Waiting and Service Time of a Unique Customer in an M/M/m Preemptive LCFS Queue with Impatient Customers

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Abstract. We study an M/M/m preemptive last-come, first-served queue with impatient customers without priority classes. We focus on the probability of service completion and abandonment as well as the waiting and service times of a *unique customer* who has the mean service and patience times that are different from those of all other customers in the steady state. The problem is formulated as the first passage times in a combination of two one-dimensional birth-and-death processes each with two absorbing states. We provide explicit expressions in terms of Laplace-Stieltjes transform of the distribution function for the time to service completion or abandonment, which is decomposed into the waiting and service times of the unique customer. A numerical example is presented in order to demonstrate the computation of theoretical formulas.

Keywords: M/M/m preemptive LCFS queue · Impatient customers First passage time

1 Introduction

We are concerned with an M/M/m queueing system with impatient customers without exogenous priority classes. Customers arrive according to a Poisson process at rate λ. The service time of each customer is exponentially distributed with mean $1/\mu$. There are m servers and a waiting room of infinite capacity. At any time, each customer present in the system is either being served or staying in the waiting room. Each customer in the waiting room leaves the system (abandons the waiting process) with probability $\theta \Delta t$ within a short time interval $(t, t + \Delta t)$. That is to say, the patience time for each customer is exponentially distributed with mean $1/\theta$. Customers never leave the system while being served before the service is completed.

It is assumed that the service to each customer is started immediately upon arrival. If all servers are busy, the arriving customer preempts the ongoing service to the customer who arrived first among those who are being served. The

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customer whose service is preempted is placed at the head of the queue in the waiting room. When one of the servers becomes available, a customer at the head of the queue, if any, is called in for service to be resumed. This discipline is equivalent to the one called "preemptive *last-in, first-out* (LIFO)" for an M/G/1 queue by Wolff [4, p. 456].

In our previous work [2,3], we studied the time interval from arrival to either service completion or abandonment, whichever occurs first, of an arbitrary customer in steady state. The problem was formulated as a combination of two one-dimensional birth-and-death processes, each with two absorbing states, for the behavior of a tagged customer. We provided explicit expressions in terms of Laplace-Stieltjes transform (LST) of the distribution function (DF) for the first passage time to service completion or abandonment, which is decomposed into the waiting time and the received service time.

In the present paper, we turn our attention to the waiting and service time of a *unique customer* who has the mean service time $1/\mu_0$ and mean patience time $1/\theta_0$ that may be different from $1/\mu$ and $1/\theta$, respectively, of other customers. It is assumed that such a customer arrives during the steady state of an M/M/m queueing system with otherwise uniformly impatient customers. We are interested in the waiting and service time of the unique customer. The analysis technique is similar to the one in [3]. Through a numerical example, we compare the probability of service completion and abandonment as well as the mean waiting and service time of the unique customer to those of other customers. For a more patient customer, we find that (i) the probability of service completion is higher, (ii) the mean time spent in the system is longer whether he abandons waiting or he gets served, and (iii) the received service time is not much different from that of other customers.

2 First Passage Time to Service Preemption or Completion from State $k, 0 \leq k \leq m-1$

We focus on a unique customer in *state* k, signifying that there are k other customers who compete with him for service at any given time in the steady state, where $k = 0, 1, 2, \ldots$. They are the customers who arrived after the unique one and have been staying in the system until that time. According to the preemptive LCFS discipline, an arriving unique customer always joins the system at state $k = 0$.

We first consider a birth-and-death process of state transitions for the unique customer in state k, $0 \leq k \leq m-1$, in which he is being served. The service to this customer, with probability one, is eventually either preempted by another customer who arrives after him or completed without preemption.

2.1 Behavior of a Unique Customer Until Service Preemption or Completion

The state transition diagram for the discrete-time, one-dimensional birth-and-death process modeling the behavior of a unique customer in service is shown in

Fig. 1. This process has m transient states $\{0, 1, 2, \ldots, m-1\}$ and two absorbing states denoted by "Pr" (state m) and "Sr", representing service preemption and service completion, respectively. The state transition probabilities and the LST of the DF for the time spent by the unique customer in state k are given by

$$\alpha_k = \frac{k\mu}{\lambda + k\mu + \mu_0}; \quad \beta_k = \frac{\mu_0}{\lambda + k\mu + \mu_0}; \quad B_k^*(s) = \frac{\lambda + k\mu + \mu_0}{s + \lambda + k\mu + \mu_0}.$$

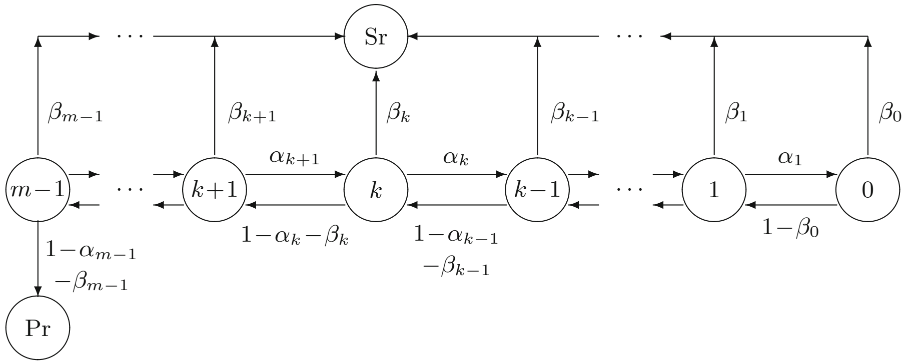

Fig. 1. State transitions for a unique customer until service preemption or completion.

2.2 LST of the DF for the Time to Service Preemption or Completion

By $H_k^*(s, \mathrm{Pr})$, we denote the joint probability of service preemption and the LST of the DF for the first passage time from state k to state m ("Pr") without reaching state "Sr". Moreover, we denote by $H_k^*(s, \mathrm{Sr})$ the joint probability of service completion and the LST of the DF for the first passage time from state k to state "Sr" without reaching state "Pr".

Applying the *first step analysis* for the discrete-time Markov chain, we have the following finite sets of equations for $\{H_k^*(s, \mathrm{Pr}); 0 \leq k \leq m-1\}$ and $\{H_k^*(s, \mathrm{Sr}); 0 \leq k \leq m-1\}$:

$$\begin{aligned}
(s + \lambda + \mu_0)H_0^*(s, \mathrm{Pr}) &= \lambda H_1^*(s, \mathrm{Pr}), \\
(s + \lambda + k\mu + \mu_0)H_k^*(s, \mathrm{Pr}) &= k\mu H_{k-1}^*(s, \mathrm{Pr}) + \lambda H_{k+1}^*(s, \mathrm{Pr}) \\
&\qquad\qquad\qquad 1 \leq k \leq m-2, \\
[s + \lambda + (m-1)\mu + \mu_0]H_{m-1}^*(s, \mathrm{Pr}) &= (m-1)\mu H_{m-2}^*(s, \mathrm{Pr}) + \lambda.
\end{aligned}$$

$$\begin{aligned}
(s + \lambda + \mu_0)H_0^*(s, \mathrm{Sr}) &= \mu_0 + \lambda H_1^*(s, \mathrm{Sr}), \\
(s + \lambda + k\mu + \mu_0)H_k^*(s, \mathrm{Sr}) &= k\mu H_{k-1}^*(s, \mathrm{Sr}) + \mu_0 + \lambda H_{k+1}^*(s, \mathrm{Sr}) \\
&\qquad\qquad\qquad 1 \leq k \leq m-2, \\
[s + \lambda + (m-1)\mu + \mu_0]H_{m-1}^*(s, \mathrm{Sr}) &= (m-1)\mu H_{m-2}^*(s, \mathrm{Sr}) + \mu_0.
\end{aligned}$$

In addition, we let $H_m^*(s, \mathrm{Pr}) \equiv 1$ and $H_m^*(s, \mathrm{Sr}) \equiv 0$. The solution can be obtained in terms of functions $\{h_k^*(s); 0 \le k \le m\}$ in the form

$$H_k^*(s, \mathrm{Pr}) = \frac{h_k^*(s)}{h_m^*(s)}; \quad H_k^*(s, \mathrm{Sr}) = \frac{\mu_0}{s+\mu_0}\left[1 - \frac{h_k^*(s)}{h_m^*(s)}\right] \qquad 0 \le k \le m.$$

2.3 Solution for $\{h_k^*(s); 0 \le k \le m\}$

A finite set of equations for $\{h_k^*(s); 0 \le k \le m\}$ is given by

$$\begin{aligned} &h_0^*(s) = 1; \quad s + \lambda + \mu_0 = \lambda h_1^*(s), \\ &(s + \lambda + k\mu + \mu_0) h_k^*(s) = k\mu h_{k-1}^*(s) + \lambda h_{k+1}^*(s) \qquad 1 \le k \le m-1, \end{aligned}$$

which can be written as the following set of recurrence relations:

$$h_k^*(s) = \frac{s + \lambda + (k-1)\mu + \mu_0}{\lambda} h_{k-1}^*(s) - \frac{(k-1)\mu}{\lambda} h_{k-2}^*(s) \qquad 2 \le k \le m.$$

The solution is given by *Cramer's formula* as the determinant of the $k \times k$ tridiagonal matrix

$$h_k^*(s) = (-1)^k$$

$$\times \begin{vmatrix} -\frac{s+\lambda+\mu_0}{\lambda} & 1 & 0 & 0 & 0 & \cdots & 0 \\ \frac{\mu}{\lambda} & -\frac{s+\lambda+\mu+\mu_0}{\lambda} & 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{2\mu}{\lambda} & -\frac{s+\lambda+2\mu+\mu_0}{\lambda} & 1 & 0 & \cdots & 0 \\ 0 & 0 & \frac{3\mu}{\lambda} & -\frac{s+\lambda+3\mu+\mu_0}{\lambda} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\frac{s+\lambda+(k-2)\mu+\mu_0}{\lambda} & 1 \\ 0 & 0 & 0 & 0 & \cdots & \frac{(k-1)\mu}{\lambda} & -\frac{s+\lambda+(k-1)\mu+\mu_0}{\lambda} \end{vmatrix}$$

for $1 \le k \le m$. Note that $h_k^*(s)$ is a kth-degree polynomial in s, the coefficient of s^k being $(1/\lambda)^k$. Thus, we obtain the probability of service preemption and completion

$$p_k\{\mathrm{Pr}\} := H_k^*(0, \mathrm{Pr}) = \frac{h_k^*(0)}{h_m^*(0)}; \quad p_k\{\mathrm{Sr}\} := H_k^*(0, \mathrm{Sr}) = 1 - \frac{h_k^*(0)}{h_m^*(0)} \qquad 0 \le k \le m.$$

In particular, we have $p_0\{\mathrm{Pr}\} = 1/h_m^*(0)$ and $p_m\{\mathrm{Pr}\} = 1$.

3 First Passage Time to Service Resumption or Abandonment from State k, $k \geq m$

We next consider another birth-and-death process of state transitions for a unique customer in state k, $k \geq m$, in which he is staying in the waiting room. With probability one, this customer, eventually, either is called in to resume his service or abandons waiting.

3.1 Behavior of a Unique Customer Until Service Resumption or Abandonment

The state transition diagram for the discrete-time, one-dimensional birth-and-death process modeling the behavior of a unique customer in the waiting room is shown in Fig. 2. The process has an infinite number of transient states $\{m, m+1, \ldots\}$ and two absorbing states denoted by "Rs" (state $m-1$) and "Ab", representing service resumption and abandonment, respectively. The state transition probabilities and the LST of the DF for the time spent by the unique customer in state k are given by

$$\alpha'_k = \frac{m\mu + (k-m)\theta}{\lambda + m\mu + (k-m)\theta + \theta_0}; \quad \beta'_k = \frac{\theta_0}{\lambda + m\mu + (k-m)\theta + \theta_0},$$
$$B'^*_k(s) = \frac{\lambda + m\mu + (k-m)\theta + \theta_0}{s + \lambda + m\mu + (k-m)\theta + \theta_0}.$$

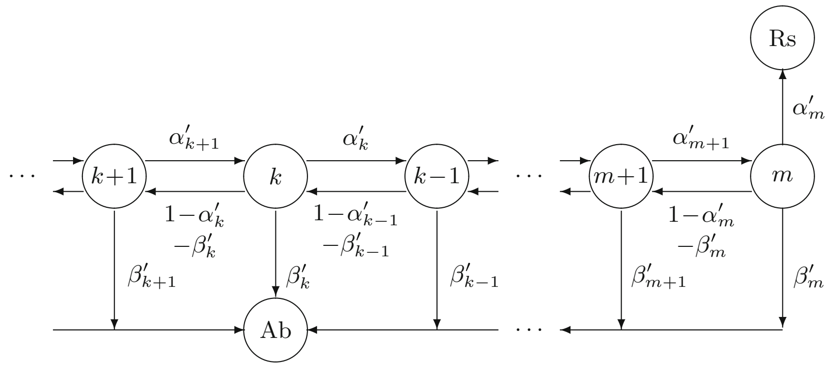

Fig. 2. State transitions for a unique customer until service resumption or abandonment.

3.2 LST of the DF for the Time to Service Resumption or Abandonment

By $W^*_k(s, \mathrm{Rs})$, we denote the joint probability of service resumption and the LST of the DF for the first passage time from state k to state $m-1$ ("Rs") without reaching state "Ab". Moreover, we denote by $W_k^*(s, \mathrm{Ab})$ the joint probability of abandonment and the LST of the DF for the first passage time from state k to state "Ab" without reaching state "Rs".

Infinite sets of equations for $\{W_k^*(s, \mathrm{Rs}); k \geq m\}$ and $\{W_k^*(s, \mathrm{Ab}); k \geq m\}$ are given by

$$\begin{aligned}
&(s+\lambda+m\mu+\theta_0)W_m^*(s,\mathrm{Rs}) = m\mu + \lambda W_{m+1}^*(s,\mathrm{Rs}),\\
&[s+\lambda+m\mu+(k-m)\theta+\theta_0]W_k^*(s,\mathrm{Rs})\\
&= [m\mu+(k-m)\theta]W_{k-1}^*(s,\mathrm{Rs}) + \lambda W_{k+1}^*(s,\mathrm{Rs}) \qquad k \geq m+1.\\
&(s+\lambda+m\mu+\theta_0)W_m^*(s,\mathrm{Ab}) = m\mu + \theta_0 + \lambda W_{m+1}^*(s,\mathrm{Ab}),\\
&[s+\lambda+m\mu+(k-m)\theta+\theta_0]W_k^*(s,\mathrm{Ab})\\
&= [m\mu+(k-m)\theta]W_{k-1}^*(s,\mathrm{Ab}) + \theta_0 + \lambda W_{k+1}^*(s,\mathrm{Ab}) \qquad k \geq m+1.
\end{aligned}$$

The solution can be obtained in terms of functions $\{G_k^*(s); k \geq m\}$ in the form

$$W_k^*(s,\mathrm{Rs}) = G_k^*(s+\theta_0); \quad W_k^*(s,\mathrm{Ab}) = \frac{\theta_0}{s+\theta_0}[1 - G_k^*(s+\theta_0)] \qquad k \geq m.$$

Thus the probability of service preemption and abandonment is given by

$$p_k\{\mathrm{Rs}\} := W_k^*(0,\mathrm{Rs}) = G_k^*(\theta_0); \quad p_k\{\mathrm{Ab}\} := W_k^*(0,\mathrm{Ab}) = 1 - G_k^*(\theta_0) \qquad k \geq m.$$

3.3 Busy Period

A *busy period* started with k ($\geq m$) customers in an M/M/m queue is the time interval, denoted by $\mathcal{G}_k$, from the instant at which there are k customers in the system (all servers are busy and $k-m$ customers are waiting) to the first instant at which any one of the servers becomes available. Let us denote by $f_{W_k}(t, \mathrm{Rs})$ and $f_{W_k}(t, \mathrm{Ab})$ the density functions of the time until service resumption and the time until abandonment, respectively, for a customer in state k, $k \geq m$. They are related with the density function $f_{\mathcal{G}_k}(t)$ for $\mathcal{G}_k$ and the probability $P\{\mathcal{G}_k > t\}$ as follows:

$$f_{W_k}(t,\mathrm{Rs}) = e^{-\theta_0 t} f_{\mathcal{G}_k}(t); \quad f_{W_k}(t,\mathrm{Ab}) = \theta_0 e^{-\theta_0 t} P\{\mathcal{G}_k > t\}.$$

The function $G_k^*(s)$ introduced in Sect. 3.2 is the LST of the DF for $\mathcal{G}_k$, $k \geq m$. The set of equations for $\{G_k^*(s), k \geq m\}$ is given by

$$\begin{aligned}
(s+\lambda+m\mu)G_m^*(s) &= \lambda G_{m+1}^*(s) + m\mu,\\
[s+\lambda+m\mu+(k-m)\theta]G_k^*(s) &= [m\mu+(k-m)\theta]G_{k-1}^*(s) + \lambda G_{k+1}^*(s) \qquad k \geq m+1.
\end{aligned}$$

Iravani and Balcıoğlu [1] provides the LST of the DF for the duration of the busy period in an M/M/m queue with an exponentially distributed service time with mean $1/(m\mu)$ as follows:

$$G_k^*(s) = \frac{\dfrac{m\mu}{s+m\mu} + \displaystyle\sum_{i=1}^{\infty}(-1)^i \psi_{i,k-m}(\lambda/\theta)\left[\prod_{j=0}^{i-1}\left(1-\frac{m\mu}{s+m\mu+j\theta}\right)\right]\frac{m\mu}{s+m\mu+i\theta}}{1+\displaystyle\sum_{i=1}^{\infty}\frac{(\lambda/\theta)^i}{i!}\prod_{j=0}^{i-1}\left(1-\frac{m\mu}{s+m\mu+j\theta}\right)} \qquad k \geq m,$$

where we have defined

$$\psi_{i,k}(x) := \sum_{j=\max\{0,i-k\}}^{i} \frac{(-x)^j}{j!}\binom{k}{i-j} \qquad i \geq 1,\ k \geq 0.$$

In particular, since $\psi_{i,0}(x) = (-x)^i/i!$, we have

$$G_m^*(s) = \frac{\dfrac{m\mu}{s+m\mu} + \displaystyle\sum_{i=1}^{\infty}\frac{(\lambda/\theta)^i}{i!}\left[\prod_{j=0}^{i-1}\left(1-\frac{m\mu}{s+m\mu+j\theta}\right)\right]\frac{m\mu}{s+m\mu+i\theta}}{1+\displaystyle\sum_{i=1}^{\infty}\frac{(\lambda/\theta)^i}{i!}\prod_{j=0}^{i-1}\left(1-\frac{m\mu}{s+m\mu+j\theta}\right)}.$$

4 Joint Distribution of the Waiting and Service Time

We are now in a position to consider the distribution of the time until departure (either by abandonment or service completion) for a unique customer in a combination of two birth-and-death processes whose state transitions are shown in Figs. 1 and 2. We note that state "Pr" in Fig. 1 is identical to state m in Fig. 2, whereas state "Rs" in Fig. 2 is identical to state $m-1$ in Fig. 1.

The time until departure consists of the *waiting time* (the time that the customer spends staying in the waiting room) and the *service time* (the time during which the customer is being served). These are not independent. Therefore, we will derive the joint LST of the DF for the waiting and service time for a unique customer who abandons waiting, denoted by $\mathcal{T}_k^*(s, s', \text{Ab})$, and for a unique customer who gets served until completion, denoted by $\mathcal{T}_k^*(s, s', \text{Sr})$. Then, we obtain the probability of abandonment and service completion, the marginal LST of the DF for the waiting time, the service time, and the total time spent in the system as follows:

$$\begin{aligned}
\mathcal{P}_k\{\text{Ab}\} &:= \mathcal{T}_k^*(0,0,\text{Ab}) \quad ; \quad \mathcal{P}_k\{\text{Sr}\} := \mathcal{T}_k^*(0,0,\text{Sr}),\\
\mathcal{W}_k^*(s,\text{Ab}) &:= \mathcal{T}_k^*(s,0,\text{Ab}) \quad ; \quad \mathcal{H}_k^*(s,\text{Ab}) := \mathcal{T}_k^*(0,s,\text{Ab}),\\
\mathcal{W}_k^*(s,\text{Sr}) &:= \mathcal{T}_k^*(s,0,\text{Sr}) \quad ; \quad \mathcal{H}_k^*(s,\text{Sr}) := \mathcal{T}_k^*(0,s,\text{Sr}),\\
\mathcal{T}_k^*(s,\text{Ab}) &:= \mathcal{T}_k^*(s,s,\text{Ab}) \quad ; \quad \mathcal{T}_k^*(s,\text{Sr}) := \mathcal{T}_k^*(s,s,\text{Sr}).
\end{aligned}$$

4.1 Waiting and Service Time Until Abandonment

We first consider the waiting and service time until abandonment for a unique customer who abandons waiting.

(1) For the unique customer being served in state k, $0 \leq k \leq m-1$, the first passage to abandonment ("Ab") consists of the following passages:
  (i) the initial passage from state k to state "Pr" in Fig. 1,
  (ii) several repetitions of the transition from state m to state "Rs" in Fig. 2, followed by the transition from state $m-1$ back to state "Pr" in Fig. 1, and
  (iii) the final passage from state m to state "Ab" in Fig. 2.

Owing to the Markovian property of state transitions, the times to take these passages in succession are independent of each other. Therefore, we get

$$\begin{aligned}
\mathcal{T}_k^*(s, s', \mathrm{Ab}) &= H_k^*(s', \mathrm{Pr}) W_m^*(s, \mathrm{Ab}) \\
&\quad + H_k^*(s', \mathrm{Pr})[W_m^*(s, \mathrm{Rs}) H_{m-1}^*(s', \mathrm{Pr})] W_m^*(s, \mathrm{Ab}) \\
&\quad + H_k^*(s', \mathrm{Pr})[W_m^*(s, \mathrm{Rs}) H_{m-1}^*(s', \mathrm{Pr})]^2 W_m^*(s, \mathrm{Ab}) + \cdots \\
&= H_k^*(s', \mathrm{Pr}) W_m^*(s, \mathrm{Ab}) \sum_{n=0}^{\infty} [W_m^*(s, \mathrm{Rs}) H_{m-1}^*(s', \mathrm{Pr})]^n \\
&= \frac{H_k^*(s', \mathrm{Pr}) W_m^*(s, \mathrm{Ab})}{1 - W_m^*(s, \mathrm{Rs}) H_{m-1}^*(s', \mathrm{Pr})} \\
&= \frac{\theta_0}{s+\theta_0} \cdot \frac{h_k^*(s')[1 - G_m^*(s+\theta_0)]}{h_m^*(s') - h_{m-1}^*(s') G_m^*(s+\theta_0)}.
\end{aligned}$$

This joint distribution leads to the marginal distributions

$$\begin{aligned}
\mathcal{W}_k^*(s, \mathrm{Ab}) &= \frac{p_k\{\mathrm{Pr}\} W_m^*(s, \mathrm{Ab})}{1 - p_{m-1}\{\mathrm{Pr}\} W_m^*(s, \mathrm{Rs})} \\
&= \frac{\theta_0}{s+\theta_0} \cdot \frac{h_k^*(0)[1 - G_m^*(s+\theta_0)]}{h_m^*(0) - h_{m-1}^*(0) G_m^*(s+\theta_0)}, \\
\mathcal{H}_k^*(s, \mathrm{Ab}) &= \frac{p_m\{\mathrm{Ab}\} H_k^*(s, \mathrm{Pr})}{1 - p_m\{\mathrm{Rs}\} H_{m-1}^*(s, \mathrm{Pr})} = \frac{h_k^*(s)[1 - G_m^*(\theta_0)]}{h_m^*(s) - h_{m-1}^*(s) G_m^*(\theta_0)}, \\
\mathcal{T}_k^*(s, \mathrm{Ab}) &= \frac{H_k^*(s, \mathrm{Pr}) W_m^*(s, \mathrm{Ab})}{1 - W_m^*(s, \mathrm{Rs}) H_{m-1}^*(s, \mathrm{Pr})} \\
&= \frac{\theta_0}{s+\theta_0} \cdot \frac{h_k^*(s)[1 - G_m^*(s+\theta_0)]}{h_m^*(s) - h_{m-1}^*(s) G_m^*(s+\theta_0)}.
\end{aligned}$$

Then we get the probability of abandonment

$$\mathcal{P}_k\{\mathrm{Ab}\} = \frac{p_k\{\mathrm{Pr}\} p_m\{\mathrm{Ab}\}}{1 - p_m\{\mathrm{Rs}\} p_{m-1}\{\mathrm{Pr}\}} = \frac{h_k^*(0)[1 - G_m^*(\theta_0)]}{h_m^*(0) - h_{m-1}^*(0) G_m^*(\theta_0)},$$

and the mean waiting and service time until abandonment

$$E[\mathcal{W}_k, \mathrm{Ab}] = \frac{1}{\theta_0}\mathcal{P}_k\{\mathrm{Ab}\} + \frac{h_k^*(0)[h_m^*(0) - h_{m-1}^*(0)]G_m'(\theta_0)}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta_0)]^2},$$

$$E[\mathcal{H}_k, \mathrm{Ab}] = [1 - G_m^*(\theta_0)]\left\{\frac{h_k^*(0)[h_m'(0) - h_{m-1}'(0)G_m^*(\theta_0)]}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta_0)]^2} - \frac{h_k'(0)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta_0)}\right\},$$

$$\begin{aligned} E[\mathcal{W}_k\mathcal{H}_k, \mathrm{Ab}] &= \frac{1}{\theta_0}E[\mathcal{H}_k, \mathrm{Ab}] \\ &+ G_m'(\theta_0)\left\{\frac{2h_k^*(0)[h_m^*(0) - h_{m-1}^*(0)][h_m'(0) - h_{m-1}'(0)G_m^*(\theta_0)]}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta_0)]^3}\right. \\ &\left. - \frac{h_k^*(0)[h_m'(0) - h_{m-1}'(0)] + h_k'(0)[h_m^*(0) - h_{m-1}^*(0)]}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta_0)]^2}\right\}, \end{aligned}$$

where $h_k'(0) := [dh_k^*(s)/ds]_{s=0}$, $0 \le k \le m$, and $G_m'(\theta_0) := [dG_m^*(s)/ds]_{s=\theta_0}$.

(2) For the unique customer waiting in state k, $k \ge m$, the first passage to abandonment ("Ab") is either
   (a) a direct passage from state k to state "Ab" in Fig. 2, or
   (b) a sequence of the following passages:
      (i) the initial passage from state k to state "Rs" in Fig. 2,
      (ii) several repetitions of the transition from state $m-1$ to state "Pr" in Fig. 1, followed by the transition from state m to state "Rs" in Fig. 2, and
      (iii) the passage from state $m-1$ to state "Pr" in Fig. 1, followed by the final passage from state m to state "Ab" in Fig. 2.

Therefore, from the Markovian property of state transitions, we get

$$\begin{aligned} \mathcal{T}_k^*(s, s', \mathrm{Ab}) &= W_k^*(s, \mathrm{Ab}) + W_k^*(s, \mathrm{Rs})H_{m-1}^*(s', \mathrm{Pr})W_m^*(s, \mathrm{Ab}) \\ &+ W_k^*(s, \mathrm{Rs})[H_{m-1}^*(s', \mathrm{Pr})W_m^*(s, \mathrm{Rs})]H_{m-1}^*(s', \mathrm{Pr})W_m^*(s, \mathrm{Ab}) \\ &+ W_k^*(s, \mathrm{Rs})[H_{m-1}^*(s', \mathrm{Pr})W_m^*(s, \mathrm{Rs})]^2H_{m-1}^*(s', \mathrm{Pr})W_m^*(s, \mathrm{Ab}) \\ &+ \cdots \\ &= W_k^*(s, \mathrm{Ab}) + W_k^*(s, \mathrm{Rs})H_{m-1}^*(s', \mathrm{Pr})W_m^*(s, \mathrm{Ab}) \\ &\qquad \times \sum_{n=0}^{\infty}[W_m^*(s, \mathrm{Rs})H_{m-1}^*(s', \mathrm{Pr})]^n \\ &= W_k^*(s, \mathrm{Ab}) + \frac{W_k^*(s, \mathrm{Rs})H_{m-1}^*(s', \mathrm{Pr})W_m^*(s, \mathrm{Ab})}{1 - W_m^*(s, \mathrm{Rs})H_{m-1}^*(s', \mathrm{Pr})} \\ &= \frac{\theta_0}{s+\theta_0}\left\{1 - \frac{[h_m^*(s') - h_{m-1}^*(s')]G_k^*(s+\theta_0)}{h_m^*(s') - h_{m-1}^*(s')G_m^*(s+\theta_0)}\right\}. \end{aligned}$$

This joint distribution leads to the marginal distributions

$$
\begin{aligned}
\mathcal{W}_k^*(s, \mathrm{Ab}) &= W_k^*(s, \mathrm{Ab}) + \frac{p_{m-1}\{\mathrm{Pr}\}W_k^*(s, \mathrm{Rs})W_m^*(s, \mathrm{Ab})}{1 - p_{m-1}\{\mathrm{Pr}\}W_m^*(s, \mathrm{Rs})} \\
&= \frac{\theta_0}{s+\theta_0}\left\{1 - \frac{[h_m^*(0) - h_{m-1}^*(0)]G_k^*(s+\theta_0)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(s+\theta_0)}\right\}, \\
\mathcal{H}_k^*(s, \mathrm{Ab}) &= p_k\{\mathrm{Ab}\} + \frac{p_k\{\mathrm{Rs}\}p_m\{\mathrm{Ab}\}H_{m-1}^*(s, \mathrm{Pr})}{1 - p_m\{\mathrm{Rs}\}H_{m-1}^*(s, \mathrm{Pr})} \\
&= 1 - \frac{[h_m^*(s) - h_{m-1}^*(s)]G_k^*(\theta_0)}{h_m^*(s) - h_{m-1}^*(s)G_m^*(\theta_0)}, \\
\mathcal{T}_k^*(s, \mathrm{Ab}) &= W_k^*(s, \mathrm{Ab}) + \frac{W_k^*(s, \mathrm{Rs})H_{m-1}^*(s, \mathrm{Pr})W_m^*(s, \mathrm{Ab})}{1 - W_m^*(s, \mathrm{Rs})H_{m-1}^*(s, \mathrm{Pr})} \\
&= \frac{\theta_0}{s+\theta_0}\left\{1 - \frac{[h_m^*(s) - h_{m-1}^*(s)]G_k^*(s+\theta_0)}{h_m^*(s) - h_{m-1}^*(s)G_m^*(s+\theta_0)}\right\}.
\end{aligned}
$$

Then we get the probability of abandonment

$$
\begin{aligned}
\mathcal{P}_k\{\mathrm{Ab}\} &= p_k\{\mathrm{Ab}\} + \frac{p_k^*\{\mathrm{Rs}\}p_{m-1}\{\mathrm{Pr}\}p_m\{\mathrm{Ab}\}}{1 - p_{m-1}^*\{\mathrm{Pr}\}p_m\{\mathrm{Rs}\}} \\
&= 1 - \frac{[h_m^*(0) - h_{m-1}^*(0)]G_k^*(\theta_0)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta_0)}
\end{aligned}
$$

and the mean waiting and service time until abandonment

$$
\begin{aligned}
E[\mathcal{W}_k, \mathrm{Ab}] &= \frac{1}{\theta_0}\mathcal{P}_k\{\mathrm{Ab}\} + [h_m^*(0) - h_{m-1}^*(0)] \\
&\quad \times \left\{\frac{G_k'(\theta_0)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta_0)} + \frac{G_k^*(\theta_0)h_{m-1}^*(0)G_m'(\theta_0)}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta_0)]^2}\right\}, \\
E[\mathcal{H}_k, \mathrm{Ab}] &= \frac{G_k^*(\theta_0)[h_m'(0)h_{m-1}^*(0) - h_m^*(0)h_{m-1}'(0)][1 - G_m^*(\theta_0)]}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta_0)]^2}, \\
E[\mathcal{W}_k\mathcal{H}_k, \mathrm{Ab}] &= \frac{1}{\theta_0}E[\mathcal{H}_k, \mathrm{Ab}] + [h_m'(0)h_{m-1}^*(0) - h_{m-1}'(0)h_m^*(0)] \\
&\quad \times \left\{\frac{G_k^*(\theta_0)G_m'(\theta_0)[h_m^*(0) - 2h_{m-1}^*(0) + h_{m-1}^*(0)G_m^*(\theta_0)]}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta_0)]^3}\right. \\
&\quad \left. - \frac{G_k'(\theta_0)[1 - G_m^*(\theta_0)]}{[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta_0)]^2}\right\},
\end{aligned}
$$

where $G_k'(\theta_0) := [dG_k^*(s)/ds]_{s=\theta_0}, k \geq m$.

4.2 Waiting and Service Time Until Service Completion

We next consider the waiting and service time until service completion for a unique customer who gets served.

(1) For the unique customer being served in state k, $0 \le k \le m-1$, we have

$$\begin{aligned}\mathcal{T}_k^*(s,s',\mathrm{Sr}) &= H_k^*(s',\mathrm{Sr}) + \frac{H_k^*(s',\mathrm{Pr})W_m^*(s,\mathrm{Rs})H_{m-1}^*(s',\mathrm{Sr})}{1-W_m^*(s,\mathrm{Rs})H_{m-1}^*(s',\mathrm{Pr})}\\ &= \frac{\mu_0}{s'+\mu_0}\left\{1-\frac{h_k^*(s')[1-G_m^*(s+\theta_0)]}{h_m^*(s')-h_{m-1}^*(s')G_m^*(s+\theta_0)}\right\}.\end{aligned}$$

This joint distribution leads to the marginal distributions

$$\begin{aligned}\mathcal{W}_k^*(s,\mathrm{Sr}) &= p_k\{\mathrm{Sr}\} + \frac{p_k\{\mathrm{Pr}\}p_{m-1}\{\mathrm{Sr}\}W_m^*(s,\mathrm{Rs})}{1-p_{m-1}\{\mathrm{Pr}\}W_m^*(s,\mathrm{Rs})}\\ &= 1-\frac{h_k^*(0)[1-G_m^*(s+\theta_0)]}{h_m^*(0)-h_{m-1}^*(0)G_m^*(s+\theta_0)},\\ \mathcal{H}_k^*(s,\mathrm{Sr}) &= H_k^*(s,\mathrm{Sr}) + \frac{p_m\{\mathrm{Rs}\}H_k^*(s,\mathrm{Pr})H_{m-1}^*(s,\mathrm{Sr})}{1-p_m\{\mathrm{Rs}\}H_{m-1}^*(s,\mathrm{Pr})}\\ &= \frac{\mu_0}{s+\mu_0}\left\{1-\frac{h_k^*(s)[1-G_m^*(\theta_0)]}{h_m^*(s)-h_{m-1}^*(s)G_m^*(\theta_0)}\right\},\\ \mathcal{T}_k^*(s,\mathrm{Sr}) &= H_k^*(s,\mathrm{Sr}) + \frac{H_k^*(s,\mathrm{Pr})W_m^*(s,\mathrm{Rs})H_{m-1}^*(s,\mathrm{Sr})}{1-W_m^*(s,\mathrm{Rs})H_{m-1}^*(s,\mathrm{Pr})}\\ &= \frac{\mu_0}{s+\mu_0}\left\{1-\frac{h_k^*(s)[1-G_m^*(s+\theta_0)]}{h_m^*(s)-h_{m-1}^*(s)G_m^*(s+\theta_0)}\right\}.\end{aligned}$$

Then we get the probability of service completion

$$\begin{aligned}\mathcal{P}_k\{\mathrm{Sr}\} &= p_k\{\mathrm{Sr}\} + \frac{p_k\{\mathrm{Pr}\}p_{m-1}\{\mathrm{Sr}\}p_m\{\mathrm{Rs}\}}{1-p_{m-1}\{\mathrm{Pr}\}p_m\{\mathrm{Rs}\}}\\ &= 1-\frac{h_k^*(0)[1-G_m^*(\theta_0)]}{h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta_0)} = 1-\mathcal{P}_k\{\mathrm{Ab}\}\end{aligned}$$

and the mean waiting and service time until service completion

$$\begin{aligned}E[\mathcal{W}_k,\mathrm{Sr}] &= \frac{h_k^*(0)[h_{m-1}^*(0)-h_m^*(0)]G_m'(\theta_0)}{[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta_0)]^2},\\ E[\mathcal{H}_k,\mathrm{Sr}] &= \frac{1}{\mu_0}\mathcal{P}_k\{\mathrm{Sr}\} - [1-G_m^*(\theta_0)]\\ &\quad\times\left\{\frac{h_k^*(0)[h_m'(0)-h_{m-1}'(0)G_m^*(\theta_0)]}{[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta_0)]^2} - \frac{h_k'(0)}{h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta_0)}\right\},\\ E[\mathcal{W}_k\mathcal{H}_k,\mathrm{Sr}] &= \frac{1}{\mu_0}E[\mathcal{W}_k,\mathrm{Sr}] - G_m'(\theta_0)\\ &\quad\times\left\{\frac{2h_k^*(0)[h_m^*(0)-h_{m-1}^*(0)][h_m'(0)-h_{m-1}'(0)G_m^*(\theta_0)]}{[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta_0)]^3}\right.\\ &\qquad\left. - \frac{h_k^*(0)[h_m'(0)-h_{m-1}'(0)]+h_k'(0)[h_m^*(0)-h_{m-1}^*(0)]}{[h_m^*(0)-h_{m-1}^*(0)G_m^*(\theta_0)]^2}\right\}.\end{aligned}$$

(2) For the unique customer waiting in state k, $k \geq m$, we have

$$\mathcal{T}_k^*(s, s', \mathrm{Sr}) = \frac{W_k^*(s, \mathrm{Rs}) H_{m-1}^*(s', \mathrm{Sr})}{1 - W_m^*(s, \mathrm{Rs}) H_{m-1}^*(s', \mathrm{Pr})}$$
$$= \frac{\mu_0}{s' + \mu_0} \cdot \frac{[h_m^*(s') - h_{m-1}^*(s')] G_k^*(s + \theta_0)}{h_m^*(s') - h_{m-1}^*(s') G_m^*(s + \theta_0)}.$$

This joint distribution leads to the marginal distributions

$$\mathcal{W}_k^*(s, \mathrm{Sr}) = \frac{p_{m-1}\{\mathrm{Sr}\} W_k^*(s, \mathrm{Rs})}{1 - p_{m-1}\{\mathrm{Pr}\} W_m^*(s, \mathrm{Rs})} = \frac{[h_m^*(0) - h_{m-1}^*(0)] G_k^*(s + \theta_0)}{h_m^*(0) - h_{m-1}^*(0) G_m^*(s + \theta_0)},$$
$$\mathcal{H}_k^*(s, \mathrm{Sr}) = \frac{p_k\{\mathrm{Rs}\} H_{m-1}^*(s, \mathrm{Sr})}{1 - p_m\{\mathrm{Rs}\} H_{m-1}^*(s, \mathrm{Pr})} = \frac{\mu_0}{s + \mu_0} \cdot \frac{[h_m^*(s) - h_{m-1}^*(s)] G_k^*(\theta_0)}{h_m^*(s) - h_{m-1}^*(s) G_m^*(\theta_0)},$$
$$\mathcal{T}_k^*(s, \mathrm{Sr}) = \frac{W_k^*(s, \mathrm{Rs}) H_{m-1}^*(s, \mathrm{Sr})}{1 - W_m^*(s, \mathrm{Rs}) H_{m-1}^*(s, \mathrm{Pr})}$$
$$= \frac{\mu_0}{s + \mu_0} \cdot \frac{[h_m^*(s) - h_{m-1}^*(s)] G_k^*(s + \theta_0)}{h_m^*(s) - h_{m-1}^*(s) G_m^*(s + \theta_0)}.$$

Then we get the probability of service completion

$$\mathcal{P}_k\{\mathrm{Sr}\} = \frac{p_{m-1}\{\mathrm{Sr}\} p_k\{\mathrm{Rs}\}}{1 - p_{m-1}\{\mathrm{Pr}\} p_m\{\mathrm{Rs}\}} = \frac{[h_m^*(0) - h_{m-1}^*(0)] G_k^*(\theta_0)}{h_m^*(0) - h_{m-1}^*(0) G_m^*(\theta_0)} = 1 - \mathcal{P}_k\{\mathrm{Ab}\},$$

and the mean waiting and service time until service completion

$$E[\mathcal{W}_k, \mathrm{Sr}] = [h_{m-1}^*(0) - h_m^*(0)]$$
$$\times \left\{ \frac{G_k'(\theta_0)}{h_m^*(0) - h_{m-1}^*(0) G_m^*(\theta_0)} + \frac{h_{m-1}^*(0) G_k^*(\theta_0) G_m'(\theta_0)}{[h_m^*(0) - h_{m-1}^*(0) G_m^*(\theta_0)]^2} \right\},$$
$$E[\mathcal{H}_k, \mathrm{Sr}] = \frac{1}{\mu_0} \mathcal{P}_k\{\mathrm{Sr}\}$$
$$- \frac{G_k^*(\theta_0)[h_m'(0) h_{m-1}^*(0) - h_{m-1}'(0) h_m^*(0)][1 - G_m^*(\theta_0)]}{[h_m^*(0) - h_{m-1}^*(0) G_m^*(\theta_0)]^2},$$
$$E[\mathcal{W}_k \mathcal{H}_k, \mathrm{Sr}] = \frac{1}{\mu_0} E[\mathcal{W}_k, \mathrm{Sr}] - [h_m'(0) h_{m-1}^*(0) - h_{m-1}'(0) h_m^*(0)]$$
$$\times \left\{ \frac{G_k^*(\theta_0) G_m'(\theta_0)[h_m^*(0) - 2h_{m-1}^*(0) + h_{m-1}^*(0) G_m^*(\theta_0)]}{[h_m^*(0) - h_{m-1}^*(0) G_m^*(\theta_0)]^3} \right.$$
$$\left. - \frac{G_k'(\theta_0)[1 - G_m^*(\theta_0)]}{[h_m^*(0) - h_{m-1}^*(0) G_m^*(\theta_0)]^2} \right\}.$$

4.3 Waiting and Service Time Until Departure

We finally consider the waiting and service time until departure (either abandonment or service completion) for a unique customer in state k ($k \geq 0$). Let

$$\mathcal{T}_k^*(s, s') := \mathcal{T}_k^*(s, s', \mathrm{Ab}) + \mathcal{T}_k^*(s, s', \mathrm{Sr}) \qquad k \geq 0$$

be the unconditional joint LST of the DF for the waiting and service time for the unique customer in state k. Then, we obtain the marginal LSTs of the DF for the waiting time, the service time, and the total time spent in the system as follows:

$$\mathcal{W}_k^*(s) := \mathcal{T}_k^*(s,0); \quad \mathcal{H}_k^*(s) := \mathcal{T}_k^*(0,s); \quad \mathcal{T}_k^*(s) := \mathcal{T}_k^*(s,s) \qquad k \geq 0.$$

(1) For the unique customer being served in state k, $0 \leq k \leq m-1$, we have

$$\mathcal{T}_k^*(s,s') = \frac{\mu_0}{s'+\mu_0} + \left(\frac{\theta_0}{s+\theta_0} - \frac{\mu_0}{s'+\mu_0}\right) \frac{h_k^*(s')[1-G_m^*(s+\theta_0)]}{h_m^*(s') - h_{m-1}^*(s')G_m^*(s+\theta_0)}.$$

This joint distribution leads to the marginal distributions

$$\begin{aligned}
\mathcal{W}_k^*(s) &= \mathcal{W}_k^*(s,\mathrm{Ab}) + \mathcal{W}_k^*(s,\mathrm{Sr}) \\
&= 1 - \frac{s}{s+\theta_0} \cdot \frac{h_k^*(0)[1-G_m^*(s+\theta_0)]}{h_m^*(0) - h_{m-1}^*(0)G_m^*(s+\theta_0)}, \\
\mathcal{H}_k^*(s) &= \mathcal{H}_k^*(s,\mathrm{Ab}) + \mathcal{H}_k^*(s,\mathrm{Sr}) \\
&= \frac{\mu_0}{s+\mu_0} + \frac{s}{s+\mu_0} \cdot \frac{h_k^*(s)[1-G_m^*(\theta_0)]}{h_m^*(s) - h_{m-1}^*(s)G_m^*(\theta_0)}, \\
\mathcal{T}_k^*(s) &= \mathcal{T}_k^*(s,\mathrm{Ab}) + \mathcal{T}_k^*(s,\mathrm{Sr}) \\
&= \frac{\mu_0}{s+\mu_0} + \left(\frac{\theta_0}{s+\theta_0} - \frac{\mu_0}{s+\mu_0}\right) \frac{h_k^*(s)[1-G_m^*(s+\theta_0)]}{h_m^*(s) - h_{m-1}^*(s)G_m^*(s+\theta_0)}.
\end{aligned}$$

The mean waiting time, service time, and total time until departure are given by

$$\begin{aligned}
E[\mathcal{W}_k] &= \frac{h_k^*(0)[1-G_m^*(\theta_0)]}{\theta_0[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta_0)]}, \\
E[\mathcal{H}_k] &= \frac{1}{\mu_0}\left\{1 - \frac{h_k^*(0)[1-G_m^*(\theta_0)]}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta_0)}\right\}, \\
E[\mathcal{T}_k] &= E[\mathcal{W}_k] + E[\mathcal{H}_k] = \frac{1}{\mu_0} + \left(\frac{1}{\theta_0} - \frac{1}{\mu_0}\right) \frac{h_k^*(0)[1-G_m^*(\theta_0)]}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta_0)}.
\end{aligned}$$

We also have

$$\begin{aligned}
E[\mathcal{W}_k\mathcal{H}_k] &= E[\mathcal{W}_k\mathcal{H}_k,\mathrm{Ab}] + E[\mathcal{W}_k\mathcal{H}_k,\mathrm{Sr}] = \frac{1}{\theta_0}E[\mathcal{H}_k,\mathrm{Ab}] + \frac{1}{\mu_0}E[\mathcal{W}_k,\mathrm{Sr}] \\
&= -\frac{h_k^*(0)[h_m^*(0) - h_{m-1}^*(0)]G_m'(\theta_0)}{\mu[h_m^*(0) - h_{m-1}^*(s)G_m^*(\theta_0)]^2} - \frac{1-G_m^*(\theta_0)}{\theta} \\
&\quad \times \left\{\frac{h_k'(0)}{h_m^*(0) - h_{m-1}^*(s)G_m^*(\theta_0)} - \frac{h_k^*(0)[h_m'(0) - h_{m-1}'(0)G_m^*(\theta_0)]}{[h_m^*(0) - h_{m-1}^*(s)G_m^*(\theta_0)]^2}\right\}.
\end{aligned}$$

(2) For the unique customer waiting in state k, $k \geq m$, we have

$$\mathcal{T}_k^*(s,s') = \frac{\theta_0}{s+\theta_0} + \left(\frac{\mu_0}{s'+\mu_0} - \frac{\theta_0}{s+\theta_0}\right) \frac{[h_m^*(s') - h_{m-1}^*(s')]G_k^*(s+\theta_0)}{h_m^*(s') - h_{m-1}^*(s')G_m^*(s+\theta_0)}.$$

This joint distribution leads to the marginal distributions

$$
\begin{aligned}
\mathcal{W}_k^*(s) &= \frac{\theta_0}{s+\theta_0} + \frac{s}{s+\theta_0} \cdot \frac{[h_m^*(0) - h_{m-1}^*(0)]G_k^*(s+\theta_0)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(s+\theta_0)}, \\
\mathcal{H}_k^*(s) &= 1 - \frac{s}{s+\mu_0} \cdot \frac{[h_m^*(s) - h_{m-1}^*(s)]G_k^*(\theta_0)}{h_m^*(s) - h_{m-1}^*(s)G_m^*(\theta_0)}, \\
\mathcal{T}_k^*(s) &= \frac{\theta_0}{s+\theta_0} + \left(\frac{\mu_0}{s+\mu_0} - \frac{\theta_0}{s+\theta_0} \right) \frac{[h_m^*(s) - h_{m-1}^*(s)]G_k^*(s+\theta_0)}{h_m^*(s) - h_{m-1}^*(s)G_m^*(s+\theta_0)}.
\end{aligned}
$$

The mean waiting time, service time, and total time until departure are given by

$$
\begin{aligned}
E[\mathcal{W}_k] &= \frac{1}{\theta_0} \left\{ 1 - \frac{h_m^*(0) - h_{m-1}^*(0)]G_k^*(\theta_0)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta_0)} \right\}, \\
E[\mathcal{H}_k] &= \frac{[h_m^*(0) - h_{m-1}^*(0)]G_k^*(\theta_0)}{\mu_0[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta_0)]}, \\
E[\mathcal{T}_k] &= E[\mathcal{W}_k] + E[\mathcal{H}_k] = \frac{1}{\theta_0} + \left(\frac{1}{\mu_0} - \frac{1}{\theta_0} \right) \frac{h_m^*(0) - h_{m-1}^*(0)]G_k^*(\theta_0)}{h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta_0)}.
\end{aligned}
$$

We also have

$$
\begin{aligned}
E[\mathcal{W}_k\mathcal{H}_k] &= E[\mathcal{W}_k\mathcal{H}_k, \mathrm{Ab}] + E[\mathcal{W}_k\mathcal{H}_k, \mathrm{Sr}] = \frac{1}{\theta_0}E[\mathcal{H}_k, \mathrm{Ab}] + \frac{1}{\mu_0}E[\mathcal{W}_k, \mathrm{Sr}] \\
&= -\frac{[h_m^*(0) - h_{m-1}^*(0)]G_k'(\theta_0)}{\mu[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta_0)]} \\
&\quad - \frac{G_k^*(\theta_0)[h_m^*(0) - h_{m-1}^*(0)]h_{m-1}^*(0)G_m'(\theta_0)}{\mu[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta_0)]^2} \\
&\quad + \frac{G_k^*(\theta_0)[h_m'(0)h_{m-1}^*(0) - h_m^*(0)h_{m-1}'(0)][1 - G_m^*(\theta_0)]}{\theta[h_m^*(0) - h_{m-1}^*(0)G_m^*(\theta_0)]^2}.
\end{aligned}
$$

(3) Recursive relations among moments of distribution for the waiting and service time.
From the explicit expressions for $\mathcal{T}_k^*(s, s', \mathrm{Ab})$, $\mathcal{T}_k^*(s, s', \mathrm{Sr})$, and $\mathcal{T}_k^*(s, s')$ given above, it can be shown that the unconditional and conditional joint LST of the DF for the waiting and service time until departure for a unique customer in state k satisfies the following relation in both cases $0 \le k \le m-1$ and $k \ge m$:

$$
\mathcal{T}_k^*(s, s') = 1 - \frac{s}{\theta_0}\mathcal{T}_k^*(s, s', \mathrm{Ab}) - \frac{s'}{\mu_0}\mathcal{T}_k^*(s, s', \mathrm{Sr}) \qquad k \ge 0.
$$

This yields the recursive relations among unconditional and conditional moments

$$
E[\mathcal{W}_k^\ell \mathcal{H}_k^{\ell'}] = \frac{\ell}{\theta_0}E[\mathcal{W}_k^{\ell-1}\mathcal{H}_k^{\ell'}, \mathrm{Ab}] + \frac{\ell'}{\mu_0}E[\mathcal{W}_k^\ell \mathcal{H}_k^{\ell'-1}, \mathrm{Sr}] \qquad \ell, \ell' = 2, 3, \dots .
$$

In particular, we have

$$
\begin{aligned}
&E[\mathcal{W}_k] = \frac{1}{\theta_0}\mathcal{P}_k\{\text{Ab}\}; \quad E[\mathcal{H}_k] = \frac{1}{\mu_0}\mathcal{P}_k\{\text{Sr}\},\\
&E[\mathcal{W}_k\mathcal{H}_k] = \frac{1}{\theta_0}E[\mathcal{H}_k,\text{Ab}] + \frac{1}{\mu_0}E[\mathcal{W}_k,\text{Sr}],\\
&E[\mathcal{W}_k^\ell] = \frac{\ell}{\theta_0}E[\mathcal{W}_k^{\ell-1},\text{Ab}]; \quad E[\mathcal{H}_k^\ell] = \frac{\ell}{\mu_0}E[\mathcal{H}_k^{\ell-1},\text{Sr}] \qquad \ell = 2,3,\dots\,.
\end{aligned}
$$

Furthermore, it follows from the relation

$$
\mathcal{T}_k^*(s) = 1 - \frac{s}{\theta_0}\mathcal{T}_k^*(s,\text{Ab}) - \frac{s}{\mu_0}\mathcal{T}_k^*(s,\text{Sr}) \qquad k \geq 0
$$

(or from $\mathcal{T}_k = \mathcal{W}_k + \mathcal{H}_k$) that

$$
\begin{aligned}
E[\mathcal{T}_k] &= \frac{1}{\theta_0}\mathcal{P}_k\{\text{Ab}\} + \frac{1}{\mu_0}\mathcal{P}_k\{\text{Sr}\},\\
E[\mathcal{T}_k^\ell] &= \frac{\ell}{\theta_0}E[\mathcal{T}_k^{\ell-1},\text{Ab}] + \frac{\ell}{\mu_0}E[\mathcal{T}_k^{\ell-1},\text{Sr}] \qquad \ell = 2,3,\dots\,.
\end{aligned}
$$

5 Numerical Example

Numerical values are shown in Table 1, where we assume $m = 5$, $\mu = 1$, $\theta = 2$, and $\lambda = 10$ for a more patient customer ($\theta_0 = 1$) and for a less patient customer ($\theta_0 = 4$) with $\mu_0 = \mu$. The performance of an arriving unique customer can be found in the row of $k = 0$ in these tables.

From the numerical results for a more patient customer, we observe the following:

- The probability of service completion is higher.
- The time spent in the system is longer whether he abandons waiting or he gets served.
- The received service time is not much different from other customers.

This observation agrees with our feeling that we had better be more patient than other customers for secure service completion, though it takes us more time.

It remains us to investigate closely the trade-off between the probability of service completion and the time spent by a unique customer who gets served depending on the degree of his patience.

Table 1. Numerical example for a unique customer.

(a) More patient customer ($\theta_0 = 1 < \theta = 2,\ \mu_0 = \mu$)								
k	$\mathcal{P}_k\{\text{Ab}\}$	$\mathcal{P}_k\{\text{Sr}\}$	$E[\mathcal{W}_k, \text{Ab}]$	$E[\mathcal{H}_k, \text{Ab}]$	$E[\mathcal{T}_k, \text{Ab}]$	$E[\mathcal{W}_k, \text{Sr}]$	$E[\mathcal{H}_k, \text{Sr}]$	$E[\mathcal{T}_k, \text{Sr}]$
0	0.49026	0.50974	0.45098	0.31260	0.76358	0.03928	0.19714	0.23642
1	0.53929	0.46071	0.49608	0.29483	0.79091	0.04320	0.16588	0.20909
2	0.59812	0.40188	0.55020	0.26861	0.81881	0.04792	0.13327	0.18119
3	0.66969	0.33031	0.61604	0.23042	0.84646	0.05365	0.09989	0.15354
4	0.75814	0.24186	0.69740	0.17503	0.87243	0.06074	0.06683	0.12757
5	0.86933	0.13067	0.79968	0.09496	0.89425	0.06964	0.03611	0.10575
6	0.91186	0.08814	0.84386	0.06379	0.90765	0.06800	0.02436	0.09235
7	0.93281	0.06719	0.86798	0.04862	0.91661	0.06483	0.01857	0.08339
8	0.94495	0.05505	0.88321	0.03984	0.92305	0.06174	0.01521	0.07965
9	0.95280	0.04720	0.89379	0.03415	0.92795	0.05901	0.01304	0.07205
10	0.95829	0.04171	0.90164	0.03018	0.93183	0.05665	0.01153	0.06817
15	0.97174	0.02826	0.92322	0.02045	0.94367	0.04852	0.00781	0.05633
20	0.97741	0.02259	0.93375	0.01635	0.95010	0.04366	0.00624	0.04990
30	0.98284	0.01716	0.94505	0.01242	0.95747	0.03779	0.00474	0.04253
(b) Equally patient customer ($\theta_0 = 2 = \theta,\ \mu_0 = \mu$)								
k	$\mathcal{P}_k\{\text{Ab}\}$	$\mathcal{P}_k\{\text{Sr}\}$	$E[\mathcal{W}_k, \text{Ab}]$	$E[\mathcal{H}_k, \text{Ab}]$	$E[\mathcal{T}_k, \text{Ab}]$	$E[\mathcal{W}_k, \text{Sr}]$	$E[\mathcal{H}_k, \text{Sr}]$	$E[\mathcal{T}_k, \text{Sr}]$
0	0.51270	0.48730	0.24299	0.30992	0.55291	0.01336	0.17738	0.19074
1	0.56396	0.43604	0.26729	0.28965	0.55693	0.01470	0.14639	0.16108
2	0.62549	0.37451	0.29645	0.26019	0.55663	0.01630	0.11432	0.13062
3	0.70034	0.29966	0.33192	0.21777	0.54969	0.01825	0.08189	0.10014
4	0.79283	0.20717	0.37576	0.15678	0.53254	0.02066	0.05039	0.07105
5	0.90911	0.09089	0.43087	0.06878	0.49965	0.02369	0.02211	0.04579
6	0.94907	0.05093	0.45368	0.03854	0.49222	0.02085	0.01239	0.03324
7	0.96686	0.03314	0.46548	0.02508	0.49056	0.01795	0.00806	0.02601
8	0.97624	0.02376	0.47252	0.01798	0.49050	0.01560	0.00578	0.02138
9	0.98181	0.01819	0.47713	0.01377	0.49090	0.01378	0.00442	0.01820
10	0.98541	0.01459	0.48038	0.01104	0.49142	0.01233	0.00355	0.01588
15	0.99293	0.00707	0.48827	0.00535	0.49362	0.00819	0.00172	0.00991
20	0.99541	0.00459	0.49146	0.00347	0.49493	0.00624	0.00112	0.00736
30	0.99732	0.00268	0.49432	0.00203	0.49635	0.00434	0.00065	0.00499
(c) Less patient customer ($\theta_0 = 4 > \theta = 2,\ \mu_0 = \mu$)								
k	$\mathcal{P}_k\{\text{Ab}\}$	$\mathcal{P}_k\{\text{Sr}\}$	$E[\mathcal{W}_k, \text{Ab}]$	$E[\mathcal{H}_k, \text{Ab}]$	$E[\mathcal{T}_k, \text{Ab}]$	$E[\mathcal{W}_k, \text{Sr}]$	$E[\mathcal{H}_k, \text{Sr}]$	$E[\mathcal{T}_k, \text{Sr}]$
0	0.52855	0.47145	0.12723	0.30714	0.43437	0.00490	0.16431	0.16921
1	0.58141	0.41859	0.13996	0.28500	0.42495	0.00540	0.13359	0.13899
2	0.64483	0.35517	0.15522	0.25314	0.40837	0.00598	0.10202	0.10801
3	0.72200	0.27800	0.17380	0.20760	0.38140	0.00670	0.07040	0.07710
4	0.81735	0.18265	0.19675	0.14250	0.33925	0.00758	0.04015	0.04773
5	0.93723	0.06277	0.22561	0.04897	0.27458	0.00870	0.01380	0.02249
6	0.97206	0.02794	0.23656	0.02180	0.25836	0.00646	0.00614	0.01260
7	0.98526	0.01474	0.24164	0.01150	0.25314	0.00467	0.00324	0.00791
8	0.99125	0.00875	0.24435	0.00682	0.25117	0.00347	0.00192	0.00539
9	0.99425	0.00565	0.24594	0.00441	0.25035	0.00265	0.00124	0.0389
10	0.99610	0.00390	0.24694	0.00304	0.24998	0.00209	0.00086	0.00294
15	0.99893	0.00107	0.24888	0.00084	0.24972	0.00085	0.00024	0.00108
20	0.99952	0.00048	0.24942	0.00037	0.24979	0.00046	0.00010	0.00057
30	0.99983	0.00017	0.24975	0.00013	0.24989	0.00020	0.00004	0.00024

(Continued)

Table 1. *(Continued)*

(d) More patient customer ($\theta_0 = 1 < \theta = 2,\ \mu_0 = \mu$)

k	$E[\mathcal{W}_k]$	$E[\mathcal{H}_k]$	$E[\mathcal{T}_k]$	$E[\mathcal{W}_k\mathcal{H}_k, \mathrm{Ab}]$	$E[\mathcal{W}_k\mathcal{H}_k, \mathrm{Sr}]$	$E[\mathcal{W}_k\mathcal{H}_k]$	$E[\mathcal{T}_k^2]$
0	0.49026	0.50974	1.00000	0.31598	0.03590	0.35187	2.0000
1	0.53929	0.46071	1.00000	0.30248	0.03556	0.22804	2.0000
2	0.59812	0.40188	1.00000	0.28117	0.03476	0.31563	2.0000
3	0.66969	0.33031	1.00000	0.25078	0.03328	0.28407	2.0000
4	0.75814	0.24186	1.00000	0.20496	0.03081	0.23577	2.0000
5	0.86933	0.13067	1.00000	0.13739	0.02682	0.16421	2.0000
6	0.91186	0.08814	1.00000	0.10788	0.02390	0.13178	2.0000
7	0.93281	0.06719	1.00000	0.09164	0.02181	0.11345	2.0000
8	0.94495	0.05505	1.00000	0.08132	0.02025	0.10157	2.0000
9	0.95280	0.04720	1.00000	0.07412	0.01904	0.09317	2.0000
10	0.95829	0.04171	1.00000	0.06876	0.01807	0.08673	2.0000
15	0.97174	0.02826	1.00000	0.05393	0.01505	0.06897	2.0000
20	0.97741	0.02259	1.00000	0.04663	0.01336	0.06001	2.0000
30	0.98284	0.01716	1.00000	0.03877	0.01144	0.05021	2.0000

(e) Equally patient customer ($\theta_0 = 2 = \theta,\ \mu_0 = \mu$)

k	$E[\mathcal{W}_k]$	$E[\mathcal{H}_k]$	$E[\mathcal{T}_k]$	$E[\mathcal{W}_k\mathcal{H}_k, \mathrm{Ab}]$	$E[\mathcal{W}_k\mathcal{H}_k, \mathrm{Sr}]$	$E[\mathcal{W}_k\mathcal{H}_k]$	$E[\mathcal{T}_k^2]$
0	0.25635	0.48730	0.74365	0.15700	0.01132	0.16832	0.93439
1	0.28198	0.43604	0.71802	0.14840	0.01112	0.15952	0.87910
2	0.31274	0.37451	0.68726	0.13565	0.01074	0.14639	0.81788
3	0.35017	0.29966	0.64983	0.11702	0.01011	0.12713	0.74997
4	0.39642	0.20717	0.60358	0.08994	0.00911	0.09905	0.67463
5	0.45456	0.09089	0.54544	0.05053	0.00755	0.05808	0.59124
6	0.47454	0.05093	0.52546	0.03405	0.00608	0.04012	0.55870
7	0.48343	0.03314	0.51567	0.02547	0.00502	0.03049	0.54257
8	0.48812	0.02376	0.51188	0.02033	0.00426	0.02459	0.53326
9	0.49090	0.01819	0.50910	0.01695	0.00371	0.02066	0.52729
10	0.49270	0.01459	0.50730	0.01456	0.00329	0.01785	0.52317
15	0.49647	0.00707	0.50353	0.00874	0.00213	0.01087	0.51345
20	0.49771	0.00459	0.50229	0.00637	0.00161	0.00798	0.50965
30	0.49866	0.00268	0.50134	0.00424	0.00111	0.00535	0.50633

(f) Less patient customer ($\theta_0 = 4 > \theta = 2,\ \mu_0 = \mu$)

k	$E[\mathcal{W}_k]$	$E[\mathcal{H}_k]$	$E[\mathcal{T}_k]$	$E[\mathcal{W}_k\mathcal{H}_k, \mathrm{Ab}]$	$E[\mathcal{W}_k\mathcal{H}_k, \mathrm{Sr}]$	$E[\mathcal{W}_k\mathcal{H}_k]$	$E[\mathcal{T}_k^2]$
0	0.13214	0.47145	0.60359	0.07776	0.00393	0.08169	0.55561
1	0.14535	0.41859	0.56394	0.07281	0.00383	0.07664	0.49046
2	0.16121	0.35517	0.51637	0.06560	0.00366	0.06927	0.42020
3	0.18050	0.27800	0.45850	0.05520	0.00340	0.05860	0.34489
4	0.20434	0.18265	0.38698	0.04022	0.00299	0.04321	0.26509
5	0.23431	0.06277	0.29708	0.01857	0.00237	0.02094	0.18228
6	0.24301	0.02794	0.27096	0.01028	0.00162	0.01190	0.15437
7	0.24632	0.01474	0.26105	0.00641	0.00113	0.00755	0.14240
8	0.24781	0.00875	0.25656	0.00435	0.00083	0.00517	0.13636
9	0.24859	0.00465	0.25424	0.00313	0.00062	0.00375	0.13296
10	0.24903	0.00390	0.25292	0.00236	0.00049	0.02850	0.13088
15	0.24973	0.00107	0.25080	0.00086	0.00019	0.00106	0.12703
20	0.24988	0.00048	0.25036	0.00045	0.00011	0.00056	0.12603
30	0.24996	0.00017	0.25013	0.00019	0.00005	0.00024	0.12543

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