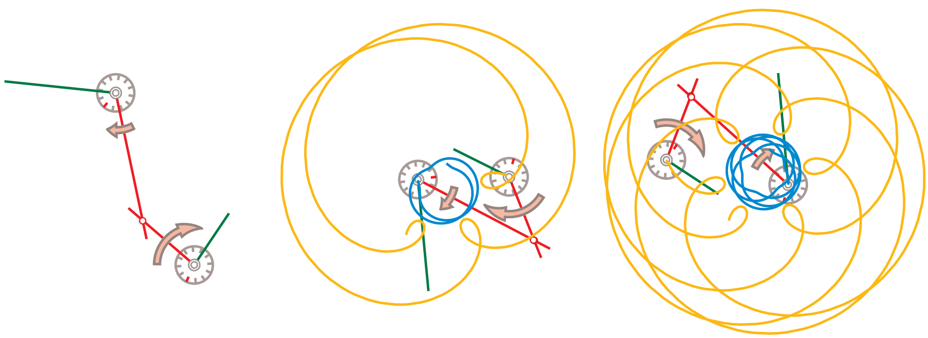

Fig. 2.40 clocks that drive each other

⊕ *Remark*: The same question is significantly more challenging to solve when two clocks that move at different speeds are linked via hinges on their minute hands (Fig. 2.40) and the whole installation is left to move on its own ... ⊕ ◂◂◂

▸▸▸ **Application**: ***counter-directed motions***

A train with an average speed of c_1 drives from A in the direction of B (distance $\overline{AB} = d$). At B, a train starts Δt hours later and drives with an average speed c_2 towards A. When do the trains meet?

Solution:
Let x and y be the respective distances that the two trains have travelled when they reach their meeting point. Thus $x + y = d$. Furthermore, let t be the time that it takes for the first train to reach the meeting point. Then,

$x = t\,c_1$ and $y = (t - \Delta t)\,c_2$. This results in $t\,c_1 + (t - \Delta t)\,c_2 = d$, and thus, $t = \dfrac{d + \Delta t\,c_2}{c_1 + c_2}$. ◀◀◀

▸▸▸ **Application**: ***rock erosion*** (Fig. 2.41)

The striking Table Mountain with an altitude of 1,087 m is a landmark of Cape Town (South Africa). It is one of the oldest mountain ranges on Earth. Its age is estimated to be an impressive 600 million years (so, it is almost 10 times as old as the Alps). It is believed that the peak was over 5 000 m at its birth. What was the average erosion per year? How high was the mountain 65 million years ago, i.e. in the period when the Alps emerged and the dinosaurs became extinct?

Fig. 2.41 Table Mountain in Cape Town with its typical "tablecloth"

Solution:

height difference	$5,000\,\text{m} - 1,000\,\text{m} = 4,000\,\text{m} = 4 \cdot 10^3\,\text{m}$
time difference	600 million years $= 6 \cdot 10^8\,a$
erosion per year	$\frac{4 \cdot 10^3\,\text{m}}{6 \cdot 10^8\,a} = \frac{40 \cdot 10^2\,\text{m}}{6 \cdot 10^8\,a} \approx 6.7 \cdot 10^{-6}\,\frac{m}{a} = 6.7 \cdot 10^{-3}\,\frac{\text{mm}}{a}$

The erosion in 65 million years $(6.5 \cdot 10^7 a)$ is then

$$6.7 \cdot 10^{-6}\,\frac{m}{a} \cdot 6.5 \cdot 10^7 a \approx 44 \cdot 10^1\,\text{m} = 440\,\text{m}.$$

The mountain was 440 m higher 65 million years ago, so a little over 1,500 m high. ◀◀◀

▸▸▸ **Application**: ***if the Antarctic ice melts . . .***

The Antarctic (the mainland has a surface area of about 12 million km^2) has only relatively recently become frozen – but now it is all the more so! The major part of the fresh water on Earth is bound in its ice sheet, which is about 2 km thick on average. By how much will the sea level rise if the ice melts completely?

Solution:

We will set the unit to kilometers. The volume of ice on the mainland is $24 \cdot 10^6\,\text{km}^3$. When ice melts, it loses about 10% of its volume, so that about

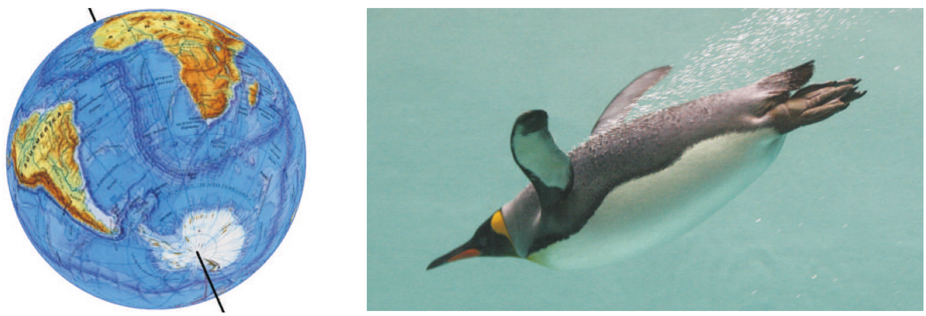

Fig. 2.42 the Antarctic and one of its inhabitants – the emperor penguin

$21 \cdot 10^6\,\mathrm{km}^3$ remains.

The Earth has a surface area of about $500 \cdot 10^6\,\mathrm{km}^2$, of which $70\% \approx 350 \cdot 10^6\,\mathrm{km}^2$ is water. Let Δ be the difference in the heights of the sea level, then

$$350 \cdot 10^6\,\mathrm{km}^2 \cdot \Delta = 21 \cdot 10^6\,\mathrm{km}^3 \Rightarrow \Delta \approx \frac{21}{350} = 0.06\,\mathrm{km} = 60\,\mathrm{m}.$$

⊕ *Remark*: 60 m is quite a lot. With this rise of sea level, whole groups of islands would disappear, large parts of Florida would be flooded, etc. The emperor penguins probably would die out – not because the temperature would become too high, but because once again – as before – mammals and reptiles would start to live on the Antarctic. These mammals and reptiles would then steal the eggs of emperor penguins and eat their defenceless pups (see Application p. 421).

Fig. 2.43 The last glacial maximum ($\approx 24{,}500$ BC): Vast ice sheets covered large parts of northern Europe, North America, Siberia, and also parts of the southern hemisphere.

Conversely, the sea level rose to this level of 60 meters about 15 million years ago (because the Antarctic was free of ice). During the ice ages that have occurred several times in the last $100{,}000$ years (where, for example, thick ice was superimposed over central and northern Europe), the sea level was 125 m (!) lower than today. Today divers in Southern France have found underground entrances of caves where Stone Age people lived! The low sea allowed a small group of people to walk across the Bering Strait from Siberia to Alaska. During that time, they had more than $22{,}000$

years to migrate to the two American continents (they joined together only 3 million years ago). ⊕ ◂◂◂

▸▸▸ Application: ***The sun dies out!***

Our sun is a giant nuclear reactor where hydrogen fuses into helium. Here 4.5 million tons of mass per second are converted into energy. The sun has a mass that is $332,000$ times the mass of Earth (Application p. 91). How long would it theoretically take to use up the total mass of the Sun?

Solution:

We try to get an idea of the given magnitudes by relating them to more familiar objects: The loss of mass per second would be – "downscaled to earthly conditions" – $1/332,000$ of $4,500,000$ tons, i.e. 13.6 tons per second. Let us reduce further down to a ball with a diameter of only 1 m. The Earth has a diameter of about $13,000\,\mathrm{km} = 13\cdot 10^6\,\mathrm{m}$. Our nickel-iron sphere with a diameter of 1 m is about $1/(13\cdot 10^6)^3$ of the mass of Earth, and per second, it would lose $13.6\cdot 10^3\mathrm{kg}/(13\cdot 10^6)^3$ of its mass, which is in milligrams

$$\frac{13.6\cdot 10^9\,\mathrm{mg}}{13^3\cdot 10^{18}} \approx 0.006\cdot 10^{-9}\,\mathrm{mg}.$$

A year has approximately 30 million seconds. Thus, the unit sphere loses

$$0.006\cdot 10^{-9}\cdot 30\cdot 10^6\,\mathrm{mg} \approx 0.2\cdot 10^{-3}\,\mathrm{mg}$$

every year or five milligrams every $5,000$ years, or 1 kg every five billion years – that is, about three tons of deadweight.

⊕ *Remark*: Studies have shown that the Sun has a 10 billion year lifespan of which it has already elapsed half of its time. Regarding the loss of mass via nuclear fusion, the limited lifespan of the Sun is not obvious: This proportion is practically negligible! Firstly, the limited lifespan is given by the uranium component of solar hydrogen, of which it only makes up about 70%, and secondly, only in the innermost part of the Sun are the temperatures high enough to propel nuclear fusion. Thus, only 10 to 20% of the solar mass is available for fusion of hydrogen helium. ⊕

⊕ *Remark*: The "fading" of the Sun as the source of all life is an old phobia of many indigenous people. Mayans and Aztecs in the historical parts of Central America made human sacrifices as a result of such fears! ⊕ ◂◂◂

▸▸▸ Application: ***diameter of a molecule***

An amount of $12\,g$ of carbon contains

$$N_A = 6.022\cdot 10^{23} \tag{2.23}$$

of ^{12}C molecules. This number is called the "Avogadro" constant (formerly *Loschmidt*'s number).

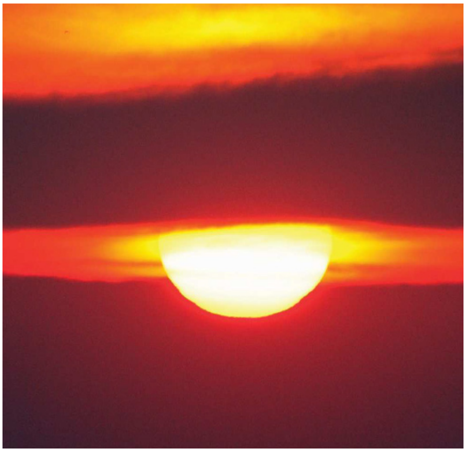

Fig. 2.44 the Sun dies out ...

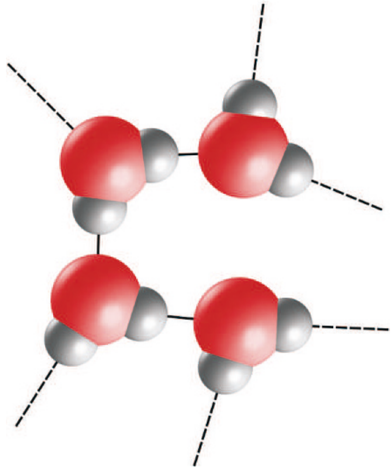

Fig. 2.45 water molecules

Water molecules consist of 1 oxygen atom and 2 hydrogen atoms (angle $\approx 105°$). The same number of molecules is contained in one mol of any substance (*mol*= molecular weight in grams).
Water (chemical formula H_2O) has a molecular weight of 18 since each water molecule contains two hydrogen atoms (each with an atomic weight of 1) and an oxygen atom (an atomic weight of 16).
So, in 18 g of water, there are N_A water molecules.
One can estimate the diameter of a water molecule.
Solution:

18 g of water has a volume of 18 cm^3. These fit into a cube with a side length of $\sqrt[3]{18}\,\text{cm} \approx 2.6\,\text{cm}$.
Suppose we pack every single water molecule into an "elementary cube" of side length d. Then, N_A of these cubes would fit into said cube with a side length of 2.6 cm. Thus, we have

$$d^3\,N_A = 18\,\text{cm}^3 \Rightarrow d = \sqrt[3]{\frac{18\,\text{cm}^3}{N_A}} = \sqrt[3]{\frac{18}{6\cdot 10^{23}}}\text{cm} = \sqrt[3]{\frac{180}{6\cdot 10^{24}}}\text{cm}$$

$$\Rightarrow\ d = \sqrt[3]{\frac{30}{10^{24}}}\text{cm} \approx 3\cdot 10^{-8}\,\text{cm} = 3\cdot 10^{-10}\text{m} = 0.3\cdot 10^{-9}\,\text{m}.$$

We have shown that the diameter of a molecule is less than one nanometer.

⊕ *Remark*: According to the above result, we can say that, in 2 g of hydrogen, there are N_A hydrogen molecules H_2. From this, one can determine the mass of a hydrogen atom, H, which used to be the so-called *atomic mass* unit until 1961 (since then, $\frac{1}{12}$ of the atomic mass of ^{12}C has been used as the legal unit): ⊕

⊕ *Remark*:

$$u = \frac{1}{2}\cdot\frac{2}{6.022\cdot 10^{23}}\,\text{g} = 0.166\cdot 10^{-23}\,\text{g} = 1.66\cdot 10^{-27}\,\text{kg}.$$

Accordingly, a helium atom has a mass of $\approx 4\,u$, one carbon atom has $12\,u$ (since exactly 1961!). An oxygen atom contains $\approx 16\,u$, a gold atom $\approx 197\,u$ etc. – each of the corresponding values are located in the "Periodic Table of Elements". The mass of the electron is (almost) negligible: An electron has a mass of $\frac{1}{1,824} u \approx 9 \cdot 10^{-31}$ kg (note that this is residual mass). ⊕ ◀◀◀

▶▶▶ Application: ***continental drift***

Pangea broke up into two roughly equal continents called *Laurasia* and *Gondwana* roughly 280 million years ago. About 150 million years ago, *Gondwana* further broke and has been drifting apart since then – as seen, for example, in the movement of South America and Africa away from each other. What used to be a daring theory (Alfred *Wegener*, 1912) can be confirmed through measurement today. Assume that the drift velocity v was reasonably constant and calculate how far Africa and South America drift apart every year or how far they drift apart every second when their current distance is about $5,000\,\mathrm{km}$. ◀◀◀

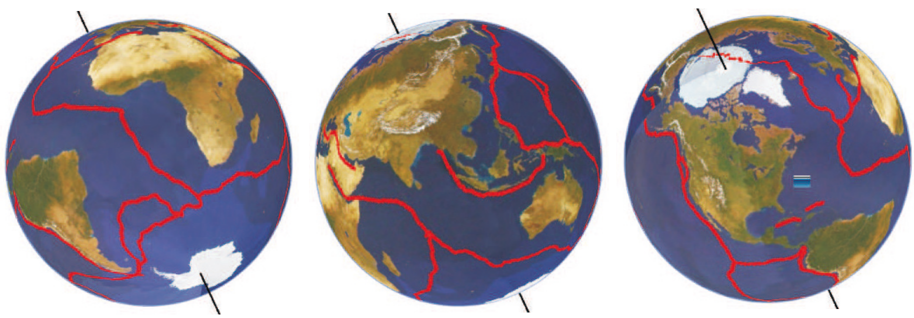

Fig. 2.46 drifting continents, tectonic plates

Solution:

$$v = \frac{5,000\,\mathrm{km}}{150 \cdot 10^6\,a} = \frac{5 \cdot 10^6\,\mathrm{m}}{150 \cdot 10^6\,a} \approx \frac{3 \cdot 10^{-2}\,\mathrm{m}}{a} = \frac{3 \cdot 10^{-2}\,\mathrm{m}}{365 \cdot 24 \cdot 3,600\,\mathrm{s}} = \frac{3 \cdot 10^{-2}\,\mathrm{m}}{3 \cdot 10^7\,\mathrm{s}} = \frac{1 \cdot 10^{-9}\,\mathrm{m}}{\mathrm{s}}.$$

We have a drift velocity per year of 3 cm, and as a result of Application p. 49, we know this involves about *three water molecules per second*, given that we know their diameters.

⊕ *Remark*: We can measure the original distance using the drift velocity of 3 cm per year by looking at the current positions (currently at 1 m accuracy) of many places in Africa or South America over large time intervals by means of GPS (Application p. 13, Application p. 363). Then, the drift velocity is obtained by calculating the average, and this becomes more accurate given the more measurements one has taken and the longer the time interval is (for example, 5 years). ⊕

▶▶▶ Application: ***a global conveyor belt as an air motor***

The enormous amount of 20 million cubic meters of salt water (which is almost half the amount of freshwater on Earth) flows every second, most of it at great depths. It flows in streams at a rate of 1 to 3 km per day repeatedly around the world (Fig. 2.47). In certain places, for example, in the Gulf of Mexico, this stream is "caught" and rises so that it heats up, quickly reaching the polar regions (the Gulf Stream!), where it cools down rapidly and descends again. Looking at the six projections in Fig. 2.47, the question arises: How long will a full cycle take?

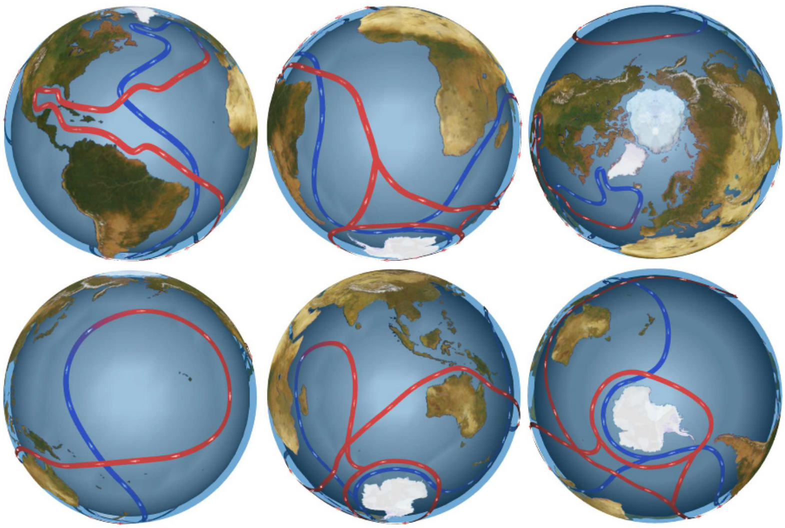

Fig. 2.47 thermohaline circulation, colloquially shown as a global conveyor belt in different views of the globe. Images using a single view of the globe can easily lead to misinterpretations of the currents (particularly around the Antarctic).

Solution:
An important preliminary remark: The circulation belt is usually depicted on a "rectangular projection" where each estimation of length – especially near the poles – leads to incorrect results.The sphere is doubly curved and cannot be unfolded distortion-free into the plane (Application p. 321). The multiple belts encircling the South Pole (bottom right) is clearly visible in the image. It is in total no more than a huge loop in the Pacific (bottom left). If we estimate the length of all flows thoroughly, then the length is roughly three times the Earth's circumference (120, 000 km). Assuming the flow covers a distance of 1 to 3 km daily, then assuming a yearly distance of 600 km – just to have a "nice number" – the cycle lasts for 200 years. ◂◂◂

▸▸▸ **Application**: ***lens power***

One can use Formula (4.29) to obtain the formula for the focal length of a thin biospheric lens (Fig. 2.48).

Solution:
Formula (4.29) applies to both spherical surfaces (radii r_1 and r_2, with refractive indices $n_1 = n$ and $n_2 = 1/n_1 = 1/n$) where Formula (4.29) is:
$\frac{1}{g} + \frac{n}{b} = \frac{n-1}{r}$, thus,

$$(1) \quad \frac{1}{g_1} + \frac{n}{b_1} = \frac{n-1}{r_1} \quad \text{and} \quad (2) \quad \frac{1}{g_2} + \frac{1/n}{b_2} = \frac{1/n-1}{r_2}.$$

Now, we can connect "in series": The image width b_1 of the first refraction is the object's distance, g_2, and the result for the second refraction is the image distance $b = b_2$ (negative!):

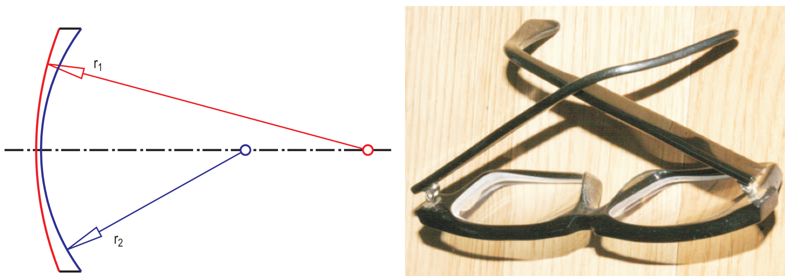

Fig. 2.48 aspherical lens: two slightly different curved spherical surfaces

$$\frac{1}{-b_1} + \frac{1/n}{b} = \frac{1/n-1}{r_2} \quad \Rightarrow \quad (3) \quad -\frac{n}{b_1} + \frac{1}{b} = \frac{1-n}{r_2}.$$

If we add (1) and (3) and set $g_1 = g$, then we obtain

$$\frac{1}{g} + \frac{1}{b} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

Objects at infinity ($g = \infty$) are mapped to the focal point (focal length f). With $1/g = 0$, we get

$$\frac{1}{f} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

Numerical example: Setting $r_1 = 16\,\text{cm}$, $r_2 = 8\,\text{cm}$, and $n = 1.4$ results in

$$\frac{1}{f} = (1.4-1)\cdot\left(\frac{1}{16\,\text{cm}} - \frac{1}{8\,\text{cm}}\right) = \frac{-0.4}{16\,\text{cm}} = -\frac{0.025}{\text{cm}} = -\frac{2.5}{\text{m}} \quad \Rightarrow \quad f = -0.4\,\text{m}.$$

The value denoted by $1/f$ (where f is measured in meters!) is called the lens power and it is expressed in diopters. Our lens has -2.5 dpt (a suitable power for a spectacle lens made for near-sighted people, Fig. 2.48 on the right).

⊕ *Remark*: The refractive power of the cornea is normally about 43 diopters (dpt) (focal length $f \approx 2.33\,\text{cm}$), the refractive index of the lens is about 19 diopters. Visual defects are expressed in positive or negative deviations. The emmetropic eye has a total of 65 diopters (focal length $f \approx 1.54\,\text{cm}$). Its value is not determined by combining the power of the lens and the cornea as discussed. With hyperopia, the rays meet behind the fovea centralis (with a positive deviation); with myopia, they meet just before (with a negative deviation). The cornea has a refractive index that can be compared with that of water. Therefore, underwater light from the cornea is hardly broken. But because the lens in these changing conditions cannot compensate enough, our vision becomes blurry when underwater. ⊕ ◂◂◂

▸▸▸ **Application**: ***depth of field (DOF) for a photographic lens*** (Fig. 2.51)
Derived from the lens equation in Application p. 16, $\frac{1}{f} = \frac{1}{b} + \frac{1}{g}$ (Formula (2.3)) follows with $b = n\,f$

$$g = \frac{n}{n-1}\,f. \tag{2.24}$$

Let us imagine the lens from Fig. 2.9a built into a camera lens. Behind the lens, the photosensitive layer will be perpendicular to the optical axis. Because of the lens equation, the only points that appear in sharp focus are those that lie in the plane γ parallel to the photo plane. Assume that for b, a tolerance of $t\,f$ is allowed. (In general, t is considered to be very small, e.g. $t = 0.01$). What is the "depth of field" in this case? How far is the object allowed to deviate from γ?

Solution:
We have $n\,f - t\,f \le b \le n\,f + t\,f$. The extreme image distances

$$b_1 = f(n+t) \text{ and } b_2 = f(n-t)$$

correspond, according to Formula (2.24), to extreme object distances

$$g_1 = \frac{n+t}{n+t-1}f, \quad g_2 = \frac{n-t}{n-t-1}f.$$

Then, the difference $g_2 - g_1$ equals the depth of field

$$s = \Big(\frac{n-t}{n-t-1} - \frac{n+t}{n+t-1}\Big)f.$$

We try to simplify the expression in parentheses by finding the common denominator

$$N = (n-t-1)(n+t-1) = [(n-1)-t][(n-1)+t] = (n-1)^2 - t^2.$$

The numerator Z is then $Z = (n-t)(n+t-1) - (n+t)(n-t-1) =$

$$= (n^2-t^2) - (n-t) - [(n^2-t^2) - (n+t)]n^2 - t^2 - n + t - [n^2 - t^2 - n - t] = 2t$$

and we have $s = \frac{Z}{N}\,f = \frac{2t}{(n-1)^2 - t^2}\,f$. If t is small, then t^2 is much smaller (for example, $t = 0.01 \Rightarrow t^2 = 0.0001$). Thus, we can write

$$s \approx \frac{2t\,f}{(n-1)^2}.$$

The result should not be interpreted rashly by assuming that a better depth of field is a larger one, i.e. the larger f, the larger s. Both t and n depend on f. Let us replace tf by t_0 (where t_0 is a constant value representing time). From the lens equation, we calculate

$$\frac{1}{b} = \frac{1}{f} - \frac{1}{g} = \frac{g-f}{fg} \Rightarrow n = \frac{b}{f} = \frac{g}{g-f} \Rightarrow n - 1 = \frac{g}{g-f} - 1 = \frac{f}{g-f}.$$

Then, we get

$$s \approx \frac{2\,t_0}{\left(\frac{f}{g-f}\right)^2} = 2\,t_0\left(\frac{g-f}{f}\right)^2 = 2\,t_0\left(\frac{g}{f} - 1\right)^2$$

and recognize:

The smaller the focal length f, or the further the object, the larger is the DOF.

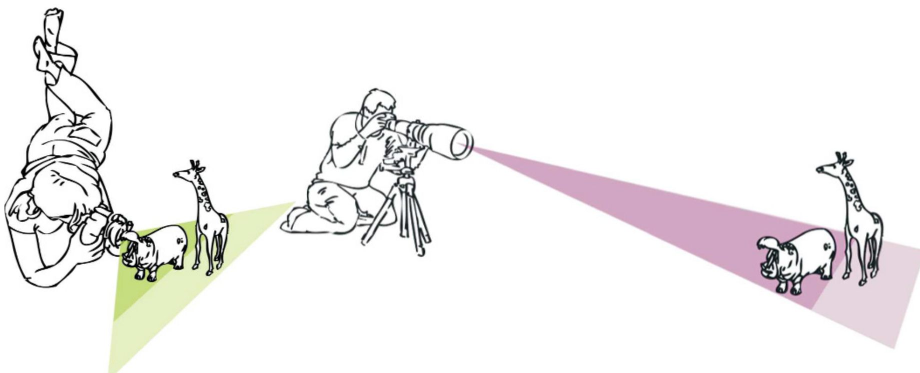

Fig. 2.49 wide-close shooting vs. telephoto shooting

Numerical example: We want to take a picture of an object that has a distance of 1.50 m, once with a wide-angle lens ($f_1 = 30\,\text{mm}$) and a second time with a telephoto lens of ($f_2 = 150\,\text{mm}$).
Let $g = 1,500\,\text{mm}$, i.e.

$$g/f_1 = 1,500\,\text{mm}/30\,\text{mm} = 50, \quad g/f_2 = 1,500\,\text{mm}/150\,\text{mm} = 10.$$

For the depth of fields, we have

$$s_1 = 2\,t_0 49^2, \quad s_2 = 2\,t_0 9^2 \Rightarrow s_1 : s_2 \approx 30 : 1.$$

Thus, we see that the telephoto lens can have a sharp image only with a *much* smaller depth range. Yet, you have to remember that such a lens allows us to capture details that might be displayed in sharp focus but they only appear tiny. Enlarging a section of the corresponding image will mostly lead to a loss of image resolution.

Fig. 2.50 depth I

Fig. 2.51 depth II

⊕ *Remark*: The lens equation is only accurate when the incident light rays do not deviate too much from the optical axis. This environment is called Gaussian space. The depth of this field can be increased very effectively by choosing a large aperture. Pictures in the macro area are very critical because they are mapped very close to the simple focal length. Fig. 2.51 shows two fighting male stag beetles (*Lucanus cervus*). Here, the depth could be achieved only through the widest aperture – this one must either use an ultra-light-sensitive film or a flash. Nevertheless, the foot joints in the front and rear area are blurred. In contrast, the blur of the background in Fig. 2.50 (garter snakes) – due to the small aperture of 2.8 and the short distance as a result of the short exposure time 1/250 s (Application p. 70) – is quite desirable. The background should be completely neutralized.
Intentional blur is an important design element in photography. In this specific case, it was important to capture the snakes' eyes and tongues in sharp focus, so that the viewer's attention is drawn to these elements of the photograph.
Even the human brain works in this manner, with the eyes acting like an "external branch". One could say that we have the advantage of a "selective perception". ⊕

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▸▸▸ **Application**: ***How long does an electron flash expose?***

The built-in flash of a DSLR cannot be arbitrarily "synchronized". One can illuminate only with relatively long exposure times (about 1/250 sec). Nevertheless, images that should actually be blurry may be engraved and appear sharper than preferred. Guess the actual exposure time by the flash of the pictures shown in Fig. 2.52.

Solution:

A ball (depicted in the middle of Fig. 2.52) is launched by a spring to a height of approximately 1.2m. At a height of 40 cm, the ball has just enough speed to make the remaining 80 cm.
With Formula 2.19 (Application p. 38), we deduce the instantaneous velocity of the ball $v = \sqrt{2gh} \approx 4\,\mathrm{m/sec} = 4,000\,\mathrm{mm/sec}$. During the time the picture was taken, the ball barely moves – perhaps 1 mm. This means that the flash exposes approximately $1/4,000$ of a second. So, under certain circumstances,

Fig. 2.52 left: the flashed (crisp) photo; middle: the test; right: without flash

one can "freeze" the blazing fast moving wings of a bumblebee (Fig. 2.52, left).

⊕ *Remark*: For comparison: 1/250 seconds (in sunlight) is definitely not sufficient to depict the wings of a bee in sharp focus (picture right). In direct sunlight, flash cannot be used to "freeze": The residual light during the synchronization time leads to the exposure of the film. Here, you have to work with an external and relatively expensive flash.
Digital cameras have no problem with flash synchronization, because they require no mechanical motion of a mirror. ⊕ ◂◂◂

▸▸▸ **Application**: ***the product of two pencils of rays*** (Fig. 2.53)
A straight line a rotates about a fixed point $A(0/0)$ with a constant angular velocity of 1 while a straight line b rotates about a fixed point $B(4/0)$ with a proportional angular velocity of a) -1, b) $+1$, c) 2. Consider the locus of all points of intersection.

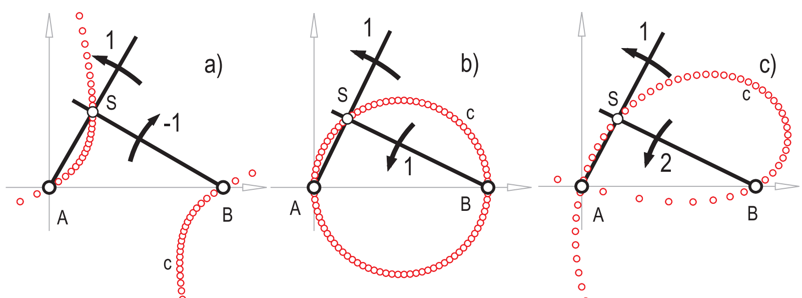

Fig. 2.53 "product" of two pencils of lines

Solution:
The overall calculation of all points is, of course, done by computer. Here, I will just say how to "draw up" the line equations: A straight line through the origin O with an inclination angle α to the x-axis has the equation $y = k_1\,x$ with $k_1 = \tan\alpha$ (or in the usual notation, of $-k_1\,x + y = 0$). A straight line

through the point $B(4/0)$ with the inclination angle β and $k_2 = \tan\beta$ has the equation $-k_2\,x + y = -4\,k_2$. The principal determinant of our $(2,2)$-system is, therefore, $D = \begin{vmatrix} -k_1 & 1 \\ -k_2 & 1 \end{vmatrix} = k_2 - k_1$. It vanishes if $k_1 = k_2$, a case which shall be excluded for the moment (since the lines are always parallel in this case). The other two determinants are

$$D_x = \begin{vmatrix} 0 & 1 \\ -4\,k_2 & 1 \end{vmatrix} = 4\,k_2 \quad \text{and} \quad D_y = \begin{vmatrix} -k_1 & 0 \\ -k_2 & -4\,k_2 \end{vmatrix} = 4\,k_1\,k_2.$$

Thus, the intersection of the lines is $S\left(\frac{4\,k_2}{k_2-k_1} / \frac{4\,k_1\,k_2}{k_2-k_1}\right)$.
Fig. 2.53 shows some specific solutions. In the general case, an equilateral hyperbola is defined as being generated by two indirectly congruent pencils of lines (Fig. 2.53a), a circle by two directly congruent pencils (Fig. 2.53b). If the ratio of the angular velocities equals $1:2$, the locus of intersection points is a cubic curve (Fig. 2.53c). If the two lines start at the same position with the ratio of the angular velocities equalling $-1:1$, then the hyperbola collapses to the bisector of AB. If the ratio of angular velocities is $1:1$ and the lines are parallel in the beginning, then they remain parallel. If the ratio equals $2:1$, then the point of intersection traces a circle (the central angle is twice the angle of circumference – see Fig. 4.52). ◂◂◂

▸▸▸ **Application**: ***supersonic speed in free-fall***

Felix *Baumgartner*'s stratosphere dive in October 2012 helps us find out how long one must fall within a (near-)vacuum in order to break the sound barrier, and it helps us to figure out the length of the distance travelled.

Solution:

Properly formulated, the problem is easily solved: The acceleration of gravity (1 g) is approximately 10 meters per second squared (at an altitude of 40 km it is still almost the same). This simply means that the instantaneous velocity increases every second by 10 meters per second. After 32 seconds, the diver is, therefore, able to reach 320 m/s, which corresponds to the speed of sound at about −20° Celsius (the speed of sound is dependent on temperature and increases by about 6 m/s given each 10° temperature increase.) Since the uniform acceleration is the average speed – equal to half the maximum speed, i.e. 160 m/s, the distance travelled is thus $32 \cdot 160 = 5,120$ meters.

⊕ *Remark*: If one theoretically accelerated (by means of rocket-power) for one year (that is 30 million seconds) with 1 g, one would reach 300,000 km/s – this is the speed of light. However, this is purely theoretical, because if you spin this idea further, you will reach superluminal speed ...

Realistically, however, just about 10 g (100 m/s speed increase per second) is bearable! At this rate, in just under two minutes, one could reach the necessary 11.2 km/s, in order to be able to escape the Earth's gravitational field (Application p. 403). ⊕

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▸▸▸ **Application**: ***A body brakes.***
A body (for example, an automobile) moves with velocity v_0 and should be brought to a standstill within d meters. What is the average deceleration a?
Solution:
For the instantaneous velocity v, the following applies in the case of a uniform deceleration a:

$$v = v_0 - a\,t. \tag{2.25}$$

For $v = 0$, this results in $t = \frac{v_0}{a}$. Inserting this variable into the formula for the distance

$$s = v_0\,t - \frac{a}{2}t^2 \tag{2.26}$$

one obtains

$$d = v_0\frac{v_0}{a} - \frac{a}{2}\left(\frac{v_0}{a}\right)^2 = \frac{v_0^2}{2a}.$$

From this, one can gather that the length of the braking distance depends on the square of the output speed. Thus, we obtain for the average deceleration

$$a = \frac{v_0^2}{2d}. \tag{2.27}$$

◂◂◂

▸▸▸ **Application**: ***bungy jumping*** (Fig. 2.54)
Bungy jumping (also “bungee jumping” or “bungi jumping”) is an “invention” of the inhabitants of New Zealand. In this daring activity, a person jumps from a fixed spot, at a great hight, to which they have been tied by an elastic rope. Until the rope length L_0 is reached, the person will fall freely. After that, the elastic rope will produce an increasing braking effect until the maximum cable length of L_{max} is reached. How long does the bungy jumper dive in absolute free fall?
To give a specific example: A jump from the Bloukrans River Bridge, South Africa (216 m height) is currently the longest possible dive: $L_0 = 90\,\text{m}$, $L_{max} = 170\,\text{m}$.
Solution:
The modified Formulas (2.25) and (2.26) for the vertical fall (initial velocity v_0, current velocity v, acceleration due to gravity $g \approx 9.81 m/s^2$) are:

$$v = v_0 + g\,t, \quad s = v_0\,t + \frac{g}{2}t^2. \tag{2.28}$$

When horizontal bounce applies, then $v_0 = 0$ (since there is no downward velocity component). The formula is valid until the time T_0, when the cable is tensioned. Then, we have

$$L_0 = 0 \cdot t + \frac{g}{2}T_0^2 \Rightarrow T_0 = \sqrt{\frac{2L_0}{g}}.$$

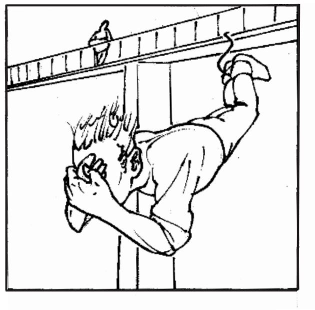

Fig. 2.54 bungy jumping

The corresponding velocity is

$$V_0 = 0 + gT_0 = \sqrt{2gL_0}. \tag{2.29}$$

In our special case ($L_0 = 90\,\text{m}$, $L_{max} = 170\,\text{m}$):
$T_0 \approx 4.28\,\text{s}$, $V_0 \approx 42\text{m/s} \approx 150\text{km/h}$

⊕ *Remark*: From the time T_0, g is reduced by the (ever-increasing) cable delay. The speed continues to increase until the rope force exceeds the bungee jumper's weight. This occurs when the rope reaches its maximum length and the jumper hangs at rest (i.e. when the rope force and gravity are in equilibrium).
We will examine the rather complex conditions in the final phase of the flight in more detail by means of differential calculus in (Application p. 274). Here, we only determine the (initially not very telling) *average* deceleration a and use it in the result of Formula (2.27) of Application p. 59. Using V_0 instead of v_0, the braking distance is the rope expansion $d = L_{max} - L_0$

$$a = \frac{V_0^2}{2(L_{max} - L_0)} = \frac{gL_0}{L_{max} - L_0}.$$

The average deceleration of the rope b is naturally greater than g, because the total acceleration consists of still-operating gravity and the counteracting deceleration of the rope.
For example, let $L_0 = 90\,\text{m}$, $L_{max} = 170\,\text{m}$:

$$a \approx 11\frac{m}{s^2} \Rightarrow b \approx 21\frac{m}{s^2} \approx 2.1g.$$

The average deceleration b of the rope is naturally not constant but it is, according to *Hook*e's law, proportional to the current cable strain $\varepsilon = \dfrac{L - L_0}{L_0}$. If b is $2.1\,g$ on average and increases linearly from 0 to a maximum value b_{max}, then b_{max} will be approximately equal to twice the average acceleration, i.e., about $4\,g$. ⊕ ◂◂◂

▸▸▸ **Application**: ***gravity through freefall***
During free fall, you are – at least in a vacuum – weightless. A person who jumps out of an airplane barely reaches speeds above 50 m/s, because air drag

and weight are balanced. A plane can overcome the barrier of air resistance by motor force. How long can one thus create weightlessness with an airplane?

Solution:

A plane cannot rise arbitrarily high because it takes a certain air density to fly. At an altitude of about 10 km, it could plunge towards the ground "like an eagle". Through proper use of the engines, constant acceleration can be achieved just as for a body in vacuum. At a certain instant, the pilot has to end the plunge and must not accelerate any further. Team and materials should not be overtaxed. From Application p. 59, we know that acceleration and deceleration have to take the same amount of time in order to reach a deceleration of 1 g. To stay on the safe side, you could thus accelerate from a height of 10 km to a height of 6 km and then be located in level flight again at a height of 2 kilometers. This would correspond to a free fall of 4,000 m. With

$$s = \frac{g}{2}\,t^2 \Rightarrow t = \sqrt{\frac{2s}{g}} \approx \sqrt{800}\,\mathrm{s} \approx 28\,\mathrm{s}$$

we compute 28 seconds of weightlessness. The maximum speed would be $v = g\,t \approx 280\,\mathrm{m/s}$, thus, below the speed of sound.

⊕ *Remark*: In fact, aircraft are used to perform physical experiments in weightlessness. An Airbus 300 is used in practice. Here weightlessness means achieving the ascent phase of a "parabolic flight": the machine faces straight up with an increasing speed and is then forced into an inverted flight parabola – this is where weightlessness occurs (the acceleration of gravity acts by reversing the parabola of the achieved speed, and so we get the same path acceleration along the opposite curve of the parabola). After the culmination point of the parabolic flight, the conditions in free fall are shown as described. ⊕ ◂◂◂

▸▸▸ **Application**: ***squaring the circle***

A square (in blue) is to be increasingly "rounded" so that the circumference or the area remain the same, as with the square shown on the left of Fig. 2.55 and even more with the one shown in the center of Fig. 2.55. In particular, the radius of the limit circle is to be determined so that the circumference or the area are equal to those of the initial square.

Solution:

Both in the circumference, as well as in the area of the circle, the circle constant π is utilized. One can, therefore, deliver no exact constructive solution to the problem, but only one through calculation:

Given the side a of the square, its circumference is $U = 4a$ and its area is $A = a^2$. With $U = 2\pi\,r$ or $A = \pi\,r^2$, we immediately have the radius of the limit circle:

$$2\pi\,r = 4a \Rightarrow r = \frac{2a}{\pi} \quad \text{or} \quad a^2 = \pi\,r^2 \Rightarrow r = \frac{a}{\sqrt{\pi}}.$$

For the intermediate circles (tangent distances y and the radius x of the rounding circles), we have:

$$4a = 4y + 4 \cdot \frac{2\pi x}{4} \quad \text{or} \quad y^2 + 4xy + 4\frac{\pi x^2}{4} = a^2.$$

With equal circumference, we have the linear condition $y = a - \frac{\pi}{2}x$; with equal area, we have the quadratic condition $y = -2x \pm \sqrt{a^2 + (4-\pi)x^2}$. The limit positions are reached at $y = 0$.

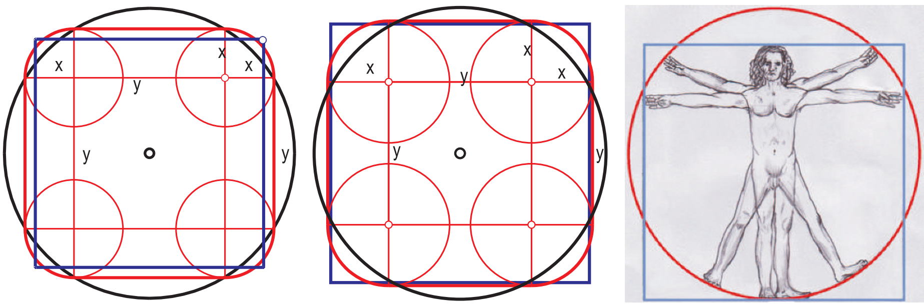

Fig. 2.55 converting a square into a circle (left: equal circumference; middle: equal area; right: an interpretation of *Leonardo*'s Vitruvian Man)

⊕ *Remark*: Leonardo da Vinci is supposedly believed to have found a graphical solution for land conversion (squaring the circle). He suggested that the radius of the circle should enlarge a thousand times (thus increasing the area to a million times larger) and this giant circle would be composed of a million equal sectors. Such a sector would be virtually indistinguishable from an isosceles triangle whose surface can be transformed into a rectangle, and then easily into a square. For a practitioner who is satisfied with a finite number of decimal places, this is a perfectly workable solution! In any case, his drawing, which is one of the most famous sketches in the world, depicts man inside a square or circle (Fig. 2.55, right). ⊕ ◂◂◂

▸▸▸ **Application**: ***quickly estimating a large area***

In Kolontar (near Ajkai in Hungary), the dam of a lake broke in October 2010. This lake was filled with toxic aluminium sludge. As the news spread across the world, it was rumoured (and this information is still available online on some websites) that an area of $40,000$ km^2 was flooded. In fact, the flooded area covered a surface of only 40 km^2 (which is still a considerable amount). As an ordinary citizen, how do you quickly assess the maximum amount of sludge brought by the flood in cubic meters or the realistic surface area of the flooded region at a given time?

Solution:

A Google Earth image (Fig. 2.56) with the plotted scale shows the dam (depicted in orange because at that time it was still filled with sewage sludge). The dam can be rapidly approximated by a polygon. Now you can convert

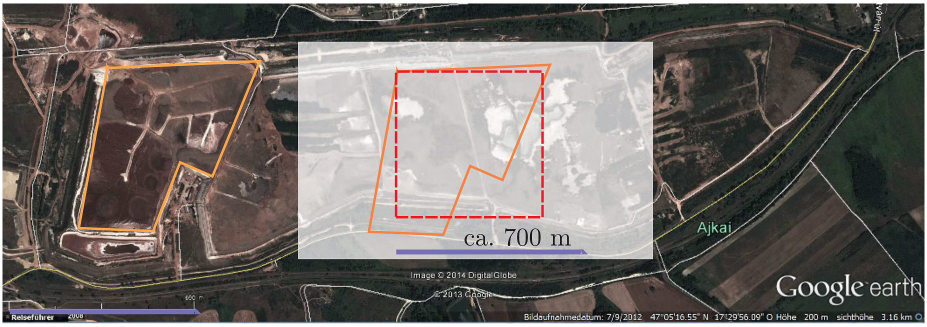

Fig. 2.56 Google Earth image with plotted scale (bottom left). The dam is in orange and approximated by a polygon. Middle: Estimated conversion of the polygon to a square (indicated by a red dashed line) of the same area.

the polygon "instinctively" and generously to a square, with no claim to great accuracy. Taking the scale of the image into account, the following can be said: If the square has 500 to 600 meters as a side length, then the area of the polygon will be approximately $300,000\ \mathrm{m}^2$. If the lake of sludge has a depth of 3 meters (a number that was bandied about and appeared realistic), the resulting area will be significantly larger as it will be covered by 1 million cubic meters of mud. With a sludge depth of 3 cm (instead of 3 m) the surface is reduced a hundredfold to $100 \cdot 300,000\ \mathrm{m}^2 \approx 30\ \mathrm{km}^2$, which is close to the actual result. ◂◂◂

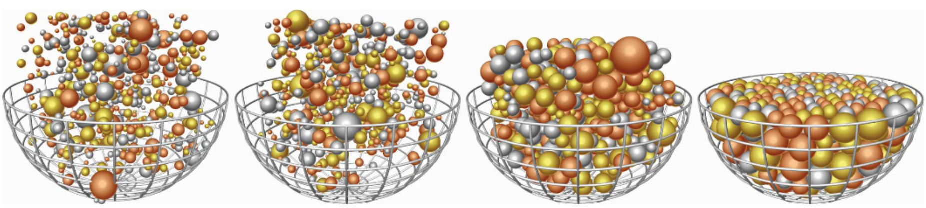

Fig. 2.57 put all together ...

▸▸▸ **Application**: ***all the gold in the world*** (Fig. 2.57)
An annual amount of $M_1 = 2,500$ tons of gold are produced worldwide. The American United States Geological Survey estimates that more than 80 percent of the current gold production of humanity was produced after 1900 and amounts to a total of $M_2 = 160,000$ tons of gold produced by human hands. This is "nothing" in relation to the estimated $M_3 = 30 \cdot 10^9$ tons produced throughout the Earth's crust (`http://www.zeit.de/2008/16/Stimmts-Gold`). How big would the respective cubes made up of the entire gold mass of M_1, M_2 and M_3be? Gold (aurum) has a density of $\varrho_{Au} = 19.2$ per dm^3.

Solution:
We compute with meters and tons. Then 19.2 tons have a volume of $1\ \mathrm{m}^3$. The

corresponding volumes are thus $V_1 = 2,500/19.2 \approx 130\text{m}^3$, $V_1 = 160,000/19.2 \approx 8,300\text{m}^3$, $V_3 = 30 \cdot 10^9/19.2 \approx 1.56 \cdot 10^9\text{m}^3$. By extracting the cubic root, we get the edge lengths of the cubes: 5 m, 20 m, and $\approx 1,160$ m.

⊕ *Remark*: Remarkably, there is a seemingly insignificant difference between the first two cubes. But the volume increases dramatically with the edge lengths of the cubes. This will be dealt with in more detail in the second chapter. ⊕ ◂◂◂

▸▸▸ **Application**: ***a crown of pure gold***

Legend says that King Hiero confronted the great all-round scientist Archimedes with a tricky question, and the answer was a matter of life or death for his goldsmith: He wanted to know if his crown was made of pure gold, but this should be determined without destroying the crown. Archimedes's answer: He brought a balance beam (Fig. 2.58) to balance between a piece of pure gold on one side and the crown on the other side. It changed upon immersion within water and was not at equilibrium, which spelt bad news for the goldsmith ...

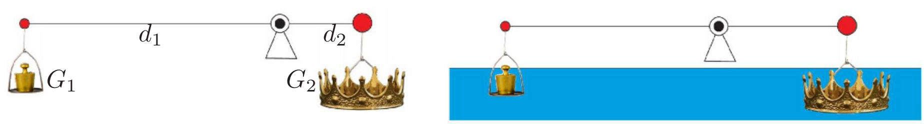

Fig. 2.58 the gold test of *Archimedes*: $G_1 : G_2 = d_2 : d_1$

Solution:

Here Archimedes combined two of his most important inventions: the law of the lever and Archimedes's principle concerning buoyancy. Suppose a calibrated weight of pure gold (weight G_1, mass $M_1 = G_1/\varrho_1$; the density $\varrho_1 = \varrho_{Au}$ being as given in the previous example). With a volume of V_2 and a density of ϱ_2, the crown has the weight $G_2 = V_2 \cdot \varrho_2 \cdot g$. If the beam balance (Fig. 2.58, left) is in equilibrium, then $G_1 : G_2 = d_2 : d_1$ from which G_2 can be deduced. If we now submerse the two weights completely in water, their buoyancy forces reduce by about the weight of either displaced amount of water. Thus, the respective density decreases indirectly by about the density of water, i.e., about 1 kg per dm^3:

$G_1^* = M_1 \cdot (\varrho_1 - 1) = G_1 \cdot (\varrho_1 - 1)/\varrho_1$ and $G_2^* = M_2 \cdot (\varrho_2 - 1) = G_2 \cdot (\varrho_2 - 1)/\varrho_2$.

With the same density $\varrho_1 = \varrho_2$, the balance remains in equilibrium. However, if $\varrho_2 < \varrho_1$, then the buoyancy of the crown is slightly higher, and as a result, the crown rises. Equilibrium is reached again only when the bearing point is moved to the left by a (measurable) distance of Δ (Fig. 2.58, right). With a little skill, you can calculate ϱ_2 from all known values, and it subsequently determines a possible silver content in the crown. This works well when using a weight G_1 that is not made of pure gold. ◂◂◂

▸▸▸ **Application**: ***swarm rules*** (Fig. 2.59)
At times, shoaling fish or flocking birds seem to "dance" in the water or in the sky. One might be excused for thinking that there is a complicated and deliberate choreography behind it. Animals often gather or travel together in large numbers. In many cases, this swarming behavior serves as a defense against predation. Yet, how can we explain the complex and intriguing motions of swarms, as these collections of animals change directions within a fraction of a second, split up into groups and then reunite?

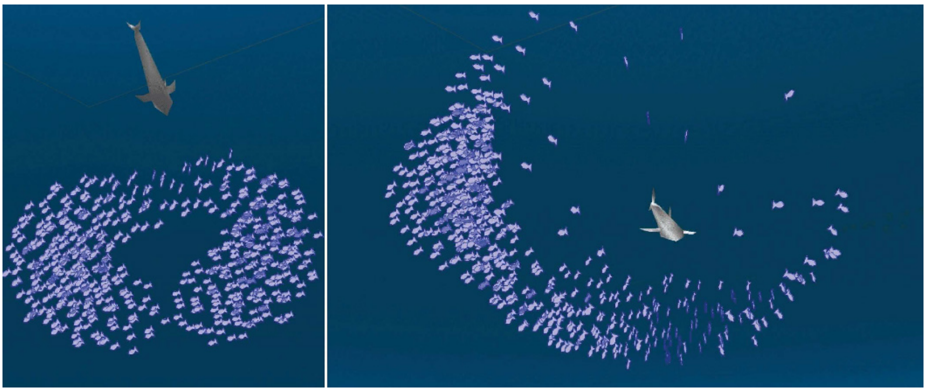

Fig. 2.59 three major rules ...

Solution:
One might assume the existence of an "alpha specimen" that determines the motion of the swarm. However, how is it possible that this individual always stays at the front of the pack? In fact, there is no such leader of the pack. All members of the swarm are equal while they are in motion and merely follow three very simple rules:
1. Move in a common direction.
2. Always keep a certain distance to your neighbours.
3. If a predator is approaching, escape.
In moments of danger, the distances between neighbours increase due to reaction time. The swarm becomes wider and may even "tear apart", but as soon as the predator has left the scene, the remaining group usually reunites. As a virtual shark attacks a swarm in a computer simulation, all individuals obey the abovementioned rules. This simulation yields extremely realistic behaviour and may, thus, be taken as heuristic "proof" that the swarm rules actually exist. What is more, predators are usually distracted by swarming behaviour, and the chances of survival are larger for the individual. ◂◂◂

▸▸▸ **Application**: ***sum of cross-sections*** (Fig. 2.60)
Thinkers as far back as Leonardo da Vinci have suspected that, as trees branch out into ever more intricate formations, their total cross-section, nevertheless, stays roughly the same. This apparent rule has been explored and

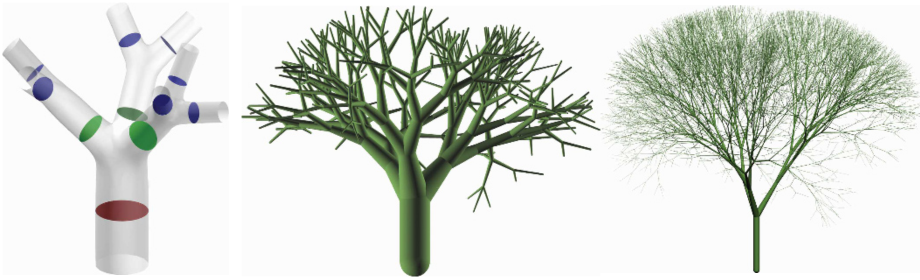

Fig. 2.60 Computer simulations based on different parameters of trunk thickness, iteration count, and branching angle. The sum of all cross-sections is constant.

refined by computer graphics engineers. Let us attempt an analysis based

Fig. 2.61 African Aloe tree

on the African Aloe tree (Fig. 2.61). It may not be exact, but it is sufficient to imagine the branches as having a locally circular cross-section. If this is true, then the following reasoning works relatively well: Wherever a first branching occurs, we select the center M of a sphere, which then includes the cross-section circles of the branches as small-circles. We can measure the radii of these circles. If the largest (lower) circle has a diameter of 1 unit (= 100%), then the smaller circles have radii of 0.57 (57%), 0.38 (38%), and 0.7 (70%). We know that the circle area increases quadratically as the radius grows linearly. In fact, our result of $0.57^2 + 0.38^2 + 0.7^2 = 0.96$ is relatively close to our expectation of $1^2 = 1$. It would seem that Leonardo's rule works well in the case of the smallest sphere. Now, we increase the sphere radius. The number of branches has grown to 25, which would suggest an average cross-section surface of 1/25 (1/5 of the maximum diameter). Leonardo's rule does not seem to apply this time, as the total cross-section seems to have grown smaller. This irregularity is even more pronounced when considering the third sphere, where the total cross-section is even smaller. ◂◂◂

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