Chapter 2
Graphs and Laplacians

2.1 Motivation

In this chapter, we are interested in exploring questions such as the following.

Question 2.1. *If a group G acts on a graph Γ, what is the relationship between the spectrum of Γ and the spectrum of the quotient Γ/G?*

If G is a group acting on graphs Γ_1 and Γ_2, then a G-equivariant map $\Gamma_1 \to \Gamma_2$ is a map that respects the action of G on these graphs.

Question 2.2. *If G is a group acting on graphs Γ_1 and Γ_2, and if there is a G-equivariant map $\Gamma_1 \to \Gamma_2$, how are the Laplacians of Γ_1 and Γ_2 related?*

2.2 Basic results

Let's start with some motivation for the definition of the Laplacian.

Recall that if $\Gamma = (V, E)$ is a graph and F is a ring, then $C^0(\Gamma, F)$ is the set of all F-valued functions on the vertex set V of Γ, and $C^1(\Gamma, F)$ is the set of all F-valued functions on the edge set E of Γ. If the graph Γ is a large square lattice grid, and if $f \in C^0(\Gamma, F)$, then the usual definition of the Laplacian,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2},$$

corresponds to the discrete *Laplacian* Q on f

$$(Qf)(v) = \sum_{w:d(w,v)=1} [f(w) - f(v)] \tag{2.1}$$

W.D. Joyner and C.G. Melles, *Adventures in Graph Theory*,
Applied and Numerical Harmonic Analysis,
https://doi.org/10.1007/978-3-319-68383-6_2

where $d(w,v)$ is the graph distance function on $V \times V$. Indeed,

$$\frac{\partial^2 f}{\partial x^2} = \lim_{\epsilon \to 0} \frac{[f(x+\epsilon,y) - f(x,y)] + [f(x-\epsilon,y) - f(x,y)]}{\epsilon^2}$$

and

$$\frac{\partial^2 f}{\partial y^2} = \lim_{\epsilon \to 0} \frac{[f(x,y+\epsilon) - f(x,y)] + [f(x,y-\epsilon) - f(x,y)]}{\epsilon^2}$$

so taking $\epsilon = 1$ gives the desired discrete analog of $f_{xx} + f_{yy}$ on the grid graph. This operator only depends on "local" properties. That is, $(Qf)(v)$ depends only on the neighbors of v in the graph Γ. It may come as a surprise to find out that Q governs a number of "global properties" of Γ as well, such as connectivity. We shall see these and other fascinating properties of Q below.

Recall from §1.1 that, given an orientation on Γ, there is a linear transformation

$$B : C^1(\Gamma, F) \to C^0(\Gamma, F)$$

given by

$$(Bf)(v) = \sum_{h(e)=v} f(e) - \sum_{t(e)=v} f(e) \tag{2.2}$$

and a dual linear transformation

$$B^* : C^0(\Gamma, F) \to C^1(\Gamma, F).$$

Recall also (see Lemma 1.1.20) that there are natural bases for the spaces $C^0(\Gamma, F)$ and $C^1(\Gamma, F)$. If we choose orderings of the vertices and edges of Γ, then the linear transformation B is given with respect to these bases by the incidence matrix, which we also denote by B. The dual B^* of this linear transformation is given by the transpose B^t of the incidence matrix.

The *vertex Laplacian* (or simply "the Laplacian") is the linear transformation $Q = Q_\Gamma : C^0(\Gamma, F) \to C^0(\Gamma, F)$ defined by

$$Q = BB^*, \tag{2.3}$$

where B is the linear transformation of Equation (2.2) above, and B^* is its dual.

The matrix representation of the Laplacian will also be denoted Q. We leave it as an exercise to show that the linear transformation $Q = BB^*$ is independent of the orientation chosen on Γ.

Exercise 2.1. Suppose that Γ is given an orientation, and denote by B the $n \times m$ incidence matrix of Γ with respect to this orientation (and some orderings of the vertex and edge sets of Γ). Denote by Q the matrix representation of the Laplacian. Show that the matrix BB^t is independent of the orientation chosen and that $BB^t = Q$.

Recall that if the graph Γ has vertex set

$$V = V_\Gamma = \{0, 1, 2, \ldots, n-1\},$$

then the (undirected, unweighted) adjacency matrix of Γ is the $n \times n$ matrix $A = (a_{ij})$, where $a_{ij} = 1$ if vertex i shares an edge with vertex j, and $a_{ij} = 0$ otherwise. There is a simple connection between the Laplacian and the adjacency matrix.

Lemma 2.2.1. *For an oriented graph Γ with unsigned adjacency matrix A, there is a natural basis of $C^0(\Gamma, F)$ for which the matrix representation of the Laplacian is given by*

$$Q = \Delta - A,$$

where Δ denotes the diagonal matrix of the degrees of the vertices of V:

$$\Delta = \begin{pmatrix} d_0 & 0 & 0 & \ldots & 0 \\ 0 & d_1 & 0 & \ldots & 0 \\ 0 & 0 & d_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & d_{n-1} \end{pmatrix},$$

where $d_i = \deg_\Gamma(i)$, for $i \in V$.

Proof. Let $Q = (Q_{ij})$. Since $Q = BB^t$, where $B = (b_{ij})$ is the incidence matrix, Q_{ii} is the inner product of the i-th row of the incidence matrix with itself. This simply counts the number of edges incident to vertex i, so $Q_{ii} = \deg_\Gamma(i)$. If $i \neq j$ then Q_{ij} is the inner product of the i-th row of the incidence matrix with the j-th row. This is nonzero if i and j are both incident to the same edge, and zero if they are not. Indeed, if i and j are both incident to edge k then either

$$b_{ik} = 1, \quad b_{jk} = -1, \quad b_{\ell,k} = 0 \text{ for all } \ell \neq i, j,$$

or

$$b_{ik} = -1, \quad b_{jk} = 1, \quad b_{\ell,k} = 0 \text{ for all } \ell \neq i, j.$$

In either case, the k-th entry of the i-th row times the k-th entry of the j-th row equals -1. Summing over all edges k gives $Q_{ij} = -a_{ij}$. □

Example 2.2.2. Consider the graph in Figure 2.1.

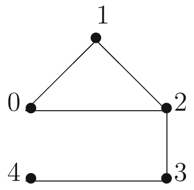

Figure 2.1: A graph with 5 vertices.

This graph has Laplacian matrix

$$Q = BB^t = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Example 2.2.3. Consider the 3×3 grid graph in Figure 2.2.

Sage can be used to calculate the Laplacian matrix of this graph.

Sage

```
sage: Gamma = graphs.GridGraph([3,3])
   ## this is the 3x3 grid graph with 9 vertices
sage: B = incidence_matrix(Gamma, 12*[-1])
sage: B
[-1 -1  0  0  0  0  0  0  0  0  0  0]
[ 0  1 -1 -1  0  0  0  0  0  0  0  0]
[ 0  0  0  1 -1  0  0  0  0  0  0  0]
[ 1  0  0  0  0 -1 -1  0  0  0  0  0]
[ 0  0  1  0  0  0  1 -1 -1  0  0  0]
[ 0  0  0  0  1  0  0  0  1 -1  0  0]
[ 0  0  0  0  0  1  0  0  0  0 -1  0]
[ 0  0  0  0  0  0  0  1  0  0  1 -1]
[ 0  0  0  0  0  0  0  0  0  1  0  1]
```

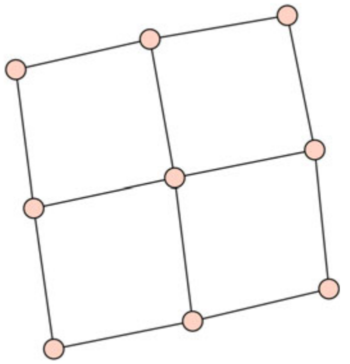

Figure 2.2: The 3×3 grid graph.

```
sage: B*transpose(B)
[ 2 -1  0 -1  0  0  0  0  0]
[-1  3 -1  0 -1  0  0  0  0]
[ 0 -1  2  0  0 -1  0  0  0]
[-1  0  0  3 -1  0 -1  0  0]
[ 0 -1  0 -1  4 -1  0 -1  0]
[ 0  0 -1  0 -1  3  0  0 -1]
[ 0  0  0 -1  0  0  2 -1  0]
[ 0  0  0  0 -1  0 -1  3 -1]
[ 0  0  0  0  0 -1  0 -1  2]
sage: Gamma.laplacian_matrix()
[ 2 -1  0 -1  0  0  0  0  0]
[-1  3 -1  0 -1  0  0  0  0]
[ 0 -1  2  0  0 -1  0  0  0]
[-1  0  0  3 -1  0 -1  0  0]
[ 0 -1  0 -1  4 -1  0 -1  0]
[ 0  0 -1  0 -1  3  0  0 -1]
[ 0  0  0 -1  0  0  2 -1  0]
[ 0  0  0  0 -1  0 -1  3 -1]
[ 0  0  0  0  0 -1  0 -1  2]
```

The "4" in the center of the Laplacian matrix illustrates the fact that there are four edges emanating from the central vertex of the 3×3 grid graph in Figure 2.2.

Lemma 2.2.4. *For any vector* $\mathbf{x} = (x_0, \ldots, x_{n-1})$*, we have*

$$\mathbf{x}^t Q\mathbf{x} = \sum_{(i,j)\in E, j>i} (x_i - x_j)^2.$$

In other words, the quantity $\mathbf{x}^t Q\mathbf{x}$ measures how far away the vector $\mathbf{x}$ is from being constant.

Proof. Indeed,

$$\mathbf{x}^t Q\mathbf{x} = \mathbf{x}^t BB^t\mathbf{x} = (B^t\mathbf{x}) \cdot (B^t\mathbf{x}) = \sum_{(i,j)\in E,\, j>i} (x_i - x_j)^2.$$

□

Example 2.2.5. Consider the tetrahedral graph, Γ. Using Sage, it is easy to verify that the identity in Lemma 2.2.4 holds in this example.

Sage

```
sage: Gamma = graphs.TetrahedralGraph()
sage: Q = Gamma.laplacian_matrix(); Q
[ 3 -1 -1 -1]
[-1  3 -1 -1]
[-1 -1  3 -1]
[-1 -1 -1  3]
sage: x0,x1,x2,x3 = var("x0,x1,x2,x3")
sage: x = vector(SR, [x0,x1,x2,x3])
sage: expand(x.dot_product(Q*x))
3*x0^2 - 2*x0*x1 + 3*x1^2 - 2*x0*x2 - 2*x1*x2 + 3*x2^2 - 2*x0*x3
  - 2*x1*x3 - 2*x2*x3 + 3*x3^2
sage: expand(x.dot_product(Q*x))-expand((x1-x0)^2+(x2-x1)^2+(x3-x2)^2
  +(x2-x0)^2+(x3-x0)^2+(x3-x1)^2)
0
```

For a given orientation of Γ, the *edge Laplacian* is the linear transformation $Q_e = Q_{e,\Gamma} : C^1(\Gamma, F) \to C^1(\Gamma, F)$ defined by

$$Q_e = B^* B, \tag{2.4}$$

where B is the linear transformation of Equation (2.2) and B^* is its dual. Unlike the vertex Laplacian, Q_e depends on the orientation.

The following proposition describes the kernel of the Laplacian matrix of a connected graph.

Proposition 2.2.6. *If Γ is a connected graph, the kernel of the Laplacian matrix Q consists of all multiples of the all 1's vector $\mathbf{1} = (1, 1, \ldots, 1)$, i.e., $\mathbf{1}$ is an eigenvector of Q corresponding to the eigenvalue 0, and the eigenspace of 0 is 1-dimensional.*

Proof. Each row of B^t contains 1 once and -1 once and all other entries of the row are 0. Thus $B^t\mathbf{1} = \mathbf{0}$, the zero vector. Furthermore, if x is a vector in the kernel of BB^t, then $x^t BB^t x = \mathbf{0}$, so $B^t x = \mathbf{0}$. But if x is in the kernel of B^t, then x takes the same value on the head and tail vertices of each edge. Since Γ is assumed to be connected, x must take the same value on all vertices of Γ. $\square$

Corollary 2.2.7. *If Γ is a connected graph, the rank of the Laplacian matrix Q is $n - 1$, where n is the number of vertices of Γ.*

Definition 2.2.8. The *spectrum* of a graph Γ is the multi-set of eigenvalues of the (unsigned) adjacency matrix $A = A_\Gamma$. We sometimes denote the spectrum of Γ by $\sigma(\Gamma)$. The *Laplacian spectrum* of a graph Γ is the set of eigenvalues of the Laplacian matrix $Q = Q_\Gamma$. The *characteristic polynomial* p_Γ of a graph Γ is the characteristic polynomial of A.

Let Γ be a simple connected graph. If Γ has n vertices, let

$$\lambda_0 = 0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1} \tag{2.5}$$

denote the eigenvalues of Q. By Lemma 2.2.4, Q is positive semidefinite. This implies $\lambda_i(Q) \geq 0$ for all i. Consequently, $\lambda_0 = 0$, because the vector $\mathbf{v}_0 = (1, 1, \ldots, 1)$ satisfies $Q\mathbf{v}_0 = \mathbf{0}$.

Lemma 2.2.9. *If Γ is a k-regular graph, and the Laplacian Q of Γ has eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$, then the adjacency matrix A of Γ has eigenvalues $k - \lambda_0, k - \lambda_1, \ldots, k - \lambda_{n-1}$.*

Proof. This is an immediate corollary of Lemma 2.2.1. $\square$

Recall, the diameter of the graph Γ is the maximum distance between any two vertices of Γ.

Lemma 2.2.10. *Let Γ be a simple connected graph with n vertices and diameter d. Then*

$$\lambda_1 \geq (dn)^{-1},$$

where λ_1 is as in Equation (2.5).

Proof. Since Q is symmetric, $\mathbb{R}^n$ has an orthonormal basis of eigenvectors of Q.

It is a consequence of well-known facts from linear algebra[1] that we have

$$\lambda_1 = \min_{\mathbf{v},\ \mathbf{v}\cdot\mathbf{1}=0} \frac{\mathbf{v}\cdot Q\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}.$$

Let $\mathbf{f} = (f_0, f_1, \ldots, f_{n-1})$ denote an eigenvector of Q satisfying

$$\lambda_1 = \frac{\sum_{(i,j)\in E}(f_j - f_i)^2}{\sum_{i\in V} f_i^2}.$$

Let j_0 denote a vertex such that $|f_{j_0}| = \max_{i\in V}|f_i|$. We know that $\mathbf{f}$ is orthogonal to the all 1's vector (which is an eigenvector for $\lambda_0 = 0$), so there exists $j_1 \in V$ such that $f_{j_0}f_{j_1} < 0$.

Let P denote a shortest path from j_0 to j_1. We denote the adjacent vertices in P by $i_0 = j_0, i_1, \ldots, i_{r-1}, i_r = j_1$. The number of edges in P is at most d, so $r \leq d$. Note

$$\begin{aligned} f_{i_r} - f_{i_0} &= f_{i_r} - f_{i_{r-1}} + f_{i_{r-1}} - f_{i_{r-2}} + \cdots + f_{i_1} - f_{i_0} \\ &= (f_{i_r} - f_{i_{r-1}}, f_{i_{r-1}} - f_{i_{r-2}}, \ldots, f_{i_1} - f_{i_0}) \cdot (1, 1, \ldots, 1) \\ &\leq ((f_{i_r} - f_{i_{r-1}})^2 + (f_{i_{r-1}} - f_{i_{r-2}})^2 + \cdots + (f_{i_1} - f_{i_0})^2)^{1/2} \cdot \sqrt{d}, \end{aligned}$$

so

$$(f_{i_r} - f_{i_{r-1}})^2 + (f_{i_{r-1}} - f_{i_{r-2}})^2 + \cdots + (f_{i_1} - f_{i_0})^2 \geq \frac{(f_{i_r} - f_{i_0})^2}{d}.$$

We therefore have

[1] See Biggs, §8c, [Bi93] for further details.

$$\begin{aligned}
\lambda_1 &= \frac{\sum_{(i,j)\in E}(f_j - f_i)^2}{\sum_{i\in V} f_i^2} \\
&\geq \frac{\sum_{(i,j)\in P}(f_j - f_i)^2}{n|f_{j_0}|^2} \\
&\geq \frac{(f_{j_1} - f_{j_0})^2}{dn|f_{j_0}|^2} \\
&\geq \frac{1}{dn},
\end{aligned} \tag{2.6}$$

as desired. □

Remark 2.2.11. (1) A similar result, for the "normalized Laplacian", can be found in Chung [Ch92], Lemma 1.9.

(2) The above argument, with a more detailed analysis, can be pushed to prove a stronger result (due to B. McKay):

$$\lambda_1 \geq \frac{4}{dn}.$$

See Mohar [Mo91b] for details. See also Spielman [Sp10].

Example 2.2.12. A barbell graph (see Figure 2.3) illustrates how good McKay's lower bound really is.

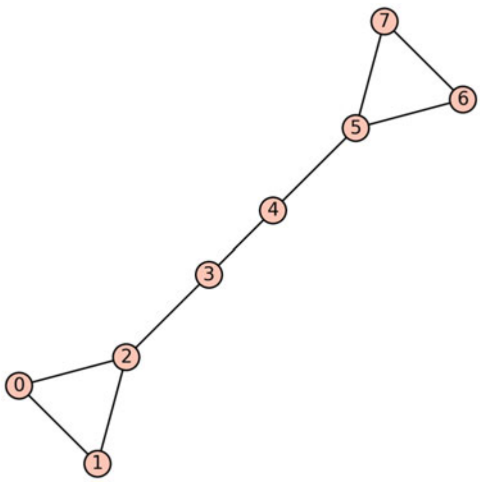

Figure 2.3: A barbell graph created using Sage.

Sage

```
sage: Gamma = graphs.BarbellGraph(3,2)
sage: d = Gamma.diameter()
sage: n = len(Gamma.vertices())
sage: Q = Gamma.laplacian_matrix()
sage: spec = Q.eigenvalues()
sage: spec.sort()
sage: spec
[0, 0.1863934973516692?, 1, 2.470683419871161?, 3, 3, 4, 4.342923082777170?]
sage: 4.0/(d*n)
0.100000000000000
```

Let Γ be a graph, and let ρ be an element of the automorphism group $Aut(\Gamma)$ of Γ. If f is an eigenfunction of Q (regarded as a vector indexed by the vertices V of Γ), we define ρf to be the eigenfunction whose entries are permuted according to the action of ρ^{-1} on V, i.e., $(\rho f)(v) = f(\rho^{-1}(v))$ for any vertex v of Γ.

Definition 2.2.13. A *representation* of G is a homomorphism $\pi : G \to GL(n, \mathbb{C})$. A vector subspace $W \subset \mathbb{C}^n$ is called *G-invariant* if $\pi(g)w \in W$ for all $g \in G$ and all $w \in W$. The restriction of π to a G-invariant subspace is known as a *subrepresentation*. A representation is said to be *irreducible* if it has only trivial subrepresentations. A *character* χ of G is a trace of a representation π, denoted

$$\chi(g) = \mathrm{tr}(\pi(g)), \qquad g \in G,$$

i.e., $\chi(g)$ is the trace of the matrix $\pi(g) \in GL(n, \mathbb{C})$.

Lemma 2.2.14. *The eigenspaces of the Laplacian of a graph Γ are representations of the automorphism group $Aut(\Gamma)$. In other words, if f is an eigenfunction of $Q = Q_\Gamma$ corresponding to an eigenvalue λ, then ρf is also an eigenfunction of Q with eigenvalue λ.*

Proof. We have

$$\begin{aligned}
Q(\rho f)(v) &= \textstyle\sum_{(v,w)\in E}(\rho f(v) - \rho f(w)) \\
&= \textstyle\sum_{(v,w)\in E}(f(\rho^{-1}(v)) - f(\rho^{-1}(w))) \\
&= \textstyle\sum_{(\rho^{-1}v,w)\in E}(f(\rho^{-1}(v)) - f(w)) \\
&= Qf(\rho^{-1}(v)) \\
&= \lambda f(\rho^{-1}(v)) \\
&= \lambda(\rho f)(v).
\end{aligned}$$

$\square$

Lemma 2.2.15. *Every row sum and column sum of Q is zero.*

Proof. Since Q is symmetric, each column sum agrees with the corresponding row sum. Indeed, in the row sum corresponding to vertex v, the degree of v is summed with a "-1" for each neighbor of v. These cancel, giving us a sum of 0, as desired. $\square$

Example 2.2.16. The *star graph* $Star_n$ is a graph on $n+1$ vertices $v_0, \ldots, v_n$ with edges (v_0, v_i), for $i = 1, \ldots, n$. For example, $Star_5$ is depicted in Figure 2.4.

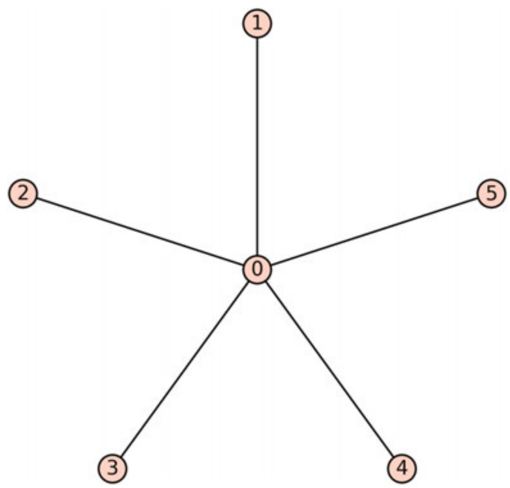

Figure 2.4: A star graph created using Sage.

Exercise 2.2. (a) Show that the eigenvalues of the adjacency matrix of $Star_n$ are $-\sqrt{n}$, 0 (with multiplicity $n-1$), $\sqrt{n}$.

(b) Show that the eigenvalues of the Laplacian matrix of $Star_n$ are 0, 1 (with multiplicity $n-1$), $n+1$.

Example 2.2.17. Consider the Paley graph on 9 vertices, Γ, depicted in Figure 1.12.

This graph has incidence matrix

$$B = \begin{pmatrix}
-1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1
\end{pmatrix}$$

and Laplacian matrix

$$Q = BB^t = \begin{pmatrix} 4 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \\ -1 & 4 & -1 & 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 4 & -1 & -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 & -1 & 4 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 0 & -1 & 4 & -1 \\ -1 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 4 \end{pmatrix}.$$

The eigenvalues of Q are

$$0, 3, 3, 3, 3, 6, 6, 6, 6.$$

This graph is just one case of a very interesting family of graphs named after Raymond Paley. More information on them is given, for example, in §5.17 below. We shall see more of this remarkable graph and its cousins, later.

Example 2.2.18. Let Cyc_n denote the cycle graph with n vertices. The characteristic polynomial of $\Gamma = Cyc_n$ is

$$p_\Gamma(x) = 2T_n(x/2) - 2,$$

where $T_n(x)$ is the n-th Chebyshev polynomial of the first kind (see Stevanovic [St14]).

For an (undirected) graph Γ, denote the eigenvalues of the Laplacian by

$$\lambda_0(Q) \leq \lambda_1(Q) \leq \cdots \leq \lambda_{n-1}(Q).$$

Recall the incidence matrix B and its transpose B^t can be regarded as homomorphisms

$$B : C^1(\Gamma, \mathbb{Z}) \to C^0(\Gamma, \mathbb{Z}) \quad \text{and} \quad B^t : C^0(\Gamma, \mathbb{Z}) \to C^1(\Gamma, \mathbb{Z}).$$

Therefore, we can regard the Laplacian $Q = BB^t$ as a homomorphism $C^0(\Gamma, \mathbb{Z}) \to C^0(\Gamma, \mathbb{Z})$.

Lemma 2.2.19. *The adjacency matrix of Γ has an eigenvector of all 1's if and only if Γ is regular.*

Proof. By definition, the sum of the entries in the i-th row of A is equal to the number of vertices which share an edge with vertex i. This is, of course, the degree of i. Let $\mathbf{1} = \mathbf{1}_n \in \mathbb{Z}^n$ denote the vector of all 1's. If $\deg(v) = \deg_\Gamma(v)$ denotes the degree of a vertex v then we have shown

$$A\mathbf{1} = \begin{pmatrix} \deg(0) \\ \deg(1) \\ \vdots \\ \deg(n-1) \end{pmatrix}.$$

The vector on the right-hand side of the above equation is a scalar vector (i.e., all the components are the same) if and only if Γ is regular. □

The *index* of Γ is the largest eigenvalue of Γ. The index has an eigenvector which consists of nonnegative components.

Lemma 2.2.20. *Suppose Γ has connected components $\Gamma_1, \ldots, \Gamma_r$.*

(a) Possibly after reordering the vertices of Γ, Q is a block diagonal matrix, where each block is the respective Laplacian matrix for a corresponding component. In other words, Q is permutation conjugate to a block diagonal matrix.

(b) If $n = |V|$, define the vector $\mathbf{v}_j \in \mathbb{R}^n$ to be the vector whose component associated to a vertex in Γ_j is $= 1$ and all other components $= 0$. Then $\mathbf{v}_j$ is an eigenvector of Q having eigenvalue 0.

Proof. This proof is left as an exercise in every other textbook on graph theory, so we should not be any different. Exercise! □

Lemma 2.2.21. *The graph Γ is connected if and only if the index of Γ occurs with multiplicity 1 and it has an eigenvector which consists of strictly positive components.*

Proof. This follows from the Perron–Frobenius Theorem: If an $n \times n$ matrix has nonnegative entries then it has a nonnegative real eigenvalue λ which has maximum absolute value among all eigenvalues. This eigenvalue λ has a nonnegative real eigenvector. If, in addition, the matrix has no block triangular decomposition (i.e., it does not contain a $k \times (n-k)$ block of 0s disjoint from the diagonal), then λ has multiplicity 1 and the corresponding eigenvector is positive. □

Exercise 2.3. Show that the multiplicity of $\lambda = 0$ as an eigenvalue of the Laplacian Q is the number of connected components in the graph.

Remark 2.2.22. Let Q^* denote a reduced Laplacian matrix (obtained by removing any row and the corresponding column of the Laplacian matrix Q) of a connected graph Γ. Then the critical group of Γ is isomorphic to

$\mathbb{Z}^{n-1}/\mathrm{Col}(Q^*)$, where $\mathrm{Col}(Q^*)$ denotes the $\mathbb{Z}$-span of the columns of Q^* and n is the number of vertices of Γ. For more details, see Proposition 4.6.2 in Chapter 4 on chip-firing.

For further reading on the topics of this section, see also Biyikogu, Leydold, and Stadler [BLS07] and Mohar [Mo91a].

2.3 The Moore–Penrose pseudoinverse

Throughout this section, we assume that the underlying graph Γ is connected.

The Moore–Penrose pseudoinverse of the Laplacian matrix Q is a type of generalized inverse of Q. There are other generalized inverses. Generalized inverses are sometimes classified by the additional properties they have, beyond that of the definition below.

After some preliminaries, we will give a construction of the Moore–Penrose pseudoinverse of the Laplacian matrix. An alternative construction will also be given (see Lemma 2.3.10).

Definition 2.3.1. If M and L are matrices such that $MLM = M$, then L is said to be a *generalized inverse* of M.

Let J be the $n \times n$ matrix, all of whose entries are 1. Let $\mathbf{1}$ be the n-vector, all of whose entries are 1.

Remark 2.3.2. Note that

- $QJ = JQ = 0$ (the all 0's matrix).
- $J^2 = nJ$.
- If x is any n-vector and $\deg(x) = \sum_{i=1}^n x_i$, then $Jx = \deg(x)\mathbf{1}$.
- In particular, $J\mathbf{1} = n\mathbf{1}$.
- If s has degree 0, then $Js = \mathbf{0}$ (the all 0's vector) since $\deg(s) = 0$.

Lemma 2.3.3. *$Q + \frac{1}{n}J$ is nonsingular.*

We will prove this lemma at the end of §2.3, after giving an alternative construction of the Moore–Penrose pseudoinverse.

Remark 2.3.4. Note that $(Q + \frac{1}{n}J)J = J = J(Q + \frac{1}{n}J)$.

Definition 2.3.5. The *Moore–Penrose pseudoinverse* of Q is defined to be

$$Q^+ = \left(Q + \frac{1}{n}J\right)^{-1} - \frac{1}{n}J.$$

Example 2.3.6. The Paley graph on 9 vertices Γ has Laplacian matrix

$$Q = \begin{pmatrix} 4 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \\ -1 & 4 & -1 & 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 4 & -1 & -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 & -1 & 4 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 0 & -1 & 4 & -1 \\ -1 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 4 \end{pmatrix},$$

whose Moore–Penrose pseudoinverse is

$$Q^+ = \begin{pmatrix} \frac{2}{9} & 0 & 0 & -\frac{1}{18} & 0 & -\frac{1}{18} & -\frac{1}{18} & -\frac{1}{18} & 0 \\ 0 & \frac{2}{9} & 0 & -\frac{1}{18} & -\frac{1}{18} & 0 & 0 & -\frac{1}{18} & -\frac{1}{18} \\ 0 & 0 & \frac{2}{9} & 0 & -\frac{1}{18} & -\frac{1}{18} & -\frac{1}{18} & 0 & -\frac{1}{18} \\ -\frac{1}{18} & -\frac{1}{18} & 0 & \frac{2}{9} & 0 & 0 & -\frac{1}{18} & 0 & -\frac{1}{18} \\ 0 & -\frac{1}{18} & -\frac{1}{18} & 0 & \frac{2}{9} & 0 & -\frac{1}{18} & -\frac{1}{18} & 0 \\ -\frac{1}{18} & 0 & -\frac{1}{18} & 0 & 0 & \frac{2}{9} & 0 & -\frac{1}{18} & -\frac{1}{18} \\ -\frac{1}{18} & 0 & -\frac{1}{18} & -\frac{1}{18} & -\frac{1}{18} & 0 & \frac{2}{9} & 0 & 0 \\ -\frac{1}{18} & -\frac{1}{18} & 0 & 0 & -\frac{1}{18} & -\frac{1}{18} & 0 & \frac{2}{9} & 0 \\ 0 & -\frac{1}{18} & -\frac{1}{18} & -\frac{1}{18} & 0 & -\frac{1}{18} & 0 & 0 & \frac{2}{9} \end{pmatrix}.$$

Proposition 2.3.7. *The Moore–Penrose pseudoinverse Q^+ has the following properties:*

i. *Q^+ is symmetric.*

ii. $Q^{++} = Q$.

iii. $\left(Q + \frac{1}{n}J\right)^{-1} = Q^+ + \frac{1}{n}J$.

iv. $JQ^+ = Q^+J = 0$ *(the all 0's matrix).*

v. $QQ^+ = Q^+Q = I - \frac{1}{n}J$.

vi. $QQ^+Q = Q$ *and* $Q^+QQ^+ = Q^+$.

vii. $Q^+ = B^+(B^+)^t$, *where* $B^+ = Q^+B$ *and B is the incidence matrix.*

Proof. The first three properties are immediate, from the definition. From property (iii), we have

$$J\left(Q + \frac{1}{n}J\right)\left(Q^+ + \frac{1}{n}J\right) = JI$$

which expands to

$$JQQ^+ + \frac{1}{n}JQJ + \frac{1}{n}J^2Q^+ + \frac{1}{n^2}J^3 = J.$$

Noting that $JQ = 0$ and $J^2 = nJ$, we obtain

$$0 + 0 + JQ^+ + J = J,$$

so that $JQ^+ = 0$. Taking transposes, and noting that both J and Q^+ are symmetric, gives $Q^+J = 0$ also.

Using property (iii) again, we have

$$\left(Q + \frac{1}{n}J\right)\left(Q^+ + \frac{1}{n}J\right) = I.$$

Expanding gives

$$QQ^+ + \frac{1}{n}QJ + \frac{1}{n}JQ^+ + \frac{1}{n^2}J^2 = I.$$

Simplifying and using property (iv) gives

$$QQ^+ + 0 + 0 + \frac{1}{n}J = I$$

so that $QQ^+ = I - \frac{1}{n}J$. Taking transposes and noting that Q and Q^+ are symmetric gives $Q^+Q = I - \frac{1}{n}J$.

Using property (v), we have

$$QQ^+Q = Q\left(I - \frac{1}{n}J\right) = Q - \frac{1}{n}QJ = Q.$$

The proof that $Q^+QQ^+ = Q^+$ is similar.

Finally, we note that $B^+(B^+)^t = Q^+BB^t(Q^+)^t = Q^+QQ^+$ (since Q^+ is symmetric) which equals Q^+ by property (vi). □

Corollary 2.3.8. *If s is a vector whose entries sum to 0, then*

$$QQ^+s = Q^+Qs = s \qquad \text{and} \qquad s^tQQ^+ = s^tQ^+Q = s^t.$$

Proof. By part (v) of Proposition 2.3.7, $QQ^+s = Q^+Qs = (I - \frac{1}{n}J)s$. When the entries of s sum to 0, $Js = 0$, so $QQ^+s = Q^+Qs = Is = s$. The proof of the second statement is similar, and it also follows from the fact that Q and Q^+ are symmetric. □

Example 2.3.9. The cycle graph on 3 vertices $\Gamma = C_3$ has Laplacian matrix

$$Q = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

whose Moore–Penrose pseudoinverse is

$$Q^+ = \begin{pmatrix} \frac{2}{9} & -\frac{1}{9} & -\frac{1}{9} \\ -\frac{1}{9} & \frac{2}{9} & -\frac{1}{9} \\ -\frac{1}{9} & -\frac{1}{9} & \frac{2}{9} \end{pmatrix}.$$

The signed incidence matrix attached to the edges

$$E = \{e_0 = (1,0), e_1 = (2,0), e_2 = (2,1)\}$$

having orientation $[1,1,1]$ is

$$B = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix},$$

so that

$$B^+ = Q^+B = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{3} \end{pmatrix}.$$

Exercise 2.4. Show that the matrices of Example 2.3.9 satisfy properties (i)-(vii) of Proposition 2.3.7. Also, show that B^+ is not a generalized inverse of B, i.e., show that $BB^+B \neq B$.

We will now give another construction of the Moore–Penrose pseudoinverse.

Since the Laplacian matrix Q is a real $n \times n$ symmetric matrix, we can choose a basis of $\mathbb{R}^n$ consisting of n orthonormal eigenvectors $w_1, w_2, \ldots, w_n$ corresponding to eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of Q. For Γ a connected graph, the rank of Q is $n-1$, so we may assume that $\lambda_1 = 0$ and $w_1 = \frac{1}{\sqrt{n}}\mathbf{1}$, i.e., every entry of w_1 is $\frac{1}{\sqrt{n}}$. The nonzero eigenvalues are all positive, since $Q = BB^t$. Let U be the orthogonal matrix whose columns are the eigenvectors $w_1, w_2, \ldots, w_n$ and let Σ be the diagonal matrix whose diagonal entries are $\lambda_1 = 0, \lambda_2, \lambda_3, \ldots, \lambda_n$. Then $UU^t = U^tU = I$ and

$$Q = U\Sigma U^t. \tag{2.7}$$

We define the pseudoinverse of the diagonal matrix Σ to be the diagonal matrix Σ^+ whose i-th diagonal entry is the reciprocal of the i-th diagonal entry of Σ, if this entry is nonzero, and zero otherwise, i.e., the diagonal entries of Σ^+ are $0, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \ldots, \frac{1}{\lambda_n}$.

We define Q^+ to be the matrix given by

$$Q^+ = U\Sigma^+U^t. \tag{2.8}$$

The matrix Q^+ is called a pseudoinverse of Q. We will show that Q^+ is equal to the Moore–Penrose pseudoinverse defined above, and is thus independent of the choice of matrix U. Note that from Equation (2.8), it is clear that Q^+ is symmetric and has the same rank $n-1$ as Q. Furthermore, $QQ^+ = Q^+Q$ and the matrices Q and Q^+ have the same eigenvectors. Thus the all 1's vector $\mathbf{1}$ is a basis for the kernel of Q^+ and $Q^+J = JQ^+ = 0$, the all 0's matrix.

Lemma 2.3.10. *The matrix Q^+ given by Equation (2.8) is the Moore–Penrose pseudoinverse, i.e.,*

$$U\Sigma^+U^t = \left(Q + \frac{1}{n}J\right)^{-1} - \frac{1}{n}J. \tag{2.9}$$

Proof. By Equation (2.7), proving Equation (2.9) is equivalent to proving

$$\left(U\Sigma^+U^t + \frac{1}{n}J\right)\left(U\Sigma U^t + \frac{1}{n}J\right) = I.$$

The left side of the previous equation expands to give

$$\begin{aligned}
U\Sigma^+U^tU\Sigma U^t &+ \frac{1}{n}U\Sigma^+U^tJ + \frac{1}{n}JU\Sigma U^t + \frac{1}{n^2}J^2 \\
&= U\Sigma^+\Sigma U^t + \frac{1}{n}UU^tQ^+J + \frac{1}{n}JQUU^t + \frac{1}{n}J \\
&= U\Sigma^+\Sigma U^t + \frac{1}{n}J
\end{aligned}$$

since $Q^+J = JQ = 0$. Let $I_{(1,1)}$ denote the matrix, all of whose entries are 0, except for the $(1,1)$-entry, which is 1. Note that $\Sigma^+\Sigma = I - I_{(1,1)}$. Then

$$\begin{aligned}
U\Sigma^+\Sigma U^t + \frac{1}{n}J &= U(I - I_{(1,1)})U^t + \frac{1}{n}J \\
&= I - UI_{(1,1)}U^t + \frac{1}{n}J.
\end{aligned}$$

It is not hard to check that $UI_{(1,1)}U^t = \frac{1}{n}J$, from which the required identity follows. □

Exercise 2.5. Prove that the formula

$$\left(Q - \frac{1}{n}J\right)^{-1} + \frac{1}{n}J$$

also gives the Moore–Penrose pseudoinverse of Q.

We will now prove Lemma 2.3.3.

Proof. Let $Q = U\Sigma U^t$ be a decomposition of Q as above. The diagonal elements of Σ are $0, \lambda_2, \lambda_3, \ldots, \lambda_n$ and the first column of U is the eigenvector $\frac{1}{\sqrt{n}}\mathbf{1}$ corresponding to $\lambda_1 = 0$. We can decompose J as $J = U\Sigma_J U^t$ where $\Sigma_J = I_{(1,1)}$ is the diagonal matrix with 1 in the $(1,1)$-entry and all other entries zero. Then $Q + \frac{1}{n}J = U\Sigma' U^t$, where Σ' is the diagonal matrix with diagonal entries $1, \lambda_2, \lambda_3, \ldots, \lambda_n$. Thus, $Q + \frac{1}{n}J$ is nonsingular. $\square$

2.4 Circulant graphs

Recall that a *circulant matrix* is a square matrix where each row vector is a cyclic shift one element to the right relative to the preceding row vector, such as

$$C = \begin{pmatrix} c_0 & c_{n-1} & \ldots & c_1 \\ c_1 & c_0 & \ldots & c_2 \\ \vdots & \vdots & & \vdots \\ c_{n-1} & c_{n-2} & \ldots & c_0 \end{pmatrix}.$$

Circulant matrices have the property that $\mathbf{v}_k = (\zeta^{jk}/\sqrt{n} \mid j = 0, \ldots, n-1)$ is an eigenvector with eigenvalue

$$\lambda_k(C) = \sum_{j=0}^{n-1} \zeta^{-jk} c_j = \sum_{j=1}^{n} \zeta^{jk} c_{n-j},$$

for each $k = 0, \ldots, n-1$.

A graph Γ is called *circulant* if its vertices can be reindexed in such a way that its adjacency matrix is a circulant matrix. For example, a cycle graph is a circulant graph.

Example 2.4.1. Consider the Möbius ladder graph on 8 vertices, Γ, depicted in Figure 2.5.

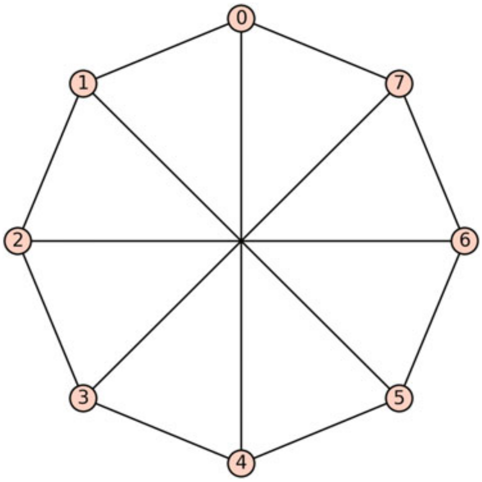

Figure 2.5: A Möbius ladder graph created using Sage.

This graph has adjacency matrix

$$A_\Gamma = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix},$$

incidence matrix

$$B = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and Laplacian matrix

$$Q = BB^t = \begin{pmatrix} 3 & -1 & 0 & 0 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 3 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & 3 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & 3 \end{pmatrix}.$$

It is a circulant graph.

2.4.1 Cycle graphs

For the cycle graph on n vertices, Γ_n, the eigenvalues are $2\cos(2\pi k/n)$, for $0 \leq k \leq n-1$. Since these are not distinct, some can occur with multiplicities.
n even: The only eigenvalues of Γ_n which occur with multiplicity 1 are 2 and -2. The eigenvalues $2\cos(2\pi k/n)$, for $1 \leq k \leq \frac{n-2}{2}$, all occur with multiplicity 2.
n odd: The only eigenvalue of Γ_n which occurs with multiplicity 1 is 2. The eigenvalues $2\cos(2\pi k/n)$, for $1 \leq k \leq \frac{n-1}{2}$, all occur with multiplicity 2.

For example, the graph Γ_8 is depicted in Figure 2.6

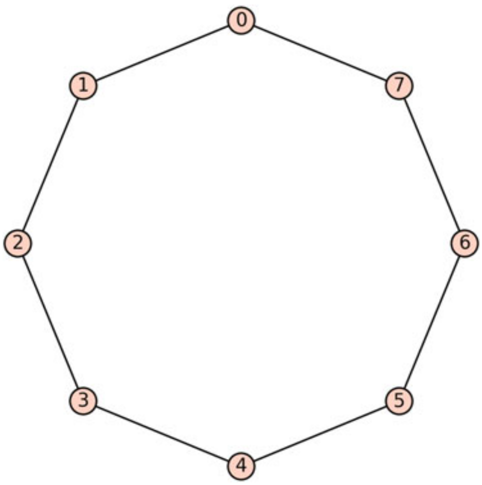

Figure 2.6: A cycle graph created using Sage.

The adjacency matrix is circulant:

$$A_{\Gamma_8} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues (counted according to their multiplicity are 2, $\sqrt{2}$, $\sqrt{2}$, 0, 0, $-\sqrt{2}$, $-\sqrt{2}$, -2.

Let Γ_1 be the cycle graph with n vertices, let Γ_2 be the cycle digraph (directed graph) with n vertices (with edges oriented counterclockwise around the cycle), let G_i denote the automorphism group of Γ_i, $i = 1, 2$.

Lemma 2.4.2. *The automorphism group G_1 of the cycle graph Γ_1 is the dihedral group of order $2n$, D_n.*

Proof. We may assume that the vertices are labeled

$$V = \{0, 1, \ldots, n-1\},$$

and that the edges are

$$E = \{(0,1), (1,2), \ldots, (n-1, 0)\}.$$

Clearly, the "rotation" (written in disjoint cycle notation) belongs to the automorphism group, i.e., $(0, 1, \ldots, n-1) \in G_1$. Clearly, the "reflection" (written in disjoint cycle notation) belongs to the automorphism group, i.e., $(0, n-1)(1, n-2) \cdots \in G_1$. The rotation and reflection generate D_n. The remainder of the proof is left as Exercise 2.6. $\square$

Exercise 2.6. Complete the proof of Lemma 2.4.2 above by showing that, for $n > 2$, there are no other automorphisms of the cycle graph Γ_1 beyond the elements in the dihedral group, D_n. (Hint: Let $g \in G_1$. Suppose $g : 0 \mapsto i$. Then it must send 1 and $n-1$ to a neighbor of i.)

Exercise 2.7. Show that the automorphism group G_2 of the cycle digraph (directed graph) Γ_2 is the cyclic group of order n, C_n.

2.4.2 Relationship to convolution operators

We identify the vertices V of a circulant graph Γ having n vertices with the abelian group of integers mod n, $\mathbb{Z}/n\mathbb{Z}$. If $\mathbb{C}$ denotes the field of complex numbers, let

$$C^0(\Gamma,\mathbb{C}) = \{f \mid f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}\}.$$

This is a complex vector space which we can identify with the vector space $\mathbb{C}^n$ via the map $f \longmapsto (f(0), f(1), \dots, f(n-1))$.

Define *convolution* by

$$\begin{array}{ccc} C^0(\Gamma,\mathbb{C}) \times C^0(\Gamma,\mathbb{C}) & \to & C^0(\Gamma,\mathbb{C}) \\ (f,g) & \longmapsto & f * g, \end{array} \tag{2.10}$$

where

$$(f * g)(k) = \sum_{\ell \in \mathbb{Z}/n\mathbb{Z}} f(\ell) g(k-\ell).$$

This is commutative: $f * g = g * f$.

Let $\zeta = \zeta_n$ denote a primitive n^{th} root of unity in $\mathbb{C}$. Recall, for $g \in C^0(\Gamma,\mathbb{C})$, the *discrete Fourier transform* $\mathcal{F}_n$ of g is defined by

$$(\mathcal{F}_n g)(\lambda) = g^\wedge(\lambda) = \sum_{\ell \in \mathbb{Z}/n\mathbb{Z}} g(\ell)\zeta^{\ell\lambda}, \qquad \lambda \in \mathbb{Z}/n\mathbb{Z}.$$

Define the *inverse discrete Fourier transform* of G by

$$(\mathcal{F}_n^{-1} G)(\ell) = G^\vee(\ell) = \frac{1}{n} \sum_{\lambda \in \mathbb{Z}/n\mathbb{Z}} G(\lambda)\zeta^{-\ell\lambda}, \qquad \ell \in \mathbb{Z}/n\mathbb{Z}.$$

The following lemma states the basic and very useful fact that the Fourier transform of a convolution is the product of the Fourier transforms.

Lemma 2.4.3. *For any $f, g \in C^0(\Gamma,\mathbb{C})$, we have*

$$(f * g)^\wedge(\lambda) = f^\wedge(\lambda) g^\wedge(\lambda).$$

Proof.

$$\begin{aligned} (f * g)^\wedge(\lambda) &= \sum_{\ell \in \mathbb{Z}/n\mathbb{Z}} \sum_{k \in \mathbb{Z}/n\mathbb{Z}} f(k) g(\ell - k)\zeta^{\ell\lambda} \\ &= \sum_{k \in \mathbb{Z}/n\mathbb{Z}} f(k) \sum_{\ell \in \mathbb{Z}/n\mathbb{Z}} g(\ell - k)\zeta^{\ell\lambda} \\ &= \sum_{k \in \mathbb{Z}/n\mathbb{Z}} f(k)\zeta^{k\lambda} \sum_{\ell' \in \mathbb{Z}/n\mathbb{Z}} g(\ell')\zeta^{\ell'\lambda} \\ &= f^\wedge(\lambda) g^\wedge(\lambda). \end{aligned}$$

$\square$

Definition 2.4.4. For $h \in C^0(\Gamma, \mathbb{C})$, define $T_h : C^0(\Gamma, \mathbb{C}) \to C^0(\Gamma, \mathbb{C})$ by

$$T_h(f) = (h \otimes f^\wedge)^\vee,$$

where $\otimes$ denotes the componentwise product of two vectors:

$$(a_0, a_1, \ldots, a_{n-1}) \otimes (b_0, b_1, \ldots, b_{n-1}) = (a_0 b_0, a_1 b_1, \ldots, a_{n-1} b_{n-1}).$$

A linear transformation $M : C^0(\Gamma, \mathbb{C}) \to C^0(\Gamma, \mathbb{C})$ of the form $M = T_h$, for some $h \in C^0(\Gamma, \mathbb{C})$, is called a *Fourier multiplier operator*.

Let $\tau : C^0(\Gamma, \mathbb{C}) \to C^0(\Gamma, \mathbb{C})$ denote the *translation map*: $(\tau f)(x) = f(x + 1)$ (addition in $\mathbb{Z}/n\mathbb{Z}$). Note τ sends $(x_0, x_1, \ldots, x_{n-1})$ to $(x_1, x_2, \ldots, x_{n-1}, x_0)$. A transformation $T : C^0(\Gamma, \mathbb{C}) \to C^0(\Gamma, \mathbb{C})$ which commutes with τ is called *translation invariant* (or translation equivariant).

Define the *convolution operator associated to* g,

$$\mathcal{T}_g : C^0(\Gamma, \mathbb{C}) \to C^0(\Gamma, \mathbb{C}),$$

by $\mathcal{T}_g(f) = f * g$.

Exercise 2.8. Show that a Fourier multiplier operator $M = T_h$ is a linear transformation of the form $\mathcal{F}_n^{-1} D \mathcal{F}_n$, where D is an $n \times n$ diagonal matrix. (Hint: The diagonal elements of D may be taken to be values of h on the elements of $\mathbb{Z}/n\mathbb{Z}$.)

Recall a matrix A is *circulant* if and only if there is an n such that $A_{k,\ell} = A_{k+1 \,(\mathrm{mod}\ n),\ \ell+1 \,(\mathrm{mod}\ n)}$, for all $0 \le k \le n-1$, $0 \le \ell \le n-1$.

The following result appears as Theorem 2.19 of Frazier [Fr99]. It characterizes the Fourier multiplier operators on $C^0(\Gamma, \mathbb{C})$.

Theorem 2.4.5. *Let* $T : C^0(\Gamma, \mathbb{C}) \to C^0(\Gamma, \mathbb{C})$ *denote a linear operator. The following statements are equivalent:*

1. *T is translation invariant.*
2. *The matrix $[T]$ representing T in the standard basis is circulant.*
3. *T is a convolution operator.*
4. *T is a Fourier multiplier operator.*
5. *The matrix B representing T in the Fourier basis is diagonal.*

We will prove the equivalence of the first three items in the following three lemmas. We leave the proof of the equivalence of the remaining items as an exercise.

Lemma 2.4.6. *The convolution operator $\mathcal{T}_g$ is translation invariant. In other words, the diagram*

$$\begin{array}{ccc} C^0(\Gamma,\mathbb{C}) & \xrightarrow{\tau} & C^0(\Gamma,\mathbb{C}) \\ {\scriptstyle \mathcal{T}_g}\big\downarrow & & {\scriptstyle \mathcal{T}_g}\big\downarrow \\ C^0(\Gamma,\mathbb{C}) & \xrightarrow{\tau} & C^0(\Gamma,\mathbb{C}) \end{array}$$

commutes, for all $g \in C^0(\Gamma,\mathbb{C})$.

Proof. For $k \in \mathbb{Z}/n\mathbb{Z}$, we have

$$\begin{aligned} \mathcal{T}_g(\tau(f))(k) &= (\tau(f) * g)(k) \\ &= \sum_{\ell \in \mathbb{Z}/n\mathbb{Z}} (\tau(f))(\ell) g(k-\ell) \\ &= \sum_{\ell \in \mathbb{Z}/n\mathbb{Z}} f(\ell+1) g(k-\ell) \\ &= \sum_{\ell' \in \mathbb{Z}/n\mathbb{Z}} f(\ell') g(k-\ell'+1) \\ &= \tau(\mathcal{T}_g(f))(k). \end{aligned}$$

□

Lemma 2.4.7. *A linear transformation $T : C^0(\Gamma,\mathbb{C}) \to C^0(\Gamma,\mathbb{C})$ is translation invariant if and only if the matrix representing it in the standard basis is circulant.*

Proof. Since $T : C^0(\Gamma,\mathbb{C}) \to C^0(\Gamma,\mathbb{C})$ is linear, it is represented by an $n \times n$ matrix

$$Tf = A\vec{f},$$

where $\vec{f} = (f(0), f(1), \ldots, f(n-1))^t$. In other words,

$$T(f)(k) = \sum_{\ell \in \mathbb{Z}/n\mathbb{Z}} f(\ell) A_{k,\ell},$$

for $k \in \mathbb{Z}/n\mathbb{Z}$. For $k \in \mathbb{Z}/n\mathbb{Z}$, we have

$$\begin{aligned} T(\tau(f))(k) &= \textstyle\sum_{\ell \in \mathbb{Z}/n\mathbb{Z}} A_{k,\ell}(\tau(f))(\ell) \\ &= \textstyle\sum_{\ell \in \mathbb{Z}/n\mathbb{Z}} A_{k,\ell} f(\ell+1) \\ &= \textstyle\sum_{\ell' \in \mathbb{Z}/n\mathbb{Z}} A_{k,\ell'-1} f(\ell'), \end{aligned}$$

where the 2^{nd} subscript of $A_{i,j}$ is taken mod n. On the other hand,

$$\tau(T(f))(k) = \sum_{\ell \in \mathbb{Z}/n\mathbb{Z}} A_{k+1,\ell} f(\ell).$$

It follows that the linear transformation T is translation invariant if and only if its matrix A is circulant: $A_{k,\ell} = A_{k+1 \pmod{n},\ \ell+1 \pmod{n}}$, for all $0 \le k \le n-1$, $0 \le \ell \le n-1$. $\square$

Lemma 2.4.8. *A linear transformation $T : C^0(\Gamma, \mathbb{C}) \to C^0(\Gamma, \mathbb{C})$ is a convolution map if and only if the matrix representing it in the standard basis is circulant.*

Proof. It is not hard to see the connection between maps given by circulant matrices and convolution operators. Suppose that the matrix representing T in the standard basis is circulant. By Lemma 2.4.7, T is also translation invariant. Define $g \in C^0(\Gamma, \mathbb{C})$ by

$$g(-k) = A_{0,k \pmod{n}}, \qquad \text{for } k \in \mathbb{Z}/n\mathbb{Z}.$$

Then

$$T(f)(0) = \sum_{\ell \in \mathbb{Z}/n\mathbb{Z}} g(-\ell) f(\ell),$$

for all $f \in C^0(\Gamma, \mathbb{C})$. Replacing f by a translation $\pmod{n}$ (since T is translation invariant and g is periodic with period n) gives

$$T(f)(k) = \sum_{\ell \in \mathbb{Z}/N\mathbb{Z}} g(k-\ell) f(\ell),$$

for all $f \in C^0(\Gamma, \mathbb{C})$. In other words, T is a convolution map. This construction can be reversed: a convolution map corresponds to a circulant matrix transformation. $\square$

Exercise 2.9. For the circulant matrices T and maps f given below, verify that $\tau T(f) = T(\tau(f))$.

(a)

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } f = (1, 2, 3, 4, 5).$$

(b)

$$T = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 0 \end{pmatrix} \text{ and } f = (1, 2, 3, 4, 5).$$

Exercise 2.10. Prove that the last two items of Theorem 2.4.5 are equivalent to the first three items.

In graph-theoretic terms, a Fourier multiplier operator is a linear transformation of the form $B_\Gamma^{-1} D B_\Gamma$, where D is an $n \times n$ diagonal matrix and B_Γ is a matrix of eigenvectors of the Laplacian of Γ.

Question 2.3. *To what extent can the result above about Fourier multiplier operators be generalized to arbitrary graphs?*

2.5 Expander graphs

Let $\Gamma = (V, E)$ be a graph, let S be a subset of V, and let

$$\partial S = \{(u, v) \in E(G) \mid u \in S, v \in V \setminus S\}$$

denote the *edge boundary* of S. This is the cocycle associated to the partition $S \cup (V \setminus S)$.

The *edge expansion* $h(\Gamma)$ is defined as

$$h(\Gamma) = \min_{0<|S|\leq \frac{|V|}{2}} \frac{|\partial S|}{|S|}.$$

We say Γ has the *expander property* when each subset $S \subset V$ has a "relatively large" edge expansion, as specified in the definition below.

The second smallest eigenvalue of the Laplacian matrix Q of Γ, $\lambda_1(Q)$, is called the *spectral gap*. For example, if $\Gamma = (V, E)$ is a k-regular graph, then it is known that all eigenvalues λ of the adjacency matrix A of Γ satisfy $-k \leq \lambda \leq k$ (so the eigenvalues of Q satisfy $0 \leq \lambda \leq 2k$, by Lemma 2.2.9). Moreover, $\lambda = k$ is an eigenvalue of A (with the all 1's vector as an eigenvector). Therefore, in the regular case, the spectral gap measures the gap from the so-called trivial eigenvalue k of A to the next one (i.e., to $k - \lambda_1(Q)$, by Lemma 2.2.9).

Definition 2.5.1. Let $\Gamma = (V, E)$ be a k-regular graph.

We call

$$\gamma_\Gamma = \frac{\lambda_1(Q)}{k},$$

the *relative spectral gap* of Γ. We say Γ is a (k, r)-*expander* if, for each $S \subset V$,

$$\frac{|\partial S|}{|S|} \geq kr\left(1 - \frac{|S|}{|V|}\right).$$

The following result can be found in Roth [Ro06], §13.3.

Theorem 2.5.2. *If Γ is a k-regular graph then Γ is a (k, r)-expander for each r with $0 \leq r \leq 1 - \gamma_\Gamma$.*

Example 2.5.3. Consider the 4-regular Paley graph on 9 vertices $\Gamma = (V, E)$, depicted in Figure 1.12.

If $S = \{0, 1, 2\}$ then

$$\partial S = \{(0, a+1), (0, 2a+2), (1, a+2), (1, 2a), (2, a), (2, 2a+1)\}.$$

The edge expansion is 2. Recall from Example 2.2.17 that the Laplacian spectrum is $\{0, 3, 3, 3, 3, 6, 6, 6, 6\}$, so $\lambda_1(Q) = 3$ and $\gamma_\Gamma = 3/4$. By the above theorem, the inequality

$$h(\Gamma) \geq kr(1 - |S|/|V|),$$

for all nonempty subsets $S \subset V$, holds for $0 \leq r \leq 1 - \gamma_\Gamma$. In this case, $k = 4$, and so $kr(1 - |S|/|V|) \leq 32r/9$.

Example 2.5.4. Consider the 6-regular graph on 16 vertices $\Gamma = (V, E)$, depicted in Figure 2.7 (Example 2.6.3 below). The eigenvalues of the adjacency matrix are

$$6, 2, 2, 2, 2, 2, 2, -2, -2, -2, -2, -2, -2, -2, -2, -2.$$

According to Sage, the edge expansion is $h(\Gamma) = 7/2$. The Laplacian spectrum is $\{0, 4, 4, 4, 4, 4, 4, 8, 8, 8, 8, 8, 8, 8, 8, 8\}$, so $\lambda_1(Q) = 4$ and $\gamma_\Gamma = 4/6 = 2/3$. By the above theorem, the inequality

$$h(\Gamma) \geq kr(1 - |S|/|V|),$$

for all nonempty subsets $S \subset V$, holds for $0 \leq r \leq 1 - \gamma_\Gamma$. In this case, $k = 6$, and so $kr(1 - |S|/|V|) \leq 15/8$.

Let $\Gamma = (V, E)$ be a connected k-regular graph with $n = |V|$ vertices, and let

$$k = \lambda_0(A) \geq \lambda_1(A) \geq \cdots \geq \lambda_{n-1}(A)$$

be the eigenvalues of its adjacency matrix $A = A_\Gamma$. If there exists $\lambda_i = \lambda_i(A)$ with $|\lambda_i| < k$, define

$$\lambda(\Gamma) = \max_{|\lambda_i| < k} |\lambda_i|.$$

We call Γ a *Ramanujan graph* if

$$\lambda(\Gamma) \leq 2\sqrt{k-1}.$$

Example 2.5.5. The above Example 2.5.4 is a 6-regular graph Γ satisfying

$$\lambda(\Gamma) = 2 \leq 2\sqrt{k-1} = 2\sqrt{5}.$$

Therefore, it is an example of a Ramanujan graph.

The excellent book by Davidoff, Sarnak, and Valette [DSV03] contains a detailed construction of an infinite family of Ramanujan graphs. For instance, if p, q are odd primes with $q > 2\sqrt{p}$ and p is not a quadratic residue $\pmod{q}$, they construct a symmetric generating set $S_{p,q}$ of $PGL(2,q)$, such that the Cayley graph $Cay(PGL(2,q), S_{p,q})$ (this notation is defined at the beginning of §2.6 below) is a $(p+1)$-regular Ramanujan graph. The details of the construction require more number-theoretic background than we have introduced here. We refer to [DSV03] for details.

2.6 Cayley graphs

Let G be a finite multiplicative group. Let $S \subset G$ be a subset which satisfies $S = S^{-1}$ and $1 \notin S$. The *Cayley graph* of (G, S) is the graph $\Gamma = Cay(G, S)$ whose vertices are $V = G$ and whose edges E are defined by those pairs (g_1, g_2) such that $g_2 g_1^{-1} \in S$.

This is a k-regular graph having degree $k = |S|$. By Lemma 2.2.1, the eigenvalues $\lambda_i(Q)$ of the Laplacian Q are related to the eigenvalues $\lambda_i(A)$ of the (unweighted) adjacency matrix by

$$\lambda_i(Q) = k - \lambda_i(A).$$

Exercise 2.11. Show that a Cayley graph $\Gamma = Cay(G, S)$ is regular with degree $|S|$.

Exercise 2.12. Show that a Cayley graph $\Gamma = Cay(G, S)$ is connected if and only if S generates G.

Example 2.6.1. The Sage commands to produce the types of Cayley graphs we are interested in are a bit tricky. You will want to be sure to select the "simple" option and to select symmetric generating sets.

Sage

```
sage: G = AdditiveAbelianGroup([6])
sage: Gp = G.permutation_group()
sage: g1, g2 = Gp.gens()
sage: S = [(g1*g2)^2,(g1*g2)^(-2),(g1*g2)^3]; S
[(3,5,4), (3,4,5), (1,2)]
sage: A = Gp.cayley_graph(generators=S,simple=True).adjacency_matrix()
sage: Gamma1 = Graph(A)
sage: AG1 = Gamma1.automorphism_group()
sage: AG1.cardinality()
12
```

Sage

```
sage: G = SymmetricGroup(3)
sage: S = G.gens()+[G.gens()[0]^(-1)]; S
[(1,2,3), (1,2), (1,3,2)]
sage: A = G.cayley_graph(generators=S,simple=True).adjacency_matrix()
sage: Gamma2 = Graph(A)
sage: AG2 = Gamma2.automorphism_group()
sage: AG2.cardinality()
12
```

These two Cayley graphs are isomorphic. With these options selected, we see that the graph has the desired symmetry. More of this example will be given later (e.g., Example 4.7.8 below).

There is also an edge-weighted analog of this definition.

For $g \in G$, the *conjugacy class* of g is the subset

$$Cl_G(g) = \{x^{-1}gx \mid x \in G\}.$$

The set of conjugacy classes will be denoted G_*. A function $f : G \to \mathbb{C}$ is called a *class function* if it is constant on conjugacy classes. In other words, f is a class function if and only if the restriction $f|_\gamma$ is a constant (possibly depending on γ), for each $\gamma \in G_*$.

Let $\alpha : G \to \mathbb{Z}$ be a given class function. The *edge-weighted Cayley graph* associated to (G, S, α), is the graph $\Gamma = Cay(G, S)$, where edge (g_1, g_2) has weight $\alpha(g_2 g_1^{-1})$. We denote this graph by $Cay(G, S, \alpha)$. By convention, if $\alpha(g_2 g_1^{-1}) = 0$ then we say that the 0-weighted edge (g_1, g_2) does not exist.

Let $n = |G|$. A subgroup H of S_n *acts* on Γ if and only if it is an automorphism group of the unweighted graph Γ^* and each graph automorphism $h \in H$ also preserves the edge weights. In particular, the H-orbit of any edge of Γ consists of edges which have the same edge weight. The set of such actions on Γ forms a subgroup of the automorphism group of Γ^*.

2.6.1 Cayley graphs on abelian groups

The eigenvalues of the Cayley graph in the abelian case are easy to determine.

Proposition 2.6.2. *Let G be a finite abelian group written multiplicatively, let $\chi : G \to \mathbb{C}^\times$ be a homomorphism of G, and let $S \subset G$ be a symmetric set. Let A be the adjacency matrix of the Cayley graph $\Gamma = Cay(G, S)$. Consider the vector $x \in \mathbb{C}^G$ such that $x_a = \chi(a)$, where $\mathbb{C}^G$ denotes the vector space of complex-valued functions on G. Then x is an eigenvector of A, with eigenvalue $\chi(S) = \sum_{s \in S} \chi(s)$.*

Proof. This follows from the proof in the non-abelian case, given in the next section. $\square$

Example 2.6.3. Consider the Cayley graph Γ of $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, with generator set $S = \{\pm(0,1), \pm(1,0), \pm(1,1)\}$. This has Laplacian matrix

$$\begin{pmatrix}
6 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\
-1 & 6 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & -1 & 6 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
-1 & 0 & -1 & 6 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\
-1 & 0 & 0 & -1 & 6 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & -1 & 6 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 & -1 & 6 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & -1 & 0 & -1 & 6 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 6 & -1 & 0 & -1 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 6 & -1 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 6 & -1 & 0 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & 6 & -1 & 0 & 0 & -1 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 6 & -1 & 0 & -1 \\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 6 & -1 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 6 & -1 \\
-1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & 6
\end{pmatrix}$$

and characteristic polynomial

$$(x-6) \cdot (x-2)^6 \cdot (x+2)^9.$$

The graph Γ is depicted in Figure 2.7.

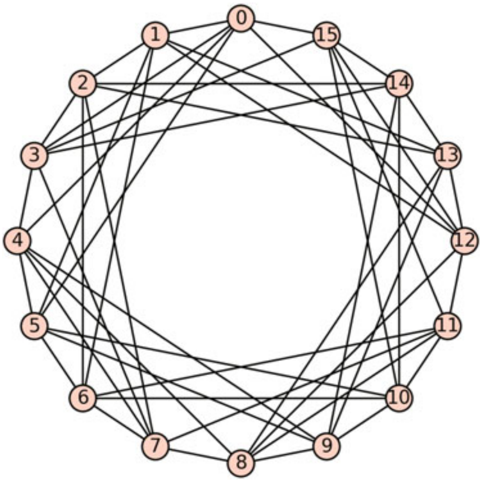

Figure 2.7: The Cayley graph of $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, with generator set $S = \{\pm(0,1), \pm(1,0), \pm(1,1)\}$, created using Sage.

Sage

```
sage: G = AdditiveAbelianGroup([4,4])
sage: GP = G.permutation_group()
sage: g0 = GP.gens()[0]
sage: g1 = GP.gens()[1]
sage: S = [g0, g1, g0^(-1), g1^(-1), (g0*g1)^(-1), g0*g1]
sage: Gamma = GP.cayley_graph(side='left', generators = S)
sage: A = Gamma.adjacency_matrix()
sage: Gamma1 = Graph(A, format = "adjacency_matrix")
sage: Gamma1.show(layout="circular", dpi = 300)
sage: Gamma1.characteristic_polynomial().factor()
(x - 6) * (x - 2)^6 * (x + 2)^9
```

2.6.2 Cayley graphs for non-abelian groups

Let $\alpha : G \to \mathbb{Z}$ be a given class function. Let $S \subset G$ be a subset which generates G satisfying $S = S^{-1}$ and $1 \notin S$. Let Γ denote the edge-weighted Cayley graph associated to (G, S, α), where edge (g_1, g_2) has weight $\alpha(g_2 g_1^{-1})$. In particular, we assume α is supported on S. In the notation above, $\Gamma = Cay(G, S, \alpha)$. We assume that the (weighted) adjacency matrix of Γ is the $|G| \times |G|$ matrix $A = (a_{g,h})$, where

$$a_{g,h} = \alpha(gh^{-1}).$$

Let G^* denote a complete set of inequivalent representations of G. We can write

$$\rho_i : G \to Aut(V_i),$$

where $V_i \cong \mathbb{C}^{d_i}$ and d_i is the degree of ρ_i, for $i = 1, \ldots, |G^*|$.

Example 2.6.4. While this is a very long example, we hope it will be useful to illustrate the ideas in the theorem below.

Let $G{=}D_6$ denote the dihedral group of order 12, written in the following order:

$$g_1 = 1, g_2 = (2,6)(3,5), g_3 = (1,2)(3,6)(4,5), g_4 = (1,2,3,4,5,6),$$
$$g_5 = (1,3)(4,6), g_6 = (1,3,5)(2,4,6), g_7 = (1,4)(2,3)(5,6),$$
$$g_8 = (1,4)(2,5)(3,6), g_9 = (1,5)(2,4), g_{10} = (1,5,3)(2,6,4),$$
$$g_{11} = (1,6,5,4,3,2), g_{12} = (1,6)(2,5)(3,4).$$

Let

$$S = \{g_4, g_{11}, g_3, g_7, g_{12}\}$$

and note S generates G, and is conjugation-invariant and closed under taking inverses. The conjugacy classes $cl_G(x)$ of G are ordered as follows:

$$cl_G(g_1), cl_G(g_2), cl_G(g_3), cl_G(g_4), cl_G(g_6), cl_G(g_8).$$

Sage

```
sage: G = DihedralGroup(6); G
Dihedral group of order 12 as a permutation group
sage: g1 = G.gens()[0]
sage: g2 = G.gens()[0]^(-1)
sage: g3 = G.gens()[1]
sage: Cg3 = [x^(-1)*g3*x for x in G]
sage: g4 = Cg3[1]; g5 = Cg3[2]
sage: S = [g1, g2, g3, g4, g5]; S
[(1,2,3,4,5,6), (1,6,5,4,3,2), (1,6)(2,5)(3,4), (1,2)(3,6)(4,5), (1,4)(2,3)(5,6)]
```

The character table of G is

	g_1	g_2	g_3	g_4	g_6	g_8
χ_1	1	1	1	1	1	1
χ_2	1	−1	−1	1	1	1
χ_3	1	−1	1	−1	1	−1
χ_4	1	1	−1	−1	1	−1
χ_6	2	0	0	1	−1	−2
χ_8	2	0	0	−1	−1	2

,

where χ_i is the i-th irreducible character of G (in the ordering given by Sage and GAP).

Define the class function $\alpha : G \to \mathbb{Z}$ by

$$\alpha(x) = \begin{cases} 1, & x \in \{(1,2,3,4,5,6),(1,6,5,4,3,2)\}, \\ 2, & x \in \{(1,2)(3,6)(4,5),(1,2)(3,6)(4,5),(1,4)(2,3)(5,6)\}, \\ 0, & \text{otherwise.} \end{cases}$$

The associated (weighted) adjacency matrix is

$$A = \begin{pmatrix}
0 & 0 & 2 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 1 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 1 \\
2 & 1 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 2 & 0 & 0 \\
1 & 2 & 0 & 0 & 2 & 1 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & 2 & 0 \\
0 & 0 & 2 & 1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 2 \\
2 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 \\
0 & 2 & 0 & 0 & 2 & 1 & 0 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & 2 & 1 \\
0 & 0 & 2 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 1 & 2 \\
1 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 2 & 0 & 0
\end{pmatrix}.$$

Sage

```
sage: def alpha(x):
          if x==S[0] or x==S[1]:
             return 1
          if x==S[2] or x==S[3] or x==S[4]:
             return 2
          return 0
sage: A = [[alpha(x*y^(-1)) for x in G] for y in G]; matrix(A)
```

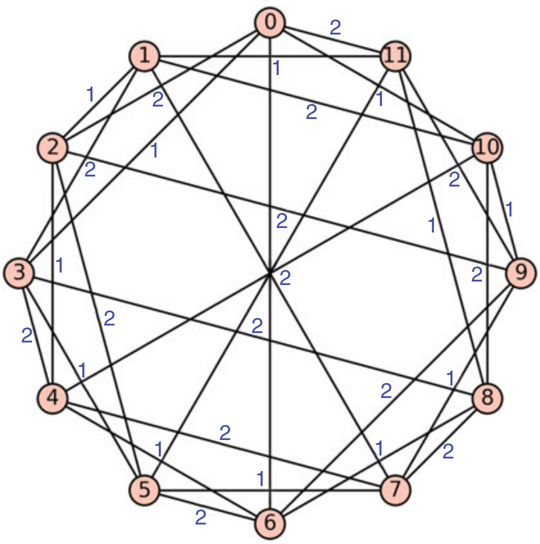

Figure 2.8: The undirected edge-weighted Cayley graph of D_6 with respect to S.

```
[0 0 2 1 0 0 2 0 0 0 1 2]
[0 0 1 2 0 0 0 2 0 0 2 1]
[2 1 0 0 1 2 0 0 0 2 0 0]
[1 2 0 0 2 1 0 0 2 0 0 0]
[0 0 1 2 0 0 1 2 0 0 2 0]
[0 0 2 1 0 0 2 1 0 0 0 2]
[2 0 0 0 1 2 0 0 1 2 0 0]
[0 2 0 0 2 1 0 0 2 1 0 0]
[0 0 0 2 0 0 1 2 0 0 2 1]
[0 0 2 0 0 0 2 1 0 0 1 2]
[1 2 0 0 2 0 0 0 2 1 0 0]
[2 1 0 0 0 2 0 0 1 2 0 0]
sage: Gamma = Graph(matrix(A), format = "adjacency_matrix", weighted=True)
sage: Gamma.show(layout="circular", dpi = 300, edge_labels=True)
sage: Gamma.automorphism_group(edge_labels=True).order()
144
sage: Gamma.automorphism_group().order()
1440
```

The weighted Cayley graph of (G, S, α) is shown in Figure 2.8.
The eigenvalues and eigenvectors of A are

$$\lambda_0 = 8, x_0 = (1,1,1,1,1,1,1,1,1,1,1,1),$$

$$\lambda_1 = -4, x_1 = (1,-1,-1,1,-1,1,-1,1,-1,1,1,-1),$$

$$\lambda_2 = 4, x_2 = (1,-1,1,-1,-1,1,1,-1,-1,1,-1,1),$$

$$\lambda_3 = -8, x_3 = (1,1,-1,-1,1,1,-1,-1,1,1,-1,-1),$$

each occurring with multiplicity 1. Note that, for $i = 0, 1, 2, 3$, we have

$$x_i = (\chi_i(g_1), \dots, \chi_i(g_{12})),$$

where χ_i is the i-th character in the character table of G. Here they are in Sage [2]:

Sage

```
sage: A*x1
(8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8)
sage: x2 = vector([1,-1,-1,1,-1,1,-1,1,-1,1,1,-1])
sage: A*x2
(-4, 4, 4, -4, 4, -4, 4, -4, 4, -4, -4, 4)
sage: x3 = vector([1,-1,1,-1,-1,1,1,-1,-1,1,-1,1])
sage: A*x3
(4, -4, 4, -4, -4, 4, 4, -4, -4, 4, -4, 4)
sage: x4 = vector([1,1,-1,-1,1,1,-1,-1,1,1,-1,1])
sage: A*x4
(-4, -6, 8, 8, -8, -4, 8, 8, -6, -4, 8, 8)
sage: x4 = vector([1,1,-1,-1,1,1,-1,-1,1,1,-1,-1])
sage: A*x4
(-8, -8, 8, 8, -8, -8, 8, 8, -8, -8, 8, 8)
sage: x5 = vector([2,0,0,1,0,-1,0,-2,0,-1,1,0])
```

[2]Some care must be taken to record the entries of x_i consistent with the way the elements of G are listed.

```
sage: A*x5
(2, 0, 0, 1, 0, -1, 0, -2, 0, -1, 1, 0)
```

Indeed, all these characters have degree 1. The remaining eigenvalues each occur with multiplicity 4. The eigenvalue $\lambda_4 = 1$, has eigenspace

$$\begin{aligned} E_{\lambda_4} = \mathrm{Span}((&1,0,0,0,0,-1,0,-1,0,0,1,0), \\ &(0,1,0,0,-1,0,-1,0,0,0,0,1), \\ &(0,0,1,0,1,0,0,0,-1,0,0,-1), \\ &(0,0,0,1,0,1,0,0,0,-1,-1,0)). \end{aligned}$$

It is easy to see that

$$(\chi_4(g_1), \ldots, \chi_4(g_{12})) = (2,0,0,1,0,-1,0,-2,0,-1,1,0)$$

is an element of E_{λ_4}. The eigenvalue $\lambda_5 = -1$, has eigenspace

$$\begin{aligned} E_{\lambda_5} = \mathrm{Span}((&1,0,0,0,0,-1,0,1,0,0,-1,0), \\ &(0,1,0,0,-1,0,1,0,0,0,0,-1), \\ &(0,0,1,0,-1,0,0,0,1,0,0,-1), \\ &(0,0,0,1,0,-1,0,0,0,1,-1,0)). \end{aligned}$$

It is easy to see that

$$(\chi_5(g_1), \ldots, \chi_5(g_{12})) = (2,0,0,-1,0,-1,0,2,0,-1,-1,0)$$

is an element of E_{λ_5}.

The following well-known result describes a way to generalize the above example. Roughly speaking, it says that if Γ is a weighted Cayley graph attached to a group G and if the weight function of Γ is given by a class function of G, then the spectrum of Γ is determined by the representations of G.

Theorem 2.6.5. *Let α be a class function, and let $\Gamma = Cay(G, S, \alpha)$, as above (so, in particular, α is supported on S). If $n = |G|$, write*

$$G = \{g_1 = 1, g_2, \ldots, g_n\}.$$

Each eigenvector of the adjacency matrix A has the form

$$(\chi(g_1), \ldots, \chi(g_n)),$$

where $\chi = \mathrm{tr}(\rho)$*, for some* $\rho \in G^*$*, with eigenvalue*

$$\lambda = \lambda_\rho = \frac{1}{d_\rho} \sum_{s \in S} \alpha(s) \mathrm{tr}(\rho)(s),$$

where d_ρ *is the degree of* ρ*. Moreover, the multiplicity of* λ *is* $\chi(1)^2$*.*

The proof below follows Brouwer and Haemers [BH11], §6.3, and Kaski [KA02], §5, and is included for the reader's convenience. See also Rockmore, Kostelec, Hordijk, and Stadler [RKHS02].

Proof. Suppose Γ has vertex set V, and $W = C^0(\Gamma, \mathbb{R}) \cong \mathbb{R}^V$ is the $\mathbb{R}$-vector space spanned by the vertices of Γ. Part of this proof applies to any matrix which commutes with the action of G, such as the (unweighted) adjacency matrix A^* or the Laplacian Q. By Schur's Lemma[3], A^* acts as a scalar on each irreducible G-invariant subspace of W. In other words, the irreducible G-invariant subspaces are eigenspaces of A^*. If A^* acts like θI on the irreducible G-invariant subspace $U = W_\chi$ with character χ, then $tr(A^* g|_U) = \theta\chi(g)$.

Since S is a union of conjugacy classes of G, the weighted adjacency matrix A commutes with the elements of G, and the previous discussion applies. The regular representation of G decomposes into a direct sum of irreducible subspaces, where for each irreducible character χ there are $\chi(1)$ copies of W_χ. To be explicit, W_χ is spanned by $v_\chi = (\chi(g_1), \ldots, \chi(g_n))$ and all its images under the G-action. On each copy A acts like θI, for some $\theta \in \mathbb{R}$, and $\dim(W_\chi) = \chi(1)$, so θ has multiplicity $\chi(1)^2$. We saw that

$$tr(Ag|_{W_\chi}) = \theta\chi(g).$$

The first entry of Av_χ is equal to $\sum_{s \in S} \alpha(s)\chi(s)$. Since this is also $\theta\chi(1)$, we have

$$\theta = \frac{1}{\chi(1)} \sum_{s \in S} \alpha(s)\chi(s) = \frac{1}{\chi(1)} tr(A|_{W_\chi}).$$

□

[3]If π is an irreducible n-dimensional representation of G and if $B \in GL(n, \mathbb{C})$ commutes with all matrices $\pi(g)$, $g \in G$, then B is a scalar matrix. A proof can be found in many textbooks on abstract algebra, e.g., [DF99], page 337.

Example 2.6.6. This is an extension of (the already very long) Example 2.6.4.

Let G and Γ be as in Example 2.6.4. The unweighted adjacency matrix is

$$A_0 = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Sage

```
sage: A0 = Gamma.adjacency_matrix()
sage: A0*v1==(5)*v1
True
sage: A0*v2==(-1)*v2
True
sage: A0*v3==(1)*v3
True
sage: A0*v4==(-5)*v4
True
sage: A0*v5==(1)*v5
True
sage: A0*v6==(-1)*v6
True
```

In other words, in this example, Sage helps us verify that, even if we use the unweighted adjacency matrix, the conclusion of the above Theorem 2.6.5 still holds.

2.7 Additive Cayley graphs

An *additive Cayley graph* (or *Cayley sum graph*) Γ with sum set S in a finite abelian group G has vertex set $V_\Gamma = G$, and two elements $g, h \in G$ are adjacent (i.e., connected by a edge) if and only if $g + h \in S$.

Let G^* denote the *dual group* of G, that is, the multiplicative group of multiplicative characters $\chi : G \to \mathbb{C}^\times$. Let $G^*_\mathbb{R}$ denote the subgroup of real-valued characters $\chi : G \to \mathbb{R}^\times$. For convenience of notation, we fix an indexing of the elements of these groups,

$$G = \{g_1, \ldots, g_n\},$$

and

$$G^* = \{\chi_1, \ldots, \chi_n\}.$$

Example 2.7.1. Consider the group $G = \mathbb{Z}/10\mathbb{Z}$ and the set $S = \{3, 5, 7\}$. The adjacency matrix of the associated additive Cayley graph is

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

and the graph itself is depicted in Figure 2.9.

If $\chi(S) = \sum_{s \in S} \chi(s)$, the values of $\chi(S)$, for $\chi \in G^*$, are listed as follows:

$$\begin{gathered} \chi_1(S) = 3,\ \chi_2(S) = e^{\frac{7}{5} i\pi} + e^{\frac{3}{5} i\pi} - 1,\ \chi_3(S) = e^{\frac{14}{5} i\pi} + e^{\frac{6}{5} i\pi} + 1, \\ \chi_4(S) = e^{\frac{21}{5} i\pi} + e^{\frac{9}{5} i\pi} - 1,\ \chi_5(S) = e^{\frac{28}{5} i\pi} + e^{\frac{12}{5} i\pi} + 1,\ \chi_6(S) = -3, \\ \chi_7(S) = e^{\frac{42}{5} i\pi} + e^{\frac{18}{5} i\pi} + 1,\ \chi_8(S) = e^{\frac{49}{5} i\pi} + e^{\frac{21}{5} i\pi} - 1, \\ \chi_9(S) = e^{\frac{56}{5} i\pi} + e^{\frac{24}{5} i\pi} + 1,\ \chi_{10}(S) = e^{\frac{63}{5} i\pi} + e^{\frac{27}{5} i\pi} - 1. \end{gathered}$$

The following result can be found in Brouwer and Haemers [BH11].

Proposition 2.7.2. *Let Γ be the additive Cayley graph with sum set S in the finite abelian group G. The spectrum of Γ consists of*

$$\{\chi(S) \mid \chi \in G^*_\mathbb{R}\} \cup \{\pm|\chi(S)| \mid \chi \in G^* - G^*_\mathbb{R}\}.$$

The eigenvector of $\chi(S)$, for $\chi \in G^*_\mathbb{R}$, is

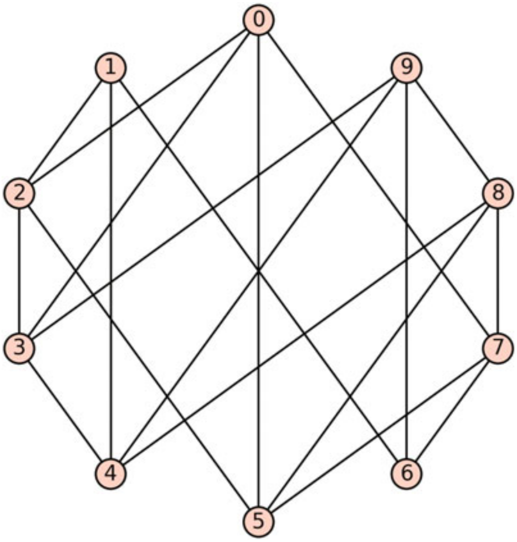

Figure 2.9: The undirected additive Cayley graph of $\mathbb{Z}/10\mathbb{Z}$ with respect to S.

$$x_\chi = (\chi(g_1), \dots, \chi(g_n)).$$

Pick $\alpha = \alpha_{S,\chi} \in \mathbb{C}^\times$ *so that* $|\chi(S)| = \alpha^2\chi(S)$. *The eigenvector of* $|\chi(S)|$, *for* $\chi \in G^* - G^*_{\mathbb{R}}$, *is*

$$x_{\chi,+} = (Re\,(\alpha\chi(g_1)), \dots, Re\,(\alpha\chi(g_n))).$$

The eigenvector of $-|\chi(S)|$, *for* $\chi \in G^* - G^*_{\mathbb{R}}$, *is*

$$x_{\chi,-} = (Im\,(\alpha\chi(g_1)), \dots, Im\,(\alpha\chi(g_n))).$$

Proof. For $x, y \in G$, define $x \sim y$ if and only if $x + y \in S$. If the vertices of Γ are denoted $1, \dots, n$, then the (unweighted) adjacency matrix A acts on $\mathbb{R}^n$ by sending a vector $x = (x_1, \dots, x_n)$ to $x' = (x'_1, \dots, x'_n)$, where

$$x'_j = \sum_{(i,j)\in E_\Gamma} x_i.$$

If $\chi : G \to \mathbb{C}^\times$ is a character of G, then

$$\sum_{y\sim x} \chi(y) = \sum_{s\in S} \chi(s-x) = \left(\sum_{s\in S} \chi(s)\right) \chi(-x) = \chi(S)\overline{\chi(x)}.$$

Since Γ is undirected, the spectrum is real.

If χ is a real character, then the computation displayed above tells us that $\mathbf{v}_\chi$ is an eigenvector of A with eigenvalue $\chi(S)$. If χ is nonreal, then $x_{\chi,+}$ and $x_{\chi,-}$ are eigenvectors with eigenvalues $|\chi(S)|$ and $-|\chi(S)|$, respectively. Indeed, the j-th component of $Ax_{\chi,+}$ is

$$\begin{aligned}
(Ax_{\chi,+})_j &= \textstyle\sum_{(i,j)\in E_\Gamma} Re(\alpha\chi(g_i)) \\
&= Re(\alpha \textstyle\sum_{(i,j)\in E_\Gamma} \chi(g_i)) \\
&= Re(\alpha \textstyle\sum_{s\in S} \chi(s-g_j)) \\
&= Re(\alpha^{-1}\overline{\chi(g_j)}\alpha^2\chi(S)) \\
&= |\chi(S)| \cdot Re(\overline{\alpha\chi(g_j)}).
\end{aligned}$$

□

An analogous definition of an additive Cayley graph associated to a subset $S \subset G$ is a graph Γ which has vertex set $V_\Gamma = G$, and two elements $g, h \in G$ are adjacent (i.e., connected by a edge) if and only if $g - h \in S$. In this case, we also require that S is symmetric: $S = -S$.

The next section is devoted to a class of graphs of this type.

2.7.1 Cayley graphs and p-ary functions

This section describes a type of Cayley graph attached to a p-ary function. More details are in Chapter 6 and in Celerier, Joyner, Melles, Phillips, and Walsh [CJMPW15].

Fix $n \geq 1$ and let $V = GF(p)^n$, where p is a prime.

If $f\colon V \to GF(p)$, then we let $f_{\mathbb{C}}\colon V \to \mathbb{C}$ be the function whose values are those of f but regarded as integers (i.e., we select the congruence class residue representative in the interval $\{0, 1, \ldots, p-1\}$). We sometimes abuse notation and often write f in place of $f_{\mathbb{C}}$.

Let f be a $GF(p)$-valued function on V such that $f(0) = 0$.

The *Cayley graph of f* is defined to be the edge-weighted directed graph

$$\Gamma_f = (GF(p)^n, E_f), \tag{2.11}$$

whose vertex set is $V = V(\Gamma_f) = GF(p)^n$ and whose set of edges is defined by

$$E_f = \{(u,v) \in GF(p)^n \mid f(u-v) \neq 0\},$$

where the edge $(u,v) \in E_f$ has weight $f(u-v)$. However, if f is even then we can (and do) regard Γ_f as a weighted (undirected) graph. We assume from this point on that f is even.

The adjacency matrix $A = A_f$ is the matrix whose entries are

$$A_{i,j} = f_{\mathbb{C}}(\eta(i) - \eta(j)), \tag{2.12}$$

where $\eta(k)$ is the p-ary representation as in (6.5). Ignoring edge weights, we let

$$A^*_{i,j} = \begin{cases} 1, & f_{\mathbb{C}}(\eta(i) - \eta(j)) \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.13}$$

Let $\mathrm{supp}(f) = \{v \in V \mid f(v) \neq 0\}$ be the *support* of f, let

$$\omega_f = |\mathrm{supp}(f)|,$$

and let

$$\sigma_f = \sum_{v \in V} f_{\mathbb{C}}(v).$$

Clearly, the vertices in Γ_f connected to $0 \in V$ are in natural bijection with the elements of $\mathrm{supp}(f)$.

Recall that, given a graph Γ and its adjacency matrix A, the spectrum

$$\sigma(\Gamma) = \{\lambda_1, \lambda_2, \ldots, \lambda_N\},$$

is the multi-set of real eigenvalues of A. When Γ is the Cayley graph of a p-ary function on $GF(p)^n$, we have $N = p^n$. Following a standard convention, we index the elements $\lambda_i = \lambda_i(A)$ of the spectrum in such a way that they are monotonically increasing. Because Γ_f is regular, the row sums of A are all σ_f, hence the all 1's vector is an eigenvector of A with eigenvalue σ_f. We will see later (Corollary 2.7.6) that $\lambda_N(A) = \sigma_f$.

Let Δ denote the identity matrix multiplied by σ_f. The Laplacian of Γ_f is the matrix $Q = \Delta - A$.

Lemma 2.7.3. *Assume f is even. As an edge-weighted graph, Γ_f is connected if and only if $\lambda_{N-1}(A) < \lambda_N(A) = \sigma_f$, where A is the adjacency matrix of Equation (2.12). If we ignore edge weights, then Γ_f is connected if and only if $\lambda_{N-1}(A^*) < \lambda_N(A^*) = \omega_f$, where A^* is the unweighted adjacency matrix in (2.13).*

Proof. We only prove the statement for the edge-weighted case.

Note that for $i = 1, \ldots, N$, $\lambda_i(Q) = \sigma_f - \lambda_{N-i+1}(A)$, since $\det(Q - \lambda I) = \det(\sigma_f I - A - \lambda I) = (-1)^n \det(A - (\sigma_f - \lambda)I)$. Thus, $\lambda_i(Q) \geq 0$, for all i. By a theorem of Fiedler [Fi73], $\lambda_2(Q) > 0$ if and only if Γ_f is connected. But $\lambda_2(Q) > 0$ is equivalent to $\sigma_f - \lambda_{N-1}(A) > 0$. □

Recall a circulant matrix is a square matrix where each row vector is a cyclic shift one element to the right relative to the preceding row vector.

Our Fourier transform matrix F is not circulant, but is "block circulant." Like circulant matrices, it has the property that $\mathbf{v}_a = (\zeta^{-\langle a,x\rangle} \mid x \in V)$ is an eigenvector with eigenvalue $\lambda_a = \hat{f}(-a)$ (something related to a value of the Hadamard transform of f). Thus, the proposition below shows that it "morally" behaves like a circulant matrix in some ways.

Proposition 2.7.4. *The eigenvalues $\lambda_a = \hat{f}(-a)$ of this matrix F are values of the Fourier transform of the function $f_{\mathbb{C}}$,*

$$\hat{f}(y) = \sum_{x\in V} f_{\mathbb{C}}(x)\zeta^{-\langle x,y\rangle},$$

and the eigenvectors are the vectors of p-th roots of unity,

$$\mathbf{v}_a = (\zeta^{-\langle a,x\rangle} \mid x \in V).$$

Proof. In $F = (F_{i,j})$, we have $F_{i,j} = f_{\mathbb{C}}(\eta(i) - \eta(j))$ for $i, j \in \{0, 1, \ldots, p^n - 1\}$. For each $a \in GF(p)^n$, let

$$\mathbf{v}_a = (\zeta^{-\langle a,\eta(i)\rangle} \mid i \in \{0, 1, \ldots, p^n - 1\}).$$

Then

$$F\mathbf{v}_a = (\sum_{y\in V} f_{\mathbb{C}}(x - y)\zeta^{-\langle a,y\rangle} \mid x \in V).$$

The entry in the i-th coordinate, where $x = \eta(i)$ is given by

$$\begin{aligned}
\textstyle\sum_{y\in V} f_{\mathbb{C}}(x - y)\zeta^{-\langle a,y\rangle} &= \textstyle\sum_{y\in V} f_{\mathbb{C}}(-y)\zeta^{-\langle a,y+x\rangle} \\
&= \zeta^{-\langle a,x\rangle} \textstyle\sum_{y\in V} f_{\mathbb{C}}(-y)\zeta^{-\langle a,y\rangle} \\
&= \zeta^{-\langle a,x\rangle} \textstyle\sum_{y\in V} f_{\mathbb{C}}(y)\zeta^{\langle a,y\rangle} \\
&= \zeta^{-\langle a,x\rangle} \hat{f}(-a).
\end{aligned}$$

Therefore, the coordinates of the vector $F\mathbf{v}_a$ are the same as those of $\mathbf{v}_a$, up to a scalar factor. Thus $\lambda_a = \hat{f}(-a)$ is an eigenvalue and $\mathbf{v}_a = (\zeta^{-\langle a,x\rangle} \mid x \in V)$ is an eigenvector. □

Corollary 2.7.5. *The matrix F is invertible if and only if none of the values of the Fourier transform of $f_{\mathbb{C}}$ vanish.*

Corollary 2.7.6. *The spectrum of the graph Γ_f is precisely the set of values of the Fourier transform of $f_{\mathbb{C}}$.*

Example 2.7.7. We take $V = GF(3)^2$ and consider an even function $f : V \to GF(3)$ given by

$$f(x_0, x_1) = -x_0^2 x_1^2 + x_0^2 + x_0 x_1 - x_1^2.$$

Its Cayley graph Γ_f has weighted adjacency matrix

$$A_w = \begin{pmatrix} 0 & 1 & 1 & 2 & 0 & 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 1 & 2 & 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 0 & 1 & 2 & 1 & 0 & 2 \\ 2 & 1 & 0 & 0 & 1 & 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 1 & 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 2 & 1 & 1 & 0 & 0 & 1 & 2 \\ 2 & 0 & 1 & 2 & 1 & 0 & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 & 2 & 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 & 2 & 1 & 1 & 0 \end{pmatrix}.$$

As a weighted graph, the adjacency spectrum is

$$8, 2, 2, -1, -1, -1, -1, -4, -4,$$

while the Laplacian spectrum is

$$12, 12, 9, 9, 9, 9, 6, 6, 0.$$

The unweighted adjacency matrix is

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

As an unweighted graph, its Laplacian has eigenvalues

$$9, 9, 6, 6, 6, 6, 6, 6, 0,$$

and the adjacency matrix has eigenvalues

$$6, 0, 0, 0, 0, 0, 0, -3, -3.$$

This function is not bent (in the sense of §6.3), yet the unweighted version of the graph Γ_f, shown in Figure 2.10, is a strongly regular graph with parameters $(9, 6, 3, 6)$. (Strongly regular is defined in §6.6.1 below.) However, the complement of Γ_f is disconnected, so Γ_f is not a primitive strongly regular graph[4].

Consider the subgraph Γ_1 of weight one edges. This has adjacency matrix

$$A_1 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix},$$

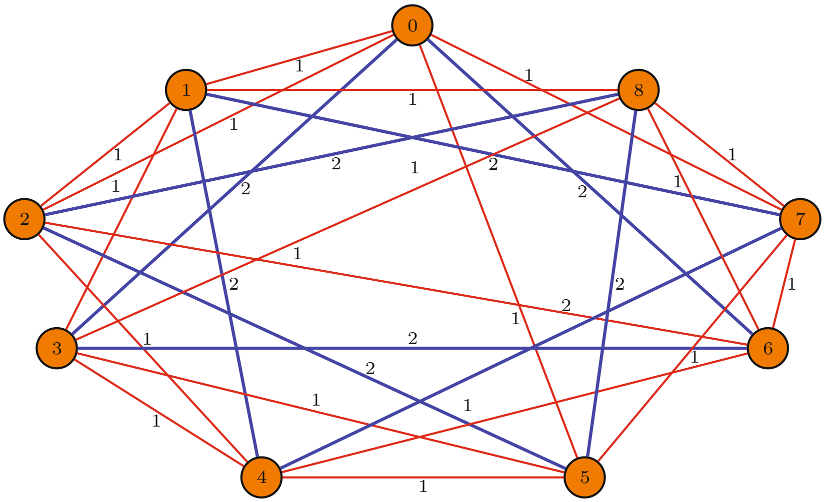

Figure 2.10: The undirected Cayley graph of an even $GF(3)$-valued function of two variables from Example 2.7.7. (The vertices are ordered as in the example.)

and is depicted in Figure 2.11.

Clearly, this is a 4-regular graph on 9 vertices. In fact, Sage allows us to verify that it is isomorphic to the Paley graph on 9 vertices.

[4] An SRG is *primitive* if both the graph and its complement are connected.

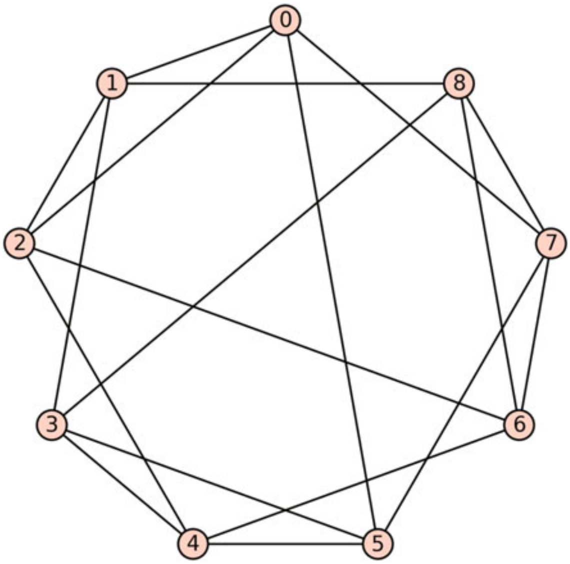

Figure 2.11: The subgraph Γ_1 of Γ is isomorphic to the Paley graph on 9 vertices.

Sage

```
sage: F = GF(3)
sage: V = F^2
sage: f = lambda x:-x[0]^2*x[1]^2+x[0]^2+x[0]*x[1]-x[1]^2
sage: Gamma = boolean_cayley_graph(f, V)
sage: Alist = [[ZZ(f(x-y)) for x in V] for y in V]; matrix(ZZ, Alist)
[0 1 1 2 0 1 2 1 0]
[1 0 1 1 2 0 0 2 1]
[1 1 0 0 1 2 1 0 2]
[2 1 0 0 1 1 2 0 1]
[0 2 1 1 0 1 1 2 0]
[1 0 2 1 1 0 0 1 2]
[2 0 1 2 1 0 0 1 1]
[1 2 0 0 2 1 1 0 1]
[0 1 2 1 0 2 1 1 0]
sage: A = matrix(GF(2), Alist); A
[0 1 1 0 0 1 0 1 0]
[1 0 1 1 0 0 0 0 1]
[1 1 0 0 1 0 1 0 0]
[0 1 0 0 1 1 0 0 1]
[0 0 1 1 0 1 1 0 0]
[1 0 0 1 1 0 0 1 0]
[0 0 1 0 1 0 0 1 1]
[1 0 0 0 0 1 1 0 1]
[0 1 0 1 0 0 1 1 0]
sage: Gamma1 = Graph(A)
sage: Gamma2 = graphs.PaleyGraph(9)
sage: Gamma1.is_isomorphic(Gamma2)
True
```

2.8 Graphs of group quotients

If $\Gamma = (V, E)$ is a graph and G a subgroup of its automorphism group, we define the *quotient graph* by G, denoted Γ/G, as follows:

1. The vertices of Γ/G are the G-orbits in V.
2. Distinct vertices $\overline{v_1}$, $\overline{v_2}$ of Γ/G are connected by an edge if and only if there is a vertex v_1 in V belonging to the orbit $\overline{v_1}$, and a vertex v_2 in V belonging to the orbit $\overline{v_2}$, for which (v_1, v_2) belongs to E.
3. Γ/G is simple.

For example, if Γ is any graph and G is any group that acts regularly[5] on Γ then Γ/G is the empty graph with one vertex.

Example 2.8.1. The graph Γ_2 depicted in Figure 2.12 is a 2-fold cover[6] of the diamond graph Γ_1 (isomorphic to the graph depicted in Figure 1.2).

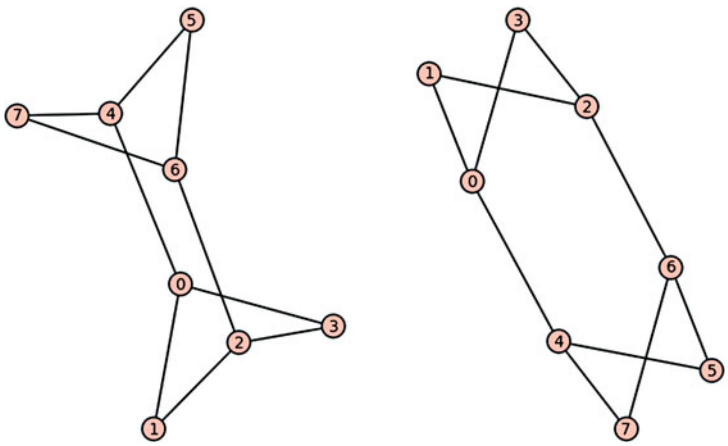

Figure 2.12: Two depictions of a twofold cover of the diamond graph.

Moreover, the automorphism group G of Γ_2 is a cyclic group of order 16 and the quotient map $\Gamma_2 \to \Gamma_1 = \Gamma_2/G_0$ is harmonic[7], where G_0 is the subgroup of order 2 generated by $(0,6)(1,5)(2,4)(3,7)$. The Laplacian of Γ_1 is

[5]The adjective "regular" is over-used in mathematics. In this case, it means that G acts transitively on Γ (i.e., each vertex can be sent to any other by some element of G) and no vertex is fixed by any element of $G - \{1\}$.

[6]A cover is defined in §3.4.

[7]In the sense of Definition 3.3.5 below.

$$Q_1 = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix},$$

and the Laplacian of Γ_2 is

$$Q_2 = \begin{pmatrix} 3 & -1 & 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 3 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 2 \end{pmatrix}.$$

Sage

```
sage: V1 = [0,1,2,3]
sage: E1 = [(0,1),(0,2),(0,3),(1,2),(2,3)]
sage: V2 = [0,1,2,3,4,5,6,7]
sage: E2 = [(0,1),(0,3),(0,4),(1,2),(2,3),(2,6),(4,5),(4,7),(6,5),(6,7)]
sage: Gamma1 = Graph([V1,E1], format='vertices_and_edges')
sage: Gamma2 = Graph([V2,E2], format='vertices_and_edges')
sage: AG = Gamma2.automorphism_group()
sage: AG.cardinality()
16
sage: AG.sylow_subgroup(2).cardinality()
16
sage: Q2 = Gamma2.laplacian_matrix()
sage: Q1 = Gamma1.laplacian_matrix()
sage: factor(Q2.charpoly())
x * (x - 4)^2 * (x - 2)^3 * (x^2 - 6*x + 4)
sage: factor(Q1.charpoly())
(x - 2) * x * (x - 4)^2
```

Recall that if Γ is a graph, we denote by $\sigma(\Gamma)$ the spectrum of Γ, i.e., the multi-set of eigenvalues of the adjacency matrix of A. Recall also that if G is a finite multiplicative group, S is a subset of G such that $S = S^{-1}$, and S does not contain the identity 1, then we denote by $Cay(G, S)$ the Cayley graph of the pair (G, S). The next proposition states that if S is also a generating set of G, and if H is a "good" subgroup of G, then the spectrum of the Cayley graph of $(H, H \cap S)$ is contained in a translate of the spectrum of the Cayley graph of (G, S).

Proposition 2.8.2. *Let G be a finite multiplicative group. Let $H \subset G$ be a normal subgroup. Assume that $S \subset G$ is a symmetric (i.e., $S = S^{-1}$) generating set with $1 \neq S$ such that (1) $H \cap S$ generates H, (2) $S - H \cap S$ generates a subgroup K disjoint from H (i.e., $H \cap K = \{1\}$), and (3) G acts by permutations on $H \cap S$ via conjugations. Then*

$$|S - H \cap S| + \sigma(Cay(H, H \cap S)) \subset \sigma(Cay(G, S)).$$

Proof. If G is abelian then this follows from Proposition 2.6.2. Indeed, in this case

$$\sigma(Cay(H, H \cap S)) = \{\chi(H \cap S) \mid \chi \in H^*\},$$

and

$$\sigma(Cay(G, S)) = \{\chi(S) \mid \chi \in G^*\}.$$

Since G is a direct product of cyclic subgroups of prime power order, H is a sub-product. By induction, we may assume without loss of generality that G is cyclic and so is H. For $\chi \in G^*$ such that $\chi|_K = 1$, we have

$$\chi(S) = \sum_{s \in S} \chi(s) = \sum_{s \in H \cap S} \chi(s) + \sum_{s \in S - H \cap S} \chi(s) = \chi(H \cap S) + |S - H \cap S|,$$

by our hypothesis on $K = G/H$.

Now, assume G is non-abelian. By Theorem 2.6.5,

$$\sigma(Cay(H, H \cap S)) = \left\{ \frac{1}{\deg(\chi)} \chi(H \cap S) \mid \chi \in H^* \right\},$$

and

$$\sigma(Cay(G, S)) = \left\{ \frac{1}{\deg(\chi)} \chi(S) \mid \chi \in G^* \right\}. \tag{2.14}$$

We need to understand the behavior of characters under restriction. The following fact can be found in Weintraub [Wi03] (Corollary 1.12, p. 105): Let $H \subset G$ be normal, $\rho : G \to Aut(V)$ an irreducible representation, $\tau : H \to Aut(W)$ an irreducible component of $Res_H^G(\rho) = \rho|_H$. Then

$$Res_H^G(\rho) = m \oplus_j \tau_j,$$

for some integer $m \geq 1$, where $\{\tau_j\}$ is a complete set of representatives of conjugates of $\tau = \tau_1$ under the action of G. It is easy to compute m if the number of conjugates $N = |\{\tau_j\}|$ is known:

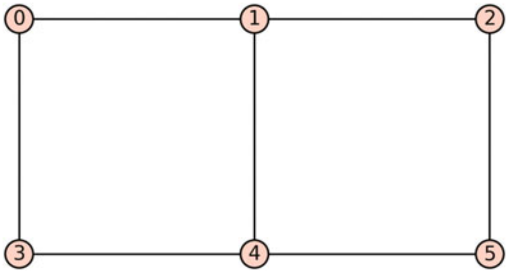

Figure 2.13: The ladder graph L_3.

$$m = \frac{1}{N}\frac{\deg(\rho)}{\deg(\tau)}.$$

We assumed that $S - H \cap S$ generates a group $K \subset G$ for which $H \cap K = \{1\}$. This implies that if $\chi = tr(\rho)$ and $\psi = tr(\tau)$, then

$$\begin{aligned}
\chi(S) &= \sum_{s\in S} \chi(s) \\
&= \sum_{s\in H\cap S} \chi(s) + \sum_{s\in S-H\cap S} \chi(s) \\
&= \chi(H\cap S) + |S - H\cap S| \deg(\chi) \\
&= m \sum_j tr(\tau_j)(H\cap S) + |S - H\cap S| \deg(\chi) \\
&= \frac{1}{N}\frac{\deg(\rho)}{\deg(\tau)} \cdot N \cdot \psi(H\cap S) + |S - H\cap S| \deg(\chi) \qquad \text{by condition (3)} \\
&= \frac{\deg(\rho)}{\deg(\tau)} \cdot \psi(H\cap S) + |S - H\cap S| \deg(\chi).
\end{aligned}$$

Dividing by $\deg(\chi) = \deg(\rho)$ and using Equation (2.14) gives the result. □

Example 2.8.3. We consider next the ladder graph of 6 vertices L_3, depicted in Figure 2.13. The automorphism group G of L_3 is the permutation group with generators $\{(0,2)(3,5), (0,3)(1,4)(2,5)\}$. Because G has only 4 elements, it can't be vertex transitive or edge transitive.

The quotient L_3/G is the connected graph with 2 vertices.

Example 2.8.4. We continue with the weighted Cayley graph Γ of $G = D_6$ in Example 2.6.4 above. The following Sage computation uses the Sage algebraic graph theory module written for this book[8]. It must be loaded before the following computations are possible.

Sage

```
sage: G = DihedralGroup(6); G
Dihedral group of order 12 as a permutation group
sage: G3 = G.sylow_subgroup(3)
sage: G3.order()
3
sage: A = [[alpha(x*y^(-1)) for x in G] for y in G]
sage: Gamma = Graph(matrix(A), format = "adjacency_matrix", weighted=True)
sage: Gamma.automorphism_group(edge_labels=True)
Permutation Group with generators [(3,10)(5,9), (2,11)(4,8),
   (1,4)(6,11), (0,1)(2,3)(4,5)(6,7)(8,9)(10,11), (0,2,5,6,9,11)(1,3,4,7,8,10)]
sage: AutGammaD12a = Gamma.automorphism_group()
sage: AutGammaD12b = Gamma.automorphism_group(edge_labels=True)
sage: AutGammaD12b.order()
144
sage: AutGammaD12a.order()
1440
sage: G9 = AutGammaD12b.sylow_subgroup(3)
sage: G9.order()
9
```

Let G_9 denote the Sylow subgroup of G. This group acts on the edge-weighted graph Γ (in the sense of §2.6). The orbits of G_9 on Γ are

$$\overline{0} = \{0, 5, 9\}, \overline{1} = \{1, 4, 8\}, \overline{2} = \{2, 6, 11\}, \overline{3} = \{3, 7, 10\},$$

and the edges in the cycle graph Γ/G_9 are $(\overline{0}, \overline{2})$ (weight 2), $(\overline{1}, \overline{3})$ (weight 2), $(\overline{1}, \overline{2})$ (weight 1), $(\overline{0}, \overline{3})$ (weight 1). The orbits of G_3 on Γ are

$$\begin{aligned} &\overline{0} = \{0\}, \overline{1} = \{1, 3, 5\}, \overline{2} = \{2, 4, 6\}, \overline{7} = \{7\}, \\ &\overline{8} = \{8\}, \overline{9} = \{9\}, \overline{10} = \{10\}, \overline{11} = \{11\}, \end{aligned}$$

and the edges in Γ/G_9 are as in Figure 2.14, where 0 corresponds to $\overline{0}$, 1 corresponds to $\overline{1}$, 2 corresponds to $\overline{2}$, 3 corresponds to $\overline{7}$, 4 corresponds to $\overline{8}$, 5 corresponds to $\overline{9}$, 6 corresponds to $\overline{10}$, and 7 corresponds to $\overline{11}$.

Example 2.8.5. Consider the Paley graph on 9 vertices, Γ, depicted in Figure 1.12.

[8]This module is available from the github site for this book.

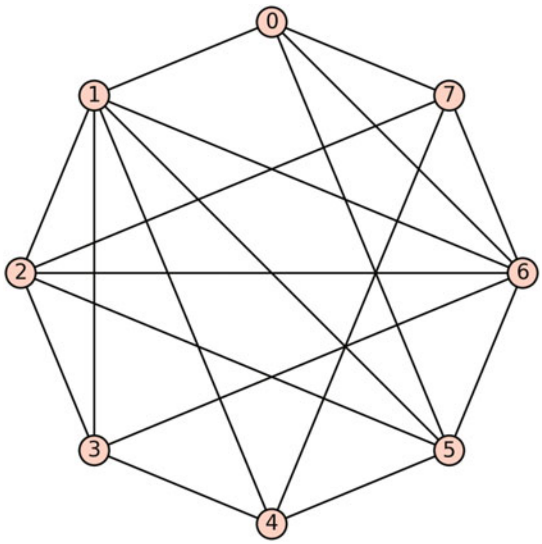

Figure 2.14: The group quotient graph of Γ/G_3.

The automorphism group G of Γ has order 72. The action of G on the vertices of Γ is transitive. The action of G on the edges of Γ is also transitive. Thus, the quotient graph Γ/G is the empty graph with one vertex.

The 2-Sylow subgroup G_2 of G is order 8. The quotient graph Γ/G_2 is described as follows. Vertices of Γ:

$$0, 1, 2, a, a+1, a+2, 2a, 2a+1, 2a+2,$$

where $a \in GF(9) - GF(3)$ is a root of the generating polynomial, x^2+2x+2. The orbits under the G_2-action:

$$\overline{0} = \{0\}, \overline{1} = \{1, 2, a+1, 2a+2\}, \overline{a} = \{a, a+2, 2a, 2a+1\}.$$

The quotient graph therefore has three vertices, $\overline{0}$, $\overline{1}$, and $\overline{a}$, with edges

$$(\overline{0}, \overline{1}), (\overline{1}, \overline{a}).$$

The quotient graph on 3 vertices, Γ/G_2, is depicted in Figure 2.15.

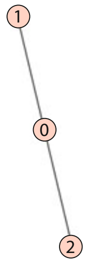

Figure 2.15: A quotient of a Paley graph created using Sage.

Definition 2.8.6. The *Cartesian graph product* of two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ is a graph $\Gamma_3 = \Gamma_1 \Box \Gamma_2$ with the following properties:

- The vertex set of Γ_3 is the Cartesian product $V_1 \times V_2$.
- Two vertices (u_1, u_2) and (v_1, v_2) of Γ_3 are connected by an edge if and only if $u_1 = v_1$ and u_2 is a neighbor of v_2 in Γ_2 or $u_2 = v_2$ and u_1 is a neighbor of v_1 in Γ_1.

Exercise 2.13. Let G denote the cyclic group of order 3, Γ_1 denote the cycle graph on 3 vertices and let Γ_2 denote the cycle graph on 4 vertices. Take G to act on Γ_1 in the obvious way and to act on Γ_2 trivially. Let $\Gamma = \Gamma_1 \Box \Gamma_2$ denote the graph product of them, depicted in Figure 2.16. Show that $\Gamma/G \cong \Gamma_2$.

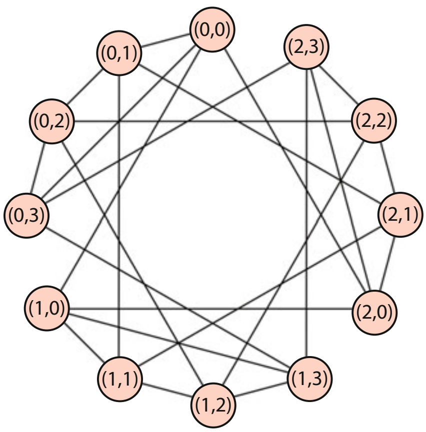

Figure 2.16: A graph product of two cycle graphs.

Exercise 2.14. Let G be a finite group, and let S be a subset of G such that $S = S^{-1}$ and $1 \notin S$. Let Γ_1 denote the Cayley graph of (G, S), let Γ_2 denote an arbitrary graph, and let $\Gamma = \Gamma_1 \Box \Gamma_2$ denote the graph product of them. Show that there is a natural action of G on Γ such that $\Gamma/G \cong \Gamma_2$.

Example 2.8.7. Consider the symmetric group on $\{1, 2, 3\}$, G generated by $S = \{(1, 2, 3), (1, 2), (1, 3, 2)\}$. The Cayley graph Γ_2 of (G, S), depicted in Figure 4.8, has automorphism group of order 12 (with G as a normal subgroup). In particular, there is an action of G on Γ_2.

Let Γ_1 denote the cycle graph on 5 vertices and let Γ_3 denote the product graph $\Gamma_3 = \Gamma_1 \Box \Gamma_2$. This has an automorphism group of order 120, with the automorphism group of Γ_1 (a cyclic group of order 5) as a normal subgroup. The following Sage code shows there is a copy of G in the automorphism group of Γ_3 such that $\Gamma_3/G \cong \Gamma_1$.

Sage

```
sage: G = SymmetricGroup(3)
sage: S = G.gens()+[G.gens()[0]^(-1)]; S
[(1,2,3), (1,2), (1,3,2)]
sage: A = G.cayley_graph(generators=S,simple=True).adjacency_matrix()
sage: Gamma1 = Graph(A)
sage: Gamma1.show(layout="spring", dpi = 300)
Launched png viewer for Graphics object consisting of 16 graphics primitives
sage: AG1 = Gamma1.automorphism_group()
sage: AG1.cardinality()
12
sage: Gamma2 = graphs.CycleGraph(5)
sage: Gamma3 = Gamma1.cartesian_product(Gamma2)
sage: AG3 = Gamma3.automorphism_group()
sage: AG3.cardinality()
120
sage: AG33 = AG3.sylow_subgroup(3)
sage: AG33.cardinality()
3
sage: AG32 = AG3.sylow_subgroup(2)
sage: AG32.cardinality()
8
sage: G0 = AG3.subgroup([AG32.list()[2],AG33.list()[1]])
sage: G0.cardinality()
6
sage: G0.is_normal(AG3)
True
sage: G0.is_abelian()
False
sage: Gamma4 = quotient_graph(Gamma3, G0)
sage: Gamma4.is_circulant()
True
sage: Gamma4.is_connected()
True
sage: len(Gamma4.vertices())
5
```

Next we give a construction of a graph with a given graph quotient.

Let G be a finite group, and let S be a subset of G such that $S = S^{-1}$ and $1 \notin S$. Let Γ_1 denote the Cayley graph of (G, S), and let Γ_2 denote an arbitrary graph with a distinguished vertex v_0. Let $\Gamma_3 = \Gamma_1 \Box \Gamma_2$ denote the Cartesian graph product of Γ_1 and Γ_2, and let Γ be the result of deleting from Γ_3 all edges of the form $((g, v), (g, v_0))$ where $v \neq v_0$.

Exercise 2.15. Show that there is a natural action of G on Γ such that $\Gamma/G \cong \Gamma_2$, where Γ is as in the construction above.

2.8.1 Example of the Biggs–Smith graph

Consider the Biggs–Smith graph Γ, encountered in Chapter 5 on graph examples. It is a 3-regular graph with 102 vertices and 153 edges, having an automorphism group $G \cong PSL(2,17)$, of order 2448, which acts regularly on it. Using Sage, it can be shown that there is an edge e_0 of Γ such that the stabilizer of the edge e_0 is a subgroup G_0 of order 16. The quotient graph $\Gamma_0 = \Gamma/G_0$, shown in Figure 2.17, is a connected graph having 10 vertices and 10 edges, which itself has an automorphism group of order 2.

Sage

```
sage: Gamma = graphs.BiggsSmithGraph()
sage: G = Gamma.automorphism_group(); G.order()
2448
sage: E = Gamma.edges(); len(E)
153
sage: V = Gamma.vertices(); len(V)
102
sage: Gamma.delete_edge(0,1)
sage: E = Gamma.edges(); len(E)
152
sage: Gamma.add_edge((0,1), label="label")
sage: E = Gamma.edges(); len(E)
153
sage: G0 = Gamma.automorphism_group(edge_labels=True); G0.order()
16
```

The stabilizer of the vertex 0 is a subgroup G_1 of order 24. The quotient graph $\Gamma_1 = \Gamma/G_1$, shown in Figure 2.18, is a tree having 8 vertices and 7 edges, which itself has an automorphism group of order 2.

Sage

```
sage: Gamma = graphs.BiggsSmithGraph()
sage: G = Gamma.automorphism_group(); G.order()
2448
sage: G1 = Gamma.automorphism_group(partition=[[0],range(1,102)])
sage: G1.order()
24
```

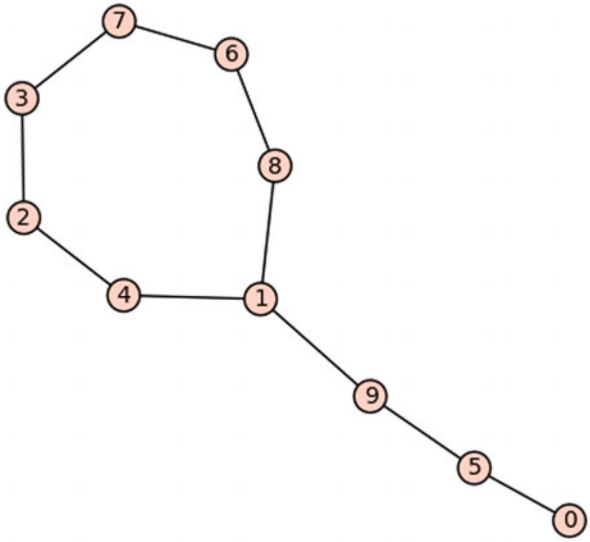

Figure 2.17: The group quotient graph of Γ/G_0.

The orbits of G_1 acting on Γ are:

$$\begin{gathered}\{0\},\{7,41,10,44,50,88,94,31\},\{16,1,101\},\\ \{17,2,100,25,36,15\},\{64,65,67,4,98,70,71,73,76,13,77,\\ 82,19,20,23,27,28,34,38,54,55,84,61,62\},\\ \{66,35,37,72,83,14,99,18,3,24,26,63\},\\ \{68,5,97,74,75,12,78,81,21,22,90,91,86,29,69,\\ 33,39,46,47,53,56,57,60,85\},\{96,6,8,9,11,79,80,87,89,92,\\ 93,30,95,32,40,42,43,45,48,49,51,52,58,59\}.\end{gathered}$$

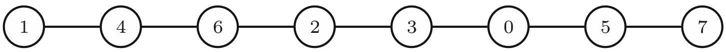

Figure 2.18: The group quotient graph of Γ/G_1.

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