

Chapter 2

Preliminaries, Basic Definitions, and Conventions

Basic concepts that appear throughout the monograph are summarized here. “W.l.o.g.” will abbreviate “without loss of generality”; “r.h.s.” will stand for “right hand-side”. The notations “:=” and “=:” indicate definition, with the defined symbol standing on the hand-side of the colon.

2.1 Links and Link Diagrams

Links are represented by diagrams; we assume diagrams are oriented (though sometimes orientation is not relevant).

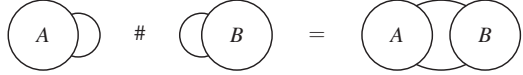
A crossing p in a link diagram D is called *reducible* (or nugatory) if it looks like on the left of Equation (2.1). D is called *reducible* if it has a reducible crossing, else it is called *reduced*. The reducing of the reducible crossing p is the move depicted on Equation (2.1). Each diagram D can be (made) reduced by a finite number of these moves.


(2.1)

We assume in the following all diagrams reduced, unless otherwise stated.

The diagram on the right of Figure 2.1 is called *connected sum* $A\#B$ of the diagrams A and B . If a diagram D can be represented as the connected sum of diagrams A and B , such that both A and B have at least one crossing, then D is called *disconnected* (or composite), else it is called *connected* (or prime). K is *prime* if whenever $D = A\#B$ is a composite diagram of K , one of A and B represents an unknotted arc (but not both; the unknot is not considered to be prime per convention).

Fig. 2.1 Connected sum of diagrams



By $c(D)$ we denote the *number of crossings* of D , $n(D)$ the *number of components* of D (or K , 1 if K is a knot), and $s(D)$ the *number of Seifert circles* of D . The *crossing number* $c(K)$ of a knot or link K is the minimal crossing number of all diagrams D of K .

The *Seifert graph* $\Gamma(D)$ of a diagram D is defined to be the graph whose vertices are the Seifert circles of D and whose edges are the crossings. The *reduced Seifert graph* $\Gamma'(D)$ is defined by removing multiple copies of an edge between two vertices in $\Gamma(D)$, so that a simple edge remains. (See [MP, MP2] for example.)

The (*Seifert*) *genus* $g(K)$ resp. *Euler characteristic* $\chi(K)$ of a knot or link K is said to be the minimal genus resp. maximal Euler characteristic of Seifert surface of K . For a diagram D of K , $g(D)$ is defined to be the genus of the Seifert surface obtained by Seifert's algorithm on D , and $\chi(D)$ its Euler characteristic. We have $\chi(D) = s(D) - c(D)$ and $2g(D) = 2 - n(D) - \chi(D)$.

We denote by $!D$ the *mirror image* of D , and $!K$ denotes the mirror image of K . Clearly $g(!D) = g(D)$ and $g(!K) = g(K)$.

Similarly we use χ_s, g_s for (smooth) slice genus and Euler characteristic.

The numbering of knots we use is as in the tables of [Ro, Appendix] for prime knots of crossing number ≤ 10 , and as in [HT] for those of crossing number 11–16. KnotScape's numbering is reorganized so that for given crossing number non-alternating knots are appended after alternating ones, instead of using “a” and “n” superscripts.

2.2 Polynomial Link Invariants

Let $X \in \mathbb{Z}[t, t^{-1}]$. The *minimal* or *maximal degree* $\min \deg X$ or $\max \deg X$ is the minimal resp. maximal exponent of t with non-zero coefficient in X . Let $\text{span}_t X = \max \deg_t X - \min \deg_t X$. The coefficient in degree d of t in X is denoted $[X]_{t^d}$ or $[X]_d$. The *leading coefficient* $\max \text{cf } X$ of X is its coefficient in degree $\max \deg X$. If $X \in \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$, then $\max \deg_{x_1} X$ denotes the maximal degree in x_1 . Minimal degree and coefficients are defined similarly, and of course $[X]_{x_1^k}$ is regarded as a polynomial in $x_2^{\pm 1}$.

Let $P(v, z)$ be the *skein polynomial* [F&, LM]. It is a Laurent polynomial in two variables of oriented knots and links. We use here the convention of [Mo], i.e. with the polynomial taking the value 1 on the unknot, having the variables v and z and satisfying the skein relation

$$v^{-1} P \left(\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right) - v P \left(\begin{array}{c} \nearrow \nearrow \\ \nwarrow \nwarrow \end{array} \right) = z P \left(\begin{array}{c} \nearrow \\ \nwarrow \end{array} \right) \left(\begin{array}{c} \nearrow \\ \nearrow \end{array} \right). \quad (2.2)$$

We will denote in each triple as in (2.2) the diagrams (from left to right) by D_+ , D_- and D_0 . For a diagram D of a link L , we will use all of the notations $P(D) = P_D = P_D(v, z) = P(L)$, etc. for its skein polynomial, with the self-suggestive meaning of indices and arguments. So we can rewrite (2.2) as

$$v^{-1}P_+(v, z) - vP_-(v, z) = zP_0(v, z). \quad (2.3)$$

The *writhe*, or (*skein*) *sign*, is a number ± 1 , assigned to any crossing in an oriented link diagram. A crossing as on the left in (2.2) has writhe 1 and is called *positive*. A crossing as in the middle of (2.2) has writhe -1 and is called *negative*. The writhe of a link diagram is the sum of writhes of all its crossings.

Let $c_{\pm}(D)$ be the number of positive, respectively negative crossings of a diagram D , so that $c(D) = c_+(D) + c_-(D)$ and $w(D) = c_+(D) - c_-(D)$.

The *Jones polynomial* [J2] V and (one variable) *Alexander polynomial* [A1] Δ are obtained from P by variable substitutions

$$\Delta(t) = P(1, t^{1/2} - t^{-1/2}), \quad (2.4)$$

and

$$V(t) = P(t, t^{1/2} - t^{-1/2}). \quad (2.5)$$

Hence these polynomials also satisfy corresponding skein relations. (In algebraic topology, the Alexander polynomial is usually defined only up to units in $\mathbb{Z}[t, t^{-1}]$; the present normalization is so that $\Delta(t) = \Delta(1/t)$ and $\Delta(1) = 1$.)

In very contrast to its relatives, the *range* of the Alexander polynomial (i.e., set of values it takes) is known. Let us call a polynomial $\Delta \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$ *admissible* if it satisfies for some natural number $n \geq 1$ the three properties

1. $t^{(n-1)/2} \Delta \in \mathbb{Z}[t^{\pm 1}]$,
2. $\Delta(t) = (-1)^{n-1} \Delta(1/t)$ and
3. $(t^{1/2} - t^{-1/2})^{n-1} \mid \Delta$ for $n > 1$, or $\Delta(1) = 1$ for $n = 1$.

It is well-known that these are exactly the polynomials that occur as (1-variable) Alexander polynomials of some n -component link.

The *Kauffman polynomial* [Ka] F is usually defined via a regular isotopy invariant $\Lambda(a, z)$ of unoriented links. We use here a slightly different convention for the variables in F , differing from [Ka, Th2] by the interchange of a and a^{-1} . Thus in particular we have for a link diagram D the relation $F(D)(a, z) = a^{w(D)} \Lambda(D)(a, z)$, where $\Lambda(D)$ is the writhe-unnormalized version of the polynomial, given in our convention by the properties

$$\Lambda\left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}\right) + \Lambda\left(\begin{array}{c} \diagdown \diagdown \\ \diagup \diagup \end{array}\right) = z \left(\Lambda\left(\begin{array}{c} \diagup \diagup \\ \diagup \diagup \end{array}\right) + \Lambda\left(\begin{array}{c} \diagdown \diagdown \\ \diagdown \diagdown \end{array}\right) \right), \quad (2.6)$$

$$\Lambda\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right) = a^{-1} \Lambda\left(\begin{array}{c} | \\ | \end{array}\right); \quad \Lambda\left(\begin{array}{c} \diagdown \\ \diagup \end{array}\right) = a \Lambda\left(\begin{array}{c} | \\ | \end{array}\right), \quad (2.7)$$

$$\Lambda\left(\bigcirc\right) = 1. \quad (2.8)$$

The *Brandt-Lickorish-Millett-Ho polynomial* [BLM] Q is given by $Q(z) = F(1, z)$.

2.3 Semiadequacy

An alternative description of V is given by the Kauffman bracket in [Ka2]. We do not need this description directly, but for self-containedness it is useful to recall the related concept of semiadequacy that was popularized in [LT].

Let D be an unoriented link diagram. A *state* is a choice of splittings of type A or B for any single crossing (see Figure 2.2). Let the A -*state* of D be the state where all crossings are A -spliced; similarly define the B -state.

We call a diagram A -(semi)adequate if in the A -state no crossing trace (one of the dotted lines in Figure 2.2) connects a loop with itself. Similarly we define B -(semi)adequate. A diagram is *semiadequate* if it is A - or B -semiadequate, and *adequate* if it is simultaneously A - and B -semiadequate. A link is adequate/semiadequate if it has an adequate/semiadequate diagram. (Here A -adequate and B -adequate is what is called $+$ adequate resp. $-$ adequate in [Th].)

We also need a part of the semiadequacy formulas for the Jones polynomial. As in [St4], for a non-negative integer i , we write V_i for $[V]_{\min \deg V + i}$ and $\bar{V}_i = [V]_{\max \deg V - i}$. (These are the $i + 1$ st and $i + 1$ st last coefficient of V in its coefficient list.) For an A - (resp. B -) semiadequate diagram, in [St4], and independently in [DL], formulas were obtained for $V_{1,2}$ (resp. $\bar{V}_{1,2}$).

From the A -state $A(D)$ of a diagram D we define two graphs. The first graph, we call it the A -graph $G(A) = G(A(D))$ has vertices for each loop in $A(D)$, and an edge between each pair of loops that are connected by a trace of at least one crossing in D . If there are at least two such traces, we call the edge *multiple*.

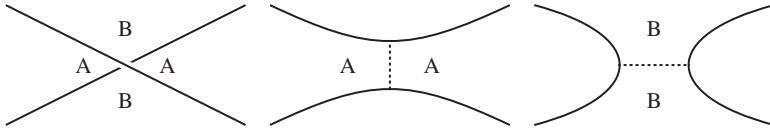
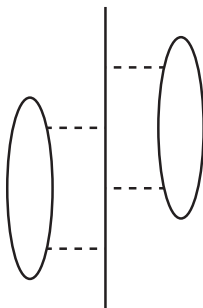


Fig. 2.2 The A - and B -corners of a crossing, and its both splittings. The corner A (resp. B) is the one passed by the overcrossing strand when rotated counterclockwise (resp. clockwise) towards the undercrossing strand. A type A (resp. B) splitting is obtained by connecting the A (resp. B) corners of the crossing. It is useful to put a “trace” of each splitted crossing as an arc connecting the loops at the splitted spot

Let $\Delta(D) = \Delta(A(D))$ be the number of cycles of length 3 (*triangles*) in $G(A(D))$. (Multiple edges do not form different triangles between the same three vertices.)

The second graph, we call it the *intertwining graph* $IG(A) = IG(A(D))$, has vertices for each multiple edge in $A(D)$, and an edge in $IG(A)$ is drawn between each pair of vertices in $IG(A)$, whose corresponding multiple traces in $A(D)$ contain traces of the form shown below.



Theorem 2.1 ([St4, DL]) Assume that D is a connected A -adequate diagram. Then $V_0 V_1 = \chi(G(A)) - 1$ and

$$V_0 V_2 = \binom{2 - \chi(G(A))}{2} + \chi(IG(A)) - \Delta(A(D)).$$

Here χ is the Euler characteristic (number of vertices minus number of edges); $V_0 = \pm 1$ by [LT]. Analogous formulas hold for B -adequate diagrams.

2.4 Braids and Braid Words

The n -string *braid group* B_n is considered generated by the Artin *standard generators* σ_i for $i = 1, \dots, n-1$. These are subject to relations of the type $[\sigma_i, \sigma_j] = 1$ for $|i - j| > 1$, which we call *commutativity relations* (the bracket denotes the commutator) and $\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$, which we call *Yang–Baxter* (or shortly *YB*) relations.

We will make one noteworthy modification of this notation. In the following σ_3 will stand for the usual Artin generator for braids of 4 or more strands (such braids are considered explicitly only in Chapters 5.1 and 7), while on 3-braids (considered in all other chapters) it will denote the “band” generator $\sigma_2\sigma_1\sigma_2^{-1} = \sigma_1^{-1}\sigma_2\sigma_1$.

It will be often convenient in braid words to write $\pm i$, in brackets, for $\sigma_i^{\pm 1}$. For example, $[21(33 - 4)^2 32]$ is an alternative writing for $\sigma_2 \sigma_1 (\sigma_3^2 \sigma_4^{-1})^2 \sigma_3 \sigma_2$.

The following definition summarizes basic terminology of braid words used throughout the monograph.

Definition 2.1 Choose a word

$$\beta = \prod_{i=1}^n \sigma_{k_i}^{l_i} \quad (2.9)$$

with $k_i \neq k_{i+1}$ and $l_i \neq 0$. We understand such words *in cyclic order*.

Call $\sigma_{k_i}^{l_i}$ the *syllables* of β . For such a syllable, let k_i be called the *index* of the syllable, and l_i its *exponent* or *length*. We call a syllable $\sigma_{k_i}^{l_i}$ *non-trivial* if $|l_i| > 1$ and *trivial* if $|l_i| = 1$.

We say n is the *syllable length* of β in (2.9). The (*word*) *length* $c(\beta)$ of β is $\sum_{i=1}^n |l_i|$, and $c_{\pm}(\beta) = \sum_{\pm l_i > 0} \pm l_i$ are the *positive/negative length* of β . A word is *positive* if $c_{-}(\beta) = 0$, or equivalently, if all $l_i > 0$. Similarly it is *almost positive* if $c_{-}(\beta) = 1$ and *negative* if $c_{+}(\beta) = 0$. The *exponent sum* of β is defined to be $[\beta] = \sum_{i=1}^n l_i$. The *index sum* of β is $\sum_{i=1}^n k_i \cdot |l_i|$ (i.e., each letter, not just syllable in β , contributes to that sum).

For 3-braid words, the k_i interchange between 1 and 2. Thus the vector of the l_i , considered up to cyclic permutations, determines the conjugacy class unambiguously. We call it the *Schreier vector*.

Let β be a positive word. We call $\tilde{\beta}$ an *extension* of β if $\tilde{\beta}$ is obtained by replacing some (possibly no) trivial syllables in β by non-trivial ones of the same index. Contrarily, we call β a *syllable reduction* of $\tilde{\beta}$. We call β *non-singular* if $[\beta]_k > 1$ for all $k = 1, \dots, n-1$, where $[\beta]_k = \sum_{k_i=k} l_i$ is the exponent sum of σ_k in β . If $[\beta]_k = 1$, we say that the syllable of index k in β is *isolated* or *reducible*.

To avoid confusion, it seems useful to clarify a priori the following use of symbols (even though we recall it at appropriate places later).

A comma separated list of integers will stand for a *sequence of syllable indices* of a braid word. The “at” sign “@”, written after such an index means that the corresponding syllable is trivial, while by an exclamation mark “!” we indicate that the syllable is non-trivial. (As the exponent for non-trivial syllables will be immaterial, it is enough to distinguish only whether the syllable is trivial or not.) If none of ! and @ is specified, we do not exclude explicitly any of either types.

A bracketed but non-comma separated list of integers will stand for a braid word. An asterisk “*” put after a letter (number) in such a word means that this letter may be repeated (it need not be repeated, but it must not be omitted). So a (possibly trivial) index-2 syllable can be written as 2^* . The expression “[23]” (within another pair of brackets) should mean a letter which is either “2” or “3”, and “[23]+” means a possibly empty sequence of letters “2” and “3”. (See pp. 33 and 49 for examples of use.)

2.5 Braid Representations of Links

By a theorem of Alexander [Al], any link is the *closure* $\hat{\beta}$ of a braid β . The *braid index* $b(L)$ of a link L is the smallest number of strands of a braid β whose closure $\hat{\beta}$ is L . See [Mo, FW, Mu]. Such β are also called *braid representations* of L . The closure operation gives for a particular braid word β a link diagram $D = \hat{\beta}$. Then we have, for example, $w(\hat{\beta}) = [\beta]$, $c(\beta) = c(\hat{\beta})$, $c_{\pm}(\beta) = c_{\pm}(\hat{\beta})$, and $s(\hat{\beta})$ is the number of strings of β (i.e., n for $\beta \in B_n$).

Many properties of braid words we will deal with relate to the corresponding properties of their link diagrams. For example, a braid word is called *positive* if it contains no σ_i^{-1} , or in other words, its closure diagram is positive. A braid is *positive* if it has a positive braid word. Likewise can be done with almost positive and negative. In a similar fashion, we say that braid is (A/B) -adequate if it has an (A/B) -adequate word representation, and a word is adequate if the link diagram obtained by its closure is adequate.

If a braid word β is written as $\sigma_1^{\pm 1} \alpha \sigma_1^p \alpha'$, where $p \in \mathbb{Z}$ and none of α and α' contains a syllable of index 1, then the diagram admits a *flype*, which exchanges the syllables of index 1 in β , so that we obtain $\sigma_1^p \alpha \sigma_1^{\pm 1} \alpha'$. This operation preserves the isotopy type of the closure link $\hat{\beta}$, but in general changes the braid conjugacy class. The phenomenon is explained in [BM]. In the context of general link diagrams, the flype has been studied also extensively, most prominently in [MT].

Alternatively to the standard Artin generators, one considers also a representation of the braid groups by means of an extended set of generators (and their inverses)

$$\sigma_{i,j}^{\pm 1} = \sigma_i \dots \sigma_{j-2} \sigma_{j-1}^{\pm 1} \sigma_{j-2}^{-1} \dots \sigma_i^{-1}$$

for $1 \leq i < j \leq n$. Note that

$$\sigma_i = \sigma_{i,i+1}. \quad (2.10)$$

A representation of a braid β , and its closure link $L = \hat{\beta}$, as word in $\sigma_{i,j}^{\pm 1}$ is called a *band representation* [BKL]. A band representation of β spans naturally a Seifert surface of the link L as in Figure 1.1: one glues disks into the strands, and connects them by half-twisted bands along the $\sigma_{i,j}$. The resulting surface is called *braided Seifert surface* of L . Bennequin proved in [Be]:

Theorem 2.2 (Bennequin) *For a 3-braid link, a minimal genus braided Seifert surface band representation always exists on a 3-braid.*

Followingly, minimal genus Seifert surface of L , which is a braided Seifert surface, is also called a *Bennequin surface* [BM2].

In this monograph we will deal exclusively with band representations in B_3 . Then we have three band generators $\sigma_{i,i \bmod 3+1}$ (where $i = 1, 2, 3$, and “mod” is taken with values between 0 and 2). With (2.10), we have $\sigma_1 = \sigma_{1,2}$ and $\sigma_2 = \sigma_{2,3}$, and

with the special meaning of $\sigma_3 \in B_3$ introduced above, $\sigma_3 = \bar{\sigma}_{1,3}$, where bar denotes the mirror image. (This mirroring is used here for technical reasons related to Xu's normal form, as explained below.)

If a band representation contains only positively half-twisted bands (i.e., no $\sigma_{i,j}^{-1}$ occur), it is called (*band-*)*positive* or *strongly quasi-positive*. A link with a strongly quasi-positive band (braid) representation is called strongly quasi-positive. Such links have an importance in connection to algebraic curves; see [Ru2].

If a band representation contains exactly one negative band, we call it *almost strongly quasi-positive*. If it contains k negative bands, we can call it *k-almost strongly quasi-positive*. Similarly we can define a *band-negative* representation and (*k-*)*almost strongly quasi-negative*, and various versions of *strongly quasi-signed* for braids and links as a common term for the corresponding types of strongly quasi-positive and strongly quasi-negative objects.

Using the skein polynomial, define a quantity by

$$MFW(L) = \frac{1}{2} (\max \deg_v P - \min \deg_v P) + 1. \quad (2.11)$$

The *Morton-Franks-Williams braid index inequality* [Mo, FW] (abbreviated as MFW) states that

$$b(L) \geq MFW(L) \quad (2.12)$$

for every link L . This inequality is often exact (i.e., an equality). The study of links where it is exact or not has occupied a significant part of previous literature. Most noteworthy is the work of Murasugi [Mu] and Murasugi-Przytycki [MP2].

The Morton-Franks-Williams inequality results from two other inequalities, due to Morton, namely that for a diagram D , we have

$$1 - s(D) + w(D) \leq \min \deg_v P(D) \leq \max \deg_v P(D) \leq s(D) - 1 + w(D). \quad (2.13)$$

Franks-Williams showed these inequalities for the case of braid representations (i.e. when $D = \hat{\beta}$ for some braid β). Later it was observed from the algorithm of Yamada [Y] and Vogel [Vo] that the braid version is actually equivalent to (and not just a special case of) the diagram version.

These inequalities were later improved in [MP2] in a way that allows to settle the braid index problem for many links (see Chapter 6 or also [Oh]).

In the special case of 3-braids (and with the special meaning of σ_3 as described above), Xu [Xu] gives a normal form of a conjugacy class in $\sigma_{1,2,3}$. By Xu's algorithm, each $\beta \in B_3$ can be written in one of the two forms

- (A) $[21]^k R$ or $L^{-1} [21]^{-k}$ ($k \geq 0$), or
- (B) $L^{-1} R$,

where L and R are positive words in $\sigma_{1,2,3}$ with (cyclically) non-decreasing indices (i.e., each σ_i is followed by σ_i or $\sigma_{i \bmod 3+1}$, with “mod” taken with values between 0 and 2). Since the form (B) must be cyclically reduced, we may assume that L and R do not start or end with the same letter. This form is the shortest word in $\sigma_{1,2,3}$ of a conjugacy class. By Theorem 2.2, the braided surface is then a minimal genus (or Bennequin) surface. (Remark: Below we will make clear each time from the context whether we use the letter L to denote the compound of Xu’s form, or a link.)

2.6 Gauß Sum Invariants

We recall briefly the definition of Gauß sum invariants. They were introduced first in [Fi] for braids, and later [Fi2, PV] for knots. It is known that all they give formulas for Vassiliev invariants.

Definition 2.2 ([Fi2]) A Gauß diagram of a knot diagram is an oriented circle with arrows connecting points on it mapped to a crossing and oriented from the preimage of the undercrossing (underpass) to the preimage of the overcrossing (overpass).

We will call a pair of crossings whose arrows intersect in the Gauß diagram a *linked pair*.

Example 2.1 As an example, Figure 2.3 shows the knot 6_2 in its commonly known projection and the corresponding Gauß diagram.

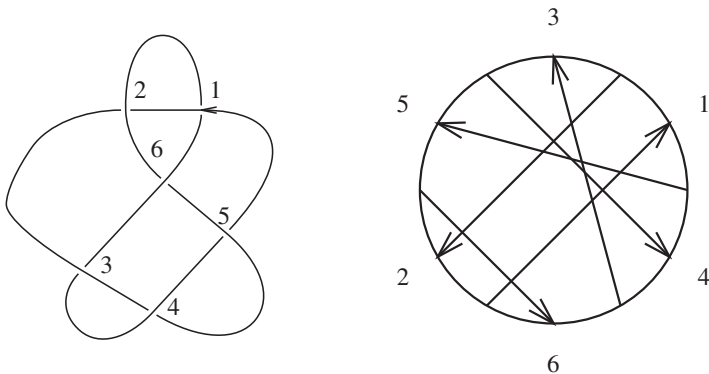


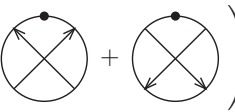
Fig. 2.3 The standard diagram of the knot 6_2 and its Gauß diagram

The simplest (non-trivial) Vassiliev knot invariant is the *Casson invariant* v_2 , with $v_2 = \Delta''(1)/2 = -V''(1)/6$, for which Polyak–Viro [PV, PV2] gave the simple Gauß sum formula

$$v_2 = \left(\text{diagram} \right). \quad (2.14)$$


Here the point on the circle corresponds to a point on the knot diagram, to be placed arbitrarily except on a crossing. (The expression does not alter with the position of the basepoint; we will hence have, and need, the freedom to place it conveniently.)

We will use the symmetrized version of (2.14) w.r.t. taking the mirror image of the knot diagram:

$$v_2 = \frac{1}{2} \left(\text{diagram}_1 + \text{diagram}_2 \right). \quad (2.15)$$


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Properties of Closed 3-Braids and Braid
Representations of Links

Stoimenow, A.

2017, X, 110 p. 89 illus., Softcover

ISBN: 978-3-319-68148-1