

Laudatio of Bertram Kostant

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with Anthony Joseph and Shrawan Kumar

The 2016 Wigner Medal has been awarded to Bertram Kostant of the Massachusetts Institute of Technology (USA) for his fundamental contributions to the representation theory of Lie algebraic systems. Many of his results have led to new developments both in Mathematics and, as emphasized here, in Theoretical Physics.

For this occasion, let me highlight some of the themes in Kostant's work directly related to particle physics: Geometric quantization, convexity, and completely integrable systems. This brief account has been prepared with the help of Anthony Joseph and Shrawan Kumar.

The fundamental problem of quantum mechanics, as inaugurated by Dirac, is the passage from Hamiltonian mechanics to unitary representations of the symmetry group. Quantum mechanics should explain why some states of some physical systems take discrete values, and was directly motivated by the quantum theory of matter—at the time new—since it is the unitary transformations that preserve the all important probability density.

Valentine Bargmann and Eugene Wigner, the first recipients in 1978 of the Wigner medal, would have been delighted by the choice of the new laureate. Indeed, in his fundamental paper *Quantization and Unitary representations* (1970) B. Kostant showed that only those Hamiltonian manifolds admitting a prequantum line bundle, now called the Kostant line bundle, are candidates for giving rise to unitary representations of the symmetry group. Applied to the Poincaré group, this provided

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a clear theoretical understanding as to why a massive elementary particle must have a discrete spin.

Convexity theorems are important in determining the domain where experiments should be done.

For completely integrable systems, the Hamiltonian equation is solvable in an explicit fashion, the classical example being that of the Kepler laws of planetary motion. The Toda lattice, originally introduced as a simple model for a one-dimensional crystal, was generalized by Kostant into a multi-dimensional completely integrable system defined for any semisimple Lie algebra. A very simple and brilliant idea of Kostant produces a maximal algebra of Poisson commuting functions. Furthermore, the representation theory of semisimple groups allows us to compute the evolution law of the system.

Let me comment in more detail on Geometric Quantization, its history and its recent developments along the lines of Kostant's theory. There were many ways, apparently very different, to construct unitary representations of Lie groups. For example, the unitary representation of the Heisenberg group in the Bargmann-Fock space of holomorphic functions on the n -dimensional complex vector space, the Borel-Weil-Bott construction of the irreducible representations of a compact Lie group K on the $\bar{\partial}$ cohomology of flag manifolds with line bundles, Kirillov's construction of unitary representations of unipotent Lie groups by polarizing coadjoint orbits, Harish-Chandra's construction of unitary representations of real semisimple Lie groups based on differential equations and induction. Kostant saw that all these constructions are part of the unique scheme of quantum mechanics: passing from a classical phase space to a Hilbert space. Kostant realized the fundamental fact that any coadjoint orbit of a Lie group gives a Hamiltonian system. These systems are the most basic ones: any Hamiltonian manifold with a transitive action of a Lie group covers a coadjoint orbit, and those that are quantizable cover an orbit satisfying some discrete integrality conditions.

Furthermore, Kostant explained quantum conditions in terms of Chern classes of line bundles: a quantizable manifold is a symplectic manifold equipped with a prequantum line bundle, now called the Kostant line bundle. It could be "quantized" as a unitary representation of the underlying Lie group of symmetry if a suitable "polarization" could be found. This separates (removes) one half of the variables of phase space, a process that encapsulated Dirac's original insight.

Building on the Bargmann-Fock realization of representations of the Heisenberg group and of the quantum harmonic oscillator, Kostant considered complex polarizations, and the notion that the corresponding Hilbert space of sections is to be found among holomorphic sections, or going into cohomological constructions among solutions of a Dirac operator.

As a first successful use of geometric quantization, Kostant (with Auslander) classified the unitary representations of real class 1 simply-connected solvable Lie groups. Geometric quantization greatly generalizes provided one allows for cohomological methods and the study of the complex structure associated to a polarization.

Kostant's study of the homology of certain nilpotent Lie algebras encompasses the Borel-Weil-Bott theorem for compact Lie groups, and is used as a fundamental tool in constructing unitary representations for any real semisimple Lie group. Finally, as shown by Duflou, and also following the deep work of many authors, notably Schmid on the discrete series, geometric quantization of admissible coadjoint orbits of maximum dimension produces most (but not all) unitary representations of any real Lie group.

Geometric quantization applies to any Hamiltonian manifold. The main intrinsic object is the Kostant line bundle, together with its connection. This provides a moment map, and a notion of reduction. The most basic pieces of geometric quantization theory are quantization of coadjoint orbits. It was shown by Meinrenken-Sjamaar how to associate to any Kostant line bundle on a manifold with a compact group of symmetry a quantum model made up of these basic pieces and reflecting the semi-classical properties obtained at the asymptotic limit.

Severe difficulties may arise in quantizing a general Hamiltonian manifold with an arbitrary symmetry group, involving the absence of a suitable polarization and the verification of unitarity. The quantization of "small" coadjoint orbits or real semisimple Lie groups are of particular interest because they lead to many relations outside of those of the Lie algebra which are often just those of a physical system. The quantization of those orbits is difficult to construct. It may seem paradoxical that it is more difficult to quantize small coadjoint orbits than orbits of maximal dimensions. This is because they are small dimensional manifolds, but with a large group of symmetries and it may not be possible to integrate the full group of symmetries with a group of symmetries of the quantized space. If the Hamiltonian space is just one point with a trivial line bundle, then the quantization is just the trivial representation of the group G .

Models of quantization are usually produced by producing several models with different groups of symmetry, and then piecing these models together. This is the way that the metaplectic representation, a representation of the full group of symmetries of the simplest phase space T^*R^n , was constructed by Segal-Shale-Weil using the uniqueness of the canonical commutation relation. The following is one of the most fundamental representations, namely, the quantization of the minimal orbit of the symplectic group. With R. Brylinski, Kostant constructed uniform Fock space models for quantizing minimal orbits. Kostant showed that the smallest non-trivial orbit (for a semisimple Lie algebra) is defined by quadratic relations, thereby giving rise to a so-called quadratic algebra. This result is of great importance. In particular, this quadratic algebra was shown to be Koszul, which meant that it could be rather readily quantized — Gerstenhaber's ghastly infinite set of quantization conditions thereby reduces to just three. Imitating this, symplectic reflection algebras were defined and have proved to be central to the understanding of several physical systems, notably the Knizhnik-Zamolodchikov equations arising in the study of quantum many-body problems.

Pursuing the work of Valentine Bargmann on the complementary series, Kostant computed a remarkable determinant (for real Lie groups) whose description still provides one of the best tests for unitarity of the complementary series. The Kostant

determinant had many other generalizations, notably by Parthasarathy-Ranga Rao-Varadarajan, Shapovalev, Jantzen and Kac. These have been used many times as a criteria for irreducibility and unitarity, and even for some infinite dimensional Lie groups.

Kostant's work on Lie algebra cohomology is an essential tool in representation theory. It was for example influential on Vogan's algebraic approach to the classification of irreducible representations of semisimple Lie groups, in particular in the success of the Atlas team in finding all unitary representations of the split form of E_8 . Let us quote Bert: "Dealing with E_8 is like looking at a diamond...from one direction, one sees $2s$ all over the place, from another direction, one sees $3s$, from a third direction, one sees $5s$,... it is magnificent, it is a symphony in the numbers 2,3 and 5".

Let us recall at this point that the classification of all irreducible unitary representations of a real Lie semisimple Lie group is still an open problem.

Can "everything" be quantized? Yes, if one abandons the idea of unitarity. Deformation quantization is in some sense an infinitesimal version of geometric quantization, and it might not be possible to integrate the symmetries. Using the powerful techniques of Feynman graphs, Kontsevich showed that deformation quantization allows us to produce a quantization of the commutation relations of any Poisson manifold as a formal series. This striking result of Kontsevich relies in part on the fundamental Hochschild-Kostant-Rosenberg theorem identifying the Hochschild homology of an affine regular algebra.

One important object of quantization is the study of the spectra of matrices. The simplest case of representation theory is to study the decomposition of a Hermitian space under the action of a Hermitian matrix. Horn-Schur showed that the diagonal of a Hermitian matrix with prescribed spectrum always lies in some convex polytope, the vertices being obtained when the matrix itself is diagonal. Convexity results are important notably in studying measurements related to quantum computers. Kostant generalizes convexity results for linear projections, and also in the context of the Iwasawa decomposition $G = KAN$ of a real Lie group. It led to a further decomposition $G = KNK$ of the latter. Moreover it provided a generalization of the Golden-Thompson rule which was widely used in the C^* algebra approach to quantum field theory.

Let us now discuss completely integrable systems. One of Kostant's most influential articles is his paper on the *Toda Lattice* in 1978. The fact that the solution of the Toda lattice problem can be solved by the representation theory of the corresponding semisimple Lie algebra is a result of astonishing beauty and significance. It was the ingenuity of Kostant who could see at the time the interplay between coadjoint orbits of the Borel subgroup, a Hamiltonian manifold with a solvable Lie group of symmetry, and invariant polynomials of the corresponding semisimple Lie algebra, two of Kostant's favorite subjects. As is the case with several of Kostant's ideas, it is a brilliant, surprising, yet very simple idea.

The Toda lattice, originally introduced as a simple model for a one-dimensional crystal, was transformed by Kostant into a multi-dimensional completely integrable system defined for any semisimple Lie algebra. The quantization of a transversal

slice, first undertaken by Kostant in the regular case and leading to the Whittaker model, was further developed by many in greater generality. This eventually allowed researchers (Premet, Losev, and others) to show that any nilpotent orbit could be quantized. When applied to Kac-Moody infinite dimensional Lie algebras, the Drinfeld-Sokolov generalization of the Toda system leads to W -algebras, the latter being important in the study of conformal field theory.

In this short talk, it is impossible to mention Kostant's various contributions to Pure Mathematics. Kostant is counted as one of the most remarkable mathematicians of the latter half of the last century in Lie Theory. Every paper by Kostant has a life of its own, being the precursor of many developments in representation theory of semisimple Lie groups and quantum groups, some developments being completely unexpected. His works and ideas have inspired innumerable mathematicians.

Each of the papers is a bright star in the dark sky of our knowledge. And, over the years, it has formed a beautiful constellation.

Thank you, Bert, for all this beautiful mathematics.

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