

Chapter 2

Core Theory

The previous chapter established that neighborhood structures with the basic propositional modal language is an interesting and well-motivated logical framework. In this chapter, I move away from questions of motivation to explore the logical theory of neighborhood structures.

The main object of study is a neighborhood model $\langle W, N, V \rangle$ in which W is a non-empty set; N assigns a collection of subsets of W to each state; and V assigns a subset of W to each atomic proposition (see Definitions 1.11 and 1.9). In order to facilitate a comparison with relational models, it is convenient to let N be a relation $N \subseteq W \times \wp(W)$ (cf. Remark 1.10). Two different definitions of truth for the modal operator can be found in the literature. In order to compare and contrast these two definitions, I introduced two different modalities (here, I give the definition of truth treating N as a relation):

- $\mathcal{M}, w \models \langle \rangle \varphi$ iff there is a $X \subseteq W$ such that $w N X$ and $X \subseteq \llbracket \varphi \rrbracket_{\mathcal{M}}$.
- $\mathcal{M}, w \models \Box \varphi$ iff there is a $X \subseteq W$ such that $w N X$ and $X = \llbracket \varphi \rrbracket_{\mathcal{M}}$.

These two modalities are equivalent when the neighborhoods are monotonic (i.e., so that if $w N X$ and $X \subseteq Y$, then $w N Y$; see the discussion in Sect. 1.2.2). It is clear from this presentation that neighborhood models generalize the standard relational models $\langle W, R, V \rangle$, where $R \subseteq W \times W$ for the basic modal language (cf. Appendix A). Indeed, much of the mathematical theory of modal logic with respect to relational structures can be adapted to the more general setting involving neighborhood structures. In particular, there is a well-behaved notion of structural equivalence between neighborhood models matching the expressivity of the basic modal language (Sect. 2.1); there is a well-developed proof theory for weak systems of modal logic (Sects. 2.3 and 2.4.3); the canonical model method for proving axiomatic completeness can be adapted to the more general setting (Sect. 2.3.2); there is a generalization of frame correspondence theory linking properties of the neighborhood relation and valid formulas (Sect. 2.5); the satisfiability problem for non-normal modal logics is decidable (Sect. 2.4.1); and there is a standard translation into first-order logic (Sect. 2.6.3).

However, there are some important differences between neighborhood semantics and relational semantics for modal logic. Two of the most striking properties are that the satisfiability problem for many non-normal modal logics is **NP**-complete as opposed to **PSPACE**-complete (Sect. 2.4.2), and that there are consistent normal modal logics that are incomplete with respect to relational semantics but complete with respect to neighborhood semantics (Sect. 2.3.3). Finally, an important theme in this chapter is the relationship between neighborhood models and other semantics for the basic modal language (Sect. 2.2).

2.1 Expressive Power and Invariance

Once a language and semantics are defined, the first steps towards a model theory is to identify an appropriate notion of *structural equivalence* between models matching the expressivity of the language. For example, the appropriate notion of structural equivalence for first-order logic is an *isomorphism* (Enderton 2001, Chap. 2). For the basic modal language \mathcal{L} , the appropriate notion of structural equivalence between relational models (Definition A.1) is a *bisimulation* (Definition A.9). In this section, I show that there is a natural notion of a bisimulation between neighborhood models.

I start with the definition of *modal equivalence*. Suppose that \mathcal{M} is a neighborhood model. I write $\text{dom}(\mathcal{M})$ for the **domain** of \mathcal{M} —i.e., the set of states in \mathcal{M} . A pair \mathcal{M}, w with $w \in \text{dom}(\mathcal{M})$ is called a **pointed model**. For each pointed model \mathcal{M}, w , let

$$\text{Th}_{\mathcal{L}}(\mathcal{M}, w) = \{\varphi \in \mathcal{L} \mid \mathcal{M}, w \models \varphi\}.$$

The set of formulas $\text{Th}_{\mathcal{L}}(\mathcal{M}, w)$ is called the **theory** of \mathcal{M}, w —i.e., the set of all modal formulas true at w in \mathcal{M} . If $\text{Th}_{\mathcal{L}}(\mathcal{M}, w) = \text{Th}_{\mathcal{L}}(\mathcal{M}', w')$, then the two situations \mathcal{M}, w and \mathcal{M}', w' are indistinguishable from the point of view of the modal language \mathcal{L} .

Definition 2.1 (*\mathcal{L} -Equivalence*) Suppose that \mathcal{M}, w and \mathcal{M}', w' are two pointed neighborhood models and \mathcal{L} is a modal language. We say that \mathcal{M}, w and \mathcal{M}', w' are **\mathcal{L} -equivalent**, denoted $\mathcal{M}, w \equiv_{\mathcal{L}} \mathcal{M}', w'$, when $\text{Th}_{\mathcal{L}}(\mathcal{M}, w) = \text{Th}_{\mathcal{L}}(\mathcal{M}', w')$. If \mathcal{L} is the basic modal language (Definition 1.6) and $\mathcal{M}, w \equiv_{\mathcal{L}} \mathcal{M}', w'$, then we say that \mathcal{M}, w and \mathcal{M}', w' are modally equivalent.

Exercise 2.1 Use Exercise 1.7 part 2 to show that for all monotonic pointed models \mathcal{M}, w and \mathcal{M}', w' and the language \mathcal{L}^{mon} from Sect. 1.2.2, $\mathcal{M}, w \equiv_{\mathcal{L}} \mathcal{M}', w'$ iff $\mathcal{M}, w \equiv_{\mathcal{L}^{\text{mon}}} \mathcal{M}', w'$.

There is a natural notion of bisimulation between *monotonic* neighborhood models. In order to facilitate a comparison with the definition of a bisimulation on relational models, I state the following definition treating the neighborhood functions as relations.

Definition 2.2 (*Monotonic Bisimulation*) Suppose that $\mathcal{M} = \langle W, N, V \rangle$ and $\mathcal{M}' = \langle W', N', V' \rangle$ are two monotonic neighborhood models. A relation $Z \subseteq W \times W'$ is a **monotonic bisimulation** provided that, whenever wZw' :

Atomic harmony: for each $p \in \mathbf{At}$, $w \in V(p)$ iff $w' \in V'(p)$.

Zig: If $wN X$, then there is an $X' \subseteq W'$ such that $w'N' X'$ and $\forall x' \in X', \exists x \in X$ such that xZx' .

Zag: If $w'N' X'$, then there is an $X \subseteq W$ such that wNX and $\forall x \in X, \exists x' \in X'$ such that xZx' .

Write $\mathcal{M}, w \Leftrightarrow \mathcal{M}', w'$ when there is a monotonic bisimulation $Z \subseteq \text{dom}(\mathcal{M}) \times \text{dom}(\mathcal{M}')$ such that wZw' .

A simple, but instructive, induction on the structure formulas shows that monotonic bisimulations preserve truth over models:

Proposition 2.1 *Suppose that Z is a monotonic bisimulation between two monotonic models $\mathcal{M} = \langle W, N, V \rangle$ and $\mathcal{M}' = \langle W', N', V' \rangle$. Then, for all $\varphi \in \mathcal{L}$, for all $w \in W$, $w' \in W'$, if wZw' , then $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}', w' \models \varphi$. That is, $\mathcal{M}, w \Leftrightarrow \mathcal{M}', w'$ implies that $\mathcal{M}, w \equiv_{\mathcal{L}} \mathcal{M}', w'$.*

It is well known that (relational) bisimulations (Definition A.9) completely characterize modal equivalence on certain classes of relational models. For instance, for all *image-finite relational models* (relational models such that for all $w \in W$ the set of states accessible from w is finite), two states are modally equivalent iff the states are bisimilar. An analogous result holds on monotonic neighborhood models. Before stating this result, it is convenient to restrict the definition of a monotonic bisimulation to *non-monotonic core* of the neighborhood models (Definition 1.1). More formally, a **monotonic core bisimulation** is similar to a monotonic bisimulation, except that the zig and zag clauses are restricted to the non-monotonic core. For instance, the zig-condition of a monotonic core bisimulation is:

Zig^{nc}: If $X_1 \in N_1^{nc}(w_1)$, then there is an $X_2 \subseteq W_2$ such that $X_2 \in N_2^{nc}(w_2)$ and $\forall x_2 \in X_2, \exists x_1 \in X_1$ such that x_1Zx_2 .

A key observation is that on core-complete monotonic models, every monotonic core bisimulation is a monotonic bisimulation, and vice versa.

Proposition 2.2 *Suppose that \mathcal{M}_1 and \mathcal{M}_2 are core-complete monotonic neighborhood models. Then, Z is a monotonic bisimulation between \mathcal{M}_1 and \mathcal{M}_2 iff Z is a monotonic core bisimulation.*

Exercise 2.2 Prove Proposition 2.2.

I can now define a class of models for which there is a perfect match between bisimilarity and modal equivalence.

Definition 2.3 (*Locally Core-Finite*) A neighborhood model $\mathcal{M} = \langle W, N, V \rangle$ is locally core-finite provided that \mathcal{M} is core-complete and for each $w \in W$, $N^{nc}(w)$ is finite, and for all $X \in N^{nc}(w)$, X is finite.

Obviously, any model with finitely many states is locally core-finite. However, a model with infinitely many states can still be locally core-finite. I first prove that there is a perfect match between bisimilarity and modal equivalence on finite monotonic neighborhood models.

Theorem 2.4 Suppose that $\mathcal{M} = \langle W, N, V \rangle$ and $\mathcal{M}' = \langle W', N', V' \rangle$ are finite monotonic models (i.e., W and W' are finite sets). Then, for all $w \in W$, $w' \in W'$, $\mathcal{M}, w \equiv_{\mathcal{L}} \mathcal{M}', w'$ iff $\mathcal{M}, w \rightleftharpoons \mathcal{M}', w'$.

Proof The right-to-left direction is Proposition 2.1 (the result holds for all monotonic neighborhood models). For the left-to-right direction, suppose that $\mathcal{M} = \langle W, N, V \rangle$ and $\mathcal{M}' = \langle W', N', V' \rangle$ are monotonic locally core-finite models. We show that modal equivalence $\equiv_{\mathcal{L}}$ is a monotonic bisimulation (to simplify the notation, write \equiv instead of $\equiv_{\mathcal{L}}$).

Suppose that $X \in N(w)$. We must show that there exists $X' \in N'(w')$ such that for all $x' \in X'$, there exists $x \in X$ such that $x \equiv x'$. Suppose not. Since both models are finite, we have $N'(w') = \{X'_1, \dots, X'_k\}$ and $X = \{x_1, \dots, x_m\}$. Thus, the assumption is that for each $i = 1, \dots, k$, there exists $x'_i \in X'_i$ such that (*) for all $x_j \in X$, $x_j \not\equiv x'_i$. Fix a set of elements $x'_i \in X'_i$ for $i = 1, \dots, k$ satisfying (*). This means that for each $i = 1, \dots, k$, for each $j = 1, \dots, m$, there is a formula φ_{ij} such that $\mathcal{M}, x_j \models \varphi_{ij}$, but $\mathcal{M}', x'_i \not\models \varphi_{ij}$. Now, we have $\mathcal{M}, x_j \models \bigwedge_{i=1, \dots, k} \varphi_{ij}$; and so,

$$X \subseteq \llbracket \bigvee_{j=1, \dots, m} \bigwedge_{i=1, \dots, k} \varphi_{ij} \rrbracket_{\mathcal{M}}$$

Let $\varphi := \bigvee_{j=1, \dots, m} \bigwedge_{i=1, \dots, k} \varphi_{ij}$. Then, $\mathcal{M}, w \models \Box \varphi$. Since $\mathcal{M}, w \equiv \mathcal{M}', w'$, we have $\mathcal{M}', w' \models \Box \varphi$. However, this is a contradiction, since there is no $i = 1, \dots, k$ such that $X'_i \subseteq \llbracket \varphi \rrbracket_{\mathcal{M}'}$. \square

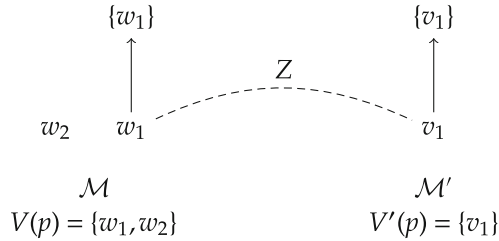
Exercise 2.3 Use Theorem 2.4 and Proposition 2.2 to prove that monotonic bisimulations characterize modal expressivity on locally core-finite monotonic neighborhood models:

Theorem 2.5 Suppose that $\mathcal{M} = \langle W, N, V \rangle$ and $\mathcal{M}' = \langle W', N', V' \rangle$ are monotonic, locally core-finite models. Then, for all $w \in W$, $w' \in W'$, $\mathcal{M}, w \equiv_{\mathcal{L}} \mathcal{M}', w'$ iff $\mathcal{M}, w \rightleftharpoons \mathcal{M}', w'$.

Theorem 2.5 shows that monotonic bisimulations capture modal expressivity on locally core-finite monotonic neighborhood models.¹ Interestingly, the above notion of a bisimulation applies only when the models are monotonic.

¹This result can be generalized to the class of *modally saturated* neighborhood models (Hansen 2003; Hansen et al. 2009).

Example 2.6 (Monotonic Bisimulations) Suppose that \mathcal{M} and \mathcal{M}' are neighborhood models, the first of which is not monotonic. Consider the relation Z pictured below between the domains of \mathcal{M} and \mathcal{M}' .



As the reader is invited to check, the dashed line satisfies all the conditions in Definition 2.2. That is, Z is a monotonic bisimulation. However, $\mathcal{M}, w_1 \not\models \Box p$ and $\mathcal{M}', v_1 \models \Box p$.

Thus, if the neighborhood models are not closed under supersets, then monotonic bisimulations do not necessarily preserve the truth of modal formulas.

Exercise 2.4 Prove that monotonic bisimulations preserves the truth of the modal language \mathcal{L}^{mon} on neighborhood models even if they are not monotonic. This suggests that it is smoother to develop a model theory for neighborhood models using the language \mathcal{L}^{mon} (cf. Hansen 2003).

The above example raises a question: What is the right notion of equivalence between *arbitrary* neighborhood models? A complete answer to this question is discussed in Hansen et al. (2009). The main idea is to use *bounded morphisms*.²

Definition 2.7 (Bounded Morphism) Suppose that $\mathcal{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, N_2, V_2 \rangle$ are two neighborhood models. If $f : W_1 \rightarrow W_2$ and $X \subseteq W_2$, then let $f^{-1}(X) = \{w \in W_1 \mid f(w) \in X\}$ be the inverse image of X . A function $f : W_1 \rightarrow W_2$ is a **bounded morphism** if

1. for all $p \in \mathbf{At}$, $w \in V_1(p)$ iff $f(w) \in V_2(p)$; and
2. for all $X \subseteq W_2$, $f^{-1}(X) \in N_1(w)$ iff $X \in N_2(f(w))$.

Bounded morphisms preserve the truth of the basic modal language.

Proposition 2.3 Suppose that $\mathcal{M} = \langle W, N, V \rangle$ and $\mathcal{M}' = \langle W', N', V' \rangle$ are two neighborhood models, and $f : W \rightarrow W'$ is a bounded morphism. Then, for all $\varphi \in \mathcal{L}$, $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}', f(w) \models \varphi$.

Proof The proof is by induction on the structure of φ . The argument for the base case and Boolean connectives is as usual. We only give the argument for the modal operator. Suppose that φ is of the form $\Box\psi$.

²The analogue of a bounded morphism for relational models is a *p-morphism* (Blackburn et al. 2001, Sect. 2.1).

The induction hypothesis is that for all $w \in W$, $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}', f(w) \models \varphi$. This means that $f^{-1}(\llbracket \varphi \rrbracket_{\mathcal{M}'}) = \llbracket \varphi \rrbracket_{\mathcal{M}_1}$ (where $f^{-1}(X) = \{w \mid f(w) \in X\}$ is the inverse image of X). Then,

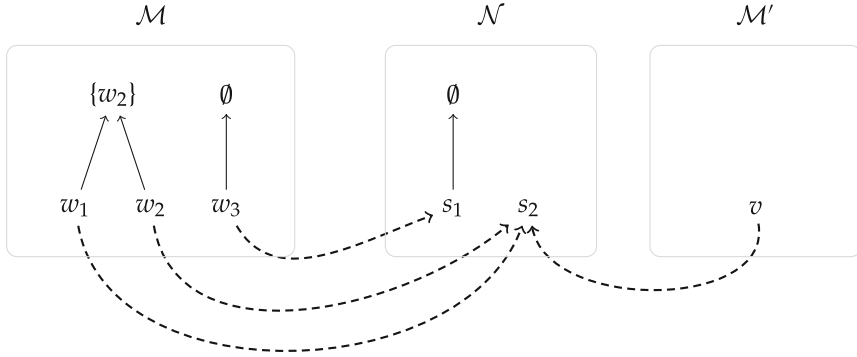
$$\begin{aligned}
 \mathcal{M}, w \models \Box\psi &\text{ iff } \llbracket \psi \rrbracket_{\mathcal{M}} \in N(w) && \text{(definition of truth)} \\
 &\text{ iff } f^{-1}(\llbracket \psi \rrbracket_{\mathcal{M}'}) = \llbracket \psi \rrbracket_{\mathcal{M}} \in N(w) && \text{(induction hypothesis)} \\
 &\text{ iff } \llbracket \psi \rrbracket_{\mathcal{M}'} \in N'(f(w)) && \text{(definition of bounded morphism)} \\
 &\text{ iff } \mathcal{M}', f(w) \models \Box\psi && \text{(definition of truth)}
 \end{aligned}$$

□

Exercise 2.5 What is the relationship between the Rudin–Keisler ordering discussed in Sect. 1.4.2 and bounded morphisms? For further comparisons between the Rudin–Keisler ordering and monotonic bisimulations, consult, Daniëls (2011).

Remark 2.8 (Bounded Morphisms vs. Monotonic Bisimulations) Note that the relation between the models in Example 2.6 is actually a function from W' to W (strictly speaking, the converse of the relation Z is the function). Let $f : W' \rightarrow W$ be the function where $f(v_1) = w_1$. It is instructive to see why this function is not a *bounded morphism* from \mathcal{M}' to \mathcal{M} . The second condition of Definition 2.7 requires that for all $X \subseteq W'$, $f^{-1}(X) \in N(w)$ iff $X \in N'(f(w))$. Let $X = \{w_1, w_2\}$. Then, $X \notin N'(f(v_1))$. However, $f^{-1}(X) = \{w_1\} \in N(w_1)$. Thus, f is not a bounded morphism. Note, also, that Z is not a bounded morphism from \mathcal{M} to \mathcal{M}' since a bounded morphism must be a total function.

I start with an illustrative example. Consider the following two neighborhood models: $\mathcal{M} = \langle W, N, V \rangle$ and $\mathcal{M}' = \langle W', N', V' \rangle$. In the first model, $W = \{w_1, w_2, w_3\}$, $N(w_1) = N(w_2) = \{w_2\}$ and $N(w_3) = \emptyset$. In the second model, $W' = \{v\}$ and $N(v) = \emptyset$. In both models, all propositional variables are false at all states. The models are pictured below. Note that w_1, w_2 and v are modally equivalent. At all three states, all formulas of the form $\Box\varphi$ are false and all formulas of the form $\Diamond\varphi$ are true. Since $N'(v) = \emptyset$, but $N(w_1) = N(w_2) \neq \emptyset$, there is no monotonic bisimulation between \mathcal{M} and \mathcal{M}' . Rather than trying to find a relationship between the two models, the idea is to show that both models can “live” inside a third model in such a way that modally equivalent states can be identified. For example, let $\mathcal{N} = \langle W'', N'', V'' \rangle$ be a neighborhood model with $W'' = \{s_1, s_2\}$, $N''(s_1) = \{\emptyset\}$, $N''(s_2) = \emptyset$, and for all atomic propositions p , $V''(p) = \emptyset$. There are bounded morphisms $f : \mathcal{M} \rightarrow \mathcal{N}$ and $g : \mathcal{M}' \rightarrow \mathcal{N}$ such that $f(w_1) = f(w_2) = g(v)$. The models and bounded morphisms are pictured below:



Definition 2.9 (*Behavioral Equivalence*) Suppose that $\mathcal{M} = \langle W, N, V \rangle$ and $\mathcal{M}' = \langle W', N', V' \rangle$ are neighborhood models, and let $w \in W$ and $w' \in W'$. Then, \mathcal{M}, w and \mathcal{M}', w' are **behaviorally equivalent** iff there is a neighborhood model $\mathcal{N} = \langle W'', N'', V'' \rangle$ and bounded morphisms $f : \mathcal{M} \rightarrow \mathcal{N}$ and $g : \mathcal{M}' \rightarrow \mathcal{N}$ such that $f(w) = g(w')$.

Proposition 2.4 Suppose that $\mathcal{M} = \langle W, N, V \rangle$ and $\mathcal{M}' = \langle W', N', V' \rangle$ are two neighborhood models. If states $w \in W$ and $w' \in W'$ are behaviorally equivalent, then for all $\varphi \in \mathcal{L}$, $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}', w' \models \varphi$.

The proof is an immediate consequence of Proposition 2.3.

Disjoint Unions

An important feature of the modal language is that the definition of truth is “local”. This feature is best exemplified by the fact that taking the *disjoint union* of models does not affect the truth of formulas at states in each component.

Definition 2.10 (*Disjoint Union*) Let $\{\langle W_i, N_i, V_i \rangle\}_{i \in I}$ be a collection of neighborhood models with disjoint sets of states. The **disjoint union** is the model $\uplus_i \mathcal{M}_i = \langle W, N, V \rangle$, where $W = \cup_{i \in I} W_i$; for all $p \in \mathbf{At}$, $V(p) = \cup_{i \in I} V_i(p)$; and

$$\text{For all } X \subseteq \bigcup_{i \in I} W_i, X \in N(w) \text{ iff } X \cap W_i \in N_i(w).$$

Proposition 2.5 Suppose that for all $i \in I$, $\mathcal{M}_i = \langle W_i, N_i, V_i \rangle$. Then, for each $i \in I$ and $w \in W_i$, for all $\varphi \in \mathcal{L}$, $\mathcal{M}_i, w \models \varphi$ iff $\uplus_i \mathcal{M}_i, w \models \varphi$.

The above proposition can be directly proved by an induction on the structure of formulas in \mathcal{L} . Alternatively, one can show that the natural embedding of each model into the disjoint union is a bounded morphism.

Exercise 2.6 Suppose that $\{\mathcal{M}_i = \langle W_i, N_i, V_i \rangle\}_{i \in I}$ is a collection of neighborhood models with disjoint sets of states and that $\uplus_i \mathcal{M}_i = \langle W, N, V \rangle$ is the disjoint union of the models. Prove that for each $i \in I$ and $w \in W_i$, for all the natural embedding

of each model \mathcal{M}_i into the disjoint union $\uplus_i \mathcal{M}_i$ (the 1-1 function that embeds each \mathcal{W}_i into $\bigcup_i \mathcal{W}_i$) is a bounded morphism.

Comparing Different Classes of Models

An important theme in this book is to compare and contrast neighborhood models with alternative models for the basic modal language. Often, the goal is to show that certain classes of neighborhood models are *modally equivalent* to some other class of models. In order to express this more formally, I need some notation. Suppose that \mathbf{M} is a class of models (such as relational models), and that \mathbf{pM} is the resulting class of pointed models—i.e., pairs \mathcal{M}, w where \mathcal{M} is a model from \mathbf{M} and w is a state from \mathcal{M} . Each class of models \mathbf{M} comes with a definition of *truth* for the basic modal language \mathcal{L} . Formally, “truth” for the modal language \mathcal{L} is a relation, denoted $\models_{\mathbf{M}}$, between pointed models from \mathbf{pM} and formulas $\varphi \in \mathcal{L}$ (write $\mathcal{M}, w \models \varphi$ when φ is true at w in \mathcal{M}). The definition of modal equivalence between neighborhood models (Definition 2.1) can be generalized to this more general setting.

Definition 2.11 (*Modal Equivalence between Classes of Models*) Suppose that \mathcal{L} is a modal language, and \mathbf{M} and \mathbf{M}' are two classes of models for \mathcal{L} . Let \mathcal{M}, w be a pointed model from \mathbf{pM} and \mathcal{M}', w' be a pointed model from \mathbf{pM}' . Say that \mathcal{M}, w is \mathcal{L} -equivalent to \mathcal{M}', w' , denoted $\mathcal{M}, w \equiv_{\mathcal{L}} \mathcal{M}', w'$, provided that $Th_{\mathcal{L}}(\mathcal{M}, w) = \{\varphi \mid \mathcal{M}, w \models_{\mathbf{M}} \varphi\} = \{\varphi \mid \mathcal{M}', w' \models_{\mathbf{M}'} \varphi\} = Th_{\mathcal{L}}(\mathcal{M}', w')$. If \mathcal{L} is the basic modal language, then we say that \mathcal{M}, w and \mathcal{M}', w' are **modally equivalent**.

A class of models \mathbf{M} is \mathcal{L} -equivalent to a class of models \mathbf{M}' provided for each pointed model \mathcal{M}, w from \mathbf{pM} , there exists a pointed model \mathcal{M}', w' from \mathbf{pM}' such that $\mathcal{M}, w \equiv_{\mathcal{L}} \mathcal{M}', w'$, and *vice versa*.

Typically, demonstrating that \mathbf{M} and \mathbf{M}' are modally equivalent involves showing how to transform models from \mathbf{M} into models from \mathbf{M}' and, conversely, how to transform models from \mathbf{M}' into models from \mathbf{M} . For instance, the following theorem is a direct consequence of Proposition 1.17 from Sect. 1.4.1.

Theorem 2.12 *The class $\mathbf{T} = \{\mathcal{M}^T \mid \mathcal{M}^T \text{ is a topological model}\}$ is modally equivalent to the class $\mathbf{MS}_4 = \{\mathcal{M} \mid \mathcal{M} \text{ is an S4 neighborhood model}\}$.*

2.2 Alternative Semantics for Non-normal Modal Logics

Neighborhood models are not the only semantics for the basic modal language. Indeed, depending on the intended interpretation of the modalities, neighborhood models may not always be the best choice of semantics for weak modal logics (cf. the discussion of logics of ability in Sect. 1.3). It is important to understand the relationship between neighborhood models and alternative semantics for the basic modal language. To keep the discussion manageable, in this section, I focus on variations of relational models (Definition A.1). Consult Venema (2007) and Chellas

(1980, Exercises 7.11, 7.42, 7.43, and 8.33) for discussions of *algebraic models*. There are also *coalgebraic models* for the basic modal language (Kupke and Pattinson 2011) that generalize neighborhood models (Hansen and Kupke 2004; Hansen et al. 2009; Venema 2007).

2.2.1 Relational Models

Let R be a relation on a non-empty set W (i.e., $R \subseteq W \times W$). For each $w \in W$, let $R(w) = \{v \mid wRv\}$, and for each $X \subseteq W$, let $R(X) = \{w \mid \exists v \in X \text{ such that } wRv\}$. So, $R(w)$ is the set of states that w can “see” via the relation R , and $R(X)$ is the set of states that can “see” some element of X (via the relation R).

Definition 2.13 (*R-Necessity*) Let R be a relation on a non-empty set W and $w \in W$. A set $X \subseteq W$ is **R -necessary at w** if $R(w) \subseteq X$. Let \mathcal{N}_w^R be the set of sets that are R -necessary at w (we simply write \mathcal{N}_w if R is clear from the context). That is, $\mathcal{N}_w^R = \{X \mid R(w) \subseteq X\}$.

The following Lemma shows that the collection of R -necessary sets for some relation R have very nice algebraic properties.

Lemma 2.1 *Let R be a relation on W . Then, for each $w \in W$, \mathcal{N}_w is augmented.*

Exercise 2.7 Prove Lemma 2.1.

Furthermore, properties of R are reflected in this collection of sets.

Observation 2.14 *Let W be a set and $R \subseteq W \times W$.*

1. *If R is reflexive, then for each $w \in W$, $w \in \cap \mathcal{N}_w$*
2. *If R is transitive, then for each $w \in W$, if $X \in \mathcal{N}_w$, then $\{v \mid X \in \mathcal{N}_v\} \in \mathcal{N}_w$.*

Proof Suppose that R is reflexive. Let $w \in W$ be an arbitrary state. Suppose that $X \in \mathcal{N}_w$. Then, since R is reflexive, wRw and, hence, $w \in R(w)$. Therefore by the definition of \mathcal{N}_w , $w \in X$. Since X was an arbitrary element of \mathcal{N}_w , $w \in X$ for each $X \in \mathcal{N}_w$. Hence, $w \in \cap \mathcal{N}_w$.

Suppose that R is transitive. Let $w \in W$ be an arbitrary state. Suppose that $X \in \mathcal{N}_w$. We must show $\{v \mid X \in \mathcal{N}_v\} \in \mathcal{N}_w$. That is, we must show $R(w) \subseteq \{v \mid X \in \mathcal{N}_v\}$. Let $x \in R(w)$. Then wRx . To complete the proof, we need only show that $X \in \mathcal{N}_x$. That is, we must show $R(x) \subseteq X$. Since R is transitive, $R(x) \subseteq R(w)$ (why?). Hence, since $R(w) \subseteq X$, $R(x) \subseteq X$. \square

Exercise 2.8 State and prove analogous results for the situations in which R is serial (for all $w \in W$, there exists a v such that wRv), Euclidean (for all $w, v, u \in W$, if wRv and wRu then vRu) and symmetric (for all $w, v \in W$, if wRv , then vRw).

Recall that a **relational frame** \mathfrak{F} is a tuple $\langle W, R \rangle$, where $W \neq \emptyset$ and $R \subseteq W \times W$; and a **relational model** for $\mathcal{L}(\mathbf{At})$ based on \mathfrak{F} is a tuple $\langle \mathfrak{F}, V \rangle$ where $V : \mathbf{At} \rightarrow \wp(W)$ is a propositional valuation function (cf. Definition A.1). Both relational models and neighborhood models can be used to provide a semantics for the basic modal language (cf. Appendix A). It should be clear that neighborhood models are more general than relational models (that is, neighborhood models satisfy more sets of formulas than relational models). The following Theorem identifies the class of neighborhood models that is modally equivalent to the class of relational models.

Theorem 2.15 *The class $\mathbf{K} = \{\mathfrak{M} \mid \mathfrak{M} \text{ is a relational model}\}$ is modally equivalent to the class $\mathbf{M}_{aug} = \{\mathcal{M} \mid \mathcal{M} \text{ is an augmented neighborhood model}\}$.*

The proof of this Theorem starts with a definition of equivalence between neighborhood and relational frames.

Definition 2.16 Let W be a nonempty set of states, $\langle W, N \rangle$ a neighborhood frame and $\langle W, R \rangle$ a relational frame. We say that $\langle W, N \rangle$ and $\langle W, R \rangle$ are **point-wise equivalent** provided that for all $X \subseteq W$, $X \in N(w)$ iff $X \in \mathcal{N}_w^R$.

Exercise 2.9 Prove that if a neighborhood frame $\mathcal{F} = \langle W, N \rangle$ and a relational frame $\mathfrak{F} = \langle W, R \rangle$ are point-wise equivalent, then for any propositional valuation $V : \mathbf{At} \rightarrow \wp(W)$, if $\mathcal{M} = \langle \mathcal{F}, V \rangle$ and $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$, then for all $w \in W$, \mathcal{M}, w and \mathfrak{M}, w are modally equivalent.

Using Exercise 2.9, the proof of Theorem 2.15 is a simple consequence of the following two Lemmas.

Lemma 2.2 *Let $\langle W, R \rangle$ be a relational frame. Then, there is a modally equivalent augmented neighborhood frame.*

Proof The proof is straightforward given Lemma 2.1: for each $w \in W$, let $N(w) = \mathcal{N}_w^R$ (where R is the relation under consideration). \square

Lemma 2.3 *Let $\langle W, N \rangle$ be an augmented neighborhood frame. Then, there is a modally equivalent relational frame.*

Proof Let $\langle W, N \rangle$ be a neighborhood frame. We must define a relation R_N on W . Since $\langle W, N \rangle$ is augmented, for each $w \in W$, $\cap N(w) \in N(w)$. For each $w, v \in W$, let $w R_N v$ iff $v \in \cap N(w)$. To show that $\langle W, R_N \rangle$ and $\langle W, N \rangle$ are equivalent, we must show that for each $w \in W$, $\mathcal{N}_w^{R_N} = N(w)$. Let $w \in W$ and $X \subseteq W$. If $X \in \mathcal{N}_w^{R_N}$, then $R_N(w) \subseteq X$. Since $R_N(w) = \cap N(w)$ and N contains its core, $R_N(w) \in N(w)$. Furthermore, since N is supplemented and $R_N(w) = \cap N(w) \subseteq X$, $X \in N(w)$. Now, suppose that $X \in N(w)$. Then, clearly, $\cap N(w) \subseteq X$. Hence, $X \in \mathcal{N}_w^{R_N}$. \square

Exercise 2.10 Suppose that \mathbf{K}_{eq} is the class of relational models $\mathfrak{M} = \langle W, R, V \rangle$, where R is an equivalence relation. Find the class of neighborhood models that is modally equivalent to \mathbf{K}_{eq} .

2.2.2 Generalized Relational Models

The next class of models is intended to provide a natural semantics for so-called *non-adjunctive logics*. These are modal logics that do not include the axiom scheme (C): $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$. Schotch and Jennings introduced a semantics for such logics in a series of papers (Schotch and Jennings 1980; Jennings and Schotch 1981, 1980).

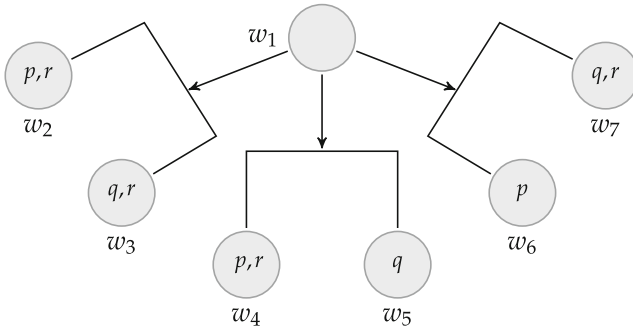
Definition 2.17 (*n-ary Relational Model*) An n -ary relational model (where $n \geq 2$) is a tuple $\langle W, R, V \rangle$, where W is a non-empty set and $R \subseteq W^n$ is an n -ary relation,³ and $V : \mathbf{At} \rightarrow \wp(W)$ is a valuation function.

So, relational models (cf. Definition A.1) are 2-ary models. The definition of truth for the basic modal language $\mathcal{L}(\mathbf{At})$ follows the usual pattern. Let $\mathfrak{M}^n = \langle W, R, V \rangle$ be an n -ary relational model and $w \in W$. The Boolean connectives are defined as usual. The clauses for the modal operators are:

- $\mathfrak{M}^n, w \models \Box\varphi$ iff for all $(w_1, \dots, w_{n-1}) \in W^{n-1}$, if $(w, w_1, \dots, w_{n-1}) \in R$, then there exists i such that $1 \leq i \leq n$ and $\mathfrak{M}^n, w_i \models \varphi$.
- $\mathfrak{M}^n, w \models \Diamond\varphi$ iff there exists $(w_1, \dots, w_{n-1}) \in W^{n-1}$ such that $(w, w_1, \dots, w_{n-1}) \in R$, and for all i such that $1 \leq i \leq n$, we have $\mathfrak{M}^n, w_i \models \varphi$.

An n -ary frame is a pair $\langle W, R \rangle$, where $R \subseteq W^n$ is an n -ary relation. The standard logical notions of satisfiability and validity are defined as usual (cf. Definition 1.13).

Example 2.18 (A 3-ary Relational Model) Let $\mathfrak{M}^3 = \langle W, R, V \rangle$ be a 3-ary relational model for the modal language generated from the atomic propositions $\mathbf{At} = \{p, q, r\}$, where $W = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$; $R = \{(w_1, w_2, w_3), (w_1, w_4, w_5), (w_1, w_6, w_7)\}$; and $V(p) = \{w_2, w_4, w_6\}$, $V(q) = \{w_3, w_5, w_7\}$ and $V(r) = \{w_2, w_3, w_7\}$. This model is depicted as follows:



According to the above definition of truth for the modal operators on n -ary relational models, we have:

³I write X^n for the n -fold cross product of the set X . That is, X^n consists of all tuples $\langle x_1, \dots, x_n \rangle$ of length n where each $x_i \in X$.

- $\mathfrak{M}^3, w_1 \models \Box p$ (and $\mathfrak{M}^3, w_1 \models \Box \neg p$);
- $\mathfrak{M}^3, w_1 \models \Box q$ (and $\mathfrak{M}^3, w_1 \models \Box \neg q$); and
- $\mathfrak{M}^3, w_1 \not\models \Box(p \wedge q)$.

The above model shows that the axiom scheme (C) is not valid on the class of 3-ary relational models. Consider, again, the model \mathfrak{M}^3 given in Example 2.18. Note that $\mathfrak{M}^3, w_1 \models \Box r$, so we have $\mathfrak{M}^3 \models \Box p \wedge \Box q \wedge \Box r$. While \Box is not “closed under conjunction”,⁴ a weaker conjunctive closure condition is satisfied: $\mathfrak{M}^3, w_1 \models \Box((p \wedge r) \vee (q \wedge r))$. The primary interest in n -ary relational models is that they can be used to study a hierarchy of weaker conjunctive closure principles. For each $n \geq 2$, define the following formula:

$$(C^n) \quad \bigwedge_{i=1}^n \Box \varphi_i \rightarrow \Box \bigvee_{1 \leq k, l \leq n, k \neq l} (\varphi_k \wedge \varphi_l).$$

So, for example, C^3 is the formula

$$(\Box \varphi_1 \wedge \Box \varphi_2 \wedge \Box \varphi_3) \rightarrow \Box((\varphi_1 \wedge \varphi_2) \vee (\varphi_2 \wedge \varphi_3) \vee (\varphi_1 \wedge \varphi_3)).$$

Exercise 2.11 Prove that the formula C^3 is valid on 3-ary relational frames.

Allen (2005) showed that every finite n -ary relational structure is modally equivalent to a finite monotonic neighborhood structure, and vice versa (cf. Arló-Costa 2005).

Theorem 2.19 *The class $\mathbf{K}^n = \{\mathfrak{M}^n \mid \mathfrak{M}^n \text{ is an } n\text{-ary relational model}\}$ is modally equivalent to the class*

$$\mathbf{M}_{mon} = \{\mathcal{M} \mid \mathcal{M} \text{ is a non-trivial monotonic neighborhood model}\}.$$

To illustrate, I give an example showing how to translate a neighborhood model into a modally equivalent n -ary relational model.

Example 2.20 (Neighborhoods to n -ary Relations) Let $\mathcal{M} = \langle W, N, V \rangle$ be a monotonic neighborhood model with $W = \{w, v\}$; $N(w) = \{\{w\}, \{v\}, \{w, v\}\}$ and $N(v) = \{\{w, v\}\}$; and $V(p) = \{w\}$ and $V(q) = \{v\}$. Note that $N^{nc}(w) = \{\{w\}, \{v\}\}$ and $N^{nc}(v) = \{\{w, v\}\}$. The first step is to add copies of the states so that each minimal neighborhood contains exactly two sets. To that end, let $\mathcal{M}' = \langle W', N', V' \rangle$ with $W' = W \cup \{w', v'\}$, where w' is a copy of w and v' is a copy of v . So, $N'(v) = N'(v') = \{\{w, v\}, \{w', v'\}\}$ and $N'(w) = N'(w') = \{\{w\}, \{v\}\}$; and $V'(p) = \{w, w'\}$ and $V'(q) = \{v, v'\}$. The second step is to construct a 3-ary relational model $\mathcal{M}^{N'} = \langle W^{N'}, R^{N'}, V^{N'} \rangle$ in which

- $W^{N'} = \{w, v, w', v'\}$;
- $R^{N'} = \{(w, w, v), (v, w, w'), (v, w, v'), (v, v, w'), (v, v, v')\}$; and
- $V^{N'} = V'$.

⁴In fact, we have $\mathfrak{M}^3, w_1 \not\models \Box(p \wedge q)$, $\mathfrak{M}^3, w_1 \not\models \Box(p \wedge r)$, and $\mathfrak{M}^3, w_1 \not\models \Box(q \wedge r)$.

Exercise 2.12 Prove the following proposition:

Proposition 2.6 *Suppose that $\mathcal{M} = \langle W, N, V \rangle$ is a finite monotonic neighborhood model such that for all $w \in W$, $N(w) \neq \emptyset$. Then, there is an n -ary relational model $\mathfrak{M}^N = \langle W^N, R^N, V^N \rangle$ that is modally equivalent to \mathcal{M} .*

The proof that every finite n -ary relational model can be transformed into a monotonic neighborhood model is more involved (consult Allen 2005).

2.2.3 Multi-relational Models

Goble (2000) used models with a set of relations as a semantics for a deontic logic in which there are possibly conflicting obligations arising from different normative sources (cf., also, Governatori and Rotolo 2005).

Definition 2.21 (*Multi-Relational Model*) Suppose that \mathbf{At} is a set of atomic propositions. A **multi-relational model** is a triple $\langle W, \mathcal{R}, V \rangle$, where W is a non-empty set; $\mathcal{R} \subseteq \wp(W \times W)$ is a set of serial relations (i.e., for all $R \in \mathcal{R}$, for all $w \in W$, there exists $v \in W$ such that $w R v$); and $V : \mathbf{At} \rightarrow \wp(W)$ is a valuation function.

The definition of truth for the basic modal language $\mathcal{L}(\mathbf{At})$ follows the usual pattern. Let $\mathfrak{M} = \langle W, \mathcal{R}, V \rangle$ be a multi-relational model and $w \in W$. Boolean connectives are defined as usual. The clauses for the modal operators are:

- $\mathfrak{M}, w \models \Box\varphi$ iff there exists $R \in \mathcal{R}$ such that for all $v \in W$, if $w R v$, then $\mathfrak{M}, v \models \varphi$.
- $\mathfrak{M}, w \models \Diamond\varphi$ iff for all $R \in \mathcal{R}$ there is a $v \in W$ such that $w R v$ and $\mathfrak{M}, v \models \varphi$.

Note that, according to the above definition, the relations in a multi-relational model $\langle W, \mathcal{R}, V \rangle$ are assumed to be serial. This means that for all states $w \in W$, for all $R \in \mathcal{R}$, the set $R(w) = \{v \mid w R v\}$ is non-empty. This assumption can be dropped, but doing so will lead to some complications. A state $w \in W$ is said to be a dead-end state with respect to a relation R provided that $R(w) = \emptyset$ (i.e., there are no states accessible from w). This means that if w is a dead-end state for a relation $R \in \mathcal{R}$ in a multi-relational model \mathfrak{M} , then $\mathfrak{M}, w \models \Box\perp$. When studying non-adjunctive logics, it is important to distinguish between situations in which $\Box\varphi \wedge \Box\neg\varphi$ is true and situations in which $\Box\perp$ is true. Ruling out dead-end states ensures that $\neg\Box\perp$ is valid.

2.2.4 Impossible Worlds

Impossible worlds were first introduced into modal logic by Saul Kripke (1965) to provide a semantics for some historically important systems of modal logic weaker

than **K**.⁵ Impossible worlds can be used in a variety of ways to weaken systems of modal logic (see Berto 2013 for a discussion). I will briefly discuss how to use impossible worlds to provide a semantics for regular modal logics. These are modal logics in which the necessitation rule is not valid (equivalently, modal logics that do not contain $\Box\top$).

Say that a world w is an **impossible world** in a model \mathfrak{M} if nothing is necessary (no formulas of the form $\Box\varphi$ are true at \mathfrak{M}, w) and everything is possible (all formulas of the form $\Diamond\varphi$ are true at \mathfrak{M}, w). The key idea is to distinguish between possible and impossible worlds in a relational model.

Definition 2.22 (*Impossible worlds*) A **relational model with impossible worlds** is a tuple $\langle W, W_N, R, V \rangle$, where W is a non-empty set of worlds; $W_N \subseteq W$; $R \subseteq W \times W$; and $V : \text{At} \rightarrow \wp(W)$.

Suppose that $\mathfrak{M} = \langle W, W_N, R, V \rangle$ is a relational model with impossible worlds. Truth for the basic modal language is defined as usual, except for the modal clause:

- $\mathfrak{M}, w \models \Box\varphi$ iff $w \in W_N$ and for all $v \in W$, if $w R v$, then $\mathfrak{M}, v \models \varphi$.

Adding impossible worlds to relational models is an elegant way to invalidate the necessitation rule (while keeping all other axioms and rules of normal modal logic intact). Consider any atomic proposition p , and suppose that $\mathfrak{M} = \langle W, W_N, R, V \rangle$ is a relational model with impossible worlds consisting of worlds $W = \{w, v\}$ and $W_N = \{w\}$ with $R = \{(w, v)\}$. Since the interpretation of the Boolean connectives is as usual at both possible and impossible worlds, we have that $p \vee \neg p$ is valid on any relational model with impossible worlds (in particular, $\mathfrak{M} \models p \vee \neg p$). However, since $v \notin W_N$, we have $\mathfrak{M}, v \not\models \Box(p \vee \neg p)$. Thus, we have $\mathfrak{M}, w \models \Box(p \vee \neg p)$; yet, since $w R v$, we have $\mathfrak{M}, w \not\models \Box\Box(p \vee \neg p)$. Thus, (Nec) is not valid over the class of relational models with impossible worlds.

There is much more to say about impossible worlds and how they can be used to model various non-normal modal logics. The interested reader is invited to consult Priest (2008) and Berto (2013), and references therein, for a more extensive discussion.

Exercise 2.13 Suppose that $\mathfrak{M} = \langle W, W_N, R, V \rangle$ is a relational model with impossible worlds. Find a neighborhood model \mathcal{M} that is modally equivalent to \mathfrak{M} .

2.3 The Landscape of Non-normal Modal Logics

I argued in Sect. 1.3 that there is interest in studying so-called *non-normal modal logics*. These are weak systems of modal logics in which one or more of the following formulas and rules are not valid.

⁵Note that Kripke called impossible worlds “non-normal”.

(Dual)	$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$	(N)	$\Box\top$
(M)	$\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$	(RM)	From $\varphi \rightarrow \psi$, infer $\Box\varphi \rightarrow \Box\psi$
(C)	$(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$	(Nec)	From φ , infer $\Box\varphi$
(K)	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$	(RE)	From $\varphi \leftrightarrow \psi$, infer $\Box\varphi \leftrightarrow \Box\psi$

There are two natural questions to ask about the above formulas and rules. First, which of them are valid on all neighborhood models? Second, are all the formulas and rules *independent*? That is, which of the axioms or rules can be *derived* from the others?

I start with the first question. There are neighborhood models that invalidate each of (M), (C), (K), (N), (RM) and (Nec). Example 1.14 is a countermodel to an instance of (M) (and also (RM)— see Lemma 2.6 below).

Observation 2.23 *There are formulas φ and ψ such that (C), $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$, and (K), $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$, are not valid on the class of all neighborhood frames.*

Proof For the first formula, consider the neighborhood model $\mathcal{M} = \langle W, N, V \rangle$ with $W = \{w, v\}$, $N(w) = \{\{w\}, \{v\}\}$, $N(v) = \{\emptyset\}$ and $V(p) = \{w\}$ and $V(q) = \{v\}$. Thus, $\mathcal{M}, w \models \Box p \wedge \Box q$, but since $\llbracket p \wedge q \rrbracket_{\mathcal{M}} = \emptyset \notin N(w)$, $\mathcal{M}, w \not\models \Box(p \wedge q)$.

For the second formula, we construct the following neighborhood model: $\mathcal{M} = \langle W, N, V \rangle$ with $W = \{w, v, s\}$, $N(w) = \{\{w\}, \{w, v, s\}\}$, $V(p) = \{w\}$ and $V(q) = \{w, v\}$. Then, $\llbracket p \rrbracket_{\mathcal{M}} = \{w\}$, $\llbracket q \rrbracket_{\mathcal{M}} = \{w, v\}$ and $\llbracket p \rightarrow q \rrbracket_{\mathcal{M}} = (\neg p \vee q)^{\mathcal{M}} = \{w, v, s\}$. Thus, we have $\mathcal{M}, w \models \Box(p \rightarrow q) \wedge \Box p$ but $\mathcal{M}, w \not\models \Box q$. \square

Exercise 2.14 Can you find a neighborhood model with a state in which all \Box -formulas are false, but all \Diamond -formulas are true? Is this possible with relational semantics?

So, which formulas *are* valid on all neighborhood frames? As noted above, Definition 1.12 ensures that \Box and \Diamond are duals. This gives the following validity:

Lemma 2.4 *The schema (Dual), $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$, is valid on any neighborhood frame.*

Proof Let $\mathcal{M} = \langle W, N, V \rangle$ be any neighborhood model. Then, using the Definition of truth, the above properties of truth sets, and basic set-theoretic reasoning, we have the following equivalences for any $w \in W$:

$$\begin{aligned}
 \mathcal{M}, w \models \Box\varphi & \text{ iff } \llbracket \varphi \rrbracket_{\mathcal{M}} \in N(w) \\
 & \text{ iff } W - (W - \llbracket \varphi \rrbracket_{\mathcal{M}}) \in N(w) \\
 & \text{ iff } W - (\llbracket \neg\varphi \rrbracket_{\mathcal{M}}) \in N(w) \\
 & \text{ iff } \mathcal{M}, w \not\models \Diamond\neg\varphi \\
 & \text{ iff } \mathcal{M}, w \models \neg\Diamond\neg\varphi
 \end{aligned}$$

Thus, $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$ is valid. \square

In addition, the inference rule (RE) is valid on the class of neighborhood frames.

Lemma 2.5 *On the class of all neighborhood frames, if $\varphi \leftrightarrow \psi$ is valid, then $\Box\varphi \leftrightarrow \Box\psi$ is valid.*

Proof The simple (and instructive!) proof is left to the reader. \square

The proof that (Dual) and (RE) are the only axiom and rules (in addition to propositional logic) needed to axiomatize the class of all neighborhood structures can be found in Sect. 2.3.2.

Let us now turn to the second question: Which axioms/rules can be *derived* from the others? I assume that the reader is familiar with the basics of Hilbert-style axiomatizations of modal logics. See Appendix A.2 for a brief introduction and Blackburn et al. (2001, Sect. 1.6) for a more extensive discussion. Let (PC) denote any axiomatization of propositional logic and (MP) denote the rule of Modus Ponens (from φ and $\varphi \rightarrow \psi$ infer ψ). The smallest (in the sense of the least number of consequences) logical system that we study in this book is **E**:

- **E** is the smallest set of formulas containing all instances of (PC) and (Dual) and is closed under the rules (RE) and (MP).

The other logical systems will be extensions of **E**. For example, the logic **EC** is the smallest set of formulas containing all instances of (PC), (Dual) and (C), and is closed under the rules (RE) and (MP). That is, **EC** extends the logic **E** by adding all instances of the axiom scheme (C). This is also the case for **EM**, **EN**, **ECM**, **EK** and **EMCN**. The logic **K** is the smallest set of formulas containing all instances of (PC), (Dual), (K), and the rules (Nec) and (MP). Note the difference between **K** and **EK**. Let **L** be any of the above logics; $\vdash_{\mathbf{L}} \varphi$ means that $\varphi \in \mathbf{S}$, and, in such a case, φ is said to be a **theorem of L**. As usual, if $\vdash_{\mathbf{L}} \varphi$, then there is a **deduction** of φ in the logic **L**.

Definition 2.24 (*Tautology*) A modal formula φ is called a **tautology** if $\varphi = (\alpha)^\sigma$ where σ is a substitution, α is a formula of propositional logic and α is a tautology.

For example, $\Box p \rightarrow (\Diamond(p \wedge q) \rightarrow \Box p)$ is a tautology because $a \rightarrow (b \rightarrow a)$ is a tautology in the language of propositional logic and

$$(a \rightarrow (b \rightarrow a))^\sigma = \Box p \rightarrow (\Diamond(p \wedge q) \rightarrow \Box p)$$

where $\sigma(a) = \Box p$ and $\sigma(b) = \Diamond(p \wedge q)$.

Definition 2.25 (*Deduction*) Suppose that **L** is an extension of **E**. A **deduction** in **L** is a finite sequence of formulas $\alpha_1, \dots, \alpha_n$ where for each $i = 1, \dots, n$ either (1) α_i is a tautology; (2) α_i is an instance of the axioms of **L**; or (3) α_i follows by Modus Ponens or the other rules of **L** from earlier formulas.⁶

⁶For Modus Ponens this means that there is $j, k < i$ such that α_k is of the form $\alpha_j \rightarrow \alpha_i$.

It is useful to introduce some terminology to classify different systems of propositional modal logic.

Definition 2.26 (*Classifying Modal Logics*) A propositional modal logic **L** is called

- a **normal modal logic** provided that it contains all instances of propositional tautologies, all instances of (**K**) and is closed under the rules (**Nec**) and (**MP**);
- a **minimal modal logic** (also called a **classical modal logic**⁷) provided that it contains all instances of propositional tautologies, all instances of (**Dual**) and is closed under the rules (**RE**) and (**MP**);
- a **monotonic modal logic** provided that it contains all instances of propositional tautologies, all instances of (**Dual**) and all instances of (**M**) and is closed under the rules (**RE**) and (**MP**);
- a **regular modal logic** provided that it contains all instances of propositional tautologies, all instances of (**Dual**), all instances of (**M**), all instances of (**C**) and is closed under the rules (**RE**) and (**MP**).

So, **K** is a normal modal logic and **E** is a non-normal modal logic (this follows from Observation 2.23). In fact, **K** is the *smallest* (in terms of the number of theorems) normal modal logic; **E** is the smallest classical modal logic; **EM** is the smallest monotonic modal logic; and **EMC** is the smallest regular modal logic (see Chellas 1980, Chap. 8 for a full discussion).

My first observation about the logics introduced above, is that one can prove a uniform substitution theorem in **E**. Given formulas $\varphi, \psi, \psi' \in \mathcal{L}$, let $\varphi[\psi/\psi']$ be the formula φ but replace *some* occurrences of ψ with ψ' (recall the definition of a substitution from Definition 1.8). For example, suppose that φ is the formula $\Box(\Diamond p \wedge \Box\Box q) \wedge \Box p$; ψ is the formula p ; and ψ' is the formula $\Box p$. Then, $\varphi[\psi/\psi']$ can be any of the following

- $\Box(\Diamond p \wedge \Box\Box q) \wedge \Box p$
- $\Box(\Diamond\Box p \wedge \Box\Box q) \wedge \Box p$
- $\Box(\Diamond p \wedge \Box\Box q) \wedge \Box\Box p$
- $\Box(\Diamond\Box p \wedge \Box\Box q) \wedge \Box\Box p$

The uniform substitution theorem states that we can always replace logically equivalent formulas.

Theorem 2.27 (Uniform Substitution) *The following rule can be derived in **E**:*

$$\frac{\psi \leftrightarrow \psi'}{\varphi \leftrightarrow \varphi[\psi/\psi']}$$

⁷This is the terminology found in Segerberg (1971) and Chellas (1980). However, this is a somewhat unfortunate name. Starting with Fitch (1948), there is a line of research studying *intuitionistic modal logics* (see, for instance, Artemov and Protopopescu (2016) for an interesting epistemic interpretation of intuitionistic modal logic touching on some of the issues discussed in this book). These are modal logics that extend intuitionistic propositional logics. In this literature, a “classical modal logic” is a modal logic that extends classical propositional logic (as opposed to intuitionistic propositional logic). One may be interested in both normal and non-normal modal logics that extend either classical or intuitionistic propositional logics.

Proof Suppose that $\vdash_E \psi \leftrightarrow \psi'$. We must show that $\vdash_E \varphi \leftrightarrow \varphi[\psi/\psi']$. First of all, note that if φ and ψ are the same formula, then either $\varphi[\psi/\psi']$ is φ (when ψ is not replaced) or $\varphi[\psi/\psi']$ is ψ' (when ψ is replaced). In the first case, $\varphi \leftrightarrow \varphi[\psi/\psi']$ is the formula $\varphi \leftrightarrow \varphi$ and so trivially, $\vdash_E \varphi \leftrightarrow \varphi[\psi/\psi']$. In the second case, $\varphi \leftrightarrow \varphi[\psi/\psi']$ is the formula $\psi \leftrightarrow \psi'$, which is derivable in **E** by assumption. Thus we may assume that φ and ψ are distinct formulas.

The proof is by induction on φ . The base case and Boolean connectives are left as an exercise for the reader. I demonstrate the modal operator. Suppose that φ is $\Box\gamma$ and $\vdash_E \psi \leftrightarrow \psi'$. The induction hypothesis is $\vdash_E \gamma \leftrightarrow \gamma[\psi/\psi']$. Using the (RE) rule, $\vdash_E \Box\gamma \leftrightarrow \Box(\gamma[\psi/\psi'])$. Note that $\Box(\gamma[\psi/\psi'])$ is the same formula as $\Box\gamma[\psi/\psi']$. Hence, we have $\vdash_E \Box\gamma \leftrightarrow \Box\gamma[\psi/\psi']$. \square

The substitution theorem is a fundamental theorem of axiom systems, and will often be implicitly used in the remainder of this book (without reference).

The next observation is that there are alternative characterizations of the logics **EM** and **EN** using the rules (RM) and (Nec), respectively.

Lemma 2.6 *The logic EM equals the logic E plus the rule (RM).*

Proof We first show that (RM) can be derived in **EM**.

- | | |
|--|------------------------------------|
| 1. $\varphi \rightarrow \psi$ | Assumption |
| 2. $\varphi \leftrightarrow (\varphi \wedge \psi)$ | From 1 using propositional logic |
| 3. $\Box\varphi \leftrightarrow \Box(\varphi \wedge \psi)$ | From 2 using (RE) |
| 4. $\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$ | Instance of (M) |
| 5. $\Box\varphi \rightarrow \Box\varphi \wedge \Box\psi$ | From 3,4 using propositional logic |
| 6. $\Box\varphi \rightarrow \Box\psi$ | From 5 using propositional logic |

Thus, (RM) is a derived rule of **EM**. The proof that (M) is derivable in the logic **E** plus the rule (RM) is left to the reader. \square

Exercise 2.15 Complete the proof of Lemma 2.6.

Lemma 2.7 *The logic EN equals the logic E plus the rule (Nec).*

Proof It is easy to see that using (Nec), we can prove $\Box\top$. To see that (Nec) is an admissible rule in the logic **EN**, suppose that $\vdash_{EN} \varphi$. We must show that $\vdash_{EN} \Box\varphi$. Using propositional reasoning, since $\vdash_{EN} \varphi$, we have $\vdash_{EN} (\top \leftrightarrow \varphi)$. Then, using the rule (RE), $\vdash_{EN} \Box\top \leftrightarrow \Box\varphi$. This means that $\vdash_{EN} \Box\top \rightarrow \Box\varphi$. Since $\vdash_{EN} \Box\top$, by (MP), $\vdash_{EN} \Box\varphi$. \square

Given the above Lemmas, it is not hard to see that the logic **EMCN** is equal to the logic **K**. The first step is to show that (K) is derivable in the logic **EMC**:

Lemma 2.8 $\vdash_{EMC} (K)$

Proof First of all, since **EMC** contains **(M)**, the rule **(RM)** is derivable (see Lemma 2.6).

- | | |
|--|------------------------------------|
| 1. $((\varphi \rightarrow \psi) \wedge \varphi) \rightarrow \psi$ | Propositional Tautology |
| 2. $\Box((\varphi \rightarrow \psi) \wedge \varphi) \rightarrow \Box\psi$ | From 1 using (RM) |
| 3. $(\Box(\varphi \rightarrow \psi) \wedge \Box\varphi) \rightarrow \Box((\varphi \rightarrow \psi) \wedge \varphi)$ | Instance of (C) |
| 4. $(\Box(\varphi \rightarrow \psi) \wedge \Box\varphi) \rightarrow \Box\psi$ | From 2,3 using propositional logic |
| 5. $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ | From 4 using propositional logic |

□

Exercise 2.16 Prove that $\vdash_{\mathbf{K}} (\mathbf{M})$ and $\vdash_{\mathbf{K}} (\mathbf{C})$.

Combining Lemma 2.8 and Exercise 2.16, we have:

Corollary 2.1 *The logic **EMCN** equals the logic **K**.*

Notice that both of the deductions that you found to solve the above exercise used the necessitation rule (or the axiom **(N)**). Using neighborhood structures, it can be shown that all deductions of **(M)** and **(C)** in **K** must use the necessitation rule (or the axiom **(N)** by Lemma 2.7). To show this, we must show that there is a neighborhood frame that validates **(K)** but not **(M)** and **(C)**.

Observation 2.28 *The axiom schemes **(M)** and **(C)** are not derivable in **EK**.*

Proof Suppose that $\mathcal{F} = \langle W, N \rangle$ is a neighborhood frame with $W = \{w, v, u, z\}$, and $N(w) = N(v) = N(u) = N(z) = \{\{w, v\}, \{w, u\}\}$. We first show that $\mathcal{F} \models \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ (the axiom scheme **(K)** is valid on \mathcal{F}). Suppose that $x \in W$ and \mathcal{M} is any model based on \mathcal{F} . Assume that $\mathcal{M}, x \models \Box(\varphi \rightarrow \psi)$ and $\mathcal{M}, x \not\models \Box\varphi$. There are two cases:

1. $\llbracket \varphi \rrbracket_{\mathcal{M}} = \{w, v\}$. Then, $\{u, z\} \subseteq \llbracket \varphi \rightarrow \psi \rrbracket_{\mathcal{M}}$. Hence, $\mathcal{M}, x \not\models \Box(\varphi \rightarrow \psi)$. This contradicts the first assumption.
2. $\llbracket \varphi \rrbracket_{\mathcal{M}} = \{w, u\}$. Then, $\{v, z\} \subseteq \llbracket \varphi \rightarrow \psi \rrbracket_{\mathcal{M}}$. Hence, $\mathcal{M}, x \not\models \Box(\varphi \rightarrow \psi)$. This contradicts the first assumption.

Both cases lead to a contradiction. Thus, since it is not possible for $\Box(\varphi \rightarrow \psi)$ and $\Box\varphi$ both to be true at a state in a model based on \mathcal{F} , the formula $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ is valid on \mathcal{F} .

Next, we show that **(C)** and **(M)** are not valid on \mathcal{F} . The same model works for both formulas. Let $\mathcal{M} = \langle W, N, V \rangle$ be a model based on \mathcal{F} with $V(p) = \{w, v\}$ and $V(q) = \{w, u\}$.

- We have $\llbracket p \rrbracket_{\mathcal{M}} = \{w, v\}$ and $\llbracket q \rrbracket_{\mathcal{M}} = \{w, u\}$ and $\llbracket p \wedge q \rrbracket_{\mathcal{M}} = \{w\}$. $\mathcal{M}, w \models \Box p \wedge \Box q$, but $\mathcal{M}, w \not\models \Box(p \wedge q)$.
- We have $\llbracket (p \vee q) \wedge p \rrbracket_{\mathcal{M}} = \{w, v\}$ and $\llbracket p \vee q \rrbracket_{\mathcal{M}} = \{w, v, u\}$. Then, $\mathcal{M}, w \models \Box((p \vee q) \wedge p)$; however, $\mathcal{M}, w \not\models \Box(p \vee q)$. Thus, $\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$ is not valid. □

- Exercise 2.17** 1. Prove that $\vdash_E \Diamond \top \leftrightarrow \neg \Box \perp$ and $\vdash_E \Box \top \leftrightarrow \neg \Diamond \perp$.
2. Prove that the following are theorems of any monotonic modal logic:
- a. $\Box \psi \rightarrow \Box(\varphi \rightarrow \psi)$ (c) $\Diamond(\varphi \rightarrow \psi) \vee \Box(\psi \rightarrow \varphi)$
 - b. $\Box \neg \varphi \rightarrow \Box(\varphi \rightarrow \psi)$ (d) $\Diamond(\varphi \wedge \psi) \rightarrow \Diamond \varphi \wedge \Diamond \psi$
3. Prove that the rule $\frac{\varphi \rightarrow \psi}{\Diamond \varphi \rightarrow \Diamond \psi}$ is *admissible* in any monotonic modal logic. (A rule is *admissible* in a logic if adding it does not change the set of theorems.)
4. Prove that the following are derivable in any regular modal logic:
- a. $\Box(\varphi \rightarrow \psi) \rightarrow (\Diamond \varphi \rightarrow \Diamond \psi)$
 - b. $\Box(\varphi \leftrightarrow \psi) \rightarrow (\Box \varphi \leftrightarrow \Box \psi)$
 - c. $(\Box \varphi \wedge \Diamond \psi) \rightarrow \Diamond(\varphi \wedge \psi)$
5. Prove that $(\Box \varphi \rightarrow \Diamond \varphi) \rightarrow \Diamond \top$ is derivable in a monotonic modal logic.
6. Find a monotonic neighborhood model \mathcal{M} with a state w such that $\mathcal{M}, w \not\models \Diamond \top \rightarrow (\Box p \rightarrow \Diamond p)$. (This shows that $\not\models_{EM} \Diamond \top \rightarrow (\Box \varphi \rightarrow \Diamond \varphi)$.)
7. Prove that $\Diamond \top \leftrightarrow (\Box \varphi \rightarrow \Diamond \varphi)$ is a theorem of every regular modal logic.

2.3.1 A Non-normal Extension of K

I conclude this section with some general comments about the definition of (normal) modal logics. By definition, a *logic* is a set of formulas (typically, instances of some collection of axiom schemes) that is closed under some inference rules. Thus, the statement “the logic \mathbf{L}_1 is contained in the logic \mathbf{L}_2 ” means that all the formulas in \mathbf{L}_1 are in \mathbf{L}_2 (i.e., $\mathbf{L}_1 \subseteq \mathbf{L}_2$). From this point of view, it is a direct consequence of Definition 2.26 that if \mathbf{L} is a normal modal logic, then \mathbf{L} contains \mathbf{K} . It is, perhaps, surprising to note that it is not the case that every logic that contains \mathbf{K} is normal. An example of such a logic was provided early on by McKinsey and Tarski (1944). I will present this logic below following the discussion in Segerberg (1971, pp. 171, 172).⁸

I start by defining a well-known normal modal logic. Let $\mathbf{S4}$ be the smallest set of formulas from $\mathcal{L}(\mathbf{At})$ that is closed under (MP) and the necessitation rule (Nec), and that contains all propositional tautologies, all instances of (K), and all instances of the following axiom schemes:

$$(T) \quad \Box \varphi \rightarrow \varphi \quad \text{and} \quad (4) \quad \Box \varphi \rightarrow \Box \Box \varphi.$$

The logics that interest us in this section contain the so-called *McKinsey axiom*:

$$(\mathbf{McK}) \quad \Box \Diamond \varphi \rightarrow \Diamond \Box \varphi.$$

Consider the following two logics:

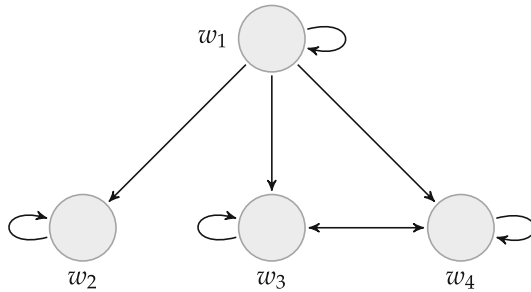
⁸This is a small digression that can easily be skipped by readers not already familiar with relational semantics for modal logic (see Appendix A for the relevant definitions).

1. **S4McK** is the smallest set of formulas that contains all instances of (Dual), (K), (T), (4) and (McK) and is closed under (MP) and (Nec).
2. **L** is the smallest set of formulas that contains all of **S4** and all instances of the axiom scheme (McK).

Clearly, **S4McK** is a normal modal logic. The logic **L** contains **S4** (and, so, also contains the logic **K**). The question is: are **L** and **S4McK** the same logic? I will argue that the two logics are distinct. In particular, $\Box(\Box\Diamond p \rightarrow \Diamond\Box p) \notin \mathbf{L}$ (of course, $\Box(\Box\Diamond p \rightarrow \Diamond\Box p) \in \mathbf{S4McK}$). Thus, **L** is a non-normal modal logic that contains **K**. To see that $\Box(\Box\Diamond p \rightarrow \Diamond\Box p) \notin \mathbf{L}$, consider the relational frame (Definition A.1) $\mathfrak{F} = \langle W, R \rangle$, where $W = \{w_1, w_2, w_3, w_4\}$, and R is the smallest reflexive relation containing

$$\{(w_1, w_2), (w_1, w_3), (w_1, w_4), (w_3, w_4), (w_4, w_3)\}.$$

This frame can be depicted as follows:



Consider the set of formulas that are true at w_1 in any relational model based on the above frame:

$$\mathbf{L}_{w_1} = \{\varphi \mid \mathfrak{M}, w_1 \models \varphi \text{ where } \mathfrak{M} \text{ is a relational model based on } \mathfrak{F}\}.$$

The following two Claims establish the fact that $\Box(\Box\Diamond p \rightarrow \Diamond\Box p) \notin \mathbf{L}$.

Claim $\mathbf{L} \subseteq \mathbf{L}_{w_1}$

Proof We must show that $\mathbf{S4} \subseteq \mathbf{L}_{w_1}$, and that every instance of (McK) is contained in \mathbf{L}_{w_1} . The fact that $\mathbf{S4} \subseteq \mathbf{L}_{w_1}$ follows from the fact that the relation in \mathfrak{F} is reflexive and transitive, and from the well-known result that **S4** is sound and complete with respect to the class of frames that are reflexive and transitive (Blackburn et al. 2001, Theorem 4.29). To see that every instance of (McK) is contained in \mathbf{L}_{w_1} , suppose that \mathfrak{M} is a relational model based on \mathfrak{F} and $\mathfrak{M}, w_1 \models \Box\Diamond\varphi$. Then, for all v , if $w_1 R v$, then $\mathfrak{M}, v \models \Diamond\varphi$. In particular, since $w_1 R w_2$, $\mathfrak{M}, w_2 \models \Diamond\varphi$. Thus, since w_2 is the only world accessible from w_2 , we must have $\mathfrak{M}, w_2 \models \varphi$. Furthermore, $\mathfrak{M}, w_2 \models \Box\varphi$ (this again follows from the fact that w_2 is the only world accessible from w_2). But this means that $\mathfrak{M}, w_1 \models \Diamond\Box\varphi$. Thus, $\mathfrak{M}, w_1 \models \Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$. \square

Claim $\Box(\Box\Diamond p \rightarrow \Diamond\Box p) \notin \mathbf{L}_{w_1}$

Proof Suppose that $\mathfrak{M} = \langle W, R, V \rangle$ is a relational model based on \mathfrak{F} with the valuation $V(p) = \{w_3\}$. Then, as the reader is invited to verify, $\mathfrak{M}, w_4 \models \Box\Diamond p$, but $\mathfrak{M}, w_4 \not\models \Diamond\Box p$. This means that $\mathfrak{M}, w_4 \not\models \Box\Diamond p \rightarrow \Diamond\Box p$, and so $\mathfrak{M}, w_1 \not\models \Box(\Box\Diamond p \rightarrow \Diamond\Box p)$. Thus, $\Box(\Box\Diamond p \rightarrow \Diamond\Box p) \notin \mathbf{L}_{w_1}$, as desired. \square

Remark 2.29 (Another Example) The example of a non-normal modal logic containing **K** presented in this section is somewhat artificial. A better motivated example of a such a logic is Solovay’s provability logic **S** (see Japaridze and de Jongh 1998, for a discussion). The normal modal logic **GL** is defined by adding the axiom scheme $\Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi$ to **K**. It is well-known that $\vdash_{\mathbf{GL}} \Box\varphi \rightarrow \Box\Box\varphi$. Solovay’s logic **S** is defined as follows: $\varphi \in \mathbf{S}$ iff $\{\Box\psi_1 \rightarrow \psi_1, \dots, \Box\psi_k \rightarrow \psi_k\} \vdash_{\mathbf{GL}} \varphi$, for some ψ_1, \dots, ψ_k . It can be shown that **S** contains **K4**, but is not closed under the rule of necessitation (Japaridze and de Jongh 1998, Sect. 2). Thus, **S** is another example of a non-normal modal logic containing **K**.

2.3.2 Completeness

In this section, I show how to adapt the standard approach for proving completeness of modal logics to prove completeness of non-normal modal logics with respect to neighborhood semantics. I assume that the reader is familiar with basic soundness and completeness results in modal logic (with respect to relational frames). See Blackburn et al. (2001, Chap. 4) for an overview. I start by reviewing some basic terminology.

2.3.2.1 Preliminaries

Recall the definition of a deduction for modal logic from (Definition 2.25 and the subsequent discussion in Sect. 2.3).

Definition 2.30 (Deduction from assumptions) Suppose that Γ is a set of formulas from \mathcal{L} ; **L** is a modal logic; and $\varphi \in \mathcal{L}$. We write $\Gamma \vdash_{\mathbf{L}} \varphi$ if there is a finitely many formulas $\alpha_1, \dots, \alpha_n \in \Gamma$ such that $\vdash_{\mathbf{L}} (\alpha_1 \wedge \dots \wedge \alpha_n) \rightarrow \varphi$.⁹

Suppose that **F** is a collection of neighborhood frames. A formula $\varphi \in \mathcal{L}$ is **valid in F**, or **F-valid**, denoted $\models_{\mathbf{F}} \varphi$, when for each $\mathcal{F} \in \mathbf{F}$, $\mathcal{F} \models \varphi$ (Definition 1.13). Given a class of frame **F**, let $\mathbf{L}(\mathbf{F}) = \{\varphi \mid \text{for all } \mathcal{F} \in \mathbf{F}, \mathcal{F} \models \varphi\}$ denote the set of formulas that are **F-valid**.

⁹I am using the definition of a deduction from assumptions found in Goldblatt (1992a, p. 17) and Blackburn et al. (2001, p. 36). See Hakli and Negri (2011) for a discussion of the issues surrounding this definition related to the deduction theorem and the proper use of inference rules.

Definition 2.31 (*Soundness*) A logic \mathbf{L} is **sound** with respect to \mathbf{F} , provided that $\mathbf{L} \subseteq \mathbf{L}(\mathbf{F})$. That is, for each $\varphi \in \mathcal{L}$, if $\vdash_{\mathbf{L}} \varphi$, then φ is valid in \mathbf{F} .

The main goal is to show that there is a semantic consequence relation between sets of formulas and formulas that is equivalent to the deduction relation.

Definition 2.32 (*Semantic Consequence*) Suppose that Γ is a set of formulas and \mathbf{F} is a class of neighborhood frames. A formula $\varphi \in \mathcal{L}$ is a **semantic consequence** with respect to \mathbf{F} of Γ , denoted $\Gamma \models_{\mathbf{F}} \varphi$, provided for each model $\mathcal{M} = \langle W, N, V \rangle$ based on a frame from \mathbf{F} (i.e., $\langle W, N \rangle \in \mathbf{F}$), for each $w \in W$, if $\mathcal{M}, w \models \Gamma$, then $\mathcal{M}, w \models \varphi$.

Remark 2.33 (*Local and Global Consequence*) The above definition of a semantic consequence is a **local consequence** relation. It is important to distinguish this from a **global consequence**: For a class of frames \mathbf{F} , let $\Gamma \models_{\mathbf{F}}^g \varphi$ provided that for each $\mathcal{F} \in \mathbf{F}$, if $\mathcal{F} \models \Gamma$, then $\mathcal{F} \models \varphi$. These two notions of semantic consequence are not equivalent. For instance, suppose that \mathbf{F} is the class of frames that contain the unit (i.e., for all $\langle W, N \rangle \in \mathbf{F}$, for all $w \in W$, $W \in N(w)$); then, $\{p\} \models_{\mathbf{F}}^g \Box p$ but $\{p\} \not\models_{\mathbf{F}} \Box p$. See Blackburn et al. (2001, Sect. 1.5) and Hakli and Negri (2011) for further discussion.

Definition 2.34 (*Strong Completeness*) A logic \mathbf{L} is **strongly complete** with respect to a class of frames \mathbf{F} , when, for each $\Gamma \subseteq \mathcal{L}$, $\Gamma \models_{\mathbf{F}} \varphi$ implies $\Gamma \vdash_{\mathbf{L}} \varphi$.

Remark 2.35 (*Weak Completeness*) A special case of the above definition is when $\Gamma = \emptyset$. A logic \mathbf{L} is **weakly complete** with respect to a class of frames \mathbf{F} , if $\models_{\mathbf{F}} \varphi$ implies $\vdash_{\mathbf{L}} \varphi$. Obviously, if \mathbf{L} is strongly complete, then \mathbf{L} is weakly complete. However, the converse is not true. There are modal logics that are weakly complete but not strongly complete. An example of such a normal modal logic can be found in the Appendix (Observation A.19). See, also, Sect. 3.3 for other examples of logics that are weakly complete but not strongly complete (cf., also, Blackburn et al. (2001), Sect. 4.8).

Let \mathbf{L} be any modal logic extending \mathbf{E} . A set of formulas Γ is said to be **L-inconsistent** if $\Gamma \vdash_{\mathbf{L}} \perp$, and **L-consistent** if it is not inconsistent.

Definition 2.36 (*Maximally Consistent Set*) A set of formulas Γ is called **maximally consistent** provided that Γ is consistent and for all formulas $\varphi \in \mathcal{L}$, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Let $M_{\mathbf{L}}$ be the set of \mathbf{L} -maximally consistent sets of formulas. Given a formula $\varphi \in \mathcal{L}$, let $|\varphi|_{\mathbf{L}}$ be the **proof set** of φ in \mathbf{L} . Formally, $|\varphi|_{\mathbf{L}} = \{\Delta \mid \Delta \in M_{\mathbf{L}} \text{ and } \varphi \in \Delta\}$. The first observation is that proof sets share a number of properties in common with truth sets.

Lemma 2.9 *Let \mathbf{L} be a logic and $\varphi, \psi \in \mathcal{L}$. Then,*

1. $|\varphi \wedge \psi|_{\mathbf{L}} = |\varphi|_{\mathbf{L}} \cap |\psi|_{\mathbf{L}}$.
2. $|\neg\varphi|_{\mathbf{L}} = M_{\mathbf{L}} - |\varphi|_{\mathbf{L}}$.
3. $|\varphi \vee \psi|_{\mathbf{L}} = |\varphi|_{\mathbf{L}} \cup |\psi|_{\mathbf{L}}$.
4. $|\varphi|_{\mathbf{L}} \subseteq |\psi|_{\mathbf{L}}$ iff $\vdash_{\mathbf{L}} \varphi \rightarrow \psi$.
5. $|\varphi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$ iff $\vdash_{\mathbf{L}} \varphi \leftrightarrow \psi$.
6. For any maximally \mathbf{L} -consistent set Γ , if $\varphi \in \Gamma$ and $\varphi \rightarrow \psi \in \Gamma$, then $\psi \in \Gamma$.
7. For any maximally \mathbf{L} -consistent set Γ , If $\vdash_{\mathbf{L}} \varphi$, then $\varphi \in \Gamma$.

Exercise 2.18 Prove Lemma 2.9.

Another standard result is *Lindenbaum's Lemma*. I leave the proof of Lindenbaum's Lemma as an exercise.¹⁰

Lemma 2.10 (Lindenbaum's Lemma) *For any \mathbf{L} -consistent set of formulas Γ , there exists a maximally \mathbf{L} -consistent set Γ' such that $\Gamma \subseteq \Gamma'$.*

Exercise 2.19 Prove Lindenbaum's Lemma.

The following useful fact about proof sets demonstrates how Lindenbaum's Lemma can be used.

Lemma 2.11 *For each $\varphi \in \mathcal{L}$, $\psi \in \bigcap |\varphi|_{\mathbf{L}}$ iff $\vdash_{\mathbf{L}} \varphi \rightarrow \psi$.*

Proof Suppose that $\vdash_{\mathbf{L}} \varphi \rightarrow \psi$. Then, for each maximally consistent set Γ , $\varphi \rightarrow \psi \in \Gamma$. Hence, since for each $\Gamma \in |\varphi|_{\mathbf{L}}$, $\psi \in \Gamma$, we have $\psi \in \bigcap |\varphi|_{\mathbf{L}}$.

Suppose that $\psi \in \bigcap |\varphi|_{\mathbf{L}}$, but it is not the case that $\vdash_{\mathbf{L}} \varphi \rightarrow \psi$. Then, $\neg(\varphi \rightarrow \psi)$ is \mathbf{L} -consistent. Using Lindenbaum's Lemma, there is a maximally consistent set Γ such that $\neg(\varphi \rightarrow \psi) \in \Gamma$. Thus, $\varphi, \neg\psi \in \Gamma$. Since $\varphi \in \Gamma$, $\Gamma \in |\varphi|_{\mathbf{L}}$. But then, $\neg\psi \in \Gamma$ contradicts the fact that $\psi \in \bigcap |\varphi|_{\mathbf{L}}$. \square

A straightforward corollary of this Lemma is the following useful fact:

Corollary 2.2 *If $\varphi \in \Gamma$ for all maximally \mathbf{L} -consistent sets Γ , then $\vdash_{\mathbf{L}} \varphi$.*

2.3.2.2 Canonical Models

Suppose that $\mathcal{M} = \langle W, N, V \rangle$ is a neighborhood model and $X \subseteq W$ any subset. A set $X \subseteq W$ is **definable** (with respect to a modal language \mathcal{L}) provided that there is a formula $\varphi \in \mathcal{L}$ such that $\llbracket \varphi \rrbracket_{\mathcal{M}} = X$. Let $\mathcal{D}_{\mathcal{M}}$ be the set of all sets that are definable in \mathcal{M} . Note that since there are only countably many formulas in \mathcal{L} , the set $\mathcal{D}_{\mathcal{M}}$ is countable (or finite if W is finite). Thus, if $\wp(W)$ is uncountable, then $\mathcal{D}_{\mathcal{M}} \neq \wp(W)$. A subset $X \subseteq M_{\mathbf{L}}$ is called a **proof set** provided that there is some formula $\varphi \in \mathcal{L}$ such that $X = |\varphi|_{\mathbf{L}}$. Again, since the modal language \mathcal{L} is countable,

¹⁰The proof is provided in the solution manual. Consult Chellas (1980) and Blackburn et al. (2001) for a discussion of this proof and a more complete discussion of maximally consistent sets.

there are only countably many proof sets. However, if \mathbf{At} is countable, then $M_{\mathbf{L}}$ is uncountable; and so, there are uncountably many subsets of $M_{\mathbf{L}}$. Thus, it is not the case that every subset of $M_{\mathbf{L}}$ is a proof set.

As usual, the states in a canonical model will be maximally consistent sets—i.e., elements of $M_{\mathbf{L}}$. A function $N_{\mathbf{L}} : M_{\mathbf{L}} \rightarrow \wp(\wp(M_{\mathbf{L}}))$ is a **canonical neighborhood function** provided that for all $\varphi \in \mathcal{L}$:

$$|\varphi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma) \text{ iff } \Box\varphi \in \Gamma.$$

So, for each maximally consistent set Γ , $N_{\mathbf{L}}(\Gamma)$ contains at least all the proof sets of the necessary formulas from Γ . The first question is: *Do any such functions actually exist?* That is, is it even possible to define a function satisfying the above condition? A problem would arise if there were proof sets $|\varphi|_{\mathbf{L}}$ and $|\psi|_{\mathbf{L}}$ such that $|\varphi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$ and a maximally consistent set with $\Box\varphi \in \Gamma$ but $\Box\psi \notin \Gamma$ (and, hence, $\neg\Box\psi \in \Gamma$). If such a situation were possible, then it would be impossible to satisfy the above condition. Fortunately, this problematic situation is ruled out in any logic containing the rule *RE*.

Lemma 2.12 *Suppose that \mathbf{L} is a logic that contains the *RE* rule and that $N_{\mathbf{L}} : M_{\mathbf{L}} \rightarrow \wp(\wp(M_{\mathbf{L}}))$ is a function such that for each $\Gamma \in M_{\mathbf{L}}$, $|\varphi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$ iff $\Box\varphi \in \Gamma$. Then, if $|\varphi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$ and $|\varphi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$, then $\Box\psi \in \Gamma$.*

Proof Let φ and ψ be two formulas such that $|\varphi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$, $\Box\varphi \in \Gamma$, and $|\varphi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$. Since $|\varphi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$, $\Box\varphi \in \Gamma$. Also, by Lemma 2.9, since $|\varphi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$, $\vdash_{\mathbf{L}} \varphi \leftrightarrow \psi$. Using the *RE* rule, $\vdash_{\mathbf{L}} \Box\varphi \leftrightarrow \Box\psi$. Hence, $\Box\varphi \leftrightarrow \Box\psi \in \Gamma$. Therefore, $\Box\psi \in \Gamma$. \square

The **canonical valuation**, $V_{\mathbf{L}} : \mathbf{At} \rightarrow \wp(M_{\mathbf{L}})$, is defined as follows. For each $p \in \mathbf{At}$, let $V_{\mathbf{L}}(p) = |p|_{\mathbf{L}} = \{\Gamma \in M_{\mathbf{L}} \text{ and } p \in \Gamma\}$. Putting everything together gives us the following definition:

Definition 2.37 (*Canonical Neighborhood Model*) Suppose that $\mathcal{M} = \langle W, N, V \rangle$ is a neighborhood model. Then, \mathcal{M} is **canonical for \mathbf{L}** provided that:

1. $W = M_{\mathbf{L}}$;
2. for each $\Gamma \in W$ and each formula $\varphi \in \mathcal{L}$, $|\varphi|_{\mathbf{L}} \in N(\Gamma)$ iff $\Box\varphi \in \Gamma$; and
3. $V = V_{\mathbf{L}}$.

For example, let $\mathcal{M}_{\mathbf{L}}^{\min} = \langle M_{\mathbf{L}}, N_{\mathbf{L}}^{\min}, V_{\mathbf{L}} \rangle$, where, for each $\Gamma \in M_{\mathbf{L}}$, $N_{\mathbf{L}}^{\min}(\Gamma) = \{|\varphi|_{\mathbf{L}} \mid \Box\varphi \in \Gamma\}$. The model $\mathcal{M}_{\mathbf{L}}^{\min}$ is easily seen to be canonical for \mathbf{L} . Furthermore, it is the minimal canonical for \mathbf{L} in the sense that for each Γ , $N_{\mathbf{L}}^{\min}(\Gamma)$ is the smallest set satisfying property 2 from Definition 2.37. Let $\mathcal{P}_{\mathbf{L}}$ be the set of all proof sets of \mathbf{L} (i.e., $\mathcal{P}_{\mathbf{L}} = \{|\varphi|_{\mathbf{L}} \mid \varphi \in \mathcal{L}\}$). The largest canonical for \mathbf{L} is the model $\mathcal{M}_{\mathbf{L}}^{\max} = \langle M_{\mathbf{L}}, N_{\mathbf{L}}^{\max}, V_{\mathbf{L}} \rangle$ with for each $\Gamma \in M_{\mathbf{L}}$, $N_{\mathbf{L}}^{\max}(\Gamma) = N_{\mathbf{L}}^{\min}(\Gamma) \cup \{X \mid X \subseteq M_{\mathbf{L}}, X \notin \mathcal{P}_{\mathbf{L}}\}$.

Lemma 2.13 (Truth Lemma) *For any consistent logic \mathbf{L} and any consistent formula φ , if \mathcal{M} is canonical for \mathbf{L} , then*

$$\llbracket \varphi \rrbracket_{\mathcal{M}} = |\varphi|_{\mathbf{L}}.$$

Proof Suppose that $\mathcal{M} = \langle W, N, V \rangle$ is a canonical model for \mathbf{L} . The proof is by induction on the structure of $\varphi \in \mathcal{L}$. The base case and the cases for the Boolean connectives are as usual and left for the reader. I give the details only for the modal case. The induction hypothesis is that $\llbracket \varphi \rrbracket_{\mathcal{M}} = |\varphi|_{\mathbf{L}}$. We must show that $\llbracket \Box \varphi \rrbracket_{\mathcal{M}} = |\Box \varphi|_{\mathbf{L}}$.

$$\begin{aligned} \Gamma \in \llbracket \Box \varphi \rrbracket_{\mathcal{M}} & \text{ iff } \llbracket \varphi \rrbracket_{\mathcal{M}} \in N(\Gamma) \text{ (Definition of truth)} \\ & \text{ iff } |\varphi|_{\mathbf{L}} \in N(\Gamma) \text{ (induction hypothesis)} \\ & \text{ iff } \Box \varphi \in \Gamma \text{ (item 2 of Definition 2.37)} \\ & \text{ iff } \Gamma \in |\Box \varphi|_{\mathbf{L}} \text{ (definition of proof sets)} \end{aligned}$$

Thus, $\llbracket \Box \varphi \rrbracket_{\mathcal{M}} = |\Box \varphi|_{\mathbf{L}}$, as desired. \square

2.3.2.3 Applications

Theorem 2.38 *The logic \mathbf{E} is sound and strongly complete with respect to the class of all neighborhood frames.*

Proof Soundness is straightforward (and, in fact, already shown in earlier exercises). As for strong completeness, I will show that every consistent set of formulas can be satisfied in some model. Before proving this, I show that this fact implies strong completeness. The proof is by contraposition. Suppose that it is not the case that $\Gamma \vdash_{\mathbf{L}} \varphi$. Then, $\Gamma \cup \{\neg \varphi\}$ is consistent. (If $\Gamma \cup \{\neg \varphi\} \vdash_{\mathbf{L}} \perp$, then, by propositional reasoning, $\Gamma \vdash_{\mathbf{L}} \neg \varphi \rightarrow \perp$. Hence, $\Gamma \vdash_{\mathbf{L}} \varphi$.) If $\Gamma \cup \{\neg \varphi\}$ is jointly true at some state in a model, then Γ cannot semantically entail φ . Thus, if $\Gamma \not\vdash_{\mathbf{L}} \varphi$, then $\Gamma \not\models_{\mathbf{F}} \varphi$ (where \mathbf{F} is the class of all neighborhood frames).

Let Γ be a consistent set of formulas. By Lindenbaum's Lemma, there is a maximally consistent set Γ' such that $\Gamma \subseteq \Gamma'$. Consider the minimal canonical model $\mathcal{M}_{\mathbf{E}}^{\min}$. By the Truth Lemma (Lemma 2.13), $\mathcal{M}_{\mathbf{E}}^{\min}, \Gamma' \models \Gamma'$. Thus, Γ is satisfiable at a state in the minimal canonical model, as desired. \square

Notice that in the above proof, the choice to use the *minimal* canonical model for \mathbf{E} was somewhat arbitrary. It is easy to see that the proof would go through if I had used $\mathcal{M}_{\mathbf{E}}^{\max}$ instead of $\mathcal{M}_{\mathbf{E}}^{\min}$. Indeed, *any* canonical model for \mathbf{E} could have been used in the above proof. The fact that there is a choice of canonical models will be useful when proving completeness for logics extending \mathbf{E} . The strategy for proving strong completeness for other non-normal modal logics discussed in Sect. 2.3 is similar to the strategy for proving strong completeness of some well-known normal modal logics, such as **S4** or **S5**. Given the above definition of a canonical model and truth lemma, all that remains is to show that a the frame of a particular canonical

model belongs to the class of frames under consideration. This argument is called *completeness-via-canonicity* in Blackburn et al. (2001). For instance, consider the logic **EC**. It is not hard to see that **C** is sound for the class of neighborhood frames that are closed under intersections (cf. Lemma 2.20). I now show that **EC** is sound and strongly complete with respect to neighborhood frames that are closed under intersections. The first step is to show that C is **canonical**¹¹ for this property.

Lemma 2.14 *If \mathbf{L} contains all instances of C , then $N_{\mathbf{L}}^{min}$ is closed under finite intersections.*

Proof Suppose that \mathbf{L} contains all instances of C . We must show that for all $\Gamma \in M_{\mathbf{L}}$, $N_{\mathbf{L}}^{min}(\Gamma)$ is closed under intersections. Suppose that $X, Y \in N_{\mathbf{L}}^{min}(\Gamma)$. By the definition of $N_{\mathbf{L}}^{min}$, $X = |\varphi|_{\mathbf{L}}$ and $Y = |\psi|_{\mathbf{L}}$ with $\Box\varphi \in \Gamma$ and $\Box\psi \in \Gamma$. Hence, $\Box\varphi \wedge \Box\psi \in \Gamma$; and so, using C , $\Box(\varphi \wedge \psi) \in \Gamma$. Thus, $|\varphi \wedge \psi|_{\mathbf{L}} \in N_{\mathbf{L}}^{min}(\Gamma)$. Therefore, $X \cap Y = |\varphi|_{\mathbf{L}} \cap |\psi|_{\mathbf{L}} = |\varphi \wedge \psi|_{\mathbf{L}} \in N_{\mathbf{L}}^{min}(\Gamma)$, as desired. \square

Given the above proof, strong completeness is straightforward.

Theorem 2.39 *The logic **EC** is sound and strongly complete with respect to the class of neighborhood frames that are closed under intersections.*

Proof The proof proceeds as in Theorem 2.38, using Lemma 2.14 to argue that the canonical frame for **EC** is closed under intersections. \square

Exercise 2.20 Prove that **EN** is sound and strongly complete with respect to neighborhood frames that contain the unit.

The proof that **EM** is strongly complete with respect to neighborhood frames that are closed under supersets is not as straightforward. Here, we need to make use of the fact that there are a number of different canonical models. The main difficulty is that $N_{\mathbf{EM}}^{min}$ is not closed under supersets.

Observation 2.40 *There is a maximally consistent set Γ such that $N_{\mathbf{EM}}^{min}(\Gamma)$ is not closed under supersets.*

Proof Let p be a propositional variable and let Γ be a maximally consistent set such that $\Box p \in \Gamma$ (such a set exists by Lindenbaum's Lemma since $\Box p$ is consistent). Then $|p|_{\mathbf{EM}} \in N_{\mathbf{EM}}^{min}(\Gamma)$. Let Y be any non-proof set that extends $|p|_{\mathbf{EM}}$ (i.e., $|p|_{\mathbf{EM}} \subsetneq Y$). To see that such a set exists, let Y' be any non-proof set such that $Y' \not\subseteq |p|_{\mathbf{EM}}$ (such a set exists since there are uncountably many subsets of $M_{\mathbf{EM}}$ but only countably many proof sets, and p is not a theorem of **EM**). Then, $Y = Y' \cup |p|_{\mathbf{EM}}$ is not a proof set since, if $Y = |\psi|_{\mathbf{EM}}$ for some formula ψ , then $Y' = |\psi \wedge \neg p|_{\mathbf{EM}}$ (why?), which contradicts the fact that Y' is not a proof set. Clearly, $Y \notin N_{\mathbf{EM}}^{min}(\Gamma)$ (why?). Then, we have $X = |p|_{\mathbf{EC}} \in N_{\mathbf{EM}}^{min}(\Gamma)$, $X \subseteq Y$, but $Y \notin N_{\mathbf{EM}}^{min}(\Gamma)$. \square

¹¹See Blackburn et al. (2001), Chap.4, for an extended discussion of canonical properties for relational models.

However, this difficulty can be easily overcome by choosing a different, better-behaved, canonical model. Recall from Sect. 1.1, that if \mathcal{F} is any collection of subsets of W , then $\mathcal{F}^{mon} = \{X \mid \text{there is a } Y \in \mathcal{F} \text{ such that } Y \subseteq X\}$. Given any model $\mathcal{M} = \langle W, N, V \rangle$, let the **supplementation** of \mathcal{M} , denoted \mathcal{M}^{mon} , be the model $\langle W, N^{mon}, V \rangle$, where for each $w \in W$, $N^{mon}(w) = (N(w))^{mon}$. The key argument is that the supplementation of the minimal canonical model is canonical for **EM**.

Lemma 2.15 *Suppose that $\mathcal{M} = (\mathcal{M}_{EM}^{min})^{mon}$. Then, \mathcal{M} is canonical for **EM**.*

Proof Suppose that $\mathcal{M} = \langle W, N, V \rangle$, where $W = M_{EM}$ and for each $\Gamma \in W$, $N(\Gamma) = (N_{EM}^{min}(\Gamma))^{mon}$, and $V = V_{EM}$. Let $\Gamma \in W$. We must show that for each formula $\varphi \in \mathcal{L}$,

$$|\varphi|_{EM} \in N(\Gamma) \text{ iff } \Box\varphi \in \Gamma.$$

The right-to-left direction is trivial since for all Γ , $N_{EM}^{min}(\Gamma) \subseteq N(\Gamma)$. Suppose that $|\varphi|_{EM} \in N(\Gamma) = (N_{EM}^{min}(\Gamma))^{mon}$. Then, there is some proof set $|\psi|_{EM} \in N_{EM}^{min}(\Gamma)$ such that $|\psi|_{EM} \subseteq |\varphi|_{EM}$. Since $|\psi|_{EM} \in N_{EM}^{min}(\Gamma)$, we have $\Box\psi \in \Gamma$. Furthermore, since $|\psi|_{EM} \subseteq |\varphi|_{EM}$, by Lemma 2.9, $\vdash_{EM} \psi \rightarrow \varphi$. Using the rule *RM* (which is admissible in **EM**), $\vdash_{EM} \Box\psi \rightarrow \Box\varphi$. Thus, $\Box\psi \rightarrow \Box\varphi \in \Gamma$. Therefore, $\Box\varphi \in \Gamma$, as desired. \square

Theorem 2.41 *The logic **EM** is sound and strongly complete with respect to the class of monotonic frames.*

Proof Left as an exercise for the reader. \square

Combining the proofs of Theorems 2.39 and 2.41 with Exercise 2.20 gives a characterization of the smallest normal modal logic **K**.

Theorem 2.42 *The logic **K** is sound and strongly complete with respect to the class of filters.*

As I explained in Sect. 1.1, not all filters are augmented. Since **K** is sound and complete with respect to the class of all relational frames (cf. Appendix A), and each relational frame corresponds to an augmented neighborhood frame (cf. Sect. 2.2.1), there is another characterization of **K**:

Exercise 2.21 Prove that **K** is sound and strongly complete with respect to the class of augmented frames.

Exercise 2.22 Prove that **S4** (see Sect. 2.3.1 for a definition) is sound and strongly complete with respect to the class of **S4** neighborhood frames (Definition 1.28).

2.3.3 Incompleteness and General Frames

In Sect. 2.2.1, we saw that the class of relational frames is modally equivalent to the class of augmented neighborhood frames. This means that if a modal logic is

complete with respect to a class of relational frames, then it must be complete with respect to the corresponding class of neighborhood frames (this follows from the fact that every relational model can be turned into a modally equivalent neighborhood model). Recall that a logic \mathbf{L} is **neighborhood complete** (resp. **Kripke complete**) provided that there is a class of neighborhood frames \mathbf{F} (resp. relational frames) such that $\mathbf{L} = \mathbf{L}(\mathbf{F}) = \{\varphi \in \mathcal{L} \mid \mathcal{F} \models \varphi \text{ for all } \mathcal{F} \in \mathbf{F}\}$. Otherwise, the logic is said to be **neighborhood incomplete** (resp. **Kripke incomplete**).

It is well known that there are Kripke incomplete modal logics—i.e., modal logics that are not the logic of any class for relational frames. Thomason (1972) provided the first consistent modal logic that is incomplete with respect to relational frames (i.e., a Kripke incomplete modal logic). See Fine (1974), Thomason (1974), van Benthem (1978), Shehtman (1977), Boolos and Sambin (1985), and Benton (2002) for other examples of Kripke incomplete modal logics. Gerson (1975b) proved that the Kripke incomplete modal logics from Thomason (1974) and Fine (1974) are also incomplete with respect to neighborhood frames (cf. also Shehtman (1980) for another example a logic incomplete with respect to neighborhood frames). Now, if a logic is non-normal, then, since there are no relational frames validating all the formulas in the logic, the logic must be Kripke incomplete. However, the situation is much more interesting. There are consistent normal modal logics that are complete with respect to a class of neighborhood frames but not complete with respect to any class of relational frames. Gerson showed this for a logic extending **S4** (Gerson 1975a) and a logic extending **T** (Gerson 1976). Other examples of modal logics that are complete with respect to neighborhood frames but incomplete with respect to relational frames are found in Shehtman (2005). The proofs of these results are beyond the scope of this book, and so I do not include them here. The interested reader is invited to consult Litak (2004, 2005) and Shehtman (2005) for further results and an overview of this fascinating literature.

A seemingly simple generalization of neighborhood frames allows us to bypass the incompleteness results mentioned above and to prove a general completeness theorem for *all* classical modal logics. There is an analogous result for relational frames, and a rich mathematical theory of so-called *general* relational frames. See Blackburn et al. (2001, Chap.5) for an overview of general relational frames and their use in the model theory of normal modal logic. It is beyond the scope of this book to go into the details of this theory. Instead, I show how to adapt the definition of a general relational frame to the neighborhood setting and prove that all classical modal logics are complete with respect to their general neighborhood frame.

Definition 2.43 (*General Neighborhood Frame*) A **general neighborhood frame** is a tuple $\mathcal{F}^g = \langle W, N, \mathcal{A} \rangle$, where W is a non-empty set of states, N is a neighborhood function, and \mathcal{A} is a collection of subsets of W closed under intersections, complements, and the m_N operator.

A valuation $V : \text{At} \rightarrow \wp(W)$ is **admissible** for a general frame $\langle W, N, \mathcal{A} \rangle$ provided, for each $p \in \text{At}$, $V(p) \in \mathcal{A}$.

Definition 2.44 (*General Neighborhood Model*) Let $\mathcal{F}^g = \langle W, N, \mathcal{A} \rangle$ be a general neighborhood frame. A general neighborhood model based on \mathcal{F}^g is a tuple $\mathcal{M}^g = \langle W, N, \mathcal{A}, V \rangle$, where V is an admissible valuation.

On general models, truth for the basic modal language is defined as in Definition 1.12. The first observation is that on a general model, the truth set of all formulas is contained in the distinguished collection of propositions.

Lemma 2.16 *Let $\mathcal{M}^g = \langle W, N, \mathcal{A}, V \rangle$ be a general neighborhood model. Then, for each $\varphi \in \mathcal{L}$, $\llbracket \varphi \rrbracket_{\mathcal{M}^g} \in \mathcal{A}$.*

Proof The proof is an easy induction over the structure of φ . □

Suppose that \mathbf{L} is a modal logic containing \mathbf{E} . It is easy to show that the set $\mathcal{A}_{\mathbf{L}} = \{|\varphi|_{\mathbf{L}} \mid \varphi \in \mathcal{L}\}$ is a Boolean algebra (i.e., closed under intersections and complements) and closed under the m_N operator. A general frame \mathcal{F}^g is called an **L-general frame**, if \mathbf{L} is valid on \mathcal{F}^g . I will show that for any modal logic \mathbf{L} containing \mathbf{E} , the canonical general frame is an **L-general frame**. A **general frame** $\langle W_{\mathbf{L}}, N_{\mathbf{L}}, \mathcal{A}_{\mathbf{L}} \rangle$ is said to be the **L-canonical general frame** provided that $\langle W_{\mathbf{L}}, N_{\mathbf{L}}, \cdot \rangle$ is the minimal **L-canonical frame** (i.e., $W_{\mathbf{L}} = M_{\mathbf{L}}$ and $N_{\mathbf{L}} = N_{\mathbf{L}}^{\min}$) and $\mathcal{A}_{\mathbf{L}} = \{|\varphi|_{\mathbf{L}} \mid \varphi \in \mathcal{L}\}$. If \mathbf{L} is a modal logic containing \mathbf{E} , then $\mathcal{F}_{\mathbf{L}}^g = \langle W_{\mathbf{L}}, N_{\mathbf{L}}, \mathcal{A}_{\mathbf{L}} \rangle$ is the **L-canonical frame** and $\mathcal{M}_{\mathbf{L}}^g = \langle W_{\mathbf{L}}, N_{\mathbf{L}}, \mathcal{A}_{\mathbf{L}}, V_{\mathbf{L}} \rangle$ is the **L-canonical general model**. Note that the minimal canonical neighborhood function, $N_{\mathbf{L}}^{\min}$, is used for *all* classical modal logics. (Compare this to the completeness proofs from Sect. 2.3.2.)

Theorem 2.45 *Suppose that \mathbf{L} is any logic containing \mathbf{E} . Then,*

$$\mathcal{F}_{\mathbf{L}}^g \models \mathbf{L}.$$

Proof Suppose that $\mathcal{F}_{\mathbf{L}}^g = \langle W_{\mathbf{L}}, N_{\mathbf{L}}, \mathcal{A}_{\mathbf{L}} \rangle$ is the **L-canonical general frame**. It is a simple exercise to adapt the proof of the Truth Lemma (Lemma 2.13) to prove a Truth Lemma for $\mathcal{M}_{\mathbf{L}}^g$: For all $\varphi \in \mathcal{L}$, $\llbracket \varphi \rrbracket_{\mathcal{M}_{\mathbf{L}}^g} = |\varphi|_{\mathbf{L}}$.

Suppose that $\varphi \in \mathbf{L}$, and V is an admissible valuation for $\mathcal{F}_{\mathbf{L}}^g$. We must show that $\mathcal{M}^g = \langle M_{\mathbf{L}}, N_{\mathbf{L}}, \mathcal{A}_{\mathbf{L}}, V \rangle$ validates φ . Since V is admissible for $\mathcal{F}_{\mathbf{L}}^g$, for each propositional letter p_i occurring in φ , $V(p_i) \in \mathcal{A}_{\mathbf{L}}$. Hence, for each p_i (there are only finitely many), $V(p_i) = |\psi_i|_{\mathbf{L}}$ for some formula $\psi_i \in \mathbf{L}$. Let φ' be φ where each p_i is replaced with ψ_i . We prove by induction on φ that

$$\llbracket \varphi \rrbracket_{\mathcal{M}^g} = \llbracket \varphi' \rrbracket_{\mathcal{M}_{\mathbf{L}}^g}.$$

The base case is $\varphi = p$. Then, $\varphi' = \psi$ for some $\psi \in \mathbf{L}$, where $V(p) = |\psi|_{\mathbf{L}} \in \mathcal{A}_{\mathbf{L}}$. Hence,

$$\begin{aligned} \Gamma \in \llbracket p \rrbracket_{\mathcal{M}^g} &\text{ iff } \Gamma \in V(p) \in \mathcal{A}_{\mathbf{L}} && \text{(definition of truth)} \\ &\text{ iff } \Gamma \in |\psi|_{\mathbf{L}} \text{ for some } \psi \in \mathbf{L} \text{ (since } V(p) = |\psi|_{\mathbf{L}}) \\ &\text{ iff } \Gamma \in \llbracket \psi \rrbracket_{\mathcal{M}_{\mathbf{L}}^g} && \text{(Truth Lemma for } \mathcal{M}_{\mathbf{L}}^g) \end{aligned}$$

The argument for the Boolean cases is as usual. Suppose that φ is of the form $\Box\gamma$. The induction hypothesis is $\llbracket\gamma\rrbracket_{\mathcal{M}^g} = \llbracket\gamma'\rrbracket_{\mathcal{M}_L^g}$. Then,

$$\begin{aligned} \Gamma \in \llbracket\Box\gamma\rrbracket_{\mathcal{M}^g} &\text{ iff } \llbracket\gamma\rrbracket_{\mathcal{M}^g} \in N_L(\Gamma) && \text{(Definition of truth)} \\ &\text{ iff } \llbracket\gamma'\rrbracket_{\mathcal{M}_L^g} \in N_L(\Gamma) && \text{(Induction hypothesis)} \\ &\text{ iff } \Gamma \in \llbracket\Box\gamma'\rrbracket_{\mathcal{M}_L^g} && \text{(Definition of truth)} \end{aligned}$$

Suppose that $\varphi \in \mathbf{L}$ and $\mathcal{M}^g = \langle W_L, N_L, \mathcal{A}_L, V \rangle$ is any general model based on the general canonical frame for \mathbf{L} . Since \mathbf{L} is closed under uniform substitution, $\varphi' \in \mathbf{L}$ where φ' is φ where each atomic proposition p in φ is replaced with the formula $\psi \in \mathbf{L}$ such that $V(p) = \{\psi\}_L \in \mathcal{A}_L$. Then, by the Truth Lemma for \mathcal{M}_L^g , φ' is valid on \mathcal{M}_L^g . Since $\llbracket\varphi\rrbracket_{\mathcal{M}^g} = \llbracket\varphi'\rrbracket_{\mathcal{M}_L^g}$, φ is valid on \mathcal{M}^g . Hence, $\mathcal{F}_L^g \models \mathbf{L}$. \square

Corollary 2.3 *Every modal logic \mathbf{L} extending \mathbf{E} is sound and strongly complete with respect to its class of general neighborhood frames.*

Consult Došen (1989) for a more extensive discussion of general frames for neighborhood structures (cf., also, Hansen 2003).

2.4 Computational Issues

Given any logical system (such as neighborhood semantics for modal logic), there are three natural computational problems that arise:

- **Model Checking Problem:** Given a (finitely represented) pointed neighborhood model \mathcal{M}, w and a formula $\varphi \in \mathcal{L}$, does $\mathcal{M}, w \models \varphi$?
- **Satisfiability Problem:** Given a formula φ , is there a model (from some class of models) that satisfies φ ? Equivalently, given a formula φ , is φ valid?
- **Model Equivalence Problem:** Given two (finitely represented) pointed neighborhood models \mathcal{M}, w and \mathcal{N}, v , do \mathcal{M}, w and \mathcal{N}, v satisfy the same modal formulas?

A variety of algorithms have been proposed (and implemented) to solve the above problems for various classes of relational structures and modal languages. Many of the same ideas can be adapted to the neighborhood setting. In Sect. 2.4.1, I show that the satisfiability problem for non-normal modal logics is decidable, and I discuss the complexity of this problem in Sect. 2.4.2. See, for example, Pauly (2002) for results about the model-checking problem for coalitional logic (cf. Sect. 1.4.5).

2.4.1 Filtrations

Suppose that $\mathcal{M} = \langle W, N, V \rangle$ is a neighborhood model and Σ is a set of formulas from \mathcal{L} . For each $w, v \in W$, write $w \sim_\Sigma v$ iff for each $\varphi \in \Sigma$, $w \models \varphi$ iff $v \models \varphi$. In

other words, $w \sim_{\Sigma} v$ iff w and v agree on all formulas in Σ . It is easy to see that \sim_{Σ} is an equivalence relation. For each $w \in W$, let $[w]_{\Sigma} = \{v \mid w \sim_{\Sigma} v\}$ be the equivalence class of \sim_{Σ} . If $X \subseteq W$, let $[X]_{\Sigma} = \{[w] \mid w \in X\}$. If Σ is clear from the context, I may leave out the subscripts. A set of sentences Σ is **closed under subformulas** provided that for all $\varphi \in \Sigma$, all subformulas of φ are in Σ . With this notation in place, I can define a filtration.

Definition 2.46 (*Filtration*) Suppose that $\mathcal{M} = \langle W, N, V \rangle$ is a neighborhood model and Σ is a set of sentences closed under subformulas. A **filtration** of \mathcal{M} through Σ is a model $\mathcal{M}^f = \langle W^f, N^f, V^f \rangle$, where

1. $W^f = [W]$
2. For each $w \in W$, for each $\Box\varphi \in \Sigma$, $\llbracket\varphi\rrbracket_{\mathcal{M}} \in N(w)$ iff $\llbracket\varphi\rrbracket_{\mathcal{M}} \in N^f([w])$
3. For each $p \in \text{At}$, $V(p) = [V(p)]$

If Σ is finite, then it is easy to see that \mathcal{M}^f will be a finite model. We need only show that this model agrees with \mathcal{M} on all formulas in Σ .

Theorem 2.47 *Suppose that $\mathcal{M}^f = \langle W^f, N^f, V^f \rangle$ is a filtration of $\mathcal{M} = \langle W, N, V \rangle$ through (a subformula closed) set of sentences Σ . Then, for each $\varphi \in \Sigma$,*

$$\mathcal{M}, w \models \varphi \text{ iff } \mathcal{M}^f, [w] \models \varphi.$$

Proof The proof is by induction on the structure of φ . The base case and Boolean connectives are straightforward. (Note that the fact that Σ is closed under subformulas is needed for to apply the induction hypothesis.) I consider only the case for the modal operator. Suppose that φ is of the form $\Box\psi$. Then, since Σ is closed under subformulas and $\Box\psi \in \Sigma$, we have $\psi \in \Sigma$. Then, the induction hypothesis implies that $\llbracket\psi\rrbracket_{\mathcal{M}} = \llbracket\psi\rrbracket_{\mathcal{M}^f}$. Thus,

$$\begin{aligned} \mathcal{M}, w \models \Box\psi &\text{ iff } \llbracket\psi\rrbracket_{\mathcal{M}} \in N(w) && \text{(Definition of truth)} \\ &\text{ iff } \llbracket\psi\rrbracket_{\mathcal{M}} \in N^f([w]) && \text{(Item 2(a) in Definition 2.46)} \\ &\text{ iff } \llbracket\psi\rrbracket_{\mathcal{M}^f} \in N^f([w]) && \text{(induction hypothesis)} \\ &\text{ iff } \mathcal{M}^f, [w] \models \Box\psi && \text{(Definition of truth)} \end{aligned}$$

The argument for the \Diamond operator is similar and left as an exercise for the reader. \square

The **finest filtration** of $\mathcal{M} = \langle W, N, V \rangle$ is defined as follows: $\mathcal{M}_{min}^f = \langle W^f, N_{min}^f, V^f \rangle$, where for all $[w] \in W^f$,

$$N_{min}^f([w]) = \{\llbracket\varphi\rrbracket_{\mathcal{M}} \mid \llbracket\varphi\rrbracket_{\mathcal{M}} \in N(w) \text{ and } \Box\varphi \in \Sigma\}.$$

Obviously, the finest filtration is, in fact, a filtration. In general, the finest filtration may not preserve the algebraic properties of the neighborhood functions in a model. For example, the finest filtration may not be closed under intersections or supersets. This can be easily rectified.

Lemma 2.17 *Suppose that $\mathcal{M} = \langle W, N, V \rangle$ is closed under supersets and Σ is a subformula closed set of formulas. If \mathcal{M}^f is the finest filtration of \mathcal{M} , then $(\mathcal{M}^f)^{mon}$ is a filtration.*

Proof Suppose that $\Box\varphi \in \Sigma$. We prove that $\llbracket \varphi \rrbracket_{\mathcal{M}} \in N(w)$ iff $\llbracket \varphi \rrbracket_{\mathcal{M}} \in (N_{min}^f([w]))^{mon}$. Suppose that $\llbracket \varphi \rrbracket_{\mathcal{M}} \in N(w)$. Then, by the definition of \mathcal{M}^f , $\llbracket \varphi \rrbracket_{\mathcal{M}} \in N_{min}^f([w])$, and so, $\llbracket \varphi \rrbracket_{\mathcal{M}} \in (N_{min}^f([w]))^{mon}$.

Now, suppose that $\llbracket \varphi \rrbracket_{\mathcal{M}} \in (N_{min}^f([w]))^{mon}$. Then, there is a ψ such that $\llbracket \psi \rrbracket_{\mathcal{M}} \in N_{min}^f([w])$ and $\llbracket \psi \rrbracket_{\mathcal{M}} \subseteq \llbracket \varphi \rrbracket_{\mathcal{M}}$. Since $\llbracket \psi \rrbracket_{\mathcal{M}} \in N_{min}^f([w])$, we have $\Box\psi \in \Sigma$ and $\llbracket \psi \rrbracket_{\mathcal{M}} \in N(w)$. We now show that $\llbracket \psi \rrbracket_{\mathcal{M}} \subseteq \llbracket \varphi \rrbracket_{\mathcal{M}}$. Suppose that $v \in \llbracket \psi \rrbracket_{\mathcal{M}}$. Then, since $\psi \in \Sigma$, $[v] \in \llbracket \psi \rrbracket_{\mathcal{M}}$. Since $\llbracket \psi \rrbracket_{\mathcal{M}} \subseteq \llbracket \varphi \rrbracket_{\mathcal{M}}$, we have $[v] \in \llbracket \varphi \rrbracket_{\mathcal{M}}$. Theorem 2.47 implies that $v \in \llbracket \varphi \rrbracket_{\mathcal{M}}$. Therefore, since $N(w)$ is closed under supersets and $\llbracket \psi \rrbracket_{\mathcal{M}} \subseteq \llbracket \varphi \rrbracket_{\mathcal{M}}$, we have that $\llbracket \varphi \rrbracket_{\mathcal{M}} \in N(w)$. This completes the proof that $(\mathcal{M}^f)^{mon}$ is a filtration. \square

Exercise 2.23 Prove analogous results for neighborhood structures that (1) are closed under intersections; (2) contain the unit; and (3) are consistent filters.

Theorem 2.47 shows that for any formula $\varphi \in \mathcal{L}$, if it is satisfiable on a neighborhood model, then it is satisfiable on a finite neighborhood model. A careful inspection of the proof of Theorem 2.47 reveals that there is a bound on the size of the finite satisfying model. In fact, this bound is a function of the number of symbols in φ . This means that the satisfiability problem for **E** is decidable (one needs only search a finite number of finite models to determine whether a formula φ has a satisfying model). Lemma 2.17 is needed to show that **EM** is decidable. In fact, the filtration method (using Lemma 2.17 and Exercise 2.23) can be used to prove the following theorem.

Theorem 2.48 *The satisfiability problems for **E**, **EM**, **EC**, **EMC**, **EN**, **EMN**, **ECN**, and **ECMN** are all decidable.*¹²

2.4.1.1 Logics with Non-iterative Axioms

A formula of modal logic is said to be **non-iterative** provided if it does not contain any modal operators inside the scope of a modal operator. For instance, $\Box p \rightarrow q$, $((p \wedge q) \rightarrow r) \rightarrow ((p \wedge q) \Box \rightarrow r)$, and $\Box(p \wedge q) \rightarrow (\Box p \wedge \Box q)$ are all examples of non-iterative formulas. Examples of iterative formulas include $\Box p \rightarrow \Box \Box p$, $\Diamond p \rightarrow \Box \Diamond p$, and $\Box(\Box p \rightarrow p)$. Suppose that **L** is a modal logic extending **E** that can be axiomatized by non-iterative axioms. Many of the logics studied in this book are examples of non-iterative logics. For instance, the non-normal logics **E**, **EM**, **EC**, and **EMC** are all non-iterative. Lewis proved that every finitely axiomatizable non-iterative logic is decidable (Lewis 1974, Theorem 2). In this section, I show how

¹²See Chellas (1980, Sects. 7.5 and 9.5) for an extended discussion.

to adapt the filtration method from the previous section to prove Lewis's general decidability result.

Suppose that φ is a formula and \mathbf{L} is a finite axiomatizable non-iterative modal logic extending \mathbf{E} . Let Σ_φ be the set of all subformulas of φ . A φ -**description** is a set D defined as follows:

$$D = X \cup \{\neg\sigma \mid \sigma \in \Sigma_\varphi - X\}$$

where $X \subseteq \Sigma_\varphi$. So, a φ -description is a set of zero or more subformulas of φ together with negations of all other subformulas of φ . Since Σ_φ is finite, each φ -description D is finite. Furthermore, there are only finitely many φ -descriptions.

For each φ -description D , choose a maximally consistent set Γ_D containing D . Note that by Lindenbaum's Lemma, every \mathbf{L} -consistent φ -description D is contained in a maximally consistent set (in general, D is contained in many maximally consistent sets). Let $W_\varphi = \{\Gamma_D \mid D \text{ is a } \varphi\text{-description}\}$. We will build a canonical model with W_φ as the set of states.

Recall that if $\vdash_{\mathbf{L}} \varphi$, then $\varphi \in \Gamma$ for all maximally \mathbf{L} -consistent sets. Furthermore, the converse is true (Corollary 2.2): If $\varphi \in \Gamma$ for *all* maximally consistent sets Γ , then $\vdash_{\mathbf{L}} \varphi$. An analogue of Corollary 2.2 holds with respect to W_φ if we restrict attention to Boolean combinations of formulas from Σ_φ (i.e., substitution instances of propositional formulas in which the atomic propositions are replaced by formulas from Σ_φ).

Lemma 2.18 *Suppose that ψ is a truth-functional combination of formulas from Σ_φ . Then, $\vdash_{\mathbf{L}} \psi$ iff $\psi \in \Gamma_D$ for all $\Gamma_D \in W_\varphi$.*

Proof Suppose that ψ is a truth-functional combination of formulas from Σ_φ . Then, if $\vdash_{\mathbf{L}} \psi$, then $\psi \in \Gamma$ for all maximally \mathbf{L} -consistent sets. Thus, in particular, $\psi \in \Gamma_D$ for all $\Gamma_D \in W_\varphi$. To prove the converse, suppose that $\psi \in \Gamma_D$ for all $\Gamma_D \in W_\varphi$. Note that for each φ -description D , since ψ is a Boolean combination of formulas from Σ_φ , it cannot be the case that both $D \cup \{\psi\}$ and $D \cup \{\neg\psi\}$ are \mathbf{L} -consistent. There are two cases:

Case 1: There is a φ -description D such that $D \cup \{\neg\psi\}$ is \mathbf{L} -consistent. Then, $D \cup \{\psi\}$ is not \mathbf{L} -consistent. Thus, $\psi \notin \Gamma_D$. This contradicts the assumption that $\psi \in \Gamma_D$ for all $\Gamma_D \in W_\varphi$.

Case 2: There is no φ -description D such that $D \cup \{\neg\psi\}$ is \mathbf{L} -consistent. Since ψ is a Boolean combination of formulas from Σ_φ and the φ -descriptions range over all possible truth-value assignments to formulas in Σ_φ , it must be the case that ψ is an instance of a propositional tautology. Hence, $\vdash_{\mathbf{L}} \psi$. \square

Before discussing the main result of this section, I adapt the proof-set notation from the previous section. For any formula ψ , let

$$|\psi|_\varphi = \{\Gamma_D \in W_\varphi \mid \psi \in \Gamma_D\}.$$

A key observation is that every element of W_φ can be associated with a modal formula. Fix an enumeration of the formulas of the language of the modal logic \mathbf{L} . For each $\Gamma_D \in W_\varphi$, let λ_D be the first (in the enumeration of all formulas) conjunction of all the formulas in D . The formula λ_D is called the **label** of Γ_D . Using these labels, we can associate a formula with every subset of W_φ . If $X \subseteq W_\varphi$ and $X \neq \emptyset$, then let $\lambda_X = \bigvee_{\Gamma_D \in X} \lambda_D$. Let $\lambda_\emptyset = \varphi \wedge \neg\varphi$. Two immediate consequences of these definitions are:

1. For each $X \subseteq W_\varphi$, λ_X is a Boolean combination of formulas from Σ_φ ; and
2. For each $X \subseteq W_\varphi$, $|\lambda_X|_\varphi = X$.

We can now define a finite φ -canonical model.

Definition 2.49 (*φ -Canonical Model*) Suppose that φ is a modal formula. A **φ -Canonical Model** is a structure $\mathcal{M}_\varphi = \langle W_\varphi, N_\varphi, V_\varphi \rangle$, where W_φ is defined as above. The neighborhood function $N_\varphi : W_\varphi \rightarrow \wp(\wp(W_\varphi))$ is defined as follows: For all $\Gamma_D \in W_\varphi$,

$$N(\Gamma_D) = \{X \mid X \subseteq W_\varphi \text{ and } \Box\lambda_X \in \Gamma_D\}$$

The valuation $V_\varphi : \mathbf{At} \rightarrow \wp(W_\varphi)$ is defined as $V_\varphi(p) = |p|_\varphi$ for all $p \in \mathbf{At}$.

Note that \mathcal{M}_φ is indeed a filtration (Definition 2.46) of the canonical model for \mathbf{L} . However, there is a stronger version of Theorem 2.47.

Proposition 2.7 *Suppose that φ is a modal formula and that $\mathcal{M}_\varphi = \langle W_\varphi, N_\varphi, V_\varphi \rangle$ is a φ -canonical model for a consistent modal logic \mathbf{L} containing \mathbf{E} . Then,*

1. *If ψ is a Boolean combination of formulas ψ_1, \dots, ψ_k such that $\llbracket \psi_i \rrbracket_{\mathcal{M}_\varphi} = |\psi_i|_\varphi$ for $i = 1, \dots, k$, then $\llbracket \psi \rrbracket_{\mathcal{M}_\varphi} = |\psi|_\varphi$; and*
2. *If (a) ψ is a Boolean combination of formulas from Σ_φ and that (b) $\llbracket \psi \rrbracket_{\mathcal{M}_\varphi} = |\psi|_\varphi$, then $\llbracket \Box\psi \rrbracket_{\mathcal{M}_\varphi} = |\Box\psi|_\varphi$.*

Proof The proof of item 1 is straightforward, and so, it is left as an exercise for the reader. We prove item 2. Suppose that (a) ψ is a Boolean combination of formulas from Σ_φ and that (b) $\llbracket \psi \rrbracket_{\mathcal{M}_\varphi} = |\psi|_\varphi$. Then,

$$\begin{aligned} \llbracket \Box\psi \rrbracket_{\mathcal{M}_\varphi} &= \{\Gamma_D \mid \llbracket \psi \rrbracket_{\mathcal{M}_\varphi} \in N_\varphi(\Gamma_D)\} \\ &= \{\Gamma_D \mid \Box\lambda_{\llbracket \psi \rrbracket_{\mathcal{M}_\varphi}} \in \Gamma_D\} \\ &= |\Box\lambda_{\llbracket \psi \rrbracket_{\mathcal{M}_\varphi}}|_\varphi \end{aligned}$$

To complete the proof, we must show that $|\Box\lambda_{\llbracket \psi \rrbracket_{\mathcal{M}_\varphi}}|_\varphi = |\Box\psi|_\varphi$. Now by assumption (b), $\llbracket \psi \rrbracket_{\mathcal{M}_\varphi} = |\psi|_\varphi$ and property 2 of labels, we have that $|\lambda_{\llbracket \psi \rrbracket_{\mathcal{M}_\varphi}}| = \llbracket \psi \rrbracket_{\mathcal{M}_\varphi} = |\psi|_\varphi$. Thus, $|\lambda_{\llbracket \psi \rrbracket_{\mathcal{M}_\varphi}}|_\varphi = |\psi|_\varphi$. This means that $\lambda_{\llbracket \psi \rrbracket_{\mathcal{M}_\varphi}} \leftrightarrow \psi \in \Gamma_D$ for all $\Gamma_D \in W_\varphi$. By assumption (a) and the definition of a label, the formula $\lambda_{\llbracket \psi \rrbracket_{\mathcal{M}_\varphi}} \leftrightarrow \psi$ is a Boolean combination of formulas from Σ_φ . By Lemma 2.18, $\vdash_{\mathbf{L}} \lambda_{\llbracket \psi \rrbracket_{\mathcal{M}_\varphi}} \leftrightarrow \psi$. Since \mathbf{L}

contains the rule (RE), we have that $\vdash_{\mathbf{L}} \Box \lambda_{\llbracket \psi \rrbracket_{\mathcal{M}_\varphi}} \leftrightarrow \Box \psi$. Applying Lemma 2.18 again, we have that $\Box \lambda_{\llbracket \psi \rrbracket_{\mathcal{M}_\varphi}} \leftrightarrow \Box \psi \in \Gamma_D$ for all $\Gamma_D \in W_\varphi$. This implies that $|\Box \lambda_{\llbracket \psi \rrbracket_{\mathcal{M}_\varphi}}|_\varphi = |\Box \psi|_\varphi$, as desired. \square

Propositions 2.18 and 2.7 lead to the following key observation.

Proposition 2.8 1. Suppose that \mathbf{L} is a consistent modal logic extending \mathbf{E} with non-iterative axioms. If φ is any formula and $\vdash_{\mathbf{L}} \psi$, then ψ is valid on the φ -canonical frame $\langle W_\varphi, N_\varphi \rangle$.
 2. If $\nVdash_{\mathbf{L}} \varphi$, then φ is not valid on the φ -canonical frame $\langle W_\varphi, N_\varphi \rangle$.

Proof Proof of item 1: Suppose that ψ is an instance of a non-iterative axiom of \mathbf{L} . We must show that ψ is valid on the φ -canonical frame $\langle W_\varphi, N_\varphi \rangle$. Let $\mathcal{M}' = \langle W_\varphi, N_\varphi, V \rangle$ be any model based on $\langle W_\varphi, N_\varphi \rangle$. We must show that $\llbracket \psi \rrbracket_{\mathcal{M}'} = W_\varphi$. Let p_1, \dots, p_k be the atomic propositions occurring in ψ , and let σ be any substitution in which for all $i = 1, \dots, k$, $\sigma(p_i) = \lambda_{\llbracket p_i \rrbracket_{\mathcal{M}'}}$. Then, ψ^σ is the formula ψ in which each atomic proposition p_i is replaced by the label of $\llbracket p_i \rrbracket_{\mathcal{M}'}$. Since ψ is an instance of a non-iterative axiom, we have that ψ^σ is also a substitution instance of a non-iterative axiom. Hence, $\vdash_{\mathbf{L}} \psi^\sigma$. By Lemma 2.11, $\psi^\sigma \in \Gamma$ for all maximally \mathbf{L} -consistent sets. Thus, $\psi^\sigma \in \Gamma_D$ for all $\Gamma_D \in W_\varphi$. This means that $|\psi^\sigma|_\varphi = W_\varphi$. Furthermore, since ψ is a non-iterative formula and for each $i = 1, \dots, k$, $\llbracket \lambda_{\llbracket p_i \rrbracket_{\mathcal{M}'}} \rrbracket_{\mathcal{M}_\varphi} = |\lambda_{\llbracket p_i \rrbracket_{\mathcal{M}'}}|_\varphi$, Proposition 2.7 implies that $\llbracket \psi^\sigma \rrbracket_{\mathcal{M}_\varphi} = |\psi^\sigma|_\varphi$. Therefore, $\llbracket \psi^\sigma \rrbracket_{\mathcal{M}_\varphi} = W_\varphi$. A simple induction on formulas shows that $\llbracket \psi^\sigma \rrbracket_{\mathcal{M}_\varphi} = \llbracket \psi \rrbracket_{\mathcal{M}'}$ (see Exercise 1.5). Hence, $\llbracket \psi \rrbracket_{\mathcal{M}'} = W_\varphi$, as desired.

Proof of item 2: Suppose that $\nVdash_{\mathbf{L}} \varphi$. Then, since φ is trivially a Boolean combination of formulas from Σ_φ , by Lemma 2.18, we have that $\llbracket \varphi \rrbracket_{\mathcal{M}_\varphi} \neq W_\varphi$. Thus, φ is not valid on $\langle W_\varphi, N_\varphi \rangle$. \square

This proposition shows that any non-iterative modal logic containing \mathbf{E} is weakly complete (Remark 2.35). Furthermore, since there is a bound on the size of W_φ , which can be calculated from φ , we only need to check finitely many¹³ models to determine if φ is derivable in \mathbf{L} .

Theorem 2.50 (Lewis 1974) Every non-iterative modal logic \mathbf{L} containing \mathbf{E} is weakly complete and decidable.

There are two ways to extend this result. First, Lewis (1974) worked with a more general modal language including modalities of arbitrary arity. It is not hard to adapt the argument of this section to this more general setting. Lewis's theorem, then, gives a weak completeness and decidability result for the minimal conditional logic of sphere model. That is, the following axiom system (expressed using the comparative possibility modality \leq discussed in Sect. 1.4.3) is sound and complete for the class of all sphere frames.

¹³Of course, there are infinitely many variations of the finite φ -canonical model. However, we can ignore irrelevant differences, such as isomorphic copies or models that differ in their interpretation of formulas not among the subformulas of φ .

- (taut) All instances of propositional tautologies
- (trans) $((\varphi \leq \psi) \wedge (\psi \leq \chi)) \rightarrow (\varphi \leq \chi)$
- (cons) $(\varphi \leq \psi) \vee (\psi \leq \varphi)$
- (dis) $((\varphi \vee \psi) \leq \chi) \rightarrow ((\varphi \leq \chi) \vee (\psi \leq \chi))$
- (RE) $\frac{\varphi \leftrightarrow \psi}{\chi \leftrightarrow \chi'}$ where χ' is χ with one or more occurrences of φ replaced with ψ .

A second generalization comes from Surendonk (2001) who showed that the argument in this section can be adapted to prove strong completeness for non-iterative logics. See, also, Schröder (2006), Pattinson and Schröder (2006) and Schröder and Pattinson (2007) for further results about non-iterative modal logics.

2.4.2 Complexity

This brief section assumes that the reader has some familiarity with computation complexity theory. As a reminder, a problem is in **NP** provided that it can be solved by a non-deterministic Turing machine that is guaranteed to halt after a number of steps that is polynomial in the size of the input. A problem is **NP**-complete provided that it is in **NP** and that any other problem in **NP** can be polynomially reduced to it. Intuitively, a problem is **NP**-complete if it is the “hardest” problem solved by a non-deterministic Turing machine in a polynomial amount of time. The other complexity class that will be mentioned is **PSPACE**. A problem is in **PSPACE** provided that it can be decided by a deterministic Turing machine that uses at most a polynomial (in the size of the input) amount of space (i.e., tape cells). A problem is **PSPACE**-complete provided that it is in **PSPACE** and that every problem in **PSPACE** can be reduced to it. It is known that every problem in **NP** is also in **PSPACE**, but the converse is still open (i.e., is every problem in **PSPACE** also in **NP**?). However, it is widely believed that **PSPACE**-complete problems are “harder” than **NP**-complete problems.

Using the above two complexity classes, we can classify how hard it is to solve the satisfiability problem for different logical systems. The satisfiability problem for propositional logic is **NP**-complete. On the other hand, as is well known, the satisfiability problem for first-order logic is undecidable (see Enderton 2001, Chap. 3 for a discussion). The satisfiability problem for normal modal logic (i.e., with respect to relational models) is **PSPACE**-complete. Interestingly, the satisfiability problem for some classes of relational models is “easier”. For example, the satisfiability problem for the class of all relational frames in which the relations are equivalence relations is **NP**-complete. Marx (2007) offers a more in-depth discussion of complexity issues in modal logic.¹⁴ What about the complexity of the satisfiability problem for neighborhood models?

¹⁴Consult Halpern and Rêgo (2007) and Spaan (1993) for discussions of the curious fact that the satisfiability problem for modal logics seems to be either **NP**-complete or **PSPACE**-hard.

The algorithm outlined in Sect. 2.4.1, which searches all finite models of a bounded size to find a satisfying model runs in double exponential time (this follows from the fact that there are exponentially many subformulas of a given formula). Vardi (1989) showed that there is a much more efficient algorithm, proving that the satisfiability problem with respect to all neighborhood frames is **NP**-complete. As in the case with relational models, this result is sensitive to the properties of the neighborhood function. Vardi (1989) also showed that the satisfiability problem for the class of neighborhood models that are closed under intersections (i.e., the class of models for the logic **EC**) is **PSPACE**-complete. Interestingly, Allen (2005) later showed that there are additional non-normal modal logics with **PSPACE**-complete satisfiability problems. Allen studied logics inspired by the generalized relational models discussed in Sect. 2.2.2. For each $n \geq 2$, let (C^n) be the follow axiom:

$$(C^n) \quad \bigwedge_{i=1}^n \Box \varphi_i \rightarrow \Box \bigvee_{1 \leq k, l \leq n, k \neq l} (\varphi_k \wedge \varphi_l).$$

Let **EMNC**^{*n*} be the logic extending **EMN** with instances of (C^n) . Allen (2005) proved that for each $n \geq 2$, the satisfiability problem for **EMNC**^{*n*} is **PSPACE**-complete.

2.4.3 Proof Theory for Non-normal Modal Logics

While there is an extensive body of research focused on developing proof calculi for normal modal logics,¹⁵ there has been much less work developing proof calculi for non-normal modal logics. Notable exceptions include Chap. 6 in Fitting (1983), the labeled tableaux systems for non-normal modal logics in Governatori and Luppi (2000) and a series of recent papers developing sequent calculi for non-normal modal logics (Gilbert and Maffezioli 2015; Girlando et al. 2016).

In this section, I briefly introduce *sequents* for non-normal modal logics. My aim in this section is to use these sequents to further illustrate the issues that arose when discussing decidability (Sect. 2.4.1) and complexity (Sect. 2.4.2), rather than providing a complete introduction to sequent calculi for non-normal modal logics.

Before defining a sequent, I discuss an elegant characterization of some non-normal modal logics (cf. Definition 2.26) using inference rules. Consider the following inference rules (where k ranges over the non-negative integers):

$$(R_k) \quad \frac{(\varphi_1 \wedge \cdots \wedge \varphi_k) \rightarrow \psi}{(\Box \varphi_1 \wedge \cdots \wedge \Box \varphi_k) \rightarrow \Box \psi}.$$

When $k = 0$ (so there are no antecedents), the above rule reduces to the rule of Necessitation (**Nec**). When $k = 1$, the above rule is the monotonicity rule (**RM**) discussed in Sect. 2.3. Then, Lemma 2.6 can be rephrased as:

¹⁵See Fitting (2006), Wansing (1998), and Negri (2011) for surveys of this literature.

A modal logic **S** is monotonic iff (R_1) is a derived rule.

This suggests the following characterization of normal and regular modal logics.

Proposition 2.9 *Suppose that **S** is a modal logic that contains **E**. Then,*

- **S** is regular iff (R_k) is a derived rule for all $k \geq 1$.
- **S** is normal iff (R_k) is a derived rule for all $k \geq 0$.

Exercise 2.24 Prove Proposition 2.9.

A careful examination of the above proof suggests a sequent-based proof calculus for non-normal modal logic. I start by reminding the reader of the definition of a sequent and the sequent rules for propositional logic.

Definition 2.51 (*Sequent*) A **sequent** is a structure $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sequences¹⁶ of modal formulas. We write Γ, φ to denote the sequence of formulas in which φ is the last element (similarly, φ, Γ is a sequence in which φ is the first element). A sequent is **valid** on a class of neighborhood frames **F** if $\bigwedge \Gamma \rightarrow \bigvee \Delta$ is valid on **F**.

As the reader is invited to verify, each of the following rules preserves validity of the sequents and, together, form a complete system for all propositional tautologies.

Definition 2.52 (*Propositional Sequent Rules*)

$$\frac{}{\Gamma, p \Rightarrow p, \Delta} (\text{axiom})$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma' \Rightarrow \Delta'} (\text{perm}) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Delta, \Lambda} (\text{weak}) \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda} (\text{cut})$$

where Γ' and Δ' are permutations of Γ and Δ , respectively.

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} (\wedge L) \quad \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta} (\wedge R)$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta} (\vee L) \quad \frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta} (\vee R)$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma, \neg \varphi \Rightarrow \Delta} (\neg L) \quad \frac{\Gamma \Rightarrow \neg \varphi, \Delta}{\Gamma, \varphi \Rightarrow \Delta} (\neg R)$$

The above rules can be used to reduce sequents containing complex modal formulas to a sequent of the following form:

$$\mathbf{p}, \Box \varphi_1, \dots, \Box \varphi_k \Rightarrow \mathbf{q}, \Box \psi_1, \dots, \Box \psi_m,$$

¹⁶I have been using capital Greek letters to denote sets of formulas (c.f. Sect. 2.3.2). For the purposes of this section, it does not matter much whether the components of a sequent are sets or sequences. However, it is standard practice to define sequents using *sequences* of formulas. So, I will adopt the convention that capital Greek letters denote sequences of formulas in this section.

where \mathbf{p} and \mathbf{q} are sequences of propositional variables. The main question when developing sequent rules for non-normal modal logics is how to further reduce the above sequent. It is simplest to start with the logic **EM**. The key observation for this logic is from van Benthem (2010):

Proposition 2.10 (van Benthem 2010) *The sequent*

$$\mathbf{p}, \Box\varphi_1, \dots, \Box\varphi_k \Rightarrow \mathbf{q}, \Box\psi_1, \dots, \Box\psi_m$$

is valid on the class of monotonic neighborhood frames if, and only if, either

- \mathbf{p} and \mathbf{q} overlap, or
- there is some $1 \leq i \leq k$ and $1 \leq j \leq m$ such that $\varphi_i \Rightarrow \psi_j$ is valid on the class of monotonic neighborhood frames.

Proof I show that if there is no overlap between \mathbf{p} and \mathbf{q} and $\varphi_i \Rightarrow \psi_j$ is not valid for each $1 \leq i \leq k$ and $1 \leq j \leq m$, then $\mathbf{p}, \Box\varphi_1, \dots, \Box\varphi_k \Rightarrow \mathbf{q}, \Box\psi_1, \dots, \Box\psi_m$ is not valid. The remaining cases of the proof are left to the reader. Suppose that for each i, j (to simplify the proof, I will write “for each i, j ” instead of “for each $1 \leq i \leq k$ and $1 \leq j \leq m$ ”), $\varphi_i \Rightarrow \psi_j$ is not valid. Then, for each i, j , there is a neighborhood model $\mathcal{M}_{ij} = \langle W_{ij}, N_{ij}, V_{ij} \rangle$ with a state $w_{ij} \in W_{ij}$ such that $\mathcal{M}_{ij}, w_{ij} \models \varphi_i \wedge \neg\psi_j$. I will show that $\mathbf{p}, \Box\varphi_1, \dots, \Box\varphi_k \Rightarrow \mathbf{q}, \Box\psi_1, \dots, \Box\psi_m$ is not valid. Without loss of generality, we can assume that the sets W_{ij} are pairwise disjoint. Let $\mathcal{M} = \langle W, N, V \rangle$ be a neighborhood model, where

- $W = \bigcup_{i,j} W_{ij} \cup \{w^*\}$ (with w^* a new state not in any W_{ij}).
- Define $V : \text{At} \rightarrow \wp(W)$ as follows: Let $V_0 : \text{At} \rightarrow \wp(\bigcup_{i,j} W_{ij})$ be the function where for all $p \in \text{At}$, $V(p) = \bigcup_{i,j} V_{ij}(p)$. Then, V is the function where for all $p \in \text{At}$:

$$V(p) = \begin{cases} V_0(p) \cup \{w^*\} & \text{if } p \text{ is on the list } \mathbf{p} \\ V_0(p) & \text{otherwise.} \end{cases}$$

- Define $N : W \rightarrow \wp(\wp(W))$ as follows: For all $w \in \bigcup_{i,j} W_{ij}$, $N(w) = \{X \mid X \subseteq W \text{ and there is a } Y \in N_{ij}(w) \text{ such that } Y \subseteq X\}$, where i, j are the unique indices such that $w \in W_{ij}$. The neighborhood at w^* is defined as follows:

$$N(w^*) = \{X \mid X \subseteq W \text{ and } w^* \in X\}.$$

It is immediate from the definition that the above model $\mathcal{M} = \langle W, N, V \rangle$ is a monotonic neighborhood model. Furthermore, by construction we have that $\mathcal{M}, w^* \models \bigwedge_{1 \leq i \leq k} \Box\varphi_i$ and $\mathcal{M}, w^* \not\models \bigvee_{1 \leq j \leq m} \Box\psi_j$. It follows from this observation and the definition of the above model that the sequent $\mathbf{p}, \Box\varphi_1, \dots, \Box\varphi_k \Rightarrow \mathbf{q}, \Box\psi_1, \dots, \Box\psi_m$ is not valid. \square

Proposition 2.10 justifies the following rule for **EM**:

$$\frac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma, \Box\varphi \Rightarrow \Delta, \Box\psi} (\Box M)$$

Sequent rules for other non-normal modal logics can be developed in a similar manner. The minimal non-normal modal logic **E** requires a simple modification of Proposition 2.10.

Proposition 2.11 *The sequent*

$$\mathbf{p}, \Box\varphi_1, \dots, \Box\varphi_k \Rightarrow \mathbf{q}, \Box\psi_1, \dots, \Box\psi_m$$

is valid on the class of neighborhood frames if, and only if, either

- **p** and **q** overlap, or
- there is some $1 \leq i \leq k$ and $1 \leq j \leq m$ such that both $\varphi_i \Rightarrow \psi_j$ and $\psi_j \Rightarrow \varphi_i$ are valid on the class of neighborhood frames.

Exercise 2.25 Prove Proposition 2.11. (Hint: Adapt the proof of Proposition 2.10.)

The corresponding sequent rule for **E** is:

$$\frac{\Gamma, \varphi \Rightarrow \Delta, \psi \quad \Gamma, \psi \Rightarrow \Delta, \varphi}{\Gamma, \Box\varphi \Rightarrow \Delta, \Box\psi} (\Box E)$$

Finally, there is an analogous result for the class of neighborhood frames closed under intersections.

Proposition 2.12 *The sequent*

$$\mathbf{p}, \Box\varphi_1, \dots, \Box\varphi_k \Rightarrow \mathbf{q}, \Box\psi_1, \dots, \Box\psi_m$$

is valid on the class of neighborhood frames that are closed under finite intersections if, and only if, either

- **p** and **q** overlap, or
- there is some $1 \leq i, j \leq k$ such that $\mathbf{p}, \Box\varphi_1, \dots, \Box\varphi_{k'}, \Box(\varphi_i \wedge \varphi_j) \Rightarrow \mathbf{q}, \Box\psi_1, \dots, \Box\psi_m$ is valid on the class of neighborhood frames closed under finite intersections (where $\Box\varphi_1, \dots, \Box\varphi_{k'}$ is the sequence $\Box\varphi_1, \dots, \Box\varphi_k$ without $\Box\varphi_i$ and $\Box\varphi_j$).

Exercise 2.26 Prove Proposition 2.12.

The resulting sequent rule for the logic **EC** is:

$$\frac{\Gamma, \Box(\varphi \wedge \psi) \Rightarrow \Delta}{\Gamma, \Box\varphi, \Box\psi \Rightarrow \Delta} (\Box C)$$

There is an important difference between the sequent rules for **E** and **EM** and the one for **EC**. Reading the rules from the bottom up, the first two rules reduce the

complexity of formulas in a sequent (i.e., the rules remove the ‘ \Box ’ from two formulas in a sequent). The sequent for **EC** behaves differently. This rule does not simplify any formulas, but, rather, reduces the length of the antecedent of the sequent. This difference has ramifications for the complexity of the proof search problem (given a formula φ , determine if there is a proof of φ). Indeed, this difference explains, in part, why the complexity of the satisfiability problem for **E** and **EM** is **NP**-complete, while it is **PSPACE**-complete for **EC**.

This is just the first steps towards a complete sequent system for non-normal modal logics. Readers interested in developing these ideas further are invited to consult Negri (2016), and the references therein.

2.5 Frame Correspondence

A central topic in the model theory of modal logic is *correspondence theory* (cf. Blackburn et al. 2001, Chap. 3). The aim of this theory is to identify (and characterize) modal formulas that correspond to interesting properties of relational frames. For instance, it is well known that a relational frame $\mathcal{F} = \langle W, R \rangle$ validates $\Box\varphi \rightarrow \varphi$ iff R is reflexive. The modal formula $\Box\varphi \rightarrow \varphi$ is said to **correspond** to the reflexivity property. See Appendix A for an introduction to correspondence theory with respect to relational structures.

Many of the ideas of correspondence theory can be adapted to the neighborhood setting. In this case, modal formulas express properties of neighborhood functions.

Definition 2.53 A modal formula $\varphi \in \mathcal{L}$ **defines a property** P of neighborhood functions if any neighborhood frame $\mathcal{F} = \langle W, N \rangle$ has property P iff \mathcal{F} validates φ .

There are some important differences when formulas are interpreted on neighborhood frames instead of relational frame. The first difference is that formulas corresponding to the same property on relational frames may correspond to different properties on neighborhood frames. For example, consider the formulas $\Diamond\top$ and $\Box\varphi \rightarrow \Diamond\varphi$. On relational frames, these formulas both define the same property: seriality (a relation R is serial provided that for each state w , there is a state v such that $w R v$).¹⁷ However, on the class of neighborhood frames, these formulas express different properties. The first formula $\Diamond\top$ is easily seen to express the fact that the empty set is not an element of the neighborhoods. That is, $\Diamond\top$ is valid on a neighborhood frame \mathcal{F} iff the empty set is not an element of any neighborhood (the proof follows immediately from the definition of truth of modal formulas). The second formula expresses a different property of neighborhood functions.

Lemma 2.19 Let $\mathcal{F} = \langle W, N \rangle$ be a neighborhood frame. Then, $\mathcal{F} \models \Box\varphi \rightarrow \Diamond\varphi$ iff \mathcal{F} is proper (i.e., if $X \in N(w)$, then $X^C \notin N(w)$).

¹⁷This makes sense since these formulas are semantically equivalent on the class of relational frames (i.e., they are true at exactly the same points in all relational models).

Proof The right-to-left direction is straightforward. For the left-to-right direction, suppose that $\mathcal{F} = \langle W, N \rangle$ is not proper. Then, there is a state $w \in W$ and a set $X \subseteq W$ such that $X \in N(w)$ and $X^C \in N(w)$. Define a model $\mathcal{M} = \langle W, N, V \rangle$ with $V(p) = X$. Then, by definition, $\mathcal{M}, w \models \Box p$, and since $\llbracket \neg p \rrbracket_{\mathcal{M}} = X^C \in N(w)$, we have $\mathcal{M}, w \models \neg \Diamond p$. Thus, $\mathcal{M}, w \not\models \Box p \rightarrow \neg \Box \neg p$; and so, $\Box \varphi \rightarrow \Diamond \varphi$ is not valid on \mathcal{F} . \square

A second difference is that modal formulas that are valid on all relational frames correspond to non-trivial properties of neighborhood functions.

Lemma 2.20 *Suppose that $\mathcal{F} = \langle W, N \rangle$ is a neighborhood frame. Then, $\mathcal{F} \models \Box \varphi \wedge \Box \psi \rightarrow \Box(\varphi \wedge \psi)$ iff \mathcal{F} is closed under finite intersections.*¹⁸

Proof Suppose that $\mathcal{F} = \langle W, N \rangle$ is a neighborhood frame that is closed under finite intersections. We must show $\mathcal{F} \models \Box \varphi \wedge \Box \psi \rightarrow \Box(\varphi \wedge \psi)$. Let $\mathcal{M} = \langle W, N, V \rangle$ be any model based on \mathcal{F} and $w \in W$. Suppose that $\mathcal{M}, w \models \Box \varphi \wedge \Box \psi$. Then, $\llbracket \varphi \rrbracket_{\mathcal{M}} \in N(w)$ and $\llbracket \psi \rrbracket_{\mathcal{M}} \in N(w)$. Since $N(w)$ is closed under finite intersections, $\llbracket \varphi \rrbracket_{\mathcal{M}} \cap \llbracket \psi \rrbracket_{\mathcal{M}} \in N(w)$. Hence, $\llbracket \varphi \wedge \psi \rrbracket_{\mathcal{M}} \in N(w)$ and, therefore, $\mathcal{M}, w \models \Box(\varphi \wedge \psi)$.

Suppose that $\mathcal{F} = \langle W, N \rangle$ is not closed under finite intersections. Then, $\langle W, N \rangle$ is not closed under binary intersections (see Lemma 1.2). Thus, there is a state $w \in W$ and two sets Y and Y' such that $Y, Y' \in N(w)$, but $Y \cap Y' \notin N(w)$. Define a valuation function so that $V(p) = Y$ and $V(q) = Y'$. This implies that $\mathcal{M}, w \models \Box p \wedge \Box q$. However, since $Y \cap Y' \notin N(w)$, $\mathcal{M}, w \not\models \Box(p \wedge q)$. \square

Lemma 2.21 *Suppose that $\mathcal{F} = \langle W, N \rangle$ is a neighborhood frame. Then, $\mathcal{F} \models \Box(\varphi \wedge \psi) \rightarrow \Box \varphi \wedge \Box \psi$ iff \mathcal{F} is closed under supersets.*

Proof The right-to-left direction is left as an exercise for the reader. Suppose that $\mathcal{F} = \langle W, N \rangle$ is not closed under supersets. Then, there are sets X and Y such that $X \subseteq Y$, $X \in N(w)$ but $Y \notin N(w)$. Define a valuation V such that $V(p) = X$ and $V(q) = Y$. Then, since $X \subseteq Y$, $\llbracket p \wedge q \rrbracket_{\mathcal{M}} = X \in N(w)$. Hence, $\mathcal{M}, w \models \Box(p \wedge q)$. However, since, $\llbracket q \rrbracket_{\mathcal{M}} = Y \notin N(w)$, we have $\mathcal{M}, w \not\models \Box q$. Hence, $\mathcal{M}, w \not\models \Box p \wedge \Box q$. \square

Lemma 2.22 *Suppose that $\mathcal{F} = \langle W, N \rangle$ is a neighborhood frame. Then, $\mathcal{F} \models \Box \top$ iff \mathcal{F} contains the unit.*

Proof Left as an exercise for the reader. \square

I conclude this brief introduction to correspondence theory by identifying properties of neighborhood functions that correspond to well-studied modal formulas.

Lemma 2.23 *Suppose that $\mathcal{F} = \langle W, N \rangle$ is a neighborhood frame such that for each $w \in W$, $N(w) \neq \emptyset$.*

¹⁸Recall that a frame $\mathcal{F} = \langle W, N \rangle$ is said to be closed under finite intersection provided that for all $w \in W$, $N(w)$ is closed under finite intersections. See the discussion after Remark 1.10.

1. $\mathcal{F} \models \Box\varphi \rightarrow \varphi$ iff for each $w \in W$, $w \in \cap N(w)$.
2. $\mathcal{F} \models \Box\varphi \rightarrow \Box\Box\varphi$ iff for each $w \in W$, if $X \in N(w)$, then $m_N(X) \in N(w)$ (recall that $m_N(X) = \{v \mid X \in N(v)\}$).

Proof Suppose that $\mathcal{F} = \langle W, N \rangle$ is a neighborhood frame. Suppose that for each $w \in W$, $w \in \cap N(w)$. Let \mathcal{M} be any model based on \mathcal{F} and $w \in W$. Suppose that $\mathcal{M}, w \models \Box\varphi$. Then, $\llbracket \varphi \rrbracket_{\mathcal{M}} \in N(w)$. Since $w \in \cap N(w) \subseteq \llbracket \varphi \rrbracket_{\mathcal{M}}$, $w \in \llbracket \varphi \rrbracket_{\mathcal{M}}$. Hence, $\mathcal{M}, w \models \varphi$. As for the converse, suppose that $w \notin \cap N(w)$. Since $N(w) \neq \emptyset$, there is an $X \in N(w)$ (note that X may be empty) such that $w \notin X$; otherwise, $w \in \cap N(w)$. Define a valuation V such that $V(p) = X$. Then, it is easy to see that $\mathcal{M}, w \models \Box p$, but $\mathcal{M}, w \not\models p$.

Suppose that for each $w \in W$, if $X \in N(w)$, then $\{v \mid X \in N(v)\} \in N(w)$. Suppose that \mathcal{M} is any model based on \mathcal{F} and $\mathcal{M}, w \models \Box\varphi$. Then, $\llbracket \varphi \rrbracket_{\mathcal{M}} \in N(w)$. Therefore, by assumption $\{v \mid \llbracket \varphi \rrbracket_{\mathcal{M}} \in N(v)\} \in N(w)$. Since $\llbracket \Box\varphi \rrbracket_{\mathcal{M}} = \{v \mid \llbracket \varphi \rrbracket_{\mathcal{M}} \in N(v)\}$, $\mathcal{M}, w \models \Box\Box\varphi$. For the other direction, suppose that there is some state $w \in W$ and set X such that $X \in N(w)$, but $\{v \mid X \in N(v)\} \notin N(w)$. Then, define a valuation V such that $V(p) = X$. It is easy to verify that $\mathcal{M}, w \models \Box p$, but $\mathcal{M}, w \not\models \Box\Box p$. \square

Exercise 2.27 Find properties on frames that are defined by the following formulas:

1. $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$
2. $\Box\varphi \vee \Box\neg\varphi$
3. $\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$

There is much more to say about correspondence theory with respect to neighborhood frames. In particular, the Sahlqvist Theorem (Sahlqvist 1975) provides a syntactic definition of a class of modal formulas, each of which corresponds to a first-order property of a relational frame.¹⁹ An interesting line of research is to explore generalizations of the Sahlqvist theorem and related algorithms (Conradie et al. 2006) for finding first-order correspondents to the class of neighborhood frames. One approach is to first translate neighborhood models and the basic modal language into a special class of relational models with an appropriate modal language (see Sect. 2.6.2 for details of such a translation). The Sahlqvist Theorem can then be applied to this special class of relational models and modal language. Consult Kracht and Wolter (1999) and Hansen (2003) for details of this approach. A second approach is to explore different generalizations of the Sahlqvist Theorem. Consult Palmigiano et al. (2016) for a generalization of the Sahlqvist Theorem to regular modal logics (Definition 2.26).

2.6 Translations

Neighborhood models generalize relational models by replacing a relation between worlds ($R \subseteq W \times W$) with a relation between worlds and sets of worlds ($N \subseteq$

¹⁹See Sects. 3.5–3.7 in Blackburn et al. (2001) for details.

$R \times \wp(W)$). In Sect. 2.2.1, we saw that the class of augmented neighborhood models is modally equivalent to the class of relational models. The central idea is that, for each augmented set $\mathcal{X} \subseteq \wp(W)$, there is a relation $R_{\mathcal{X}} \subseteq W \times W$ defined as follows: $w R_{\mathcal{X}} v$ iff $v \in \bigcap \mathcal{X}$. The definition of $R_{\mathcal{X}}$ makes sense only if the collection of subsets \mathcal{X} is augmented. In this section, I show that there are more subtle connections between neighborhood models (and non-normal modal logics) and relational models (and normal modal logics).

The first connection is that *every* collection of subsets $\mathcal{X} \subseteq \wp(W)$ can be associated with an ordering $\preceq_{\mathcal{X}} \subseteq W \times W$. For each $w, v \in W$, let $w \preceq_{\mathcal{X}} v$ iff for all $X \in \mathcal{X}$, if $w \in X$, then $v \in X$. That is, v is ranked at least as high as w by $\preceq_{\mathcal{X}}$ provided that every set from \mathcal{X} containing w also contains v . This is a well-known definition in point-set topology (it is called the *specialization ordering*), the study of preferences (Andreka et al. 2002; Liu 2011) and multi-criteria decision-making (Dietrich and List 2013; de Jongh and Liu 2009). In Sect. 2.6.1, I use this ordering to facilitate a rigorous comparison between the evidence models from Sect. 1.4.4 and *plausibility models*, which are well known in the study of modal logics of belief and belief revision.

The second connection is based on neighborhood models themselves. On the face of it, the definition of truth of the modal operator on a neighborhood model seems to be a *second-order* statement since it asserts the existence of a subset implying either that a formula is true or that it is equal to the truth set of a formula. However, appearances are deceiving. By treating the neighborhoods as states in a larger model, there is a way to translate every neighborhood model into a relational model. Building on this idea, it can be shown that every non-normal modal logic can be *simulated* by a normal modal logic (Sect. 2.6.2), and that there is a translation of non-normal modal logic into first-order logic (Sect. 2.6.3).

2.6.1 From Neighborhoods to Orders

A **plausibility ordering** on a set of states W is a reflexive and transitive relation $\preceq \subseteq W \times W$. The intended meaning of $w \preceq v$ is that “(according to the agent) world v is at least as plausible as w ”. Plausibility models are widely used as a semantics for modal logics of belief (van Benthem 2004; Baltag and Smets 2006b,a; Girard 2008 and deontic logics Hansson (1990); van Benthem et al. (2014)).

Definition 2.54 (*Plausibility model*) A **plausibility model** is a tuple $\mathfrak{M} = \langle W, \preceq, V \rangle$ where W is a finite nonempty set; $\preceq \subseteq W \times W$ is a reflexive and transitive relation on W ; and $V : \text{At} \rightarrow \wp(W)$ is a valuation function. If \preceq is also *connected* (for each $w, v \in W$, either $w \preceq v$ or $v \preceq w$), then we say that \mathfrak{M} is a **connected plausibility model**. A pair \mathfrak{M}, w where w is a state is called a **pointed (connected) plausibility model**.

When two worlds w and v cannot be compared by the plausibility ordering (for an agent), the interpretation is that the agent has either accepted contradictory evidence or lacks enough evidence to compare the two states.²⁰

A number of different modal languages have been used to reason about plausibility structures. For instance, let $\mathcal{L}^{pl}(\mathbf{At})$, where \mathbf{At} is the set of atomic propositions, be the smallest set of formulas generated by the following grammar:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid [B]\psi \mid [\leq]\varphi \mid [A]\varphi$$

where $p \in \mathbf{At}$. For each $\bigcirc \in \{B, \leq, A\}$, let $\langle \bigcirc \rangle\varphi$ be defined as $\neg[\bigcirc]\neg\varphi$. Before defining truth for this language, I need some notation. For $w, v \in W$, write $w < v$ if $w \leq v$ and $v \not\leq w$. For $X \subseteq W$, let

$$Max_{\leq}(X) = \{w \in X \mid \text{there is no } v \in X \text{ such that } w < v\}.$$

For each set X , $Max_{\leq}(X)$ is the set of most plausible worlds in X (i.e., the maximal elements of X according to the plausibility order \leq).²¹ Suppose that $\mathfrak{M} = \langle W, \leq, V \rangle$ is a plausibility model with $w \in W$. Truth of the Boolean connectives and atomic propositions is defined as usual. I give only the clauses for the modal operators:

- $\mathfrak{M}, w \models [B]\varphi$ iff $Max_{\leq}(W) \subseteq \llbracket \varphi \rrbracket_{\mathcal{M}}$
- $\mathfrak{M}, w \models [\leq]\varphi$ iff for all $v \in W$, if $w \leq v$ then $\mathcal{M}, v \models \varphi$
- $\mathfrak{M}, w \models [A]\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$.

So, φ is believed provided that φ is true throughout all of the most plausible states. There is much more to say about plausibility structures, their relationship with theories of belief revision, and the modal logic of beliefs (see van Benthem 2011 and Pacuit 2013a for discussions). In the remainder of this section, I focus on the relationship between plausibility models and neighborhood models.

From Plausibility Structures to Neighborhood Structures

There is a natural subset space associated with every plausibility model.

Definition 2.55 (*Upwards Closed Sets*) Suppose that \leq is a plausibility ordering on a set of states W (i.e., a reflexive and transitive relation on W). The upwards closure of a set X , denoted $\uparrow X$, is the set

$$\uparrow X = \{v \in W \mid \text{there is a } w \in X \text{ such that } w \leq v\}.$$

A set X is \leq -**closed** when $\uparrow X \subseteq X$. Let $\mathcal{F}_{\leq} = \{\uparrow X \mid X \subseteq W\}$ be the set of \leq -closed sets.

²⁰Swanson (2011) has an extensive discussion of incomparability when modeling conditionals.

²¹To keep things simple, I assume that the set of worlds is finite, so this maximal set always exists. One needs a (converse) well-foundedness condition to guarantee this when there are infinitely many states.

Exercise 2.28 If \preceq is a plausibility ordering on W , then \mathcal{F}_{\preceq} is closed under non-empty intersections. (Why do I need to specify non-empty intersections?) Is \mathcal{F}_{\preceq} closed under supersets?

Using the above notation, there is a straightforward way to turn any plausibility model into a neighborhood structure. Let $\mathfrak{M} = \langle W, \preceq, V \rangle$ be a plausibility model. The associated neighborhood model is the model $\mathcal{M}^{\preceq} = \langle W^{\preceq}, N^{\preceq}, V^{\preceq} \rangle$, where $W^{\preceq} = W$; for each $w \in W$, $N^{\preceq}(w) = \mathcal{F}_{\preceq}$; and $V^{\preceq} = V$. Thus, the associated neighborhood model \mathcal{M}^{\preceq} has a *uniform* neighborhood function (each state is associated with the same collection of sets).

Remark 2.56 A more general definition of plausibility models is possible in which each state is associated with a different plausibility ordering. That is, for each $w \in W$, \preceq_w is a plausibility ordering on W . In this case, the neighborhoods may vary at each state: $N^{\preceq}(w) = \mathcal{F}_{\preceq_w}$. I focus on a single, global plausibility ordering to simplify the discussion.

The next step is to show that every plausibility model $\mathfrak{M} = \langle W, \preceq, V \rangle$ is *equivalent* to the corresponding neighborhood model \mathcal{M}^{\preceq} . Here, we face a problem that did not arise in the previous sections. The notion of equivalence between classes of models discussed in Sect. 2.1 assumes that there is a single underlying language that can be interpreted on *both* classes of models. However, I have not given a definition of truth for the language \mathcal{L}^{pl} over neighborhood models.

One solution is to find a translation between an appropriate language interpreted over neighborhood models and \mathcal{L}^{pl} . Before defining such a translation, I note an important fact about the language \mathcal{L}^{pl} : We can restrict attention to formulas from \mathcal{L}^{pl} that include only the $[\preceq]$ and $[A]$ modalities.

Fact 2.57 *On finite plausibility models, the belief modality $[B]$ is definable in terms of the $[A]$ and $[\preceq]$ modalities. That is,*

- $[B]\varphi \leftrightarrow [A](\preceq)[\preceq]\varphi$ is valid on finite plausibility models.²²

The proof is straightforward given the following observation. The set of maximal elements in a plausibility model can be partitioned into *final clusters*:

Definition 2.58 (*Final Cluster*) Let $\mathfrak{M} = \langle W, \preceq, V \rangle$ be a plausibility model. A **final cluster** in \mathcal{M} is a set $X \subseteq \text{Max}_{\preceq}(W)$ that is maximal and completely connected: for any $x, y \in X$, $x \preceq y$ and $y \preceq x$, and there is no $v \in W$ such that $w < v$ for some $w \in X$.

In a connected plausibility model, there is only one largest final cluster: the set $\text{Max}_{\preceq}(W)$. However, when the order is not connected, there may be disjoint final clusters. Using this terminology, (on finite models) $[B]\varphi$ is true provided that φ is true throughout all final clusters.

²²This was first discussed by Boutilier (1992).

Exercise 2.29 Prove Fact 2.57.

Let \mathcal{L}_0^{pl} be the sublanguage of \mathcal{L}^{pl} without the belief modality $[B]$. The appropriate language for neighborhood models is $\mathcal{L}^{(\cdot, [A])}$ —the propositional modal language generated by adding the neighborhood modality $\langle \cdot \rangle$ and the universal modality $[A]$ to a propositional language. The definition of truth for formulas of the form $\langle \cdot \rangle \varphi$ is given in Sect. 1.2.2 (Definition 1.2.2), and the clause for the universal modality $[A]$ is exactly as it is for the plausibility models given above. I can now define the translation between these two languages.

Definition 2.59 (*\leq -translation*) The translation $tr_{\leq} : \mathcal{L}^{(\cdot, [A])} \rightarrow \mathcal{L}_0^{pl}$ is defined by induction on formulas in $\mathcal{L}^{(\cdot, [A])}$:

- for each $p \in \mathbf{At}$, $tr_{\leq}(p) = p$;
- $tr_{\leq}(\neg\varphi) = \neg tr_{\leq}(\varphi)$ and $tr_{\leq}(\varphi \wedge \psi) = tr_{\leq}(\varphi) \wedge tr_{\leq}(\psi)$;
- $tr_{\leq}([A]\varphi) = [A](tr_{\leq}(\varphi))$; and
- $tr_{\leq}(\langle \cdot \rangle \varphi) = \langle A \rangle [\leq](tr_{\leq}(\varphi))$.

Proposition 2.13 Let $\mathfrak{M} = \langle W, \leq, V \rangle$ be a plausibility model. For any $\varphi \in \mathcal{L}^{(\cdot, [A])}$ and state $w \in W$,

$$\mathfrak{M}, w \models tr_{\leq}(\varphi) \text{ iff } \mathcal{M}^{\leq}, w \models \varphi.$$

The proof is straightforward and left to the reader. However, this is a weak result. The conclusion is simply that every plausibility model “contains” a neighborhood model. Furthermore, anything that can be expressed in the language $\mathcal{L}^{(\cdot, [A])}$ can be translated into the language \mathcal{L}^{pl} . Of course, this translation is not surjective. That is, there are formulas of \mathcal{L}^{pl} that are not the translation of some formula from $\mathcal{L}^{(\cdot, [A])}$. So, we cannot conclude that for every plausibility model, there is a *modally equivalent* neighborhood model (at least with respect to the language \mathcal{L}^{pl}).

From Neighborhood Models to Plausibility Models

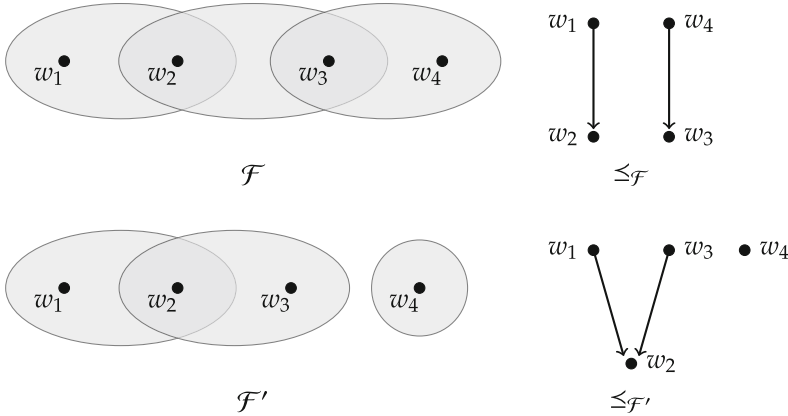
There is also a natural way to define a plausibility ordering given any subset space. The approach is to use the so-called *specialization order*, a notion that occurs in point-set topology (cf. Sect. 1.4.4) and in recent theories of relation merge (cf. Andreka et al. 2002; Liu 2011).

Definition 2.60 (*Specialization Order*) Suppose that $\langle W, \mathcal{F} \rangle$ is a subset space. Define $\leq_{\mathcal{F}} \subseteq W \times W$ as follows:

$$w \leq_{\mathcal{F}} v \text{ iff for all } X \in \mathcal{F}, \text{ if } w \in X, \text{ then } v \in X.$$

The intuition is that v is “at least as special” as w provided that every set in \mathcal{F} that contains w also contains v . If \mathcal{F} is a set of evidence as in the evidence models from Sect. 1.4.4, then $w \leq_{\mathcal{F}} v$ means that every piece of evidence that supports w (i.e., contains w) also supports v , though there might be some pieces of evidence that

support v but not w . To make this definition a bit more concrete, here is a simple illustration.



Of course, the relational properties of $\leq_{\mathcal{F}}$ depend on the algebraic properties of \mathcal{F} . However, all specialization orders are reflexive and transitive:

Observation 2.61 *Suppose that $\langle W, \mathcal{F} \rangle$ is a subset space. Then, $\leq_{\mathcal{F}}$ is a transitive and reflexive relation on W .*

Proof Suppose that $w \leq_{\mathcal{F}} v$ and $v \leq_{\mathcal{F}} y$. Suppose, also, that $X \in \mathcal{F}$ and $w \in X$. Then, since $w \leq_{\mathcal{F}} v$, we have $v \in X$. Since, $v \leq_{\mathcal{F}} y$, we have $y \in X$. Thus, $w \leq_{\mathcal{F}} y$. Clearly, $\leq_{\mathcal{F}}$ is reflexive. \square

The examples given above show that, in general, the specialization order $\leq_{\mathcal{F}}$ is not connected.

- Exercise 2.30**
1. What property of \mathcal{F} guarantees that the specialization ordering $\leq_{\mathcal{F}}$ is connected? Recall that a relation $R \subseteq X \times X$ is **connected** if every pair of distinct elements from X is related. That is, for all $x, y \in X$, if $x \neq y$, then either $x R y$ or $y R x$.
 2. Suppose that \preceq is the reflexive, transitive closure of $\{(w, v), (v, x), (w, y), (y, x)\}$ (that is, \preceq is the smallest relation containing this set that is reflexive and transitive). Find a subset space \mathcal{F} on $W = \{w, v, y, x\}$ such that $\preceq = \leq_{\mathcal{F}}$.

There are (at least) two different ways to associate a plausibility model with a neighborhood structure $\mathcal{M} = \langle W, N, V \rangle$. The first is to assign to each state $w \in W$, a plausibility ordering $\leq_{N(w)}$. Strictly speaking, unless N is a constant function, this is not a plausibility model according to Definition 2.54 since each state is assigned a different plausibility ordering (cf. Remark 2.56). Rather than pursuing this line of thought, I focus on the relationship between plausibility models and the evidence models from Sect. 1.4.4. It turns out that the precise relationship is subtle. Following the discussions in van Benthem et al. (2014) and van Benthem and Pacuit (2011, Sect. 5), I briefly discuss this relationship in the remainder of this section, focusing on uniform evidence models.

The first step is to combine the two languages \mathcal{L}^{ev} and \mathcal{L}^{pl} . Let $\mathcal{L}^{epi}(\text{At})$ be the smallest set of formulas generated by the following grammar:

$$p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid \langle \rangle\varphi \mid [\leq]\varphi \mid [B]\varphi \mid [A]\varphi$$

where $p \in \text{At}$. A model for this language includes relations for the $[\leq]$ and $[B]$ modalities (cf. Appendix A) and a neighborhood function for the $\langle \rangle$ modality ($[A]$ is the universal modality). Given such a model, the definition of truth for formulas from \mathcal{L}^{epi} follows the usual pattern (cf. Definition A.3). For example, suppose that $\mathcal{M} = \langle W, E, \leq, B, V \rangle$, where E is a neighborhood function, $B \subseteq W \times W$, $\leq \subseteq W \times W$ and $V : \text{At} \rightarrow \wp(W)$ is a valuation function. The definition of truth for the modalities at $w \in W$ is:

- $\mathcal{M}, w \models \langle \rangle\varphi$ iff there is an $X \in E(w)$ such that for all $v \in X$, $\mathcal{M}, v \models \varphi$.
- $\mathcal{M}, w \models [\leq]\varphi$ iff for all $v \in W$, if $w \leq v$, then $\mathcal{M}, v \models \varphi$.
- $\mathcal{M}, w \models [B]\varphi$ iff for all $v \in W$, if $w B v$, then $\mathcal{M}, v \models \varphi$.
- $\mathcal{M}, w \models [A]\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$.

Given the intended interpretation of the modalities in \mathcal{L}^{epi} , it is natural to impose the following constraints on a model $\langle W, E, \leq, B, V \rangle$:

1. for each $w \in W$, $\emptyset \notin E(w)$ and $W \in E(w)$;
2. for all $w, v, u \in W$, if $w \leq v$ and $w \in X \in E(u)$, then $v \in X$; and
3. if $w \leq v$ and $u B w$, then $u B v$.

To see the importance of these constraints, consider a model $\mathcal{M} = \langle W, \mathcal{E}, \leq_{\mathcal{E}}, B, V \rangle$, where \mathcal{E} is a uniform evidence function, $\leq_{\mathcal{E}}$ is the specialization order defined from \mathcal{E} and $B \subseteq W \times W$. It is easy to find such a model with states w, v and u such that $w B v$ and $v \leq_{\mathcal{E}} u$; yet it is not the case that $w B u$. For example, let $W = \{w, v\}$ and $\mathcal{E} = \{W\}$, and $B = \{(w, w)\}$. Then, $w B w$ and $w \leq_{\mathcal{E}} v$; yet it is not the case that $w B v$. However, this does not happen on intended models:

Definition 2.62 (*Transforming Evidence Models*) Let $\mathcal{M} = \langle W, \mathcal{E}, V \rangle$ be a uniform evidence model (recall Definition 1.35). An **extended evidence model** is a structure $\mathcal{M}^{\Delta} = \langle W, \mathcal{E}, B_{\mathcal{E}}, \leq_{\mathcal{E}}, V \rangle$, where

- $w B_{\mathcal{E}} v$ iff $v \in \bigcap \mathcal{X}$ for some scenario \mathcal{X} from \mathcal{E} ; and
- $w \leq_{\mathcal{E}} v$ iff for any $X \in \mathcal{E}$, if $w \in X$, then $v \in X$.

The following exercise illustrates key properties satisfied by extended evidence models.

Exercise 2.31 Suppose that $\mathcal{M} = \langle W, \mathcal{E}, V \rangle$ is a uniform evidence model, and let $\leq_{\mathcal{E}}$ be defined as above. For each $w \in W$, let $\mathcal{E}[w] = \{X \in \mathcal{E} \mid w \in X\}$. Prove the following two statements (cf. van Benthem et al. 2014, Lemma 4).

1. For each $w \in W$, if $w \in \bigcap \mathcal{X}$ for some scenario $\mathcal{X} \subseteq \mathcal{E}$, then w is $\leq_{\mathcal{E}}$ -maximal. Furthermore, if \mathcal{X} is a non-empty scenario from \mathcal{E} , then $\mathcal{X} = \mathcal{E}[w]$ for $\leq_{\mathcal{E}}$ -maximal state w .

2. Suppose that \mathcal{M} is also flat. For each $w \in W$, if w is $\leq_{\mathcal{E}}$ -maximal, then $w \in \bigcap \mathcal{X}$ for some scenario $\mathcal{X} \subseteq \mathcal{E}$. Furthermore, if w is $\leq_{\mathcal{E}}$ maximal, then $\mathcal{E}[w]$ is a scenario.
3. Using items 1 and 2 to prove that $[A](\leq)[\leq]\varphi \rightarrow [B]\varphi$ is valid on extended uniform evidence models (that is, if $\mathcal{M} = \langle W, \mathcal{E}, V \rangle$ is a uniform evidence model, then the formula is valid on \mathcal{M}^{Δ}). Furthermore, over the class of models that are, moreover, flat, the formula $[A](\leq)[\leq]\varphi \leftrightarrow [B]\varphi$ is valid (cf. Fact 2.57).

Consult van Benthem et al. (2014) for the precise relationship between evidence models, extended evidence models and plausibility models.

2.6.2 The Normal Translation

In this section, I explore a deeper connection between neighborhood models and relational models. The key observation is that neighborhood models are just a special type of relational models.

The states in these special relational models are divided into two sorts: the elements of a non-empty set W and the subsets from W . For any set W , let $W^{\circ} = W \cup \wp(W)$. (It is also important to assume that $W \cap \wp(W) = \emptyset$.) There are two natural relations on W° corresponding to the “element of” and “not element of” relations between states and subsets. It will be convenient to use the converse of the “(not) element of” relation:

- $R_{\supseteq} \subseteq \wp(W) \times W$ with $R_{\supseteq} = \{(U, w) \mid w \in W, U \in \wp(W), w \in U\}$.
- $R_{\not\supseteq} \subseteq \wp(W) \times W$ with $R_{\not\supseteq} = \{(U, w) \mid w \in W, U \in \wp(W), w \notin U\}$.

Two remarks about the above relations are in order. First, $R_{\not\supseteq}$ is *not* the complement of R_{\supseteq} (with respect to $W^{\circ} \times W^{\circ}$). That is $R_{\not\supseteq} \neq (W^{\circ} \times W^{\circ}) - R_{\supseteq}$. This is, because, for example, $(w, v) \in (W^{\circ} \times W^{\circ}) - R_{\supseteq}$, where $w, v \in W$, but $(w, v) \notin R_{\not\supseteq}$. Of course, it is true that $R_{\not\supseteq} = (\wp(W) \times W) - R_{\supseteq}$.

Second, there are other relations that can be studied in this context. For instance, one can include the subset relation $R_{\subseteq} \subseteq \wp(W) \times \wp(W)$ with $R_{\subseteq} = \{(U, V) \mid U, V \in \wp(W), U \subseteq V\}$ in the model. A *relational-neighborhood* model includes a third relation between states W and $\wp(W)$ (which is intended to represent a neighborhood function).

Definition 2.63 (Relational-Neighborhood Model) Suppose that W is a non-empty set of states and At is a set of atomic propositions. A **relational neighborhood model** on W is a tuple $\langle W^{\circ}, R_N, R_{\supseteq}, R_{\not\supseteq}, V \rangle$, where $W^{\circ} = W \cup \wp(W)$, $R \subseteq W \times \wp(W)$, $R_{\supseteq} \subseteq \wp(W) \times W$ with $R_{\supseteq} = \{(U, w) \mid w \in W, U \in \wp(W), w \in U\}$, $R_{\not\supseteq} \subseteq \wp(W) \times W$ with $R_{\not\supseteq} = \{(U, w) \mid w \in W, U \in \wp(W), w \notin U\}$, and $V : \text{At} \rightarrow \wp(W)$ is a valuation function.

The definition of the valuation function in a relational-neighborhood function highlights the two-sorted aspect of these models. Since the range of the valuation

function is the set of states W , the atomic propositions (and, hence, all non-modal formulas) are “state formulas” that can express properties of the set of states W .²³ This also means that atomic propositions are interpreted as false at all subsets of W .²⁴ A more standard approach would be to let $V : \text{At} \rightarrow \wp(W^\circ)$, allowing atomic propositions to be interpreted at both states and subsets.

There is a natural modal language associated with the above models. Let \mathcal{L}_2 be the smallest set of formulas generated by the following grammar:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid [\exists]\varphi \mid [\nexists]\varphi \mid [R]\varphi$$

where $p \in \text{At}$. Suppose that $\mathfrak{M} = \langle W^\circ, R_\exists, R_\nexists, R, V \rangle$ is a relational-neighborhood model with $W^\circ = W \cup \wp(W)$. Truth of formulas $\varphi \in \mathcal{L}_2$ at elements $x \in W^\circ$ is defined as follows:

- $\mathfrak{M}, x \models p$ iff $x \in W$ and $x \in V(p)$.
- $\mathfrak{M}, x \models \neg\varphi$ iff $x \in W$ and $\mathfrak{M}, x \not\models \varphi$.
- $\mathfrak{M}, x \models \varphi \wedge \psi$ iff $x \in W$, $\mathfrak{M}, x \models \varphi$, and $\mathfrak{M}, x \models \psi$.
- $\mathfrak{M}, x \models [R]\varphi$ iff $x \in W$ and for all $y \in W^\circ$, if $x R y$, then $\mathfrak{M}, y \models \varphi$.
- $\mathfrak{M}, x \models [\exists]\varphi$ iff $x \in \wp(W)$ and for all $y \in W^\circ$, if $x R_\exists y$, then $\mathfrak{M}, y \models \varphi$.
- $\mathfrak{M}, x \models [\nexists]\varphi$ iff $x \in \wp(W)$ for all $y \in W^\circ$, if $x R_\nexists y$, then $\mathfrak{M}, y \models \varphi$.

According to the above definition, subsets of W behave similarly to impossible worlds (cf. Sect. 2.2.4) since all non-modal formulas and formulas of the form $[R]\varphi$ are false at these points. This means that, for instance, $p \vee \neg p$ is false at points $X \in \wp(W)$. In order to express the two-sorted nature of the states in a relational-neighborhood model, it is sometimes convenient to include a special proposition St with the fixed interpretation $V(\text{St}) = W$. Given the above remark, if $p \in \text{At}$, then we can *define* St to be $p \vee \neg p$.

It is not hard to see that every neighborhood model can be transformed into a relational-neighborhood model. Suppose that $\mathcal{M} = \langle W, N, V \rangle$ is a neighborhood model. The associated relational-neighborhood model is $\mathcal{M}^\circ = \langle W^\circ, R_\exists, R_\nexists, R_N, V^\circ \rangle$, where, for all $p \in \text{At}$, $V^\circ(p) = V(p)$ and

- $R_N = \{(w, U) \mid w \in W, U \in \wp(W), U \in N(w)\}$.

To simplify the notation, I write ‘ $[N]\varphi$ ’ instead of ‘ $[R_N]\varphi$ ’.

The main observation of this section is that there is a translation of the basic modal language \mathcal{L} into \mathcal{L}_2 that preserves truth.

Definition 2.64 (*Normal Translation*) The **normal translation** of the basic modal language \mathcal{L} is a function $\text{nt} : \mathcal{L} \rightarrow \mathcal{L}_2$ defined by induction on the structure of $\varphi \in \mathcal{L}$:

- $\text{nt}(p) = p$.
- $\text{nt}(\neg\varphi) = \neg\text{nt}(\varphi)$.

²³This means that atomic propositions are *nominals* (cf. Sect. 3.1 and Areces and ten Cate 2007) with respect to elements in the domain of the model that correspond to subsets.

²⁴Another option would be to let atomic propositions be *undefined* at all $U \in \wp(W)$.

- $\text{nt}(\varphi \wedge \psi) = \text{nt}(\varphi) \wedge \text{nt}(\psi)$.
- $\text{nt}(\Box\varphi) = \langle N \rangle([\exists]\text{nt}(\varphi) \wedge [\nexists]\neg\text{nt}(\varphi))$.

The next proposition shows that the translation works as expected.

Proposition 2.14 *Suppose that $\mathcal{M} = \langle W, N, V \rangle$ is a neighborhood model. For all $\varphi \in \mathcal{L}$, for all $w \in W$,*

$$\mathcal{M}, w \models \varphi \text{ iff } \mathcal{M}^\circ, w \models \text{nt}(\varphi).$$

Proof The proof is by induction on $\varphi \in \mathcal{L}$. The proof for the base case and the boolean connectives is straightforward. I give the proof only for the case in which φ is of the form $\Box\psi$. Note that the induction hypothesis is as follows:

IH for all $w \in W$, $\mathcal{M}, w \models \psi$ iff $\mathcal{M}^\circ, w \models \text{nt}(\psi)$.

Suppose that $w \in W$ with $\mathcal{M}, w \models \Box\psi$. We must show that $\mathcal{M}^\circ, w \models \text{nt}(\Box\psi)$. That is, we must show that $\mathcal{M}^\circ, w \models \langle N \rangle([\exists]\text{nt}(\psi) \wedge [\nexists]\neg\text{nt}(\psi))$. Suppose that $X = \llbracket \psi \rrbracket_{\mathcal{M}}$. By the construction of \mathcal{M}° , since $\llbracket \psi \rrbracket_{\mathcal{M}} \in N(w)$, we have that $w R_N X$. Thus, we must show that $\mathcal{M}^\circ, X \models [\exists]\text{nt}(\psi) \wedge [\nexists]\neg\text{nt}(\psi)$. The induction hypothesis implies that

$$(*) \text{ for all } v \in W, v \in \llbracket \psi \rrbracket_{\mathcal{M}} \text{ iff } \mathcal{M}^\circ, v \models \text{nt}(\psi).$$

By the construction of \mathcal{M}° , we have for all $x \in W \cup \wp(W)$, $X R_\exists x$ iff $x \in W$ and $x \in X$. Furthermore, by (*), for all $v \in W$, if $v \in X$, then $\mathcal{M}^\circ, v \models \text{nt}(\psi)$. This implies that $\mathcal{M}^\circ, X \models [\exists]\text{nt}(\psi)$. A similar argument shows that $\mathcal{M}^\circ, X \models [\nexists]\neg\text{nt}(\psi)$. Thus, $\mathcal{M}^\circ, w \models \langle N \rangle([\exists]\text{nt}(\psi) \wedge [\nexists]\neg\text{nt}(\psi))$, as desired.

Suppose that $\mathcal{M}^\circ, w \models \text{nt}(\Box\psi)$. That is, suppose that

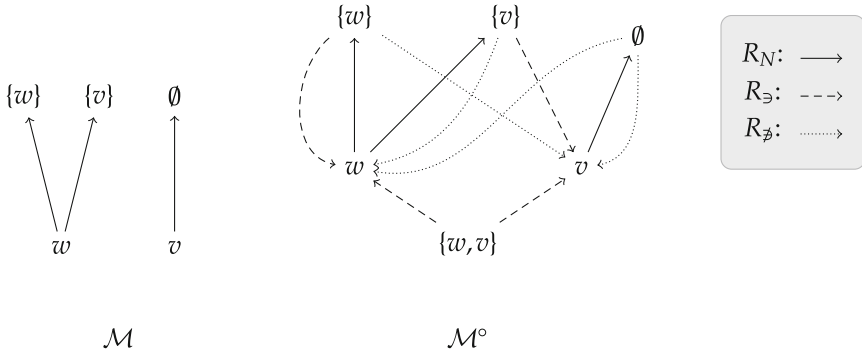
$$\mathcal{M}^\circ, w \models \langle N \rangle([\exists]\text{nt}(\psi) \wedge [\nexists]\neg\text{nt}(\psi)).$$

Then, there is some $x \in W \cup \wp(W)$ such that $w R_N x$ and

$$\mathcal{M}^\circ, x \models [\exists]\text{nt}(\psi) \wedge [\nexists]\neg\text{nt}(\psi).$$

By construction of \mathcal{M}° , since $w R_N x$, $x = X \subseteq W$ with $X \in N(w)$. Thus, in order to show that $\mathcal{M}, w \models \Box\psi$, we must show that $X = \llbracket \psi \rrbracket_{\mathcal{M}}$. Suppose that $v \in X$. By the definition of R_\exists , we have $X R_\exists v$ (and $v \in W$). Since $\mathcal{M}^\circ, X \models [\exists]\text{nt}(\psi)$, we have $\mathcal{M}^\circ, v \models \text{nt}(\psi)$. By the induction hypothesis, $\mathcal{M}, v \models \psi$. Hence, $X \subseteq \llbracket \psi \rrbracket_{\mathcal{M}}$. Conversely, suppose that $v \notin X$. Then, by the definition of R_\exists , we have $X R_\exists v$ (and $v \in W$). Since $\mathcal{M}^\circ, X \models [\nexists]\neg\text{nt}(\psi)$, we have $\mathcal{M}^\circ, v \models \neg\text{nt}(\psi)$. Since $\mathcal{M}^\circ, v \not\models \text{nt}(\psi)$ and $v \in W$, by the induction hypothesis, $\mathcal{M}, v \not\models \psi$. Hence, $\llbracket \psi \rrbracket_{\mathcal{M}} \subseteq X$. Therefore, since $X \in N(w)$ and $X = \llbracket \psi \rrbracket_{\mathcal{M}}$, we have that $\mathcal{M}, w \models \Box\psi$, as desired. \square

To illustrate the above translation, let $\mathcal{M} = \langle W, N, V \rangle$ be a neighborhood model with $W = \{w, v\}$, $N(w) = \{\{w\}, \{v\}\}$, $N(v) = \{\emptyset\}$, and $V(p) = \{w\}$. The relational model \mathcal{M}° is given below (the solid arrows correspond to the R_N relation; the dashed arrows correspond to the R_\exists relation; and the dotted arrows correspond to the R_\neq relation). To simplify the comparison between the models, I also draw the neighborhood model \mathcal{M} .



Note that $\mathcal{M}, w \models \Box p$ and $\mathcal{M}, v \models \Box \perp$. As the reader is invited to check, the following statements are true:

- $\mathcal{M}^\circ, w \models \langle N \rangle([\exists]p \wedge [\neq]\neg p)$ and $\mathcal{M}^\circ, v \not\models \langle N \rangle([\exists]p \wedge [\neq]\neg p)$; and
- $\mathcal{M}^\circ, v \models \langle N \rangle([\exists]\perp \wedge [\neq]\top)$ and $\mathcal{M}^\circ, w \not\models \langle N \rangle([\exists]\perp \wedge [\neq]\top)$.

Exercise 2.32 Find normal translations of: $\Box(p \vee \neg p)$, $\Box p \vee \Box \neg p$, and $\neg \Box p \vee \Box p$.

The normal translation can be simplified when the neighborhood models are closed under supersets.

Definition 2.65 (*Monotonic translation*) The **monotonic translation** of the basic modal language \mathcal{L} is a function $\text{mt} : \mathcal{L} \rightarrow \mathcal{L}_2$ defined by induction on \mathcal{L} , where the base case and Boolean connectives are as in Definition 2.64, and the modal clause is:

$$\text{mt}(\Box \varphi) = \langle N \rangle[\exists] \text{mt}(\varphi).$$

Proposition 2.15 Suppose that $\mathcal{M} = \langle W, N, V \rangle$ is a monotonic neighborhood model. For all $\varphi \in \mathcal{L}$, for all $w \in W$,

$$\mathcal{M}, w \models \varphi \text{ iff } \mathcal{M}^\circ, w \models \text{mt}(\varphi)$$

Exercise 2.33 Prove Proposition 2.15 (it is similar to the proof of Proposition 2.14).

Kracht and Wolter (1999) use the normal and monotonic translations to show that all non-normal modal logics can be *simulated* by a (two-sorted) normal modal logic (cf. Gasquet and Herzig 1996).

Exercise 2.34 (Kracht and Wolter 1999) Use the fact the satisfiability problem for multi-modal normal modal logic is decidable and the above translation to prove that the satisfiability problem for the non-normal modal logic **E** is decidable. What can be concluded about the complexity of the satisfiability problem for **E**?

Exercise 2.35 Suppose that \mathcal{M}_1 and \mathcal{M}_2 are monotonically-bisimilar neighborhood models (Definition 2.2). Does this imply that \mathcal{M}_1° and \mathcal{M}_2° are relationally-bisimilar (cf. Definition A.9)? Is the converse true (i.e., if \mathcal{M}_1° and \mathcal{M}_2° are relationally-bisimilar, then \mathcal{M}_1 and \mathcal{M}_2 are monotonically-bisimilar)? Answer the same questions for neighborhood models that are not monotonic.

2.6.3 The Standard Translation

Building on the normal translation from the previous section and the well-known *standard translation* of normal modal logic into first-order logic, in this section, I show that non-normal modal logic can be viewed as a fragment of *first-order logic*. In the remainder of this section, I assume that the reader is familiar with first-order logic (specifically, I assume familiarity with the syntax and semantics of first-order logic, the notion of an *isomorphism* and basic model-theoretic results). Consult Enderton (2001) for the necessary background.

I start by defining a first-order language that can be interpreted on relational-neighborhood models (Definition 2.63). It will be convenient to work with a *two-sorted* first-order language. Formally, there are two sorts, $\{\mathbf{w}, \mathbf{n}\}$. Terms of the first sort (\mathbf{w}) are intended to represent states, whereas terms of the second sort (\mathbf{n}) are intended to represent neighborhoods (i.e., subsets of sort \mathbf{w}). I assume that there are countable sets of variables of each sort. To simplify notation, I use the following conventions: $x, y, x', y', x_1, y_2, \dots$ denote variables of sort \mathbf{w} (*world variables*), and $u, v, u', v', u_1, v_1, \dots$ denote variables of sort \mathbf{n} (*neighborhood variables*).

The language is built from a signature containing a unary predicate P_i (of sort \mathbf{w}) for each $i \in \mathbb{N}$, a binary relation symbol N relating elements of sort \mathbf{w} to elements of sort \mathbf{n} , and a binary relation symbol E relating elements of sort \mathbf{n} to elements of sort \mathbf{w} . The intended interpretation of xNu is “ u is a neighborhood of x ”, and the intended interpretation of uEx is “ x is an element of u ”. The language \mathcal{L}_{fo} is built from the following grammar:

$$x = y \mid u = v \mid P_i x \mid xNu \mid uEx \mid \neg\varphi \mid \varphi \wedge \psi \mid \exists x\varphi \mid \exists u\varphi$$

where $i \in \mathbb{N}$; x and y are state variables; and u and v are neighborhood variables. The usual abbreviations (e.g. \forall for $\neg\exists\neg$) apply. Note that $\exists x\varphi$ means that “*there is an element of sort \mathbf{w} that satisfies φ* ” (similarly, $\exists u\varphi$ means that “*there is an element of sort \mathbf{n} that satisfies φ* ”).

Formulas of \mathcal{L}_{fo} are interpreted in two-sorted first-order structures $\mathcal{M} = \langle D, \{P_i \mid i \in \mathbb{N}\}, R, E \rangle$, where $D = D^{\mathbf{w}} \cup D^{\mathbf{n}}$ (and $D^{\mathbf{w}} \cap D^{\mathbf{n}} = \emptyset$), each $P_i \subseteq D^{\mathbf{w}}$, $R \subseteq D^{\mathbf{w}} \times$

D^n and $E \subseteq D^n \times D^w$. The relation R is the interpretation of the binary relational symbol N .²⁵ As usual, the equality symbol is always interpreted as equality on the appropriate domain. In addition, the usual definitions of free and bound variables apply. Truth of sentences (formulas with no free variables) $\varphi \in \mathcal{L}_{fo}$ in a structure \mathcal{M} (denoted $\mathcal{M} \models \varphi$) is defined as expected. If x is a free state variable in φ (denoted $\varphi(x)$), then $\mathcal{M} \models \varphi[w]$ means that φ is true in \mathcal{M} when $w \in D^w$ is assigned to x . Note that $\mathcal{M} \models \exists x \varphi$ iff there is an element $w \in D^w$ such that $\mathcal{M} \models \varphi[w]$. If Γ is a set of \mathcal{L}_{fo} -formulas, and \mathcal{M} is an \mathcal{L}_{fo} -model, then $\mathcal{M} \models \Gamma$ means that for all $\gamma \in \Gamma$, $\mathcal{M} \models \gamma$. Given a class \mathbf{K} of \mathcal{L}_{fo} -models, the *semantic consequence relation over \mathbf{K}* is denoted $\models_{\mathbf{K}}$. That is, for a set of \mathcal{L}_{fo} -formulas $\Gamma \cup \{\varphi\}$, $\Gamma \models_{\mathbf{K}} \varphi$, if for all $\mathcal{M} \in \mathbf{K}$, $\mathcal{M} \models \Gamma$ implies that $\mathcal{M} \models \varphi$.

The basic modal language and neighborhood models can be translated into the first-order setting as follows.

Definition 2.66 (*First-Order Translations of Neighborhood Models*) Suppose that $\mathcal{M} = \langle W, N, V \rangle$ is a neighborhood model. The **first-order translation** of \mathcal{M} is the structure $\mathcal{M}^\bullet = \langle D, \{P_i \mid i \in \mathbb{N}\}, R_N, R_\exists \rangle$, where

- $D = D^w \cup D^n$ with $D^w = W$, $D^n = N[W] = \bigcup_{w \in W} N(w)$.
- $P_i = V(p_i)$ for each $p_i \in \mathbf{At}$.
- $R_N = \{(w, X) \mid w \in D^w, X \in N(w)\}$.
- $R_\exists = \{(X, w) \mid w \in D^w, w \in X\}$.

Definition 2.67 (*Standard Translation*) The **standard translation** of the basic modal language $\mathcal{L}(\mathbf{At})$ (where $\mathbf{At} = \{p_i \mid i \in \mathbb{N}\}$ is a countable set of atomic propositional variables) is a family of functions $st_x : \mathcal{L}(\mathbf{At}) \rightarrow \mathcal{L}_{fo}$ defined as follows: $st_x(p_i) = P_i x$, $st_x(\neg\varphi) = \neg st_x(\varphi)$, $st_x(\varphi \wedge \psi) = st_x(\varphi) \wedge st_x(\psi)$, and

$$st_x(\Box\varphi) = \exists u(xNu \wedge (\forall y(uEy \leftrightarrow st_y(\varphi))))$$

Exercise 2.36 Prove that the standard translation preserves truth. That is, prove the following lemma.

Lemma 2.24 Let $\mathfrak{M} = \langle W, N, V \rangle$ be a neighborhood model and $\varphi \in \mathcal{L}$. For each $w \in W$, $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}^\bullet \models st_x(\varphi)[w]$.

So, every neighborhood model can be associated with an \mathcal{L}_{fo} -model that preserves truth of the basic modal language (using the standard translation). However, it is not the case that every \mathcal{L}_{fo} -structure is the translation of a neighborhood model.²⁶ Fortunately, it is possible to axiomatize the class of translations of neighborhood models. Let $\mathbf{N} = \{\mathcal{M} \mid \mathcal{M} \cong \mathcal{M}^\bullet \text{ for some neighborhood model } \mathcal{M}\}$, where $\mathcal{M} \cong \mathcal{N}$ means that there is an **isomorphism**²⁷ between \mathcal{M} and \mathcal{N} , and let **NAX** be the following axioms:

²⁵I am not using ' N ' since that is used to denote neighborhood functions.

²⁶This should be contrasted with the standard translation of relational models for normal modal logic. Consult Blackburn et al. (2001), Sect. 2.4, for details.

²⁷In this context, an isomorphism between \mathcal{L}_{fo} -models $\mathcal{M} = \langle D, \{P_i \mid i \in \mathbb{N}\}, R, E \rangle$ and $\mathcal{M}' = \langle D', \{P'_i \mid i \in \mathbb{N}\}, R', E' \rangle$ is a 1-1 and onto function $f : D \rightarrow D'$ satisfying the structural conditions: $w \in P_i$ iff $f(w) \in P'_i$, $w R u$ iff $f(w) R' f(u)$ and $u E w$ iff $f(u) E' f(w)$.

- (A1) $\exists x(x = x)$
- (A2) $\forall u \exists x(x N u)$
- (A3) $\forall u \forall v (\neg(u = v) \rightarrow \exists x((u E x \wedge \neg v E x) \vee (\neg u E x \wedge v E x)))$

It is not hard to see that if \mathcal{M} is a neighborhood model, then $\mathcal{M}^\bullet \Vdash \mathbf{NAX}$. The next result states that **NAX** completely characterizes the class **N**.

Proposition 2.16 *Suppose \mathcal{M} is an \mathcal{L}_{fo} -model and $\mathcal{M} \Vdash \mathbf{NAX}$. Then, there is a neighborhood model \mathcal{M}_\bullet such that $\mathcal{M} \cong (\mathcal{M}_\bullet)^\bullet$.*

Proof Let $\mathcal{M} = \langle D^w \cup D^n, \{P_i \mid i \in \omega\}, R, E \rangle$ be an \mathcal{L}_{fo} -model such that $\mathcal{M} \Vdash \mathbf{NAX}$. We will construct a neighborhood model $\mathcal{M}_\bullet = \langle W, N, V \rangle$ such that $\mathcal{M} \cong (\mathcal{M}_\bullet)^\bullet$. First, define a map $v : D^n \rightarrow \wp(D^w)$ by $v(u) = \{w \in D^w \mid u E w\}$ and let $W = D^w$. Since $\mathcal{M} \Vdash (\mathbf{A1})$, $W \neq \emptyset$. Now define for each $w \in W$ and each $X \subseteq W$: $X \in N(w)$ iff there is a $u \in D^n$ such that sNu and $X = v(u)$, and define for all $i \in \mathbb{N}$, $V(p_i) = \{w \in W \mid \mathcal{M} \models P_i[w]\}$. Then, \mathcal{M}_\bullet is clearly a well-defined neighborhood model. The proof is concluded if we can show that the map

$$f : D^w \cup D^n \rightarrow W \cup \bigcup_{w \in W} N(w),$$

defined as $f(w) = w$ for $w \in D^w$ and $f(u) = v(u)$ for $u \in D^n$, is an isomorphism from \mathcal{M} to $(\mathcal{M}_\bullet)^\bullet = \langle W \cup N[W], \{P'_i \mid i \in \omega\}, R_N, R_\supset \rangle$ (cf. Definition 2.66).

First, it follows directly from $\mathcal{M} \Vdash (\mathbf{A3})$, that v is injective. Second, by the definition of v and the $(\cdot)^\bullet$ -construction, the range of v , denoted $\text{rng}(v)$, contains $\bigcup_{w \in D^w} v(w) = \bigcup_{w \in W} N(w)$. The inclusion $\text{rng}(v) \subseteq \bigcup_{w \in W} N(w)$ follows from the assumption that $\mathcal{M} \Vdash (\mathbf{A2})$ since this implies that, for every $u \in D^n$, there is a $w \in D^w$ such that $v(u) \in N(w)$. The structural conditions follow directly by construction: For all $i \in \mathbb{N}$, $w \in P_i$ iff $w \in V(p_i)$ iff $w \in P'_i$. Similarly, for all $w \in D^w$ and all $u \in D^n$: $w R u$ iff $v(u) \in N(w)$ iff $w R_N v(u)$, and $u E w$ iff $w \in v(u)$ iff $v(u) R_\supset w$. \square

Thus, in a precise way, models in **N** can be viewed as neighborhood models. This means that (monotonic) bisimulations, bounded morphisms, disjoint unions, and other model constructions on neighborhood models (cf. Sect. 2.1) can be applied to models in **N**. For instance, an \mathcal{L}_{fo} -formula $\alpha(x)$ is said to be **invariant under behavioral equivalence** (Definition 2.9) on the class **N** iff for all w -elements w from \mathcal{M} and w -elements v from \mathcal{N} , if \mathcal{M}_\bullet, w and \mathcal{N}_\bullet, v are behaviorally equivalent, then $\mathcal{M} \models \alpha[w]$ iff $\mathcal{N} \models \alpha[v]$ (invariance under monotonic bisimulations can be defined similarly). Furthermore, Proposition 2.16 implies that we can work relative to **N** while still preserving nice first-order properties such as compactness and the existence of countably saturated models.

Using the translation defined in this section, I can clarify the relationship between non-normal modal logic and first-order logic. For normal modal logic, the seminal *van Benthem characterization theorem* (see Blackburn et al. 2001, Sect. 2.4, for details) shows that normal modal logic is the bisimulation invariant fragment of

first-order logic (with respect to relational models). Pauly (1999) generalized this result to monotonic modal logics, showing that, over the class of \mathbf{N}^{mon} of \mathcal{L}_{fo} -models isomorphic to monotonic neighborhood models, $\alpha(x)$ is equivalent to a translation of a basic modal formula iff $\alpha(x)$ is invariant under monotonic bisimulations (see, also, Hansen 2003). A similar result can be shown for *all* non-normal modal logics using the notion of behavioral equivalence.

Theorem 2.68 (Hansen et al. 2009) *Suppose that \mathbf{N} is the class of \mathcal{L}_{fo} -structures isomorphic to the translation of neighborhood models and that $\alpha(x)$ is a \mathcal{L}_{fo} -formula. Then, $\alpha(x)$ is equivalent to a translation of a basic modal formula (with respect to \mathbf{N}) iff $\alpha(x)$ is invariant under behavioral equivalence.*

Consult Hansen et al. (2009), Hansen (2003) and Kracht and Wolter (1999) for further model-theoretic results about non-normal modal logic.



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