

Chapter 2

Solitons and Homotopy

This chapter attempts to guide the readers through several concepts related to homotopy using explicit examples. Many familiar and some unfamiliar topological objects will be introduced along the way. Kinks in one space dimension and vortex in two dimensions are both characterized by the homotopy of a circle. Skyrmions in two, and hedgehogs in three dimensions are objects tied to the homotopy of a sphere. Tony Skyrme's original vision of treating elementary particles by invoking the topology of a three-sphere only arose in three dimensions. Although magnetic systems in condensed matter have not quite realized Skyrme's vision, ample sightings of two-dimensional skyrmions exist today.

2.1 Kink in One Dimension

Consider a pair of real-valued fields $\mathbf{n} = (n_1, n_2)$ that depends on time t and one spatial coordinate x . A simple Lagrangian that is symmetric under the $O(2)$ rotation $\mathbf{n} \rightarrow \mathcal{R}\mathbf{n}$ can be constructed from the combinations of $(\partial_\mu \mathbf{n}) \cdot (\partial_\mu \mathbf{n})$ and $\mathbf{n} \cdot \mathbf{n}$ terms ($\mu = t, x$):

$$L = \frac{1}{2} \int dx [(\partial_t \mathbf{n})^2 - (\partial_x \mathbf{n})^2 - m^2 \mathbf{n} \cdot \mathbf{n}], \quad (2.1)$$

where, for convenience, all the dimensional factors are set equal to unity. The associated equation of motion is a linear partial differential equation,

$$(-\partial_t^2 + \partial_x^2 - m^2)\mathbf{n} = 0, \quad (2.2)$$

which qualifies the Lagrangian (2.1) as an example of a *linear* field theory. The particular example we have here is known as the Klein-Gordon equation.

The action $S = \int dt L$ remains invariant under the translation $t \rightarrow t + \delta t$, and $x \rightarrow x + \delta x$. Both of these symmetries lead to corresponding conservation laws in accordance with Noether's theorem,

$$\begin{aligned} \partial_t [(\partial_t \mathbf{n})^2 + (\partial_x \mathbf{n})^2 + m^2 \mathbf{n}^2] + \partial_x [-2\partial_t \mathbf{n} \cdot \partial_x \mathbf{n}] &= 0 \quad (t \rightarrow t + \delta t), \\ \partial_t [-2\partial_t \mathbf{n} \cdot \partial_x \mathbf{n}] + \partial_x [(\partial_t \mathbf{n})^2 + (\partial_x \mathbf{n})^2 - m^2 \mathbf{n}^2] &= 0 \quad (x \rightarrow x + \delta x), \end{aligned} \quad (2.3)$$

and can be derived by multiplying the equation of motion (2.2) by $\partial_t \mathbf{n}$ and $\partial_x \mathbf{n}$, respectively.

On the other hand, what we would really like to consider in this and the following few sections are *nonlinear* field theories. One can certainly construct a nonlinear field theory by adding an interaction term such as $(\mathbf{n} \cdot \mathbf{n})^2$ to the linear model above. Alternatively, one can instead explore a more drastic route, such as that followed by T. Skyrme, in which the nonlinearity is introduced by a *constraint* on the field:

$$\mathbf{n} \cdot \mathbf{n} = n_1^2 + n_2^2 = 1. \quad (2.4)$$

Now, the magnitude of the field must be unity everywhere in spacetime, which makes $\mathbf{n}$ a point on the unit circle, S^1 . Owing to the constraint, the actual degrees of freedom of the field are reduced from two (n_1 and n_2) to just one (i.e., angle θ). One can thus express $\mathbf{n}$ in terms of the angle θ in a way that automatically takes care of the constraint,

$$\mathbf{n} = (\cos \theta, \sin \theta), \quad (2.5)$$

and construct a theory in terms of the “free” dynamical variable θ . In doing so, $(\partial_\mu \mathbf{n}) \cdot (\partial_\mu \mathbf{n})$ reduces to $(\partial_\mu \theta)^2$, while $\mathbf{n} \cdot \mathbf{n}$ becomes a constant. The Lagrangian (2.1) then turns into

$$L = \frac{1}{2} \int dx [(\partial_t \theta)^2 - (\partial_x \theta)^2], \quad (2.6)$$

and the two conservation laws derived in (2.3) are accordingly reduced to

$$\begin{aligned} \partial_t [(\partial_t \theta)^2 + (\partial_x \theta)^2] + \partial_x [-2\partial_t \theta \cdot \partial_x \theta] &= 0 \quad (t \rightarrow t + \delta t), \\ \partial_t [-2\partial_t \theta \cdot \partial_x \theta] + \partial_x [(\partial_t \theta)^2 + (\partial_x \theta)^2] &= 0 \quad (x \rightarrow x + \delta x). \end{aligned} \quad (2.7)$$

Skyrme, in a series of publications starting from the late 50s,¹ noted that another kind of conservation law, quite unrelated to the symmetry principle, existed for the

¹Skyrme's idea of a topological field theory, topological conservation law, and their application to elementary particles have been developed over half a dozen papers [1–5]. Although Ref. [2] receives the most citation from modern readers, it is worth looking through his earlier publication Ref. [1] that dealt with the one-dimensional example, where he must have developed much of his core intuition. The solution of what is nowadays known as the sine-Gordon model is also found in this early paper.

kinds of nonlinear field theories described by (2.6). This new kind of conserved two-current (i.e., one time, one space) could be constructed as

$$J_\alpha = \frac{1}{2\pi} \varepsilon_{\alpha\beta} \varepsilon_{ab} n_a \partial_\beta n_b. \quad (2.8)$$

Why there is the factor 2π in the denominator will be explained shortly. The first of the two antisymmetric tensors, $\varepsilon_{\alpha\beta}$, refers to the spacetime index (t, x) , while the second tensor ε_{ab} refers to the internal indices of the field $\mathbf{n} = (n_1, n_2)$. On account of the manner in which this current was defined, the spacetime in question can only be two-dimensional, i.e., (1+1)-dimensional. The topological current given above thus reduces to one temporal (J_t), and one spatial (J_x) component:

$$J_t = \frac{1}{2\pi} \partial_x \theta, \quad J_x = -\frac{1}{2\pi} \partial_t \theta, \quad (2.9)$$

with the obvious consequence that $\partial_t J_t + \partial_x J_x = 0$. This is the continuity equation for the putative “charge” density J_t and “current” density J_x . Furthermore, the conservation of the topological current can be proven without recourse to the explicit parametrization (2.5) by noting

$$\partial_\mu J_\mu = \frac{1}{2\pi} \varepsilon_{\mu\nu} \varepsilon_{ab} \partial_\mu n_a \partial_\nu n_b = \frac{1}{\pi} \frac{\partial[n_1, n_2]}{\partial[t, x]}. \quad (2.10)$$

That is, the divergence $\partial_\mu J_\mu$ is simply the Jacobian of the mapping from the two-dimensional coordinates (t, x) to (n_1, n_2) . One learns in calculus that the Jacobian measures the area of a small covered patch in the target space, in this case (n_1, n_2) , as the base space coordinates (t, x) cover an area $\Delta t \Delta x$. However, since $\mathbf{n}$ can only trace out a trajectory on a circle due to the constraint $n_1^2 + n_2^2 = 1$, there is no area element to be covered in the target space and the Jacobian has to be zero.

We would like to get a better understanding of the nature of the conserved charge, so with this in mind, if we write $J_t = \rho = (1/2\pi) \partial_x \theta$ as a density, the total “charge” associated with a particular field configuration $\theta(x, t)$ is

$$Q(t) = \int_{-\infty}^{\infty} \rho(x, t) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \partial_x \theta(x, t) dx = \frac{1}{2\pi} [\theta(\infty, t) - \theta(-\infty, t)]. \quad (2.11)$$

On account of the conservation law just derived, this charge has to be a constant of motion, $Q(t) = Q$. Let us imagine that we impose the periodic boundary condition, requiring $\mathbf{n}(\infty, t) = \mathbf{n}(-\infty, t)$. Then, the angular variable $\theta(\infty, t)$ needs to match the value at $x = -\infty$, i.e., $\theta(-\infty, t)$, up to a multiple of 2π :

$$\theta(\infty, t) = \theta(-\infty, t) + 2\pi N, \quad N = \text{integer}. \quad (2.12)$$

Thanks to the division by 2π in the definition (2.8), the charge of the system is nothing other than the integer winding number accumulated across the one-dimensional strip: $Q = N$.

The argument that led to the association of the conserved charge Q with the winding number N was purely mathematical, in the sense that no knowledge of the Hamiltonian or the dynamics of the field was required to reach the conclusion; such an argument is referred to as *topological*. Clearly, the topological conservation law is different from symmetry-originated conservations in that one does not even need to know the Hamiltonian or its internal symmetries. Skyrme's epiphany was to associate the topological number in a nonlinear field theory with the quantum number of sub-atomic particles. The traditional viewpoint of quantum field theory is that an elementary particle is described by a field variable, and the particle's Lagrangian is written out in terms of such a field variable. In Skyrme's interpretation, on the other hand, the field itself is not an elementary particle but a particular configuration of the field with a nonzero topological number is. In essence, Skyrme proposed the radical view that a particle is an "emergent feature" of some underlying field configuration, rather than being defined by the field itself.

In order to qualify a certain object of field theory as a particle, however, one must do more than simply show the existence of a conserved integer charge. A particle is a point-like object after all, and that feature is expected of any "reasonable" model of elementary particles. To help us build our intuition, let us consider a simple field configuration, reasonably localized in space, and carrying a nonzero charge $Q = 1$:

$$\theta(x) = \pi \tanh(x/\lambda). \quad (2.13)$$

This configuration varies from $-\pi$ at $x = -\infty$ to $+\pi$ at $x = +\infty$, regardless of the value for λ , which represents the width over which there is a significant variation of the angle θ . The width λ also covers the extent of the particle density $\rho(x)$. Although the topological charge Q of the soliton is independent of λ , its energy does depend on λ ,

$$H = \frac{1}{2} \int (\partial_x \theta)^2 dx = \frac{\pi^2}{2\lambda^2} \int \left[\coth\left(\frac{x}{\lambda}\right) \right]^2 dx = \frac{\pi^2}{2\lambda} \int (\coth y)^2 dy. \quad (2.14)$$

Since the energy diminishes as the width λ is increased, the most energetically stable soliton configuration is thus the one where $\lambda = \infty$, which is unfortunately nothing like a localized object one expects of a particle! In fact, the argument is even more general than this explicit example suggests, since any function $\theta(x) = f(x/\lambda)$ that is typified by a certain width λ will be subject to the same dimensional relation:

$$H = \frac{1}{2\lambda} \int_{-\infty}^{\infty} (\partial_y f(y))^2 dy \propto \frac{1}{\lambda}. \quad (2.15)$$

This is an instance of what is known as the Derrick–Hobart theorem [6, 7]. In a simple field theory such as $H = (1/2) \int dx (\partial_x \theta)^2$, the soliton does enjoy the topological

protection of its charge, but not the kind of energetic stability that would preserve its locality in space.

In order to ensure that a given nontrivial soliton configuration remains energetically stable as well as topologically protected, some additional work needs to be done. To be precise, additional terms are required to prevent the width λ from diverging, so that these additional terms cause the energy to have a nonmonotonic dependence on λ . A magnetic analogy can help decide just which term will get the job done. Taking $\mathbf{n} = (n_1, n_2)$ as a two-dimensional vector defining the magnetic orientation, one can think of adding additional energy terms through an external field $-\mathbf{B} \cdot \mathbf{n}$, higher powers of the gradient $(\partial_\mu \mathbf{n})^4$, or simply higher powers of $\mathbf{n}$. For now we focus on the last possibility, and also impose that the additional terms have symmetries under the time-reversal $\mathbf{n} \rightarrow -\mathbf{n}$ and rotation by 90° , $(n_1, n_2) \rightarrow (n_2, -n_1)$. The lowest-order term satisfying these conditions is $n_1^4 + n_2^4$. With this new term, the Lagrangian reads

$$\begin{aligned} L &= \frac{1}{2} \int dx \left[(\partial_t \mathbf{n})^2 - (\partial_x \mathbf{n})^2 - \frac{1}{8} m^2 (n_1^4 + n_2^4) \right] \\ &= \frac{1}{2} \int dx \left[(\partial_t \theta)^2 - (\partial_x \theta)^2 - \frac{1}{8} m^2 (1 - \cos 4\theta) \right]. \end{aligned} \quad (2.16)$$

Viewing $V(\theta) \propto (1 - \cos 4\theta)$ as a potential energy for θ , there is an obvious pinning effect that tries to fix the orientation angle of $\mathbf{n}$ at multiples of $\pi/2$.

With this interaction, the modified equation of motion is

$$(\partial_t^2 - \partial_x^2)\theta + \frac{1}{4}m^2 \sin 4\theta = 0. \quad (2.17)$$

This is the celebrated sine-Gordon equation, one of the best-known examples of a nonlinear field equation with a known exact solution. The “sine” part of the nomenclature comes from the sine term in the equation, while the “Gordon” is derived from the earlier Klein-Gordon equation, which is the linearized version obtained by replacing $\sin 4\theta$ with 4θ .² The sine-Gordon equation reduces to the well-known pendulum equation $\partial_t^2 \theta + m^2 \sin 4\theta = 0$, in the uniform limit $\theta(x, t) = \theta(t)$. On the other hand, if we look for a static solution, $\theta(x, t) = \theta(x)$, the equation permits an exact solution (Fig. 2.1)

$$\begin{aligned} -\partial_x^2 \theta + m^2 \sin 4\theta &= 0 \\ \theta(x) &= \tan^{-1}[\exp(mx)]. \end{aligned} \quad (2.18)$$

Depending on the sign of m , $\theta(x)$ evolves from 0 to $\pi/2$ (for $m > 0$) as a quarter-charged soliton, or from $\pi/2$ to 0 as an anti-soliton. The $Q = \pm 1/4$ charge can easily be re-labeled as integers ± 1 by defining 4θ as the new angle θ . Equally popular names for one-dimensional topological solitons and anti-solitons are the “kink” and

²To do justice to Skyrme, this equation should really be called Klein-Gordon-Skyrme equation.

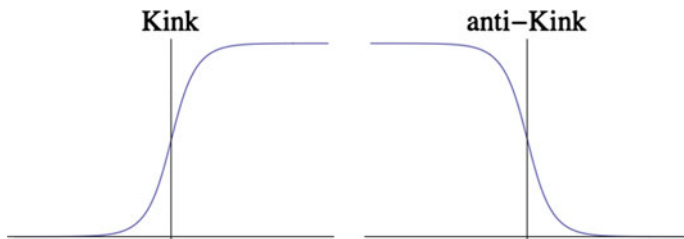


Fig. 2.1 Kink and anti-kink profiles

“anti-kink”. The nature of the kink solution is such that it connects one minimum of the potential $V(\theta) \propto -\cos 4\theta$ to another. The full time-dependent solution is obtained by the boost: $x \rightarrow \gamma(x \pm vt)$, where v is less than unity, and γ is the Lorentz factor $\gamma = (1 - v^2)^{-1/2}$. The kink solution carries a well-defined topological number Q , a well-defined shape $\theta(x \pm vt)$ that remains constant over all times, and therefore embodies the character of an elementary particle.

We have seen that the topological charge of the kink was obtained by integrating $\partial\theta/\partial x$ over the entire one-dimensional space. This space can be wrapped into a ring by identifying the two ends of a chain of length L as equal. On the other hand, the angle θ also refers to a point on the unit circle, and the whole function $\theta(x)$ can be viewed as a mapping from one circle to another:

$$\theta : S^1 \rightarrow S^1. \quad (2.19)$$

The first space with coordinate x of this mapping is called the “base” space, while the second space is called the target space. We note that the winding number embodied by the function $\theta(x)$ expresses the number of times the target space S^1 is covered, as the base space S^1 is traversed only once. By its nature, the answer must be an integer. Mathematicians have expressed such a phenomenon with the notation

$$\pi_1(S^1) = \mathbb{Z}, \quad (2.20)$$

where the subscript 1 in π_1 signifies that the base space is a circle S^1 , while the S^1 in the argument signifies that the unit circle S^1 is also the target space of field θ . Finally, $\mathbb{Z}$ signifies that an integer number characterizes the nature of the wrapping.

A few years after Skyrme completed his vision to regard a kink in the one-dimensional field profile as a particle, he was able to successfully generalize the kink $\leftrightarrow$ particle correspondence to three space dimensions in a tour-de-force demonstration of his mathematical prowess. Unlike Skyrme, we mere mortals will instead learn how this was achieved by increasing the space dimension by one dimension at a time, each time only slightly generalizing the strategy garnered from the study of the reduced dimensionality systems.

2.2 Vortices in Two Dimensions

The two-component unit vector field in one-space, one-time dimension supported a topological object called the kink. The topology underlying the quantization of charge was that of mapping S^1 to S^1 . In this section, another topological object formally characterized by the same homotopy $\pi_1(S^1)$ will be studied. This time, it is a singular object existing in one higher space dimension than before, and the circle forming the base space corresponds to an ordinary circle $x^2 + y^2 = R^2$ (or a closed path topologically equivalent to such a circle) in two dimensions. We will not be concerned with the time dependence of the field.

In this case, the unit vector field $\mathbf{n}(x, y)$ depends on two-dimensional coordinates (x, y) , and if one walks along a circle covering the base space S^1 and records how the $\mathbf{n}$ -vector field defined on that circle winds, there comes a realization that the answer must again be an integer, thanks to the same homotopy consideration $\pi_1(S^1) = \mathbb{Z}$. Suppose a particular base circle yielded a nonzero winding number $+1$ for the $\mathbf{n}$ -vector field defined on it. Then this number, being a topological integer, must remain the same as one shrinks or expands the diameter of the circle, since the $\mathbf{n}$ -field can only vary smoothly (by assumption) and a smooth change cannot bring about a change of one integer to another integer.

This kind of field configuration with a nonzero winding number in two dimensions is called a vortex. An explicit example for $N = +1$ is

$$\mathbf{n}_v(x, y) = \left(\frac{x}{r}, \frac{y}{r} \right). \quad (2.21)$$

For any fixed radius $r = \sqrt{x^2 + y^2} = R$ this reduces to $\mathbf{n} = (\cos \varphi, \sin \varphi)$, where the cylindrical angle is defined by $\varphi = \tan^{-1}(y/x)$. In fact this is just an identity map from one unit circle to another, $\mathbf{n}(\hat{r}) = \hat{r}$, where $\hat{r}$ indicates the position on the base circle S^1 . In order to create a vortex with winding number N , one can use the field configuration

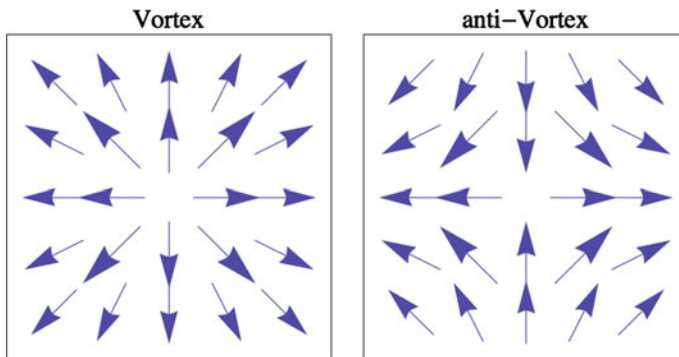


Fig. 2.2 Vortex and anti-vortex configurations in two dimensions

$$\mathbf{n}_v = (\cos[N\varphi], \sin[N\varphi]), \quad (2.22)$$

where N is any positive or negative integer. Configurations with negative values of N are called anti-vortices (Fig. 2.2). We note that any other vortex or anti-vortex configuration with the same topological number N must be able to smoothly deform into the form given above.

In order to compute the topological number for the field configuration (2.22), we adopt the same formula for the topological current as in (1+1)-dimensions, i.e., (2.9). Instead of t and x components, however, we now have x and y components of the current:

$$J_x = \frac{1}{2\pi} \partial_y \theta, \quad J_y = -\frac{1}{2\pi} \partial_x \theta. \quad (2.23)$$

The topological current density associated with a vortex configuration with winding number N follows from $\theta = N\varphi = N \tan^{-1}(y/x)$:

$$\mathbf{J} = (J_x, J_y) = \frac{N}{2\pi} \left(\frac{x}{r^2}, \frac{y}{r^2} \right). \quad (2.24)$$

As we know from vector calculus, the two-dimensional divergence of $\mathbf{J}$ is a delta function, $\nabla \cdot \mathbf{J} = N\delta^2(\mathbf{r})$, and hence, the integral of the divergence, by way of Stokes' theorem, gives

$$\int dx dy \nabla \cdot \mathbf{J} = \oint (J_x dy - J_y dx) = \frac{1}{2\pi} \oint d\theta = N. \quad (2.25)$$

This is the same integer winding number as the one obtained from the one-dimensional kink in (2.11).

The singularity in the vortex configuration arises from the fact that the winding has to occur more and more rapidly as the radius of the loop is reduced to zero. No such operation is required for the kink, which winds around a loop of fixed size. The singular nature of the vortex configuration is manifest in the divergent strength of the current density $|\mathbf{J}| \sim 1/r$, and can be seen from an energetic point of view since the natural energy functional for the $\mathbf{n}$ -vector field in two space dimensions is

$$H = \frac{1}{2} \int dx dy (\nabla \theta)^2 = 2\pi^2 \int dx dy \mathbf{J}^2. \quad (2.26)$$

Since $|\mathbf{J}| \sim 1/r$, the energy density of a vortex configuration is proportional to $1/r^2$, which leads to the logarithmic singularity of the integral both with respect to the linear dimension L of the sample being integrated over, and with respect to the core radius a down to which the integration loop shrinks: $E_v \sim N^2 \log(L/a)$.

One can go on to write down a multi-vortex configuration thanks to the additive property of the angle θ , i.e., $\theta(x, y) = \sum_i \theta(x - X_i, y - Y_i)$. Here, $\mathbf{R}_i = (X_i, Y_i)$ indicates the core of each vortex carrying the winding number N_i , and one may

write $\theta_i(x - X_i, y - Y_i) = N_i \tan^{-1}[(y - Y_i)/(x - X_i)]$. The topological continuity equation then becomes

$$\nabla \cdot \mathbf{J} = \sum_i N_i \delta^2(\mathbf{r} - \mathbf{R}_i), \quad (2.27)$$

with a source term appearing now on the right. This is just the two-dimensional version of the Gauss's law in electrodynamics. Indeed, the topological charge N_i plays the same role as the electric charge, strengthening the view that a topological object (in this case a vortex) behaves like an elementary particle. Quantized vortices are familiar objects in quantum condensates such as superfluids, superconductors, and most recently, in quantum gases. The fundamental dynamics of vortex motion will be discussed in Chap. 4.

2.3 Skyrmion in Two Dimensions

2.3.1 Formal Aspects

The elements of the two-component unit vector field $\mathbf{n}$ correspond to elements of the U(1) group, which is otherwise known as the SO(2) group. This correspondence to the U(1) group may be seen explicitly since an arbitrary $\mathbf{n}$ -vector can adopt an equivalent complex notation, $n_1 + in_2 = e^{i\theta}$, which is an element of the U(1) group. Similarly, the correspondence to the SO(2) group arises because an arbitrary $\mathbf{n}$ -vector may be written as the result of the SO(2) rotation matrix,

$$\mathcal{R}(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

acting on the reference unit vector $\mathbf{n}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, thus establishing a one-to-one correspondence between a vector $\mathbf{n}$ and an element of the SO(2) group. The two spaces (i.e., the space of two-component unit vectors and space of SO(2) rotation matrices) may therefore be considered to be equivalent. In other words, the two groups are isomorphic: $U(1) \simeq SO(2)$.

After our thorough investigation of topological objects arising from the two-component unit vector field (kinks and vortices), it seems natural to consider the topological field theory of a three-component unit vector field $\mathbf{n} = (n_1, n_2, n_3)$. Each field vector $\mathbf{n}$ defines a point on the unit sphere S^2 (which is just another name for the soccer ball). Just as an SO(2) rotation of the reference vector could produce an arbitrary unit vector on the circle, an arbitrary point on the two-sphere is produced by the SO(3) rotation of the reference vector $\mathbf{n}_0 = (0, 0, 1)$. A simple way to see this is to operate on $\mathbf{n}_0$ directly with the general SO(3) matrix

$$\mathcal{R}(\alpha, \beta, \gamma) = e^{-i\alpha S^z} e^{-i\beta S^y} e^{-i\gamma S^z}. \quad (2.28)$$

The first operation $e^{-i\gamma S^z}$ on $\mathbf{n}_0$ produces no effect at all, and the remaining operations yield

$$e^{-i\alpha S^z} e^{-i\beta S^y} \mathbf{n}_0 = \mathbf{n} = (\sin \beta \cos \alpha, \sin \beta \sin \alpha, \cos \beta), \quad (2.29)$$

a point on S^2 .

It should be cautioned, however, that the space of unit vectors on S^2 is not the same as the space of $\text{SO}(3)$ rotation matrices. While the full set of Euler rotations $\mathcal{R}(\alpha, \beta, \gamma)$ forms an $\text{SO}(3)$ group, as we just showed, an element of this group acting on the reference vector $\mathbf{n}_0$ produces the same vector $\mathbf{n}$, an element of S^2 , no matter what value is chosen for the angle γ . It is as if different elements of $e^{-i\gamma S^z}$, which are called the “fiber”, can be “bundled” together and treated as one element, say an element with $\gamma = 0$. The bundled collection of fibers is known as the “fiber bundle”, or the “coset space”, and is denoted by G/H , where, in this particular example, G is the $\text{SO}(3)$ rotation group and H is its $\text{U}(1)$ subgroup. The unit sphere S^2 is thus equivalent to the coset space $\text{SO}(3)/\text{U}(1)$,³

$$S^2 \simeq \text{SO}(3)/\text{U}(1). \quad (2.30)$$

In defining higher-dimensional topological numbers, one needs to enlarge the base space as well as the target space of the vector field. The base space in which kinks were found was one-dimensional; however, the smallest base space in which a stable topological configuration of an $\mathbf{n} \in S^2$ vector can exist is two-dimensional. The two-dimensional Euclidean space of (x, y) coordinates, called $\mathbb{R}^2$ in the math vocabulary, is not the same as the two-sphere. Nevertheless, if the vector field $\mathbf{n}$ assumes the same value, say $\mathbf{n}_0$, as $R = \sqrt{x^2 + y^2} \rightarrow \infty$, it becomes possible to treat the entire circumference of the two-dimensional Euclidean plane as effectively one point. The idea is similar to the process of wrapping a sheet of dumpling pastry into a sphere. The dumpling-making process is complete when the perimeters of the original flat sheet are merged together at the apex of a finished dumpling. Instead of using an actual flour sheet, we will show how to transform $\mathbb{R}^2$ into S^2 by the procedure called the stereographic projection. With both the base and target spaces established as that of S^2 , the topological number is the number of times the target sphere is wrapped around as one walks around the base sphere. A standard homotopy result $\pi_2(S^2) = \mathbb{Z}$ ensures that the integer winding number characterizes the mapping. In fact, for any S^d to S^d map in d dimensions we have

$$\pi_d(S^d) = \mathbb{Z}. \quad (2.31)$$

³A general theorem in topology is that $\text{SO}(n+1)/\text{SO}(n) \simeq S^n$, and $\text{SU}(n+1)/\text{SU}(n) \simeq S^{2n+1}$. The case we are considering here is $n = 2$.

It is a common practice to refer to topologically nontrivial field configurations obtained from sphere-to-sphere mapping in two or three space dimensions as the skyrmion, although Skyrme himself never bothered to worry about the $d = 2$ case.

Another possible characterization of the two-dimensional space may be achieved by imposing periodic boundary conditions in both the x and y directions, thereby turning $\mathbb{R}^2$ into a two-torus $\mathbb{T}^2$. At first sight, the homotopy group for the mapping of two-torus to the two-sphere seems more involved than that of S^2 to S^2 . Luckily, there is a theorem stating that all homotopy mappings from the torus $\mathbb{T}^2$ to a target manifold can be classified as two independent π_1 mappings and one π_2 mapping [8]. Since $\pi_1(S^2) = 0$, the only non-trivial homotopy map from the two-torus to the two-sphere is still $\pi_2(S^2) = \mathbb{Z}$, the same result that we obtained from compactifying $\mathbb{R}^2$ directly into S^2 .

2.3.2 First Encounter with Skyrmions

With the formal issue of the order parameter space out of the way, we may start asking what kind of interesting topological structures are available for a vector $\mathbf{n}$ residing on S^2 . In line with the homotopy result $\pi_2(S^2) = \mathbb{Z}$, we first seek to construct a topological structure in two dimensions explicitly.

Guided by our previous exercise in (1+1) dimensions, we begin by writing a topological current density vector

$$J_\alpha = \frac{1}{8\pi} \varepsilon_{\alpha\beta\gamma} \varepsilon_{abc} n_a \partial_\beta n_b \partial_\gamma n_c. \quad (2.32)$$

We note that the number of spacetime indices α, β, γ and field components a, b, c have both increased by one over the previous (1+1)-dimensional case. The topological continuity equation $\partial_\alpha J_\alpha = 0$ follows from the fact that

$$\partial_\alpha J_\alpha \propto \frac{\partial[n_1, n_2, n_3]}{\partial[t, x, y]}. \quad (2.33)$$

This quantity once again vanishes due to the constraint $\mathbf{n}^2 = 1$, which reduces the number of the field's degrees of freedom by one. As before, the conserved charge is defined as the integral of the J_t component, given by

$$J_t = \frac{1}{8\pi} \varepsilon_{abc} n_a (\partial_x n_b \partial_y n_c - \partial_y n_b \partial_x n_c) = \frac{1}{4\pi} \mathbf{n} \cdot \left(\frac{\partial \mathbf{n}}{\partial x} \times \frac{\partial \mathbf{n}}{\partial y} \right). \quad (2.34)$$

The spatial components of the topological current may be obtained by the cyclic permutation of t, x, y . The two-dimensional integral of this, $Q = \int dx dy J_t(x, y, t)$, is a constant of motion on account of the topological conservation law, $\partial_\alpha J_\alpha = 0$. The integer number Q counts how many times the $\mathbf{n}$ -vector covers the target sphere

while the (x, y) coordinate space is covered. In the literature, the field configurations with nonzero values of Q are called skyrmions, or sometimes, baby-skyrmions to distinguish them from their three-dimensional cousins.

Expressing the unit vector field with spherical coordinates, $\mathbf{n} = (\sin \beta \cos \alpha, \sin \beta \sin \alpha, \cos \beta)$, yields an expression for the topological charge density which can prove useful:

$$J_t = \frac{1}{4\pi} \sin \beta \left(\frac{\partial \beta}{\partial x} \frac{\partial \alpha}{\partial y} - \frac{\partial \beta}{\partial y} \frac{\partial \alpha}{\partial x} \right) = \frac{1}{4\pi} \frac{\partial[\alpha, \cos \beta]}{\partial[x, y]}. \quad (2.35)$$

The right-hand side of this equation is simply the Jacobian of the coordinate transformation from two-dimensional Euclidean coordinates (x, y) to the spherical coordinates $(\alpha, \cos \beta)$, divided by 4π , the volume of the unit sphere.

In our (1+1)-dimensional example, the periodic boundary condition effectively reduced the one-dimensional real axis $\mathbb{R}^1$ into a circle, S^1 . The quantization of the topological charge Q then followed naturally from the homotopy rule $\pi_1(S^1) = \mathbb{Z}$. Pursuant to this guide and the formal discussion of Sect. 2.3.1, we may map the two-dimensional Euclidean space $\mathbb{R}^2$ into a sphere S^2 by imposing the condition

$$\mathbf{n}(r \rightarrow \infty, t) = \mathbf{n}_\infty \quad \left(r = \sqrt{x^2 + y^2} \right). \quad (2.36)$$

The explicit construction of the topological configuration then proceeds in two steps. The first step is the one-to-one mapping of the two-dimensional Euclidean space to S^2 : $(x, y) \rightarrow \hat{r} \in S^2$. Next, one establishes the identity map $\mathbf{n}(\hat{r}) = \hat{r}$, which should generate a topological configuration with unit winding number automatically. Combining the two mappings, one finds the desired configuration with unit winding number

$$\mathbf{n}(x, y) = \hat{r}(x, y). \quad (2.37)$$

The procedure that accomplishes the first task is called the stereographic projection, and is sketched in Fig. 2.3. A sphere with diameter R sits on the Euclidean plane with its tangent point located at its south pole. Each point in the plane has the cylindrical coordinate representation $(x, y) = r(\cos \varphi, \sin \varphi)$, while each point on the sphere has the spherical coordinate representation $R(\sin \beta \cos \alpha, \sin \beta \sin \alpha, \cos \beta)$. Under the stereographic projection, the angle φ of the Euclidean plane is identified with the azimuthal angle α of the sphere, i.e., $\alpha = \varphi$, and $(\cos \varphi, \sin \varphi) = (x/r, y/r)$. Meanwhile, the polar angle β is mapped to the radius r in the plane by $r/R = \cot(\beta/2)$, as shown in Fig. 2.3. Since

$$\cos \frac{\beta}{2} = \frac{r}{\sqrt{r^2 + R^2}}, \quad \sin \frac{\beta}{2} = \frac{R}{\sqrt{r^2 + R^2}}, \quad (2.38)$$

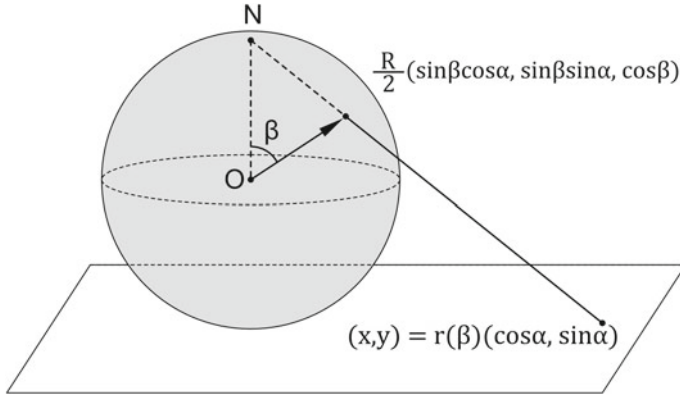


Fig. 2.3 Stereographic projection to the two-dimensional Euclidean plane from a sphere of diameter R . Each point on the sphere is connected in a one-to-one manner to a point on the plane by a ray emanating from the north pole of the sphere. The mapping produces a $Q = -1$ anti-skyrmion

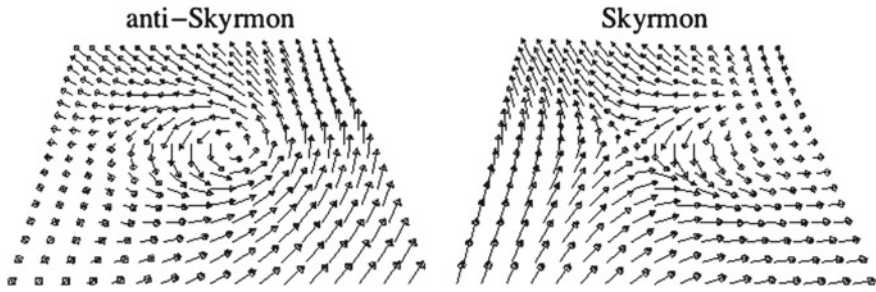


Fig. 2.4 Anti-skyrmion ($Q_s = -1$) and skyrmion ($Q_s = +1$) configurations

we have $\cos \beta = (r^2 - R^2)/(r^2 + R^2)$ and $\sin \beta = 2rR/(r^2 + R^2)$, and the desired mapping is (Fig. 2.4)

$$\mathbf{n}_s(x, y) = \hat{r}(x, y) = \left(\frac{2Rx}{r^2 + R^2}, \frac{2Ry}{r^2 + R^2}, \frac{r^2 - R^2}{r^2 + R^2} \right). \quad (2.39)$$

This is the skyrmion field for a unit winding number, and we can also see that the asymptotic condition (2.36) is satisfied, with $\mathbf{n}_\infty = (0, 0, 1)$. Physical realizations of this kind of spin configuration in magnetic materials are the subject of Chap. 3.

With the construction of skyrmions with unit winding number out of the way, one may ask how to construct skyrmion configurations with an arbitrary integer winding number N . In the previous section on vortices, we had $\mathbf{n}_v = (\cos[N\varphi], \sin[N\varphi])$ for a vortex with winding number N . The corresponding formula for skyrmions can be built upon this expression,

$$\begin{aligned}\mathbf{n}_s &= (\sin[f(r)]\mathbf{n}_v, \cos[f(r)]) \\ &= (\sin[f(r)]\cos[N\varphi], \sin[f(r)]\sin[N\varphi], \cos[f(r)]). \end{aligned} \quad (2.40)$$

The skyrmion solution expressed in (2.39) is a special case of this more general case, with $N = 1$ and a particular choice of the radial function $f(r)$, while the vortex configuration corresponds to having $f(r) = \pi/2$ everywhere.

The topological density of the skyrmion configuration (2.40) can be readily computed:

$$\rho_s(x, y) = \frac{1}{4\pi} \mathbf{n}_s \cdot (\partial_x \mathbf{n}_s \times \partial_y \mathbf{n}_s) = \frac{N}{4\pi r} f'(r) \sin[f(r)]. \quad (2.41)$$

Integrated over all space, the topological charge becomes

$$Q_s = \frac{N}{2} \int_0^\infty dr f'(r) \sin[f(r)] = \frac{N}{2} \int_0^\infty dr \frac{dn^z}{dr} = N \cdot \frac{n^z(0) - n^z(\infty)}{2}. \quad (2.42)$$

Depending on whether the vector field varies from north-to-south or south-to-north as r is increased from the origin to infinity, the skyrmion charge Q_s equals the vortex's winding number N up to a sign, i.e., $Q_s = \pm N$. In magnetic literature, the two integer quantities N and $P = [n^z(0) - n^z(\infty)]/2$ are called the vorticity and the polarity, respectively. The skyrmion charge Q_s can be viewed as the product of the two integers

$$Q_s = N \cdot P. \quad (2.43)$$

As the formula suggests, skyrmions of higher topological number can be produced by employing structures with larger vorticities N , or higher polarities P . For instance, the skyrmion state obtained from the stereographic mapping has $Q_s = -1$, due to the fact that $N = 1$ but $P = -1$. Although the expression (2.39) is often said to represent a skyrmion, it is really an anti-skyrmion according to our definition of topological charge.

There is a lesson to be learned from the way the skyrmion solution was written using the vortex solution as a “seed” in (2.40). Recall that the planar character of the spin was the cause of the singularity in our earlier discussion of the vortex solution. The skyrmion configuration lifts this singularity, using the larger space afforded to the vector $\mathbf{n}$, by tilting it out of the plane. As a result, the energy of the skyrmion remains finite. We will shortly use this trick once again in order to generate a smooth topological configuration in three dimensions. This will be none other than Skyrme's original vision of the skyrmion.

It was possible to derive the kink solution in (1+1)-dimensions as the saddle-point solution of a specific Hamiltonian. How about the two-dimensional skyrmion? Is there a (classical) Hamiltonian whose saddle-point solution describes the skyrmion? A first guess is, as in the one-dimensional problem,

$$H = \frac{1}{2} \int dx dy [(\partial_x \mathbf{n})^2 + (\partial_y \mathbf{n})^2]. \quad (2.44)$$

This kind of model is called the non-linear σ -model, and at first glance, looks like a free quadratic theory. However, this is really not the case due to the $\mathbf{n}^2 = 1$ constraint. Furthermore, it is a scale-invariant theory, in the sense that the coordinate change $(x, y) \rightarrow a(x, y)$ leaves the Hamiltonian invariant.

Feeding the skyrmion configuration (2.40) into the Hamiltonian (2.44) yields the skyrmion energy

$$E_s = \pi \int_0^\infty r dr \left[(f')^2 + \frac{N^2}{r^2} (\sin f)^2 \right]. \quad (2.45)$$

The saddle-point equation following from this energy functional,

$$2(rf')' + \frac{N^2}{r^2} \sin 2f = 0, \quad (2.46)$$

determines the shape of the profile function $f(r)$ subject to the boundary conditions at $r = 0$ and $r = \infty$. In particular, the skyrmion solution (2.39), previously obtained from the stereographic projection, is a valid solution of this equation with $(\sin f, \cos f) = (2rR, r^2 - R^2)/(r^2 + R^2)$, regardless of the choice of R . The energy of the skyrmion remains insensitive to R , in keeping with the Derrick-Hobart theorem.

We encountered a similar problem in the construction of the stable kink solution in one dimension. As we saw earlier, the way to beat it was to introduce an additional interaction that could give rise to a nonmonotonic dependence of the energy on the scale parameter R of the topological soliton. In the same manner, we ask what kind of additional terms in the Hamiltonian would fix the radius R ? One can begin with the Zeeman interaction that adds the following energy to the skyrmion (assuming $\mathbf{B} = B\hat{z}$)

$$\begin{aligned} E_Z &= -\mathbf{B} \cdot \int \mathbf{n} dx dy = -2\pi B \int_0^\infty r \cos \left[f \left(\frac{r}{R} \right) \right] dr \\ &= -2\pi B R^2 \int_0^\infty x \cos[f(x)] dx. \end{aligned} \quad (2.47)$$

This is a quadratic function of R with the sign of the curvature depending on the sign of the integral, and will not in any case yield a stable skyrmion radius. We may try another energy form, this time the quartic term, $\int (n_1^4 + n_2^4 + n_3^4) dx dy$; however, dimensional analysis reveals that this too shows a quadratic dependence on R , and so even the combined Zeeman and quartic energy contributions will not be able to fix the skyrmion radius uniquely. By now it should be clear that besides the Zeeman and/or higher-order anisotropy energies, one also needs an energy that

depends linearly on R . In Chap. 3 we will show that Dzyaloshinskii-Moriya (DM) interaction in chiral magnets is a critical new interaction that adds just such a linear dependence to the energy.

2.4 Hedgehogs in Three Dimensions

Earlier we discussed how the smooth topological texture in one dimension (kink) and the singular topological texture in two dimensions (vortex) are both described by the same homotopy map $\pi_1(S^1)$. In this section, the analogy is elevated to one higher dimension in an effort to construct a singular topological object in three dimensions that shares the homotopy $\pi_2(S^2)$ with the two-dimensional skyrmion.

Take a sphere $x^2 + y^2 + z^2 = R^2$ embedded in a three-dimensional space and follow the trajectory of the S^2 vector field $\mathbf{n}$ as one walks about the surface of such sphere. A configuration with non-zero integer winding number must exist due to the homotopy relation $\pi_2(S^2) = \mathbb{Z}$. An obvious example is the identity map

$$\mathbf{n}_h = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right), \quad r = \sqrt{x^2 + y^2 + z^2}. \quad (2.48)$$

This object goes by the amicable name of a “hedgehog”, on account of its similarity to the pet of the same name in an irate mood. The topological current density (2.32) of the hedgehog works out to be (replace t with z)

$$\mathbf{J} = (J_x, J_y, J_z) = \frac{1}{4\pi} \frac{\mathbf{r}}{r^3}. \quad (2.49)$$

Viewing $\mathbf{J}$ as a magnetic field evokes the image of a magnetic monopole sitting at the origin. For this reason, some authors prefer to call this configuration a “monopole” rather than a hedgehog.

The energy of the hedgehog (assuming the Hamiltonian to be the nonlinear σ -model),

$$E = \frac{1}{2} \int d^3\mathbf{r} [(\partial_x \mathbf{n})^2 + (\partial_y \mathbf{n})^2 + (\partial_z \mathbf{n})^2] = 4\pi \int r^2 dr \left(\frac{1}{r^2} \right), \quad (2.50)$$

diverges as the linear size of the integral $E \sim L$. Besides, the topological current density of the hedgehog satisfies the divergence condition

$$\nabla \cdot \mathbf{J} = \delta^3(\mathbf{r}), \quad (2.51)$$

which suggests its interpretation as a point particle. The surface integral of $\mathbf{J}$, or equivalently the volume integral of $\nabla \cdot \mathbf{J}$, serves as the conserved charge of the hedgehog. In many ways, the hedgehog is the three-dimensional version of the vortex, and an anti-hedgehog state is obtained by simply reversing the arrows, $\mathbf{n}_h \rightarrow -\mathbf{n}_h$.

Owing to their singular nature, hedgehogs must co-exist with an equal number of anti-hedgehogs.

Together with the skyrmion of the previous section, we have constructed one nonsingular and one singular configuration of the three-component vector field $\mathbf{n}$, both governed by the same homotopy $\pi_2(S^2)$, in two- and three-dimensional spaces, respectively. Later, when we try to construct a smooth topological configuration in three-dimensional space, the singular field $\mathbf{n}_h$ will be weighted by a function $\sin[f(r)]$ that vanishes at the origin and at infinity, while a fourth component $n_4 = \cos[f(r)]$ will be included to form a four-component unit vector

$$\mathbf{n}_s = (\sin[f(r)]\mathbf{n}_h, \cos[f(r)]). \quad (2.52)$$

As in the construction of the two-dimensional skyrmion out of the two-dimensional vortex, the idea here is to adopt the singular configuration from the lower $\mathbf{n}$ -vector space ($\mathbf{n} \in S^2$) to construct a smooth topological configuration in the higher dimensional space ($\mathbf{n} \in S^3$). Chapter 3 includes a discussion of the hedgehog spin lattice in a three-dimensional model of chiral magnet.

2.5 Skyrmions in Three Dimensions

Armed with these exercises in lesser dimensions, it is time to construct a topological object in three spatial dimensions out of a four-component field $\mathbf{n} = (n_1, n_2, n_3, n_4)$ subject to the unit modulus constraint $\mathbf{n} \cdot \mathbf{n} = 1$. The topological current,

$$J_\alpha = \frac{1}{12\pi^2} \varepsilon_{\alpha\beta\gamma\delta} \varepsilon_{abcd} n_a \partial_\beta n_b \partial_\gamma n_c \partial_\delta n_d, \quad (2.53)$$

is an obvious generalization of the previous topological currents, (2.8) and (2.32).

It is not easy to find examples of a physical system described by a four-component unit vector field in classical physics. However, as noted by Skyrme in his papers, in quantum physics the four numbers $\mathbf{n} = (n_1, n_2, n_3, n_4)$ subject to the unit modulus constraint are equivalent to an element of the $SU(2)$ group. One can verify this statement by arranging the four elements of $\mathbf{n}$ in matrix form as

$$U(\mathbf{n}) = i(n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3) + n_4 = \begin{pmatrix} n_4 + in_3 & in_1 + n_2 \\ in_1 - n_2 & n_4 - in_3 \end{pmatrix}. \quad (2.54)$$

Here, the three matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the familiar Pauli matrices. In order to see that this is an element of the $SU(2)$ group in a transparent fashion, we parameterize the four components of the $\mathbf{n}$ vector as

$$\mathbf{n} = (\mathbf{m} \sin \gamma, \cos \gamma), \quad (2.55)$$

where $\mathbf{m}$ is the unit vector on S^2 . The matrix $U(\mathbf{n})$ then assumes the familiar form

$$U = \cos \gamma + i \sin \gamma [\mathbf{m} \cdot \boldsymbol{\sigma}] = \exp(i\gamma [\mathbf{m} \cdot \boldsymbol{\sigma}]), \quad (2.56)$$

of an element of the $SU(2)$ group. Since $SU(2)$ forms a group, so do the points on S^3 .

There is an equally good way to express the topological current (2.53) in terms of the $SU(2)$ group element,

$$J_\alpha = -\frac{1}{24\pi^2} \varepsilon_{\alpha\beta\gamma\delta} \text{Tr} \left[(U^\dagger \partial_\beta U) (U^\dagger \partial_\gamma U) (U^\dagger \partial_\delta U) \right], \quad (2.57)$$

which one often finds in the field theory literature. The angular representation of the unit vector

$$\begin{aligned} \mathbf{n} &= (\mathbf{m} \sin \gamma, \cos \gamma) \\ &= (\sin \gamma \sin \beta \cos \alpha, \sin \gamma \sin \beta \sin \alpha, \sin \gamma \cos \beta, \cos \gamma) \end{aligned} \quad (2.58)$$

yields yet another equivalent expression for the topological current

$$J_D = \frac{1}{2\pi^2} \varepsilon_{ABCD} \sin \beta (\sin \gamma)^2 (\partial_A \alpha) (\partial_B \beta) (\partial_C \gamma), \quad (2.59)$$

where we have introduced capital Roman letters to avoid a notational overlap with the symbol for the angles.

The topological charge follows from integrating the temporal component of the topological current J_t over the three-dimensional space. Following the strategy suggested from the earlier treatments, we may express a three-dimensional skyrmion by

$$\mathbf{n}_s = \left(\sin[f(r)] \frac{x}{r}, \sin[f(r)] \frac{y}{r}, \sin[f(r)] \frac{z}{r}, \cos[f(r)] \right). \quad (2.60)$$

Employing the radial dependence given by

$$(\sin f(r), \cos f(r)) = \left(\frac{2rR}{r^2 + R^2}, \frac{r^2 - R^2}{r^2 + R^2} \right), \quad r = \sqrt{x^2 + y^2 + z^2}, \quad (2.61)$$

in connection to the two-dimensional stereographic projection, actually yields the three-dimensional projection of $\mathbb{R}^3$ to the three-sphere

$$(x, y, z) \in \mathbb{R}^3 \rightarrow \left(\frac{2xR}{r^2 + R^2}, \frac{2yR}{r^2 + R^2}, \frac{2zR}{r^2 + R^2}, \frac{r^2 - R^2}{r^2 + R^2} \right) \in S^3. \quad (2.62)$$

Having constructed a viable skyrmion configuration (2.60), one can calculate its topological charge density,

$$J_t(r) = -\frac{1}{2\pi^2} \frac{(\sin f)^2 f'}{r^2}, \quad (2.63)$$

and the associated topological charge

$$\begin{aligned} Q &= \int_0^\infty J_t(r) 4\pi r^2 dr = \frac{2}{\pi} \int_{f(\infty)}^{f(0)} (\sin f)^2 df \\ &= \frac{1}{\pi} \int_{f(\infty)}^{f(0)} (1 - \cos 2f) df \\ &= \frac{f(0) - f(\infty)}{\pi} - \frac{\sin[2f(0)] - \sin[2f(\infty)]}{2\pi}. \end{aligned} \quad (2.64)$$

Let us think about the appropriate boundary conditions for the radial function $f(r)$. The purpose of introducing $f(r)$ was to ensure that the singularity of the hedgehog solution at the origin become invisible, by taking $\sin[f(0)] = 0$. The other property expected of $f(r)$ is to ensure that there is no variation in the $\mathbf{n}$ -field at spatial infinity to suppress diverging energy, and to ensure this condition, we must take $\sin[f(\infty)] = 0$. In other words, both $f(0)$ and $f(\infty)$ must be some multiples of π , and as a result, the second term in the last line of (2.64) must vanish, ensuring that

$$Q = \frac{f(0) - f(\infty)}{\pi} \quad (2.65)$$

is an integer.

We further note that the skyrmion configuration (2.60) may be expressed in an equivalent SU(2) form by

$$U_s = \exp(if(r)\hat{r} \cdot \boldsymbol{\sigma}) = \cos[f(r)] + i\hat{r} \cdot \boldsymbol{\sigma} \sin[f(r)]. \quad (2.66)$$

As mentioned earlier, in condensed matter physics, neither the 2×2 unitary matrix nor the unit vector on S^3 is an easily observable quantity. In magnetic models, physical spins have three components at the most. There is, however, a remarkable way to turn an SU(2) field configuration into a corresponding configuration of the vector field on S^2 through a process called the Hopf fibration in mathematics. We will discuss this procedure in the next section, after arming ourselves with some more language related to the CP^1 mapping.

2.6 CP¹ Theory

Let us begin with some mathematical preliminary. A collection of $2n$ real numbers, x_1 through x_{2n} , can be reorganized as a collection of n complex numbers by pairing the number as follows, $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$, etc. If the initial set of real numbers were subject to the unit modular constraint, $\sum_{i=1}^{2n} x_i^2 = 1$, the space of such numbers defines the hypersphere S^{2n-1} . The simplest instance of this occurs for $n = 1$, in which case a point on S^1 is identifiable with a complex number z of unit modulus. For $n = 2$, the four real numbers pair up to give

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + ix_2 \\ x_3 + ix_4 \end{pmatrix}. \quad (2.67)$$

The normalization condition $\mathbf{z}^\dagger \mathbf{z} = 1$ suggests the possibility of interpreting $\mathbf{z}$ as the wavefunction of a spin-1/2 particle, in which case one also knows that the overall phase does not have a physical impact. This means that a second $\mathbf{z}'$, related to $\mathbf{z}$ by an overall phase factor $\mathbf{z}' = e^{if} \mathbf{z}$, may be viewed as the “same state” as $\mathbf{z}$. This is akin to identifying all the points on the original hypersphere S^3 , connected by some $U(1) \simeq S^1$ operation, as one element. This bundling process gives rise to the coset space S^3/S^1 , which is the space of allowed wave functions of the two-component spinor. Another name for this space is the “Complex Projective space”, or CP space for short. Obviously, the process of pairing up $2n$ real numbers into n complex numbers (or an n -component wave function), and considering the overall phase to be irrelevant, can continue for arbitrary n . The spaces constructed in such a manner, called CP^{n-1} , are given by

$$\text{CP}^{n-1} \simeq S^{2n-1}/S^1, \quad (2.68)$$

in accordance with the argument presented above.

In the previous chapter, the explicit coordinate representation of the CP¹ field

$$\mathbf{z} = \begin{pmatrix} \cos[\beta/2] \\ e^{i\alpha} \sin[\beta/2] \end{pmatrix} \quad (2.69)$$

was used in order to express the geometric phase of the spin action. There, we also saw that the substitution $\mathbf{n} = \mathbf{z}^\dagger \boldsymbol{\sigma} \mathbf{z}$ mapped the spin Hamiltonian into one in terms of $\mathbf{z}$. In particular, the nonlinear σ -model $(\partial_\mu \mathbf{n}) \cdot (\partial_\mu \mathbf{n})$ has an appealing form when written in terms of the CP¹ field:

$$\begin{aligned} (\partial_\mu \mathbf{n}) \cdot (\partial_\mu \mathbf{n}) &= (\partial_\mu \beta)^2 + \sin^2 \beta (\partial_\mu \alpha)^2 \\ &= 4(\partial_\mu \mathbf{z}^\dagger)(\partial_\mu \mathbf{z}) - 4([\partial_\mu \mathbf{z}^\dagger] \mathbf{z})(\mathbf{z}^\dagger \partial_\mu \mathbf{z}). \end{aligned} \quad (2.70)$$

Utilizing our previous definition of the emergent gauge field $a_\mu = -i\mathbf{z}^\dagger [\partial_\mu \mathbf{z}] = +i(\partial_\mu \mathbf{z}^\dagger) \mathbf{z}$, it is straightforward to show that the nonlinear σ -model is equivalent

to another form,

$$(\partial_\mu \mathbf{n}) \cdot (\partial_\mu \mathbf{n}) = 4 \left([\partial_\mu + i a_\mu] \mathbf{z}^\dagger \right) \left([\partial_\mu - i a_\mu] \mathbf{z} \right) = 4 (D_\mu \mathbf{z})^\dagger (D_\mu \mathbf{z}). \quad (2.71)$$

A distinctive feature in rewriting the nonlinear σ -model in terms of the CP¹ field is the appearance of a covariant derivative $D_\mu = \partial_\mu - i a_\mu$, with an abelian gauge field a_μ .

What about the topological current? Is it possible to re-write the (2+1)-dimensional topological current J_α in Eq. (2.32) as a function of the CP¹ field? Given the identity (1.50) previously listed in Chap. 1, the answer is yes, and the current is given by

$$J_\alpha = \frac{1}{4\pi} \varepsilon_{\alpha\beta\gamma} \mathbf{n} \cdot (\partial_\beta \mathbf{n} \times \partial_\gamma \mathbf{n}) = \frac{1}{2\pi} \varepsilon_{\alpha\beta\gamma} \partial_\beta a_\gamma. \quad (2.72)$$

Writing out each component explicitly, the (2+1)-dimensional topological current, expressed in terms of CP¹, is given by

$$\mathbf{J} = \frac{1}{2\pi} (\partial_x a_y - \partial_y a_x, \partial_y a_t - \partial_t a_y, \partial_t a_x - \partial_x a_t) \equiv \frac{1}{2\pi} (b_z, -e_y, e_x). \quad (2.73)$$

In the last expression, we have identified the temporal component of the spacetime current with the “magnetic field” b_z , and identified the two spatial components with the two “electric field” components $(-e_y, e_x)$. The assignment is clearly in keeping with their respective definitions in ordinary electrodynamics. The topological current proves to be none other than the emergent electric and magnetic field in (2+1)-dimensions! The conservation law of the topological current $\partial_\alpha J_\alpha = 0$ then consequently becomes the familiar Faraday’s law:

$$\partial_t b_z - \partial_x e_y + \partial_y e_x = 0. \quad (2.74)$$

Thanks to the CP¹ mapping, the theory of topological currents finds an interesting analogy in (2+1)-dimensional electrodynamics.

The two-dimensional skyrmion configuration of charge $Q = N$ found earlier in terms of $\mathbf{n}_s$ (2.40) has a corresponding CP¹ solution

$$\mathbf{z}_s = \begin{pmatrix} \cos[f(r)/2] \\ e^{iN\phi} \sin[f(r)/2] \end{pmatrix}.$$

We will make extensive use of this CP¹ wavefunction in later chapters when trying to discuss the emergent fields associated with both stationary and moving skyrmions.

In (2.72), we showed how to express the topological three-current in terms of the CP¹ field. In fact, it seemed that all the quantities originally expressed in terms of the O(3) vector $\mathbf{n}$ could find an elegant alternative expression in terms of the CP¹ field. We now ask an analogous question for the O(4) vector $\mathbf{n} \in S^3$. How many of the $\mathbf{n}$ -labeled quantities will find an elegant expression in terms of the CP¹ field? Given

that a $SU(2)$ matrix may be viewed as a rotation matrix in spinor space, we let an arbitrary $SU(2)$ element, e.g., (2.56), act on the reference spinor $\mathbf{z}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, to get

$$\mathbf{z} = U\mathbf{z}_0 = \begin{pmatrix} \cos \gamma + i \sin \gamma \cos \beta \\ i e^{i\alpha} \sin \beta \sin \gamma \end{pmatrix}. \quad (2.75)$$

We see that all three angular variables (α, β, γ) remain present in $\mathbf{z}$, and so this remains faithful representation of the $SU(2)$ matrix. This is the $O(4)$ analogue of (2.69), which expressed a given $O(3)$ vector as a CP^1 field. Given this CP^1 representation of $SU(2)$, we may invoke the Hopf map $\mathbf{n} = \mathbf{z}^\dagger \boldsymbol{\sigma} \mathbf{z}$ to calculate the corresponding three-component vector field

$$\begin{aligned} \mathbf{n} &= (2[n_1 n_3 - n_2 n_4], 2[n_1 n_4 + n_2 n_3], n_3^2 + n_4^2 - n_1^2 - n_2^2) \\ &= \left(2(m_x m_z \sin \gamma - m_y \cos \gamma) \sin \gamma, \right. \\ &\quad \left. 2(m_x \cos \gamma + m_y m_z \sin \gamma) \sin \gamma, \right. \\ &\quad \left. \cos^2 \gamma + (m_z^2 - m_x^2 - m_y^2) \sin^2 \gamma \right). \end{aligned} \quad (2.76)$$

Although this is only a three-component vector, all three angles featured in the $SU(2)$ matrix still feature in what is (hopefully) a faithful representation of the $SU(2)$ element.

Continuing with our analysis, we now ask if the topological four-current (2.53) can find an elegant expression using the CP^1 field (2.75). It turns out that by using the gauge field $\mathbf{a} = -i\mathbf{z}^\dagger \nabla \mathbf{z}$ and its curl, one can prove the interesting identity

$$\mathbf{a} \cdot (\nabla \times \mathbf{a}) = -2 \sin \beta (\sin \gamma)^2 \nabla \alpha \cdot (\nabla \beta \times \nabla \gamma) = -4\pi^2 J_t. \quad (2.77)$$

The result implies that, in terms of the CP^1 gauge field $\mathbf{a}$, the topological charge in three dimensions derived from the four-component unit modular field becomes the simple-looking integral

$$Q = -\frac{1}{4\pi^2} \int dx dy dz \mathbf{a} \cdot (\nabla \times \mathbf{a}). \quad (2.78)$$

Let us pause for a minute to mull over the meaning of the formula we have just derived. It was mentioned that the three-dimensional skyrmion is hard to realize in condensed matter systems because its construction requires a four-component unit vector field, or a $SU(2)$ matrix field, neither of which are readily available physical fields in condensed matter systems. However, we know that any given unit vector field on S^2 can be decomposed as a CP^1 field by reverse-engineering the Hopf map $\mathbf{n} = \mathbf{z}^\dagger \boldsymbol{\sigma} \mathbf{z}$. In turn, this CP^1 field engenders a gauge field $\mathbf{a} = -i\mathbf{z}^\dagger \nabla \mathbf{z}$. If the gauge field configuration was such that the integral on the right-hand side of (2.78) yielded

an integer answer, then by inverting the logic we could claim that the original spin configuration of $\mathbf{n} \in S^2$ was an “incarnation” of the three-dimensional skyrmion! This is possible thanks to the equivalence of the topological number written in terms of the four-component field (2.53), to the one written in terms of what essentially amounts to the three-component field (2.78). This phenomenon is known as the *Hopf fibration*.

As an example, we can apply the observation to the actual skyrmion structure defined earlier. Acting on the reference CP^1 field $\mathbf{z}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with U_s [see (2.66)] yields the CP^1 field for the three-dimensional skyrmion

$$\mathbf{z}_s = U_s \mathbf{z}_0 = \begin{pmatrix} \cos[f(r)] + i \sin[f(r)]z/r \\ i \sin[f(r)](x + iy)/r \end{pmatrix}. \quad (2.79)$$

The classical spin configuration $\mathbf{n}_s$ derived from this CP^1 configuration is given by,

$$\begin{aligned} \mathbf{n}_s &= \mathbf{z}_s^\dagger \boldsymbol{\sigma} \mathbf{z}_s \\ &= \left(2 \left[\frac{xz}{r^2} \sin[f(r)] - \frac{y}{r} \cos[f(r)] \right] \sin[f(r)], \right. \\ &\quad \left. 2 \left[\frac{x}{r} \cos[f(r)] + \frac{yz}{r^2} \sin[f(r)] \right] \sin[f(r)], \right. \\ &\quad \left. \cos^2[f(r)] + \frac{z^2 - x^2 - y^2}{r^2} \sin^2[f(r)] \right). \end{aligned} \quad (2.80)$$

One can also show

$$\mathbf{a}_s \cdot \nabla \times \mathbf{a}_s = \frac{2 \sin^2[f(r)] f'(r)}{r^2}. \quad (2.81)$$

Note how this topological density matches the earlier one (2.63), derived by way of the S^3 vector field.

In mathematics, the integer representing the skyrmion number (2.78) is known as the Hopf index. The integral formula (2.78) was discovered by Whitehead [9] as a way to compute the Hopf index explicitly for a given field configuration. Experimentally, observation of the spin structure $\mathbf{n}_s$ (2.80) in a three-dimensional magnet would constitute the discovery of three-dimensional skyrmion structures in condensed matter systems. Such a spin structure is also known as Shankar’s monopole [10], and refers to a topological structure realizing the non-trivial $\pi_3(\text{SO}(3)) = \mathbb{Z}$ homotopy class. (Fig. 2.5)

This remark on the homotopy class of Shankar’s monopole requires a little more explanation. The $\text{SU}(2)$ unitary rotation given by (2.66), acting on the reference CP^1 field $\mathbf{z}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, can be thought of as the $\text{SO}(3)$ rotation

$$\mathcal{R}_s = \exp(2if(r)\mathbf{J} \cdot \hat{r}) \quad (2.82)$$

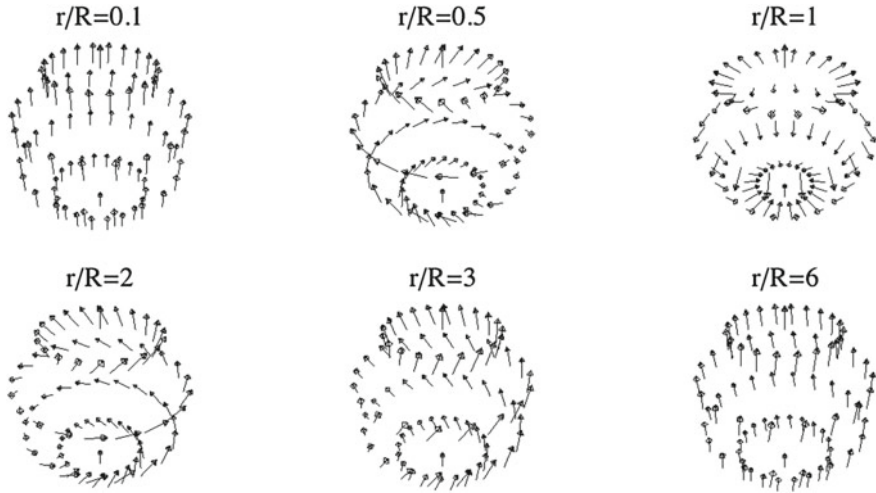


Fig. 2.5 Explicit realizations of Shankar's monopole (2.80) with $f(r) = 2\pi R^2/(r^2 + R^2)$. The spin orientations for a number of fixed values of r/R are shown

acting on the reference spin $\mathbf{n}_0 = (0, 0, 1)$. The result of this operation has already been expressed in (2.80), and we used the same boundary conditions as defined earlier, i.e., both $f(0)$ and $f(\infty)$ are multiples of π . These conditions for the $\mathbf{n}$ vector field imply that $\mathbf{n}(0) = \mathbf{n}(\infty) = \mathbf{n}_0$ for the $\mathbf{n}$ vector field. Thus, the vector field $\mathbf{n}_s = \mathcal{R}_s \mathbf{n}_0$ is an explicit realization of the nontrivial $\pi_3(\text{SO}(3))$ homotopy. The emergent magnetic field $\mathbf{b}_s = \nabla \times \mathbf{a}_s$ associated with the Shankar monopole is also very intricate,

$$\begin{aligned} \mathbf{b}_s = \frac{1}{r^5} & \left(2 \sin f [x z r \sin f - r^2 (x z \cos f + y r \sin f) f'], \right. \\ & 2 \sin f [y z r \sin f - r^2 (y z \cos f - x r \sin f) f'], \\ & \left. 2 z^2 r (\sin f)^2 + (r^2 - z^2) r^2 f' \sin 2f \right). \end{aligned} \quad (2.83)$$

Yet another name, the *hopfion*, is associated with the same topological structure in the literature since it realizes a nonzero Hopf index.

One key issue remains to be considered before concluding this chapter—that of finding a model Hamiltonian which supports a hopfion configuration as a metastable state. One such model Hamiltonian, suggested by Faddeev and Niemi [11], consists of simply adding the field energy term to the nonlinear σ -model.

$$H = \frac{1}{2} \int d^3 \mathbf{r} \sum_{\mu=1}^3 (\partial_\mu \mathbf{n})^2 + g \int d^3 \mathbf{r} (\nabla \times \mathbf{a})^2. \quad (2.84)$$

Such a Hamiltonian is not so unrealistic; after all the curl $\nabla \times \mathbf{a}$ is related to the spin texture through

$$\frac{1}{2} \mathbf{n} \cdot (\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n}) = \varepsilon_{\mu\nu\lambda} (\nabla \times \mathbf{a})_\lambda. \quad (2.85)$$

Finally, we note that a recent breakthrough in creating the hopfion structure in a chiral ferromagnetic fluid and chiral liquid crystal have been reported in Refs. [12, 13]. No realization of a hopfion structure in a hard chiral ferromagnet has been reported to date.

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