

Chapter 2

Spinor Helicity Formalism

2.1 Introduction

In recent years, great advances in the study of scattering amplitudes [1–4] have been made through the use of the “spinor helicity method,” which is at its core the fact that there is a local isomorphism $SO(1, 3) = SU(2)_L \times SU(2)_R$. That the Lorentz symmetry group is decomposable like this allows us to replace any four-vector p^μ with a bispinor $p_{a\dot{a}} = p_\mu \sigma_{a\dot{a}}^\mu$ by contracting with the Pauli matrices.¹ This is especially convenient for null vectors since $p^2 = \det p_{a\dot{a}}$: if $p^2 = 0$, then $p_{a\dot{a}}$ is a rank one matrix, meaning it can be written as an outer product of two (commuting) spinors, $p_{a\dot{a}} = \text{sgn}(p^0) p_a p_{\dot{a}}$.²

This notation is especially convenient for electromagnetism with massless matter, and we will see that the helicities of the particles are nicely encoded in the use of the two spinor types. We may also use the $SU(2)$ invariant tensors ε_{ab} and $\varepsilon_{\dot{a}\dot{b}}$ to form Lorentz-invariant bilinear forms; for $p_{a\dot{a}} = p_a p_{\dot{a}}$ and $q_{a\dot{a}} = q_a q_{\dot{a}}$, we define the angle and square bracket spinors via:

$$\langle pq \rangle \equiv p^a q_a = p^a \varepsilon_{ab} q^b = -\varepsilon_{ba} p^a q^b = -p_b q^b = -p_a \varepsilon^{ab} q_b = -\langle qp \rangle \quad (2.1.1)$$

$$[pq] \equiv p_{\dot{a}} q^{\dot{a}} = p_{\dot{a}} \varepsilon^{\dot{a}\dot{b}} q_{\dot{b}} = -\varepsilon^{\dot{b}\dot{a}} p_{\dot{a}} q_{\dot{b}} = -p^{\dot{b}} q_{\dot{b}} = -p^{\dot{a}} \varepsilon_{\dot{a}\dot{b}} q^{\dot{b}} = -[qp]. \quad (2.1.2)$$

We may also form Lorentz contractions³:

¹The van der Waerden notation here assigns undotted indices to $SU(2)_L$ and dotted to $SU(2)_R$.

²We will tend to use the same variable for the four-vector, the bispinor, and the spinors. Which is meant should be clear from context.

³There are occasional notational discrepancies in the literature; here, we follow the ${}^a_a, {}_{\dot{a}}^{\dot{a}}$ contraction convention, which is more common. Other common notations:

$$k_a = |k\rangle = |k^+\rangle = u_+(k) = v_-(k), \quad k^{\dot{a}} = |k] = |k^-\rangle = u_-(k) = v_+(k),$$

$$2p \cdot q = (\varepsilon^{ab} p_a q_b)(\varepsilon^{\dot{a}\dot{b}} p_{\dot{a}} q_{\dot{b}}) = \langle qp \rangle [pq] = \langle pq \rangle [qp]. \quad (2.1.3)$$

Numerous other identities are listed in Appendix B.

A null four-vector should have three degrees of freedom (say, a normalization scale and two angles in 3-space), as we can see from a component-wise expression

$$p_a = \sqrt{2p^0} \begin{pmatrix} -\sin \frac{\theta}{2} e^{-i\phi} \\ \cos \frac{\theta}{2} \end{pmatrix}. \quad (2.1.4)$$

For real momentum, $p_{\dot{a}} = (p_a)^*$, so that $\langle pq \rangle = [pq]^*$ and each is essentially the square root of $2p \cdot q$, up to a phase. However, an extraordinary amount of additional insight is gained from deforming p^μ to be complex and only enforcing its reality at the end. In that case the angle and square spinors are independent.⁴

The last piece of translating from Lorentz to spinor notation involves the polarization vectors of external photons. These can be fixed uniquely by demanding that transversality $\varepsilon(k) \cdot k = 0$ is maintained, that $\varepsilon^\mu(k)$ is a proper Lorentz four-vector, that $\varepsilon^\mu(k)$ is dimensionless, and that gauge freedom is still present. If we shift $\varepsilon^\mu(k) \rightarrow \varepsilon^\mu(k) + \eta k^\mu$ with η an arbitrary number, transversality and masslessness require no change in the amplitude (this is essentially a consequence of the Ward identity). We may then construct the helicity polarization vectors

$$\begin{aligned} \varepsilon_\mu^+(k) &= \frac{q^a \sigma_{\mu ab} k^b}{\sqrt{2} q^a k_a}, & \varepsilon_\mu^-(k) &= \frac{q_{\dot{a}} \bar{\sigma}_\mu^{\dot{a}b} k_b}{\sqrt{2} q_{\dot{a}} k^{\dot{a}}} \\ \Rightarrow \quad \varepsilon_{a\dot{a}}^- &= \sqrt{2} \frac{k_a q_{\dot{a}}}{[kq]}, & \varepsilon_{a\dot{a}}^+ &= \sqrt{2} \frac{q_a k_{\dot{a}}}{\langle qk \rangle}. \end{aligned} \quad (2.1.5)$$

Note the arbitrary reference spinor q : this is an indication of gauge freedom, and shifting $q \rightarrow q + \eta k$ recovers the desired properties. As we are free to choose the reference momentum q independently for each polarization vector, it is convenient to take one of the positive helicity momenta as reference for the negative helicity polarizations, and vice versa.⁵

If we extend the Zwanziger formalism to include the magnetic polarization vectors of (1.3.28), we may see that

$$\varepsilon_{Ba\dot{a}}^- = i\sqrt{2} \left(\frac{k_a q_{\dot{a}}}{[kq]} + \frac{k_a k_{\dot{a}} [qn]}{[kq][nk]} \right), \quad \varepsilon_{Ba\dot{a}}^+ = -i\sqrt{2} \left(\frac{q_a k_{\dot{a}}}{\langle qk \rangle} + \frac{k_a k_{\dot{a}} \langle nq \rangle}{\langle qk \rangle \langle kn \rangle} \right), \quad (2.1.6)$$

$$k^\mu = \langle k | = \langle k^- | = \bar{u}_-(k) = \bar{v}_+(k), \quad k_{\dot{a}} = [k] = \langle k^+ | = \bar{u}_+(k) = \bar{v}_-(k)$$

⁴Or one can keep p^μ real and switch to spacetime signature $(+, -, +, -)$, in which case $|p\rangle$ and $[p]$ are real and independent.

⁵This procedure has made calculating tree-level QCD helicity amplitudes more of a joy than a pain [4].

with

$$\begin{aligned}\varepsilon_A^\pm \cdot \varepsilon_A^\pm &= \varepsilon_B^\pm \cdot \varepsilon_B^\pm = \varepsilon_A^\pm \cdot \varepsilon_B^\pm = 0, \\ \varepsilon_A^\pm \cdot \varepsilon_B^\mp &= \mp i, \varepsilon_A^\pm \cdot \varepsilon_A^\mp = \varepsilon_B^\pm \cdot \varepsilon_B^\mp = -1.\end{aligned}\quad (2.1.7)$$

For calculational ease, it is common to perform a BCFW ij -shift with a complex reference momentum q^μ : [5]

$$p_i \rightarrow p_i(z) = p_i + qz, \quad p_j \rightarrow p_j(z) = p_j - qz, \quad z \in \mathbb{R}. \quad (2.1.8)$$

Maintenance of transversality requires

$$\varepsilon_i^-(z) = \varepsilon_j^+(z) = q, \quad \varepsilon_i^+(z) = q^* - zp_j, \quad \varepsilon_j^-(z) = q^* + zp_i. \quad (2.1.9)$$

It is a nontrivial fact that physical amplitudes remain finite even as $z \rightarrow \infty$ (a strong constraint on the form amplitudes may have), so in this sense even infinite gauge transformations are permitted, reminiscent of supersymmetry where the gauge group may be extended through complexification; here, $U(1)_\mathbb{R} \rightarrow U(1)_\mathbb{C}$, a non-compact group [6].

In our spinor notation, we see that if we define

$$z = \frac{\langle nq \rangle}{\langle qk \rangle \langle kn \rangle}, \quad \bar{z} = \frac{[nq]}{[qk][kn]}, \quad (2.1.10)$$

the magnetic polarization vectors are

$$\varepsilon_B^+ = -i(\varepsilon^+ + \sqrt{2}zk), \quad \varepsilon_B^- = i(\varepsilon^- - \sqrt{2}\bar{z}k). \quad (2.1.11)$$

Here we can see that the magnetic polarization bispinors of specific helicity are essentially $\pm i$ times the corresponding electric bispinors, up to a gauge transformation shift z or $\bar{z}$.

2.2 Little Group Scaling

It is useful to observe that the bispinor p_{aa} is invariant under the scaling

$$|p\rangle \rightarrow t|p\rangle, \quad |p] \rightarrow t^{-1}|p], \quad (2.2.1)$$

known as *little group scaling*.⁶ For real momenta, we require t to be a complex phase, but for general momenta t can be any nonzero complex number. Because the angle brackets correspond to helicity $h = -1/2$ particles and the square brackets to $h = +1/2$ particles, we can see that under little group scaling, an arbitrary n -particle amplitude as a function of the various spinors scales as

$$A_n(\dots, \{t_i|i\rangle, t_i^{-1}|i], h_i\rangle, \dots) = t_i^{-2h_i} A_n(\dots, \{|i\rangle, |i], h_i\rangle, \dots). \quad (2.2.2)$$

From the polarization vector forms discussed above, we see that this is true even for spin one particles (it holds for spin 2 as well), and that the polarizations are invariant under little group scaling of the reference spinor, as they must be.

The great utility of the spinor method is being able to bootstrap our way to more complicated amplitudes from lower-point results, and it turns out that the 3-point amplitude is completely fixed by little group scaling. To see why, we realize that to impose momentum conservation⁷ $p_1^\mu + p_2^\mu + p_3^\mu = 0$, we have [2]

$$\langle 12 \rangle [21] = 2p_1 \cdot p_2 = (p_1 + p_2)^2 = p_3^2 = 0, \quad (2.2.3)$$

so either $\langle 12 \rangle = 0$ or $[12] = 0$. Permuting through the indices demonstrates that if $\langle 12 \rangle$ and the other angle brackets are non-vanishing, then $|1\rangle \propto |2\rangle \propto |3\rangle$. Conversely, if $[12]$ and its permutations are non-vanishing, then $|1\rangle \propto |2\rangle \propto |3\rangle$. Consequently, a non-vanishing on-shell 3-point amplitude for massless particles can *only* depend on angle or square brackets, never both. Also, such an amplitude is only nonzero for complex momenta.⁸

To see that the 3-particle kinematics fixes the 3-point amplitudes completely, consider the case where A_3 depends only on angle brackets:

$$A_3(1^{h_1} 2^{h_2} 3^{h_3}) = c \langle 12 \rangle^{x_{12}} \langle 23 \rangle^{x_{23}} \langle 31 \rangle^{x_{31}}. \quad (2.2.4)$$

Little group scaling forces

$$-2h_1 = x_{12} + x_{31}, \quad -2h_2 = x_{12} + x_{23}, \quad -2h_3 = x_{31} + x_{23}, \quad (2.2.5)$$

hence

$$A_3(1^{h_1} 2^{h_2} 3^{h_3}) = c \langle 12 \rangle^{h_3-h_1-h_2} \langle 23 \rangle^{h_2-h_1-h_3} \langle 31 \rangle^{h_1-h_2-h_3}. \quad (2.2.6)$$

⁶The *little* or *isotropy* group is the group of transformations that leave the momentum of an on-shell particle fixed. For a massless particle in a frame where $p^\mu = (E, 0, 0, E)$, we can see that the little group is $SO(2) = U(1)$. (Technically, it's $E(2)$, the Euclidean group in two-dimensions, but for angular momentum concerns we only worry about the rotational subgroup.)

⁷We will always take all particles outgoing for symmetry.

⁸Unless it is a constant, as in ϕ^3 theory, or in a $(+, -, +, -)$ spacetime signature.

If we had used square brackets instead, we would arrive at

$$A_3(1^{h_1} 2^{h_2} 3^{h_3}) = c[12]^{h_1+h_2-h_3} [23]^{h_1+h_3-h_2} [31]^{h_2+h_3-h_1}. \quad (2.2.7)$$

To determine which version to use, we must also appeal to locality: the proper mass dimension of a 3-point amplitude should be $[\text{mass}]^1$, and this should come from positive momentum dimension. A term like $AA_{\square}A$ would have negative momentum mass dimension, coming as it does from a nonlocal Lagrangian.

2.3 Massive Spinor Helicities

The helicity method can also handle massive four-vectors at the expense of an extra reference spinor. Any vector may be decomposed as [7]

$$p^\mu = p_\perp^\mu + \beta q^\mu, \quad \beta = \frac{p^2}{2q \cdot p_\perp} \quad (2.3.1)$$

where both $p_\perp$ and q are null and $q \cdot p_\perp = q \cdot p \neq 0$.⁹ These two massless vectors may then undergo the usual spinor decomposition, giving us the massive bispinors

$$p_{a\dot{a}} = p_a p_{\dot{a}} + \frac{p^2}{\langle pp' \rangle [p' p]} p'_a p'_{\dot{a}} = |p\rangle [p] + \frac{p^2}{\langle pp' \rangle [p' p]} |p'\rangle [p'] \quad (2.3.2)$$

$$p^{\dot{a}a} = p^{\dot{a}} p^a + \frac{p^2}{\langle pp' \rangle [p' p]} p'^{\dot{a}} p'^a = |p\rangle \langle p| + \frac{p^2}{\langle pp' \rangle [p' p]} |p'\rangle \langle p'|. \quad (2.3.3)$$

While these may not be written as outer products of separate spinors, we may introduce a hybrid notation [9]

$$|I^\pm\rangle = |i\rangle \pm \frac{m_i}{[ii']} |i'\rangle, \quad |I^\pm| = |i| \pm \frac{m_i}{\langle ii'\rangle} |i'\rangle \quad (2.3.4)$$

$$[I^\pm] = [i] \pm \frac{m_i}{\langle ii'\rangle} \langle i'|, \quad \langle I^\pm| = \langle i| \pm \frac{m_i}{[ii']} [i'] \quad (2.3.5)$$

to aid in the construction of amplitudes that involve massive spinors; their form belies their relation to Dirac spinors, which are simply direct sums of massless spinors.

⁹Boels [8] uses the “flat” notation $p_\perp = p^\flat$. When constructing spinors for massive vectors, we will assign $|i\rangle = (p_{i\perp})_a$ and $|i'\rangle = (q_i)_a$, etc. for notational ease.

An industry has been built around bootstrapping higher n -point, loop, and dimensional results from such humble beginnings. An amazing result is the Parke–Taylor amplitude [10] for the n -point scattering of gluons (appropriately color-ordered) in the Maximally Helicity Violating (MHV) channel:

$$A_n(1^+, \dots, i^-, \dots, j^-, \dots, n^+) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}, \quad (2.3.6)$$

a result that would otherwise require over 10,000 terms of calculation in the traditional Feynman diagram prescription for just $n = 5$. Other similar results, such as the vanishing of all-alike or all-but-one alike helicities for tree-level $n > 3$

$$A_n(1^+, 2^+, \dots, n^+) = 0, \quad A_n(1^+, \dots, i^-, \dots, n^+) = 0, \quad (2.3.7)$$

greatly ease the computations of many-gluon scattering amplitudes. Higher order scattering and next-to-maximally-helicity-violating (or in general N^K MHV) amplitudes may be computed with the on-shell BCFW [5] recursion relations, both at tree and loop levels, and exceptional simplifications take place in the supersymmetric limit for planar ($N \rightarrow \infty$) $SU(N)$ gauge theories [11]. Recently, deep connections have been made between the analytic and local structures of amplitudes and mathematical objects such as Grassmania and volumes of “amplitudhedra,” multi-dimensional polytopes. Unitarity and locality themselves may even be emergent phenomena from a more fundamental underlying geometry [12]. It is an exciting field that will surely continue to produce more interesting connections in the coming years.

With the advent of Dirac monopoles and dyons, it was realized that gauge theories can quite consistently accommodate line singularities in the vector potential and that monopoles modify the global structure of the electromagnetic gauge group (namely, that it must be compact: $U(1)$). This can also be understood if the electric charge operator Q were a generator of a non-Abelian compact gauge group that had been spontaneously broken to $U(1)_{\text{EM}}$. It is this idea that we turn to next.

References

1. A. Zee, *Quantum Field Theory in a Nutshell*, 2nd edn. (Princeton University, Press, Princeton, 2010), p. 486
2. H. Elvang, Y.t. Huang, Scattering amplitudes. [hep-th/1308.1697](https://arxiv.org/abs/hep-th/1308.1697)
3. R. Boels, K.J. Larsen, N.A. Obers, M. Vonk, MHV, CSW and BCFW: field theory structures in string theory amplitudes. *J. High Energy Phys.* **0811**, 015 (2008). [hep-th/0808.2598](https://arxiv.org/abs/hep-th/0808.2598)
4. L.J. Dixon, A brief introduction to modern amplitude methods. [hep-ph/1310.5353](https://arxiv.org/abs/hep-ph/1310.5353); Calculating scattering amplitudes efficiently, in *Boulder (1995), QCD and Beyond*, pp. 539–582. [hep-ph/9601359](https://arxiv.org/abs/hep-ph/9601359)

5. E. Witten, Perturbative gauge theory as a string theory in twistor space. *Commun. Math. Phys.* **252**, 189 (2004). [hep-th/0312171](#); R. Britto, F. Cachazo, B. Feng, New recursion relations for tree amplitudes of gluons. *Nucl. Phys. B* **715**, 499 (2005). [hep-th/0412308](#); R. Britto, F. Cachazo, B. Feng, E. Witten, Direct proof of tree-level recursion relation in Yang-Mills theory. *Phys. Rev. Lett.* **94**, 181602 (2005). [hep-th/0501052](#)
6. S.P. Martin, A Supersymmetry primer. *Adv. Ser. Direct. High Energy Phys.* **21**, 1 (2010) [*Adv. Ser. Direct. High Energy Phys.* **18**, 1 (1998)] [hep-ph/9709356](#); J. Terning, *Modern Supersymmetry: Dynamics and Duality*. International Series of Monographs on Physics, vol. 132 (Oxford University Press, Oxford, 2006)
7. S. Dittmaier, Weyl-van der Waerden formalism for helicity amplitudes of massive particles. *Phys. Rev. D* **59**, 016007 (1998). [hep-ph/9805445](#)
8. R. Boels, Covariant representation theory of the Poincare algebra and some of its extensions. *J. High Energy Phys.* **1001**, 010 (2010). [hep-th/0908.0738](#)
9. J. Kuczmarski, SpinorsExtras - Mathematica implementation of massive spinor-helicity formalism. [hep-ph/1406.5612](#)
10. S.J. Parke, T.R. Taylor, An amplitude for n Gluon scattering. *Phys. Rev. Lett.* **56**, 2459 (1986)
11. F. Cachazo, P. Svrcek, E. Witten, MHV vertices and tree amplitudes in gauge theory. *J. High Energy Phys.* **0409**, 006 (2004). [hep-th/0403047](#); J.M. Drummond, J. Henn, G.P. Korchemsky, E. Sokatchev, Dual superconformal symmetry of scattering amplitudes in $N=4$ super-Yang-Mills theory. *Nucl. Phys. B* **828**, 317 (2010). [hep-th/0807.1095](#); A. Brandhuber, P. Heslop, G. Travaglini, MHV amplitudes in $N=4$ super Yang-Mills and Wilson loops. *Nucl. Phys. B* **794**, 231 (2008). [hep-th/0707.1153](#); Z. Bern, L.J. Dixon, D.A. Kosower, R. Roiban, M. Spradlin, C. Vergu, A. Volovich, The two-loop Six-Gluon MHV amplitude in maximally supersymmetric Yang-Mills theory. *Phys. Rev. D* **78**, 045007 (2008). [hep-th/0803.1465](#)
12. N. Arkani-Hamed, J. Trnka, The Amplituhedron. *J. High Energy Phys.* **1410**, 030 (2014). [hep-th/1312.2007](#); N. Arkani-Hamed, J. Trnka, Into the amplituhedron. *J. High Energy Phys.* **1412**, 182 (2014). [hep-th/1312.7878](#); N. Arkani-Hamed, A. Hodges, J. Trnka, Positive amplitudes in the amplituhedron. *J. High Energy Phys.* **1508**, 030 (2015). [hep-th/1412.8478](#)

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