

Chapter 2

Nonsense and Proscription

Abstract This chapter identifies as develops a few salient facets of the relationship between Parry-type deductive systems and the field of ‘logics of nonsense.’ Of particular importance is Dmitri Bochvar’s ‘internal’ nonsense logic Σ_0 , establishing a strong connection between such logics of nonsense and Parry systems more generally. We observe that two $\vdash$ -Parry subsystems of Σ_0 —Harry Deutsch’s S_{fde} and Frederick Johnson’s **RC**—can be considered to be the products of two particular ‘strategies’ of eliminating problematic inferences from Bochvar’s system. For example, this chapter shows that Deutsch’s S_{fde} and Johnson’s **RC** can be interpreted as the Parry-type subsystems of Bochvar’s nonsense logic generated by taking *paraconsistent* and *connexive* fragments of Σ_0 , respectively.

In this chapter, we examine the relationship between the logics of nonsense of Bochvar and Halldén and the containment logics in the neighborhood of William Parry’s **PAI**. We detail two strategies for manufacturing containment logics from nonsense logics—taking either connexive and paraconsistent fragments of such systems—and show how systems determined by these techniques have appeared as Frederick Johnson’s **RC** and the system S_{fde} independently discovered by Harry Deutsch and Carlos Oller. In particular, we prove that Johnson’s system is precisely the intersection of Bochvar’s Σ_0 and Graham Priest’s non-symmetrized connexive logic and that the Deutsch-Oller system lies just beneath the intersection of Σ_0 and Priest’s paraconsistent **LP**. We conclude by examining the Deutsch-Oller system in more depth, giving it a characterization in terms of **LP** and showing that it plays the same role to Harry Deutsch’s paraconsistent containment logic **S** that Aleksandr Zinov’ev’s S_1 plays with respect to **PAI**.

2.1 Introduction and Semantical Preliminaries

A close cousin to containment logics—although the shared genetics may not be immediately clear—is the class of so-called ‘logics of nonsense,’ such as the sys-

tems described by Åqvist [1], Bochvar [5], and Halldén [24]. The general motivation for such systems is the thesis that formal systems must have something to say about statements that are taken to be ‘nonsense’ or ‘meaningless.’ Bochvar and Halldén each proposed solutions to the semantical paradoxes by calling the problematic sentences—*e.g.*, the Liar or Curry sentences—‘meaningless’ and offered their systems as means to proceed in formal logic while still allowing for such a semantical category. Granted that some syntactic objects are indeed meaningless in this way, these types of systems provide an additional semantic value beyond truth and falsity and formalize logics flexible enough to account for meaningless formulae.

2.2 Nonsense Logics

Logics of nonsense are logical systems which aim to reconcile a theory of deduction with the thesis that some statements are meaningless or nonsense, many of which are summarized in Krystyna Piróg-Rzepecka’s [37]. If there are indeed meaningless statements—and such statements cannot be said to be true or false—then the classical, bivalent logic championed by Gottlob Frege and Bertrand Russell is inadequate to give an account of the inferential status of such statements.

The possibility of grammatical yet meaningless statements neither true nor false arises frequently in philosophical contexts. For example, one type of a purportedly meaningless statement is a so-called *category mistake*, *e.g.*, a statement such as ‘the square root of Socrates is irrational’ in which a predicate (‘the square root of x is irrational’) is applied to an object (Socrates) in an apparently nonsensical fashion. The statement is apparently grammatical; whether it is *meaningful* is less clear. It is arguably plausible to suggest that such statements are indeed nonsense—grammatical yet non-significant—and thus demand that a correct theory of deduction be flexible enough to give accounts of meaningless statements. Logics of nonsense profess to give such a correct theory.

Unlike relevant or constructive logics, there is no unifying formal property delimiting the class of logics of nonsense; what determines this family of deductive systems is the common goal of giving an account of deduction in light of meaningless statements. Even supposing that such an account is necessary, the progenitors of nonsense logic had differing positions on many technical questions, such as the proper ontological category of meaningless statements or whether a nonsensical semantic value ought to be designated.

The proponents of logics of nonsense, chief among them being Dmitri Bochvar and Sören Halldén, agree that the classical propositional calculus is ill-equipped to deal with statements that are *meaningless* or *nonsense* and fail to take a value of either true or false. Yet a theory of meaninglessness presupposes a theory of meaning and meaning is an extraordinarily opaque concept. As the theories we will survey in this chapter were developed against the backdrop of problems of analytic philosophy, we will focus on appearances of the notion of meaninglessness since the publication of Russell and Whitehead’s *Principia Mathematica*. Of those, we restrict our attention

to three cases that may be thought to necessitate a theory of deduction capable of handling meaningless statements.

To be clear, arriving at a theory of deduction accounting for the category of meaningless statements is not some esoteric task. Hans Reichenbach wrote of Russell's suggestion that such a category be considered in the following terms:

It is the basic idea of [Russell's] theory that the division of linguistic expressions into true and false is not sufficient, that a third category must be introduced which includes meaningless expressions. It seems to me that this is one of the deepest and soundest discoveries of modern logic. [42, p. 37]

A brief note on nomenclature before proceeding: While the following is not uniformly observed by proponents of nonsense logics, the distinction between syntax and semantics demands that some attention is paid to terminology.

We use the terms 'sentence,' 'statement,' and 'formula' to denote a syntactic item, a certain type of string of symbols. The term 'proposition' is used to denote a semantic or intensional item corresponding to the meaning of the sentence. This usage is by no means standard; e.g., the positivists at times used the term 'statement' to refer only to a *meaningful* string of symbols, the term 'pseudo-statement' being awarded to the remainder of syntactic items. In this chapter, we will remain ontologically neutral, putting aside the question of whether a 'meaningless proposition' is a contradiction in terms.

As logics of nonsense were first described in order to address problems of meaningfulness in early twentieth century analytic philosophy, we will survey three occasions in which meaningfulness or nonsense emerge in this tradition.

2.2.1 Semantic Paradoxes

Semantic paradoxes have been discussed in one form or another since at least Epimenides of Knossos. A very simple version is the Liar sentence, the statement 'this sentence is false': its truth seems to entail its falsehood while its falsehood entails its truth. The instance of such paradoxes that drove the development of Bochvar and Halldén's systems was presented in Whitehead and Russell's *Principia Mathematica*, in which such paradoxes of self-reference are dismissed by appeal to a *syntactic* notion of meaningfulness.

We need not rehearse the formalism of the *Principia* to describe the problem. In *Introduction to Mathematical Philosophy*, Russell gives a sketch of the type of semantical paradox which he is interested in solving and how the theory of types is intended to resolve it. In the background is the assumption that for any property *P*, there exists a class of all objects of which *P* is true. The particular paradox is this:

From... the assemblage [class] of all classes which are not members of themselves. This is a class: is it a member of itself or not? If it is, it is one of those classes which are not members of themselves, *i.e.*, it is not a member of itself. If it is not, it is not one of those classes that are not members of themselves, *i.e.*, it is a member of itself. [44, p. 136]

By the Principle of Excluded Middle, either this class—which Russell calls ‘ κ ’—is a member of itself or not; yet that each entails the contradiction that both $\kappa \in \kappa$ and $\text{not-}\kappa \in \kappa$ implies that the statement ‘ $\kappa \in \kappa$ ’ is both true and false. This is problematic because classically, a contradiction entails all propositions and hence all sentences are true in this theory of classes.

Among the various solutions to this problem described by Russell is one in which problem cases are cleared away *syntactically* at the level of *language*. By iteratively constructing the formal language in which we work, Russell shows that one can banish self-reference in the language itself. In the type of language Russell describes, the statement ‘such-and-such a class is a member of itself’ can be prevented from entering the language at every stage. In such a setting, self-referential statements are *syntactically ill-formed* and are therefore meaningless.

In Russell’s words,

a statement which appears to be about a class will only be significant [meaningful] if it is capable of translation into a form in which no mention is made of the class. [44, p. 137]

The class κ cannot be defined without such self-reference; the term ‘class of all classes not members of themselves’ is thus a *pseudo-name*, a syntactical object that does not denote. And according to Russell’s solution,

a sentence or set of symbols in which such pseudo-names occur in wrong ways is not false, but strictly devoid of meaning. The supposition that a class is, or that it is not, a member of itself is meaningless in just this way. [44, p. 137]

The resolution offered by Russell thus appears to demand that certain statements (or statement-like objects) are neither true nor false but are rather *nonsense*. Hence, the statement ‘ $\kappa \in \kappa$ ’ is not both true and false and the problem is purportedly resolved.

Importantly, this resolves apparent nonsense by appeal to syntax, i.e., the recursive rules for a language should prevent ‘ $\kappa \in \kappa$ ’ from ever appearing and it is thus an *ill-formed* string of symbols. In the introduction to Wittgenstein’s *Tractatus Logico-Philosophicus*, Russell writes that ‘a logically perfect language has rules of syntax which prevent nonsense’ [50, p. 8], i.e., a *correct* account of language would dissolve occasions of nonsense before they could even arise.

2.2.2 Positivism and Verifiability

The early twentieth century philosophical movement known as *logical positivism* gives a further appearance of a precise treatment of meaningfulness or nonsense both related to and contemporary with the issues raised by Russell.

A central theme in logical positivism is the *verifiability* or *empiricist criterion of meaning*. Hempel describes this criterion as:

A sentence makes a cognitively meaningful assertion, and thus can be said to be either true or false, only if it is either (1) analytic [logically true] or self-contradictory [logically false] or (2) capable, at least in principle, of experiential test. [25, p. 108]

We will set aside the nuances of such a principle, such as the feasibility of verification or the necessity of falsifiability as these are all variations on the same theme. Note however, that a criterion of *meaningfulness* gives rise to a criterion of *meaninglessness* as well; those statements not satisfying the criterion will be meaningless.

Importantly, a number of claims fail to meet these criteria; wielding the verifiability criterion precludes a great number of statements from being counted as meaningful propositions, e.g., ethical, theological, and metaphysical theses are judged to be nonsense. Much of the effort of the logical positivists was hence directed at defusing (or, more emphatically, ‘eliminating’) philosophical traditions such as ethics or metaphysics, employing the criterion to dismiss their central theses and points of debate as meaningless.

Rudolph Carnap, for example, initially distinguishes between three types of meaningless statements (or ‘pseudo-statements’): Those meaningless in virtue of containing a meaningless term such as ‘good,’ those meaningless in virtue of being ill-formed, and those meaningless in virtue of ‘type confusion.’ In the first case, a sentence such as, ‘This is teavy’ is meaningless because an artificial term like ‘teavy’ is nonsensical and thus cannot be employed in an empirical test of the statement.

More importantly, Carnap’s examples for the second case and third cases are ‘Caesar is and’ and ‘Caesar is a prime number.’ The former is clearly ill-formed as this string cannot be formed by the usual rules of English syntax. According to Carnap, the latter is meaningless in virtue of the fact that the predicate ‘...is a prime number’ can ‘be neither affirmed nor denied of a person.’[7, p. 68]

Interestingly, in many of the logical positivists’ theories, nonsensical statements of the third kind are in fact instances of the second kind of nonsense, that is, despite appearing to be syntactically well-constructed sentences, they are *ill-formed*. In a perfect language, something implicitly similar to Russell’s typing ought to occur, so that the verb phrase ‘...is a prime number’ would fail to syntactically apply to a noun phrase such as ‘Caesar.’

Carnap writes:

If, e.g., nouns were grammatically subdivided into several kinds of words, according as they designated properties of physical objects, of numbers etc., then the words ‘general’ and ‘prime number’ would belong to grammatically different word-categories, and [‘Caesar is a prime number’] would be just as linguistically incorrect as [‘Caesar is and’]. In a correctly constructed language, therefore, all nonsensical sequences of words would be of the kind of [‘Caesar is and’]. [7, p. 68]

That we fail to recognize this fact is diagnosed as an artifact of the imperfections of our own natural language. Hence, from the standpoint of the logical positivists, as for Russell, meaninglessness tends to be reducible to ill-formedness. But this is not the only resolution available.

2.2.3 Category Mistakes

Prior to this syntactical observation, there is a sense in which both Russell's rejection of the meaningfulness of ' $\kappa \in \kappa$ ' and Carnap's dismissal of the statement 'Caesar is a prime number' are instances of a more general notion. Suggesting that the propositional function $\hat{x}.x \in \kappa$ cannot be applied to κ or that 'is a prime number' is not the sort of predicate which may be asserted of a man suggests that the subject is of the *wrong category*; these are each occasions of making a *category mistake*.

Gilbert Ryle introduces this term in his 1949 book *The Concept of Mind*, defining the making of a category mistake as the treating of objects 'as if they belonged to one logical type or category (or range of types or categories when they actually belong to another.' [45, p. 16] The primary philosophical thesis is that, contra Descartes, taking the language and intuitions behind our experience of the *physical* world and applying them to the *mental* leads to illicit inferences. As the physical and mental are of different types, predicates applying to the former are not merely *false* of the latter, but lead to meaningless statements.

Importantly, throughout the work, Ryle continually associates making a category mistake with uttering nonsense.

It is nonsense to speak of knowing, or not knowing, this clap of thunder or that twinge of pain, this coloured surface or that act of drawing a conclusion or seeing a joke; these are accusatives of the wrong types to follow the verb 'to know.' [45, p. 161]

As a result, on Ryle's account, Cartesian philosophy is not merely *false*, it is literally *nonsense*.

Following Ryle, the theory of category mistakes has been taken up by a number of authors independently of the questions raised by Russell or Carnap. Importantly, in the literature on category mistakes (also, 'type crossings') the emphasis on syntactical ill-formedness of such statements is eschewed in favor of more semantically-oriented analyses.

Works such as Theodore Drange's *Type Crossings* ([16]) and Shalom Lappin's *Sorts, Ontology, and Metaphor* ([29]) tend to assume that such statements are semantically evaluable. There are, to be sure, debates concerning *how* to evaluate such statements, but it tends to be taken for granted that the problematic statements are, in general, well-formed. Note that this does not necessarily demand a novel logic of nonsense nor a new semantic value. It is perfectly coherent to either assign these statements values of truth and falsity at random or uniformly evaluate them as true or false.

Drange's own account, for example, is that such category mistakes (which he calls 'type crossings') are well-formed and express propositions, albeit propositions that are 'unthinkable.' On Drange's account, there is no way that one can conceive of a state affairs in which a proposition such as that expressed by 'Caesar is a prime number' turns out true. This does *not* entail that the sentence is meaningless, although it does bear the consequence that 'Caesar is a prime number' is false.

This position—that nonsensical sentences are false—is like Russell and Carnap's appeal to syntax in that it precludes a need for a logic of nonsense.

2.2.4 Many-Valued Semantics for Two Nonsense Logics

Bochvar and Halldén’s systems each distinguish two *types* of connectives: On the one hand are the connectives whose truth functions that output a ‘nonsense’ value whenever one or more of their arguments contain a ‘nonsense’ value. The semantical value of nonsense is thus ‘infectious’ or ‘contaminating’ with respect to such connectives, a property that Åqvist colorfully labels the ‘doctrine of the predominance of the atheoretical element’ in [1]. Such connectives—described by Bochvar and Halldén as ‘internal’ or ‘classical’—are identified with the operations employed in, e.g., the *Principia Mathematica*. The languages employed by Bochvar and Halldén complement these connectives with so-called ‘external’ connectives whose corresponding truth functions map *all* arguments to ‘meaningful’ values, i.e., either truth or falsity. For example, Halldén intends for his unary ‘meaningfulness’ connective $+$ to evaluate meaningless statements as false and to evaluate meaningful statements as true.

For present purposes, we look at the fragments of Bochvar and Halldén’s logics corresponding to only these ‘internal’ connectives. By ‘ Σ_0 ’ and ‘ C_0 ,’ we denote the systems that [10] describes as the ‘classical fragments’ of the nonsense logics of Bochvar and Halldén, i.e., the systems restricted to ‘internal’ negation, disjunction, and conjunction. Consequence with respect to the systems Σ_0 and C_0 can be defined by a standard account of many-valued semantics. We will follow the presentation in [6] and consider binary consequence relations induced by logical matrices.

Definition 2.2.1 A logical matrix $\mathfrak{M}$ for $\mathcal{L}_{\text{zdf}}$ is a 5-tuple $\langle \mathcal{V}_{\mathfrak{M}}, \mathcal{D}_{\mathfrak{M}}, f_{\mathfrak{M}}^{\neg}, f_{\mathfrak{M}}^{\wedge}, f_{\mathfrak{M}}^{\vee} \rangle$ where:

- $\mathcal{V}_{\mathfrak{M}}$ is a nonempty set of *truth values*
- $\mathcal{D}_{\mathfrak{M}} \subseteq \mathcal{V}_{\mathfrak{M}}$ is a nonempty set of *designated values*
- $f_{\mathfrak{M}}^{\neg}$ is a unary truth function on $\mathcal{V}_{\mathfrak{M}}$
- $f_{\mathfrak{M}}^{\wedge}$ and $f_{\mathfrak{M}}^{\vee}$ are binary truth functions on $\mathcal{V}_{\mathfrak{M}}$

Definition 2.2.2 Let $\mathfrak{M} = \langle \mathcal{V}_{\mathfrak{M}}, \mathcal{D}_{\mathfrak{M}}, f_{\mathfrak{M}}^{\neg}, f_{\mathfrak{M}}^{\wedge}, f_{\mathfrak{M}}^{\vee} \rangle$. Then an $\mathfrak{M}$ valuation v is a function $v : \mathbf{At} \rightarrow \mathcal{V}$ extended to $\mathcal{L}_{\text{zdf}}$ by the recursive scheme:

- $v(\neg A) = f_{\mathfrak{M}}^{\neg}(v(A))$
- $v(A \wedge B) = f_{\mathfrak{M}}^{\wedge}(v(A), v(B))$
- $v(A \vee B) = f_{\mathfrak{M}}^{\vee}(v(A), v(B))$

Definition 2.2.3 A logical matrix $\mathfrak{M}$ characterizes a consequence relation for $\mathbf{L}$ if

$\Gamma \vdash_{\mathbf{L}} A$ holds iff for all $\mathfrak{M}$ valuations v such that $v[\Gamma] \in \mathcal{D}_{\mathfrak{M}}$, also $v(A) \in \mathcal{D}_{\mathfrak{M}}$.

In the sequel, when $\mathbf{L}$ is a deductive system characterized by $\mathfrak{M}$ we will slightly abuse notation and conflate $\mathbf{L}$ with $\mathfrak{M}$ so that, e.g., we will call an $\mathfrak{M}$ valuation an “ $\mathbf{L}$ valuation.”

Definition 2.2.4 Σ_0 —the classical fragment of Bochvar’s Σ —is the consequence relation induced by the matrix $\mathfrak{M}_{\Sigma_0} = \langle \mathcal{V}_{\Sigma_0}, \mathcal{D}_{\Sigma_0}, f_{\Sigma_0}^{\neg}, f_{\Sigma_0}^{\wedge}, f_{\Sigma_0}^{\vee} \rangle$ where $\mathcal{V}_{\Sigma_0} = \{t, u, f\}$ and $\mathcal{D}_{\Sigma_0} = \{t\}$. The truth-functions $f_{\Sigma_0}^{\neg}$, $f_{\Sigma_0}^{\wedge}$, and $f_{\Sigma_0}^{\vee}$ are represented by the matrices:

$f_{\Sigma_0}^{\neg}$		$f_{\Sigma_0}^{\wedge}$	t	u	f	$f_{\Sigma_0}^{\vee}$	t	u	f
t	f	t	t	u	f	t	t	u	t
u	u	u	u	u	u	u	u	u	u
f	t	f	f	u	f	f	t	u	f

We also may note that the matrices provided are equivalent to the *weak tables* of Kleene. It is fair to think of the classical fragment of Σ_0 as the weak logic described—and rejected—by Kleene in [28, p. 334].

The logic C_0 —the classical fragment of Halldén’s C without the unary meaningfulness operator—differs from Σ_0 only with respect to its set of designated values.

Definition 2.2.5 C_0 is the consequence relation induced by the matrix $\mathfrak{M}_{C_0} = \langle \mathcal{V}_{C_0}, \mathcal{D}_{C_0}, f_{C_0}^{\neg}, f_{C_0}^{\wedge}, f_{C_0}^{\vee} \rangle$ where:

- $\mathcal{V}_{C_0} = \mathcal{V}_{\Sigma_0}$
- $\mathcal{D}_{C_0} = \{t, u\}$
- $f_{C_0}^{\circ} = f_{\Sigma_0}^{\circ}$ for $\circ \in \{\neg, \wedge, \vee\}$

Now, given Halldén’s “...is meaningful” operator $+$ and Bochvar’s “...is true” operator T , one can embed classical logic within the full systems; hence, the $\mathbf{PP}^+$ will not hold in C or Σ . Even in the classical fragments Σ_0 and C_0 *without* projection operators, this property fails. However, in special cases, the $\mathbf{PP}^+$ holds and Addition fails; moreover, studying *why* the $\mathbf{PP}^+$ fails is instructive and yields a road map of sorts for transforming logics of nonsense into containment logics.

An observation important to this end is that with respect to a logic of nonsense, four theses jointly entail the $\mathbf{PP}^+$. Recall that when v is a valuation and Γ is a set of formulae, $v[\Gamma]$ represents the image of Γ under v . Then:

Observation 2.2.1 *Suppose that in a semantical presentation of a logic L*

1. “nonsense” values are infectious, i.e., for any n -tuple of truth values $\mathbf{v}$ in which a nonsense value appears and an n -ary truth-function f , $f(\mathbf{v})$ is a nonsense value,
2. “nonsense” values are not designated,
3. every set of formulae Γ has a valuation v such that $v[\Gamma] \subseteq \mathcal{D}_L$, and
4. $\Gamma \models_L B$ is read as “every valuation assigning all $A \in \Gamma$ designated values also assigns B a designated value”

Then L obeys the $\mathbf{PP}^+$.

Proof Suppose that L enjoys the above four properties and suppose for contradiction that $\Gamma \models_L B$ while some atom in B is not found in any $A \in \Gamma$. Let C be an atom witnessing this fact. Now, Γ [from 3] has a valuation v in which all $A \in \Gamma$

are designated. Consider a valuation v' identical to v except for its mapping C to a nonsense value. Since $C \notin \mathbf{At}[\Gamma]$, all $A \in \Gamma$ remain designated. Since $C \in \mathbf{At}(B)$, B is assigned a nonsense value by v' [from 1] and such a value is not designated [from 2]. Given the traditional, semantic reading of $\Gamma \models_{\mathbf{L}} B$ [from 4], we infer that $\Gamma \not\models_{\mathbf{L}} B$. $\square$

The $\mathbf{PP}^+$ fails in the classical fragment of Halldén's system because the meaningless value is designated.¹ To wit, it can be easily checked that $A \models_{C_0} A \vee B$. In Bochvar's system this inference fails in general—that A is true *does not* entail that $A \vee B$ is true as B , after all, could be meaningless, rendering the disjunction meaningless. Nevertheless, the $\mathbf{PP}^+$ fails in Bochvar's system. Σ_0 does not tolerate contradictions, *i.e.*, contradictions cannot take a designated value, and hence, $A \wedge \neg A \models_{\Sigma_0} B$ holds vacuously. The $\mathbf{PP}^+$ holds, on the other hand, for consistent premises.

This clearly lays out a means to construct a Parry system from a logic of nonsense. The central question is that of the inferential status of sets of formula Γ which have no valuations mapping their formulae to designated values; the existence of such sets prevents Σ_0 from enjoying the $\mathbf{PP}^+$. We may consider two strategies for weakening Σ_0 to a nonsense logic. One strategy is to *inferentially quarantine* such sets of formula by allowing nothing to be inferred from contradictory premises; this entails rewriting the usual rules for turnstile. A second strategy is to *homogenize* formulae so that all non-empty sets not only have models, but that inconsistent sets will maintain a similar inferential behavior to that of sets of consistent formulae.

2.3 Two Strategies for Containment

The relationship between nonsense logics and containment logics is underscored by the ways in which Parry logics can be generated from nonsense logics. To illustrate, we will consider Bochvar's Σ_0 and provide two strategies to yield a fragment that qualifies as a containment logic. The first strategy is to consider what may be thought of as a *connexive fragment* of Σ_0 and the second is to consider a *paraconsistent* fragment. In Chap. 4, we will add a third strategy by showing the intuitionistic, implicational fragment of Σ_0 is also a containment logic.

2.3.1 Containment Through Connexivity: Johnson's RC

Parry's AI was not the only cousin of (or competitor to) relevant logics to receive space in Anderson and Belnap's [2]. Additionally, pages were set aside to provide an account and examination of *connexive logics*, although the systems described therein—due to Storrs McCall—are distinct from the connexive logics we will employ in the sequel.

¹Cf. [24, p. 47] for Halldén's explanation and defense of this feature.

What we wish to show in this section is that by employing connexive principles along the lines of [40], one may make use of the proof of Observation 2.2.1 to generate a containment logic from a logic of nonsense. Indeed, what we will show is that such a system has already appeared as the containment logic **RC** introduced by Frederick Johnson in [26] and that it is the intersection of the classical fragment of Bochvar's Σ_0 and a connexive logic described by Graham Priest in [40].

The characteristic feature of connexive logics is the satisfaction of a pair of theses governing the behavior of implication, *Aristotle's Thesis*:

AT[→] $\neg(A \rightarrow \neg A)$

and *Boethius' Thesis*:

BT[→] $\neg[(A \rightarrow B) \wedge (A \rightarrow \neg B)]$

Similar principles² can be captured as metalinguistic statements as well:

AT⁺ For all A , $A \not\vdash \neg A$

BT⁺ For all A, B , if $A \vdash B$ then $A \not\vdash \neg B$

Now, there is a subtle distinction between the two formulations of these theses. That the symbol “ $\neg$ ” appears twice in **AT**[→] suggests that each instance is a species of the same type of negation, yet this is not necessarily the case with respect to its metalinguistic counterpart **AT**⁺. The metalinguistic negation indicated by $\not\vdash$ and the object language negation symbolized by $\neg$ may very well diverge in meaning. We must thus content ourselves with the claim that **AT**[→] and **AT**⁺ are similar, rather than identical, principles.

Proposals abound as to how to properly motivate connexive logics, ranging from the thesis that such systems capture the subjunctive conditional (defended by Richard Angell in [3], where **AT**[→] is called the “principle of subjunctive contrariety”) to the thesis that negation “cancels” or “annihilates” an affirmation (described, but *not* defended, by Priest in [40]). McCall's [32] and Heinrich Wansing's [49] provide thorough surveys of the history, philosophy, and motivation of connexive principles; for a deeper discussion of these matters, the reader is referred to these sources.

To tie this to the strategy of inferential quarantine, note that there is an apparently very obvious motivation for why one might expect **AT**⁺ and **BT**⁺ to hold. With respect to contingent formulae—those formulae having a model in which they are verified and one in which they are not—classical logic satisfies these principles.

Observation 2.3.1 *If A and B are classically contingent, then if $A \models_{\text{CL}} B$ then $A \not\models_{\text{CL}} \neg B$*

Proof Suppose that $A \models_{\text{CL}} \neg B$; then from $A \models_{\text{CL}} B$ and $A \models_{\text{CL}} \neg B$, we may infer that $A \models_{\text{CL}} B \wedge \neg B$. This can only hold if A is itself a contradiction, from which we infer that it is not the case that both A and B are classically contingent. $\square$

²**BT** is typically stated as $(A \rightarrow B) \rightarrow \neg(A \rightarrow \neg B)$ in the literature on connexive logic. Priest's formulation from [40] (which we employ in this chapter) has been called “Strawson's Thesis” due to P.F. Strawson's endorsement of the principle in [47]. Priest's formulation bears a strong resemblance to the formula $(A \rightarrow B) \supset \neg(A \rightarrow \neg B)$, called “weak Boethius' Thesis” by Pizzi and Williamson in [38].

Observation 2.3.2 *If A is classically contingent then $A \not\models_{\text{CL}} \neg A$*

Proof Immediate from Observation 2.3.1, substituting A for B and noting that $A \models_{\text{CL}} A$. $\square$

Implicitly employing these observations, Priest introduced a pair of connexive logics—with “plain” and “symmetrized” versions—in [40]. We will call these $\mathbf{P}_N$ and $\mathbf{P}_S$, respectively, and will consider their respective consequence relations to be defined over the language $\mathcal{L}_+$ from Definition 1.1.1.

The systems share a model structure and we will thus define models for $\mathbf{P}_N$ and $\mathbf{P}_S$ in tandem:

Definition 2.3.1 Models for $\mathbf{P}_N$ and $\mathbf{P}_S$ are 3-tuples $\langle W, g, V \rangle$, where W is a set of points such that $g \in W$ and V is a function mapping $\mathbf{At}$ to subsets of W .

As the two systems interpret the conditional connective differently, we must define distinct forcing relations,³ defined identically for all cases with the exception of the truth condition for $\rightarrow$. Following [40], we represent the condition peculiar to the symmetrized system $\mathbf{P}_S$ in square brackets:

- $w \Vdash A$ iff $w \in V(A)$ for $A \in \mathbf{At}$
- $w \Vdash \neg A$ iff $w \not\Vdash A$
- $w \Vdash A \wedge B$ iff $w \Vdash A$ and $w \Vdash B$
- $w \Vdash A \vee B$ iff $w \Vdash A$ or $w \Vdash B$
- $w \Vdash A \rightarrow B$ iff $\begin{cases} \exists w' \in W \text{ such that } w' \Vdash A, \\ \forall w' \in W, \text{ if } w' \Vdash A \text{ then } w' \Vdash B \\ \text{[and } \exists w' \in W \text{ such that } w' \not\Vdash B \end{cases}$

We will call the relation for $\mathbf{P}_N$ (without the clause in square brackets) $\Vdash_{\mathbf{P}_N}$ and that for $\mathbf{P}_S$ (with the clause in square brackets) $\Vdash_{\mathbf{P}_S}$.

We are thus now able to define the notion of validity for the two systems.

Definition 2.3.2 $\mathbf{P}_N$ validity

$$\Gamma \models_{\mathbf{P}_N} A \text{ if } \begin{cases} \text{there is an } \mathcal{M} \text{ such that for all } B \in \Gamma, g \Vdash_{\mathbf{P}_N} B \\ \text{for all } \mathcal{M} \text{ such that for all } B \in \Gamma, \mathcal{M}, g \Vdash_{\mathbf{P}_N} B, \text{ also } g \Vdash_{\mathbf{P}_N} A \end{cases}$$

$\mathbf{P}_S$ validity is defined in an analogous fashion, substituting $\Vdash_{\mathbf{P}_S}$ for $\Vdash_{\mathbf{P}_N}$.

Priest’s approach has appeared in various forms in other contexts; *e.g.*, David Lewis offers a conditional connective $\Box\Rightarrow$ in [31] that determines a weak subsystem of Priest’s system $\mathbf{P}_N$. In [38], Claudio Pizzi and Timothy Williamson also indirectly

³In a number of works (*e.g.*, [22] and [46]), Melvin Fitting and Raymond Smullyan have detailed the intimate relationship between Cohen’s forcing introduced in [8] and [9] and the relation of truth-at-a-world in Kripke models. The term “forcing relation” is frequently used to describe truth-at-a-world in models with possible worlds, even in contexts in which the strict analogy with Cohen forcing is lost.

describe another subsystem of Priest's P_N , although its semantics are couched in terms of a conditional logic rather than a logic of strict implication (cf. [18]).

A further (and isolated) appearance of this approach is found in Frederick Johnson's containment logic **RC** described in [26]. Johnson was interested in identifying a simple and natural means of precluding C.I. Lewis' famous argument for the principle of *explosion* found in [30], where explosion is the validity of an inference to an arbitrary formula from a contradiction. Concerned with the apparent *irrelevance* of the consequent to the antecedent in such an inference, Johnson aligned his system—described as “syntactic relevance entailment”—with the field of relevant logics rather than with containment or connexive logics. Neither of the latter themes is mentioned in the paper. Even in the later [27], in which a related system is introduced, that the system enjoys the PP^+ is mentioned only *en passant*.

The system **RC** is semantically described by recalling the logical matrix $\mathfrak{M}_{\Sigma_0}$ from Definition 2.2.4, in which t the only designated value.

Definition 2.3.3 Consequence in the system **RC** is defined so that:

$$\Gamma \models_{\mathbf{RC}} A \text{ if } \begin{cases} \text{there is a } \Sigma_0 \text{ valuation } v \text{ such that for all } B \in \Gamma, v(B) = t \\ \text{for all } \Sigma_0 \text{ valuations } v \text{ s.t. for all } B \in \Gamma, v(B) = t, \text{ also } v(A) = t \end{cases}$$

Although it is probably clear that **RC** is a subsystem of both Σ_0 and P_N , we are able to obtain an even stronger result:

Observation 2.3.3 $\mathbf{RC} = P_N \cap \Sigma_0$

Proof We first note that the matrices Johnson provides for **RC** are Bochvar's matrices for Σ_0 . As validity in Σ_0 is a necessary condition for validity in **RC**, $\mathbf{RC} \subseteq \Sigma_0$.

Moreover, if the inference $\Gamma \models A$ is **RC** valid, then we may infer a number of things. For one, we require that Γ must be non-empty. Were it empty, then all Σ_0 valuations would vacuously map each of its members to t ; by the definition of validity, this would entail that all Σ_0 valuations map A to t , *i.e.*, that A is a theorem of Σ_0 . But Σ_0 has no theorems. Furthermore, we infer that there exists a Σ_0 valuation v by which all formulae in $\Gamma \cup \{A\}$ are designated. Any such valuation, however, restricted to $\mathbf{At}[\Gamma]$ is classical, *i.e.*, the image of $\mathbf{At}[\Gamma]$ under v is $\{t, f\}$. (Otherwise, granted the infectiousness of u , $v(B) = u$ for some $B \in \Gamma$.) As $v[\Gamma]$ depends only on the values assigned to $\mathbf{At}[\Gamma]$, we may construct a function v' such that

$$v'(B) = \begin{cases} v(B) & \text{if } B \in \mathbf{At}[\Gamma] \\ f & \text{otherwise} \end{cases}$$

The range of v' is $\{t, f\}$ and v' is thus a classical valuation mapping all formulae in Γ to t , which is just to say that Γ is classically consistent. Additionally, as Σ_0 is a subsystem of classical logic, Γ classically entails A . From these two considerations, we infer that $\Gamma \models_{P_N} A$, whence $\mathbf{RC} \subseteq P_N$.

Suppose that an inference $\Gamma \vdash A$ is both P_N - and Σ_0 valid. Then $\Gamma \vdash_{\Sigma_0} A$ holds either vacuously or it does not. The inference *cannot* hold vacuously; were it to do so, then there would be no Σ_0 valuations granting every $B \in \Gamma$ a designated value and thus, *a fortiori*, no classical valuations. But this would imply that Γ is classically a contradiction, entailing that $\Gamma \not\vdash_{P_N} A$ and contradicting the hypothesis. Hence, there *is* a Σ_0 valuation mapping all $B \in \Gamma$ to designated values and in all such valuations A receives a designated value; but this is just to say that $\Gamma \vdash_{RC} A$. $\square$

It is extraordinarily interesting that the conjunction of two unrelated theses concerning implication—that of formally accommodating meaningfulness and that of cancellation

negation—should prove equivalent to an entirely distinct intuition, that of Johnson.

The system RC is not without problems. Most notable of these is that, as in P_N , the inference $A \vdash_{RC} A$ is not valid. While the account given by Priest of P_N makes some sense of the failure of this inference, it is not clear that Priest’s story serves to resolve such a pathology in the context of RC.

Quarantining the problematic cases is not the only strategy; we have also mentioned a strategy of *homogenizing* inference. Merely providing all sets of sentences with a model is of little use if such models are trivial; rather, we may want a way to maintain nontrivial yet inconsistent models. This can be performed by taking a *paraconsistent* fragment of a nonsense logic.

2.3.2 Containment Through Paraconsistency: The System S_{fde}

As Parry was a student of Lewis, it is not surprising that many of the “paradoxes” of implication, e.g., the principle of explosion, were of concern to him. As noted in the case of Johnson, such an inference is in some quarters taken to be suspicious due to a lack of relevance between the antecedent and consequent.

As shown in Observations 1.1.1 and 1.1.2, the relationship between Parry’s system and relevant logic is a clear one: Relevant logics enjoy the *variable-sharing property*, establishing all Parry systems as relevant logics. But if we take the notion of relevance as a desideratum seriously, even in PAI, there are theorems in which apparent irrelevance arises. The PAI theorem

$$((A \wedge \neg A) \wedge B) \rightarrow \neg B$$

might arouse—and has indeed aroused—similar suspicions. Harry Deutsch describes this as “the fallacy of making too much of one small, if nasty, mistake” [12, p. 139] and asserts that this principle is as suspicious as the principle of explosion.

Carlos Oller essentially rediscovers this perceived shortcoming, diagnosing what he calls the “the paradoxes of Parry’s analytic implication” [34, p. 93] in the first-degree fragment of PAI:

$$A \wedge \neg A \wedge B \models_{\text{PAI}_{f_{\text{fde}}}} \neg B$$

Deutsch and Oller independently introduced a four-valued logic in order to rectify such perceived pathologies by further weakening Parry’s system.⁴ The system has appeared by a number of names, e.g., Deutsch calls the system “g” when it is first introduced in [11], “ $\mathcal{D}_{f_{\text{fde}}}$ ” in [13], and “ $\mathcal{S}_{f_{\text{fde}}}$ ” (which we ourselves will adopt in the sequel) in [14] while Oller introduces it with identical matrices in as “AL” in [34].

Definition 2.3.4 $\mathcal{S}_{f_{\text{fde}}}$ is the first-degree logic induced by the matrix

$$\langle \mathcal{V}_{\mathcal{S}_{f_{\text{fde}}}}, \mathcal{D}_{\mathcal{S}_{f_{\text{fde}}}}, f_{\mathcal{S}_{f_{\text{fde}}}}^{\neg}, f_{\mathcal{S}_{f_{\text{fde}}}}^{\wedge}, f_{\mathcal{S}_{f_{\text{fde}}}}^{\vee} \rangle$$

where $\mathcal{V}_{\mathcal{S}_{f_{\text{fde}}}} = \{t, b, u, f\}$ and $\mathcal{D}_{\mathcal{S}_{f_{\text{fde}}}} = \{t, b\}$ and the functions $f_{\mathcal{S}_{f_{\text{fde}}}}^{\neg}, f_{\mathcal{S}_{f_{\text{fde}}}}^{\wedge}, f_{\mathcal{S}_{f_{\text{fde}}}}^{\vee}$ are defined by the following matrices:

$f_{\mathcal{S}_{f_{\text{fde}}}}^{\neg}$		$f_{\mathcal{S}_{f_{\text{fde}}}}^{\wedge}$		$f_{\mathcal{S}_{f_{\text{fde}}}}^{\vee}$	
t	f	t	t b u f	t	t t u t
b	b	b	b b u f	b	t b u b
u	u	u	u u u u	u	u u u u
f	t	f	f f u f	f	t b u f

Although the conclusion that $\mathcal{S}_{f_{\text{fde}}}$ enjoys the $\mathbf{PP}^{\perp}$ is proven in [34], it will be instructive to rehearse our own proof.

Observation 2.3.4 $\mathcal{S}_{f_{\text{fde}}}$ enjoys the $\mathbf{PP}^{\perp}$

Proof A brief inspection of the matrices for $\mathcal{S}_{f_{\text{fde}}}$ will establish that u is infectious in the sense of Observation 2.2.1, while a glance at $\mathcal{D}_{\mathcal{S}_{f_{\text{fde}}}}$ shows that u is not designated.

Moreover, every set of formulae Γ has an $\mathcal{S}_{f_{\text{fde}}}$ valuation in which all $B \in \Gamma$ take designated values. The map $\nu' : A \mapsto b$ for all $A \in \mathbf{At}$ assigns a designated value to all atoms. By inspecting the matrices for $\mathcal{S}_{f_{\text{fde}}}$, we can easily observe that this propagates through the language, assigning every formula the value of b. With the language itself having a model, each of its subsets has a model.

By Observation 2.2.1, this entails that the $\mathbf{PP}^{\perp}$ holds for $\mathcal{S}_{f_{\text{fde}}}$. $\square$

That consideration of logics of nonsense played a role in proving that $\mathcal{S}_{f_{\text{fde}}}$ enjoys the $\mathbf{PP}^{\perp}$ is no coincidence. $\mathcal{S}_{f_{\text{fde}}}$ is a logic of nonsense; indeed, it is a subsystem of Σ_0 .

⁴While the position outlined by Oller against the Parry “paradoxes” is clear, it is also obvious that inferences such as

$$B \models_{\mathcal{S}_{f_{\text{fde}}}} B \vee \neg B$$

are correct modulo $\mathcal{S}_{f_{\text{fde}}}$, although such inferences appear to run afoul of the spirit of Deutsch and Oller’s complaint.

Observation 2.3.5 $S_{fde} \subseteq \Sigma_0$

Proof By examining the matrices appearing in Definitions 2.2.4 and 2.3.4, one may confirm that every Σ_0 valuation is also an S_{fde} valuation. Hence, if Γ entails A modulo S_{fde} the same can be said *a fortiori* for Σ_0 . $\square$

Recall that a logic is *paraconsistent* if explosion—the inference $A \wedge \neg A \vdash B$ —is not a valid inference in that logic. Also, recall that it was explosion that most clearly prevented Σ_0 from enjoying the $\mathbf{PP}^+$, because $A \wedge \neg A$ had no models at all. Just as employing connexive principles to eliminate this case generates a containment logic, so, too, does relaxing Σ_0 to a paraconsistent logic yield a containment logic.

A paradigmatic paraconsistent logic is the system $\mathbf{LP}$ introduced by Priest in [39].

Definition 2.3.5 $\mathbf{LP}$ is the first-degree logic induced by the matrix $\mathfrak{M}_{\mathbf{LP}}$:

$$\langle \mathcal{V}_{\mathbf{LP}}, \mathcal{D}_{\mathbf{LP}}, f_{\mathbf{LP}}^{\neg}, f_{\mathbf{LP}}^{\wedge}, f_{\mathbf{LP}}^{\vee} \rangle$$

with truth values $\mathcal{V}_{\mathbf{LP}} = \{t, b, f\}$ and designated values $\mathcal{D}_{\mathbf{LP}} = \{t, b\}$. The truth functions $f_{\mathbf{LP}}^{\neg}, f_{\mathbf{LP}}^{\wedge}, f_{\mathbf{LP}}^{\vee}$ are defined by the following matrices:

$f^{\neg}$		$f^{\wedge}$		$f^{\vee}$	
t	f	t	$t \ b \ f$	t	$t \ b \ f$
b	b	b	$b \ b \ f$	b	$t \ b \ b$
f	t	f	$f \ f \ f$	f	$t \ b \ f$

Clearly, S_{fde} is a subsystem of $\mathbf{LP}$, as we may easily prove.

Observation 2.3.6 $S_{fde} \subseteq \mathbf{LP}$

Proof Examining the matrices shows that every $\mathbf{LP}$ valuation is an S_{fde} valuation. Hence, if some property holds for all S_{fde} valuations it holds *a fortiori* for all $\mathbf{LP}$ valuations as well. Hence, if $\Gamma \vdash_{S_{fde}} A$ then $\Gamma \vdash_{\mathbf{LP}} A$, i.e., $S_{fde} \subseteq \mathbf{LP}$. $\square$

Corollary 2.3.1 $S_{fde} \subseteq \Sigma_0 \cap \mathbf{LP}$

Proof Immediate from Observations 2.3.5 and 2.3.6. $\square$

This result is encouraging but, although we come close, we do not enjoy the attractive alignment that we found in Observation 2.3.3, notably, there are some inferences both Σ_0 valid and $\mathbf{LP}$ valid.

Observation 2.3.7 $S_{fde} \neq \Sigma_0 \cap \mathbf{LP}$

Proof Observe that both $A \wedge \neg A \vdash_{\Sigma_0} A \vee B$ and $A \wedge \neg A \vdash_{\mathbf{LP}} A \vee B$. In the former case, there is no Σ_0 valuation granting $A \wedge \neg A$ a designated value and the inference is satisfied vacuously; in the latter cases, that $A \wedge \neg A$ is designated entails that A is also designated, whence $A \vee B$ is designated. Clearly, this inference fails to satisfy the $\mathbf{PP}^+$ and is thus not a valid S_{fde} inference. $\square$

What is especially interesting about this is that the inference witnessing the inequality between S_{fde} and $\Sigma_0 \cap LP$ holds in the latter systems for entirely different reasons.⁵

With respect to a first-degree logic L , use the notation L_{PP} to denote the class of L valid inferences satisfying the PP^+ , *i.e.*, the system defined by

$$\Gamma \models_{L_{PP}} A \text{ iff } \Gamma \models_L A \text{ and } \mathbf{At}(A) \subseteq \mathbf{At}[\Gamma]$$

We may think of this as the “analytic fragment” of L . Then we are able to correctly characterize S_{fde} :

Observation 2.3.8 $S_{fde} = LP_{PP}$

Proof For left-to-right, we note that Observations 2.3.5 and 2.3.6 entail that any S_{fde} entailment is valid in LP_{PP} .

For right-to-left, suppose that $\Gamma \models_{LP} A$ and $\mathbf{At}(A) \subseteq \mathbf{At}[\Gamma]$. Note that S_{fde} valuations come in two varieties: those whose restrictions to $\mathbf{At}[\Gamma \cup \{A\}]$ are LP valuations and those that are not, *i.e.*, those in which for some $B \in \mathbf{At}[\Gamma \cup \{A\}]$, $v(B) = n$. By hypothesis, for all valuations of the former type in which all formulae of Γ are designated, A is likewise designated—this is precisely what $\Gamma \models_{LP} A$ means. With respect to the latter type, by hypothesis, $\mathbf{At}(A) \subseteq \mathbf{At}[\Gamma]$, and hence, some formula $B \in \Gamma$ has a constituent atom valued at n . By the “infectiousness” of this value, it follows that $v(B) = n$. Hence, in any such valuation, some $B \in \Gamma$ fails to take a designated values, *i.e.*, the *only* valuations in which all $B \in \Gamma$ take designated values are the LP -like valuations. But we have assumed that in such valuations, A takes a designated value when all formulae in Γ do. $\square$

These observations will come into play again shortly, as we make a deeper examination of S_{fde} and its role in paraconsistent Parry systems in general.

2.4 The Role of S_{fde} in Paraconsistent Parry Systems

While Johnson’s RC is rather anomalous, playing no role with respect to the broader family of containment logics, the Deutsch-Oller system S_{fde} plays a central role in the structure of paraconsistent Parry systems. To observe this, we offer, with minor notational deviations, the semantics for PAI discovered by Kit Fine in [21]. We first define a PAI model:

Definition 2.4.1 A PAI model is an ordered 5-tuple $\langle W, R, C, \Gamma, V \rangle$ with the following interpretations:

- W is a non-empty set of points
- R is a transitive, reflexive relation on W

⁵We will encounter a similar phenomenon in the sequel when we describe the axiom **Safety** that is a hallmark of the first-degree fragment of R –Mingle.

- $\mathbf{C}$ is a set $\{\mathbf{C}_w : w \in W\}$ such that $\mathbf{C}_w = \langle C_w, \circ_w \rangle$ is a lower semilattice for all $w \in W$
- Γ is a set $\{\gamma_w : w \in W\}$ such that γ_w maps each element of $\mathbf{At}$ to an element of $\mathbf{C}_w$, extended through the language by $\gamma_w(A) = \gamma_w(B_0) \circ_w \dots \circ_w \gamma_w(B_n)$, where each $B_i \in \mathbf{At}(A)$
- V is a pair of functions $\langle V^+, V^- \rangle$ mapping all elements of $\mathbf{At}$ to $\wp(W)$ with the condition that for all $A \in \mathbf{At}$, $V^+(A)$ and $V^-(A)$ are pairwise disjoint and exhaust W

The elements of W may retain the usual interpretation of possible worlds while the intended interpretation of the elements of a set C_w are the “concepts” that occur at world w .

Define $a \leq_w b$ as $a \circ_w b = b$. Then we may describe a pair of forcing relations, defined and interpreted as follows:

Definition 2.4.2 In a PAI model, the positive relation $\Vdash^+$ can be thought of as holding when a formula is *true* at a point:

- $w \Vdash^+ A$ iff $w \in V^+(A)$ for $A \in \mathbf{At}$
- $w \Vdash^+ \neg A$ iff $w \Vdash^- A$
- $w \Vdash^+ B \wedge C$ iff $w \Vdash^+ B$ and $w \Vdash^+ C$
- $w \Vdash^+ B \vee C$ iff $w \Vdash^+ B$ or $w \Vdash^+ C$
- $w \Vdash^+ B \rightarrow C$ iff $\begin{cases} \forall u \text{ such that } wRu, \gamma_u(C) \leq_u \gamma_u(B), \text{ and} \\ \forall u \text{ such that } wRu \text{ and } u \Vdash^+ B, u \Vdash^+ C \end{cases}$

Similarly, the negative relation $\Vdash^-$ may be read as holding when a formula is *false* at some point:

- $w \Vdash^- A$ iff $w \in V^-(A)$ for $A \in \mathbf{At}$
- $w \Vdash^- \neg A$ iff $w \Vdash^+ A$
- $w \Vdash^- B \wedge C$ iff $w \Vdash^- B$ or $w \Vdash^- C$
- $w \Vdash^- B \vee C$ iff $w \Vdash^- B$ and $w \Vdash^- C$
- $w \Vdash^- B \rightarrow C$ iff $\begin{cases} w \nVdash^+ B \rightarrow C, \text{ or} \\ w \Vdash^+ B \text{ and } w \Vdash^- C \end{cases}$

The notation employed here is inspired by Wansing’s [48] in which a pair of forcing relations—one positive, one negative—is defined. Also note that a deeper analogy with Wansing’s logics $\mathbf{I}_j\mathbf{C}_k$ introduced in [48] is available. The conditions are virtually identical to Wansing’s treatment of Nelson’s N of [33]. Wansing observes that there isn’t necessarily a privileged interpretation of the *falsity* condition of an implicational formula and offers four distinct approaches to evaluating falsity of a conditional at a point or possible world.⁶ Just as Nelson’s logic of constructible falsity admits such variations, we could just as easily give the same treatment to Deutsch’s S by selecting alternative falsity conditions for the conditional.

⁶Also see [17], [19], and [20] for more discussion on the theme of falsity conditions for conditionals.

We say that a formula is true in a model— $\mathfrak{M} \models A$ —if for all points w in that model, $\mathfrak{M}, w \Vdash A$.

Definition 2.4.3 PAI validity

$\Gamma \models_{\text{PAI}} A$ if for every PAI model $\mathfrak{M}$ if for all $B \in \Gamma$, $\mathfrak{M} \models B$ then $\mathfrak{M} \models A$

An interesting observation is that the first-degree fragment of PAI is effervescent, popping up repeatedly in the literature. The first-degree fragment has been independently discovered by no fewer than four authors. In addition to Parry himself, the system was described by Zinov'ev as the system S_1 in [51], as Parks-Clifford's first-degree Z in [35], and was also labeled NDR in [27] when rediscovered by Frederick Johnson.

An important relationship holds between PAI, S_1 , and the classical propositional calculus CL. In regard to a logic L defined over a language including an intensional conditional connective $\rightarrow$, let L_{fde} denote the first-degree fragment of L , i.e., for a finite, non-empty set of formulae Γ and formula A with no appearances of $\rightarrow$, $\Gamma \models_{L_{\text{fde}}} A$ iff $\models_L \bigwedge \Gamma \rightarrow A$.

Observation 2.4.1 $\Gamma \models_{S_1} A$ iff $\Gamma \models_{\text{CL}} A$ and $\text{At}(A) \subseteq \text{At}[\Gamma]$

Proof That $S_1 = \text{CL}_{\text{PP}}$ is well established; the reader is referred to proofs in [51] or [27]. $\square$

The correspondence between S_1 and PAI_{fde} has been asserted on several occasions. With respect to Zinov'ev's work, that $S_1 = \text{PAI}_{\text{fde}}$ has been observed in [43] (in which S_1 is called “ZV”) while Parry asserts in [36] that PAI_{fde} is characterized by the equivalent bipartite condition. In neither case is this assertion proven, however, so it is prudent to provide proof here.

Observation 2.4.2 S_1 is the first-degree fragment of Parry's PAI, i.e., $A \models_{S_1} B$ iff $\models_{\text{PAI}} A \rightarrow B$

Proof By Observation 2.4.2, we are free to equate S_1 with CL_{PP} ; that $\text{PAI}_{\text{fde}} = \text{CL}_{\text{PP}}$ can be easily seen by considering an arbitrary PAI model and a point in that model. For left-to-right, consider a first-degree entailment $A \rightarrow B$, where A and B are zeroth-degree formulae; if $A \rightarrow B$ is a theorem of PAI then, as a subsystem of CL, it is a theorem of CL. That PAI obeys the $\text{PP}^\rightarrow$ entails that $\text{At}(B) \subseteq \text{At}(A)$.

For right-to-left, as CL is the “internal” logic of every point w , that $w \Vdash A$ entails that $w \Vdash B$. Moreover, as $\text{At}(B) \subseteq \text{At}(A)$, $\gamma_w(B) \leq_w \gamma_w(A)$ for any point w . Hence, at any point w' , $w' \Vdash A \rightarrow B$. $\square$

In Sect. 2.3.2, we referred to a critique of PAI shared by both Deutsch and Oller. We have seen how Oller responded; Deutsch, influenced by the semantical picture laid out by Fine in [21], detailed three fully intensional (i.e., higher degree) systems of

paraconsistent containment logic, S , S' , and S'' over the course of several papers: [12], [14], and [15].

In the above semantics for PAI , there was a qualification on the functions V^+ and V^- that for any atom A , $V^+(A) \cap V^-(A) \neq \emptyset$ and $V^+(A) \cup V^-(A) = W$. Relaxing this requirement would permit either an atom to be simultaneously true and false at a point or to be neither true nor false at a point, *i.e.*, would yield a paraconsistent or paracomplete logic. The semantics presented earlier with this restriction relaxed to permit paraconsistency corresponds to Deutsch's S .

Definition 2.4.4 An S model is defined by rehearsing the conditions from Definition 2.4.1 while relaxing the condition on V^+ and V^- to the weaker clause that

$$\text{For all } A \in \mathbf{At}, V^+(A) \cup V^-(A) = W$$

Definition 2.4.5 Validity in S is defined by the following scheme:

$$\Gamma \models_S A \text{ if for every } S \text{ model } \mathcal{M} \text{ if for all } B \in \Gamma, \mathcal{M} \models B \text{ then } \mathcal{M} \models A.$$

We can make a few further observations concerning the relationship between S_{fde} and other containment logics. In analogy to the fact that $S_1 = CL_{pp}$, Observation 2.3.8 shows that $S_{fde} = LP_{pp}$. For one, this enables us to provide a characterization of S_{fde} along the lines of Observation 2.3.3.

Corollary 2.4.1 $S_{fde} = S_1 \cap LP$

A further analogy may be made, however, between S_{fde} and Deutsch's S . That S_{fde} is the first-degree fragment of S is reflected by our choice of notation and is asserted by Deutsch in [13] and [14]. However, this assertion receives no proof and we thus provide a proof here.

Observation 2.4.3 S_{fde} is the first-degree fragment of Deutsch's S , *i.e.*, $A \models_{S_{fde}} B$ iff $\models_S A \rightarrow B$

Proof We recall that in Deutsch's semantics, for each point w , all atoms are given valuations of either $\{t\}$, $\{f\}$, or $\{t, f\}$. With small changes in notation, for negation, conjunction, and disjunction, Deutsch provides the following:

$$\begin{aligned} (\neg t) \quad t &\in v_w(\neg A) \text{ iff } f \in v_w(A) \\ (\neg f) \quad f &\in v_w(\neg A) \text{ iff } t \in v_w(A) \\ (\wedge t) \quad t &\in v_w(A \wedge B) \text{ iff } t \in v_w(A) \text{ and } t \in v_w(B) \\ (\wedge f) \quad f &\in v_w(A \wedge B) \text{ iff } f \in v_w(A) \text{ or } f \in v_w(B) \\ (\vee t) \quad t &\in v_w(A \vee B) \text{ iff } t \in v_w(A) \text{ or } t \in v_w(B) \\ (\vee f) \quad f &\in v_w(A \vee B) \text{ iff } f \in v_w(A) \text{ and } f \in v_w(B) \end{aligned}$$

Essentially, that $t \in v_w(A)$ and that $f \in v_w(A)$ in Deutsch's original presentation correspond to $w \Vdash^+ A$ and $w \Vdash^- A$, respectively.

Let h be a function equating the values of v_w with LP truth values so that singleton truth values in $\mathbf{S}$ are equated with their elements in $\mathbf{S}_{\text{fde}}$, *i.e.*,

$$h(v_w(A)) = \begin{cases} \mathbf{t} & \text{if } v_w(A) = \{\mathbf{t}\} \\ \mathbf{b} & \text{if } v_w(A) = \{\mathbf{t}, \mathbf{f}\} \\ \mathbf{f} & \text{if } v_w(A) = \{\mathbf{f}\} \end{cases}$$

We may note by a simple induction that the “internal logic” of a point is precisely LP. This is to say that if v_w is a valuation mapping atoms to $\wp(\{\mathbf{t}, \mathbf{f}\})$, then for a first-degree formula, not only is $h \circ v_w$ an LP valuation, but for any zeroth-degree formula B and a truth value $\mathbf{v}$, $\mathbf{v} \in v_w(B)$ *iff* $h(v_w(B)) = h(\mathbf{v})$.

Suppose that $\models_{\mathbf{S}} A \rightarrow B$. Then, in every point w in every model, at all points w' such that wRw' , $\gamma_{w'}(B) \leq_{w'} \gamma_{w'}(A)$. Moreover, if A takes a designated value at w' , *i.e.*, $\mathbf{t} \in v_{w'}(A)$, then B takes a designated value at w' . If at all points in all models does $\gamma_{w'}(B) \leq_{w'} \gamma_{w'}(A)$ then $\mathbf{At}(B) \subseteq \mathbf{At}(A)$. Suppose there is an atom $D \in \mathbf{At}(B)$ not in $\mathbf{At}(A)$; then for some model, one could assign $\gamma_{w'}(D)$ to be an element $d \in \mathbf{C}_{w'}$ such that $\gamma_{w'}(A) \not\leq_{w'} d$, whence $\gamma_{w'}(B) \not\leq_{w'} \gamma_{w'}(A)$. Moreover, if at all points in all models a zeroth-degree formula A entails a zeroth-degree formula B , then whenever $\mathbf{t} \in v_w(A)$, also $\mathbf{t} \in v_w(B)$. By the above considerations, however, this is just to say that in every LP valuation in which A is designated, B is designated, *i.e.*, $A \models_{\text{LP}} B$. By Observation 2.3.8, these two observations mean that $A \models_{\mathbf{S}_{\text{fde}}} B$.

On the other hand, if $A \models_{\mathbf{S}_{\text{fde}}} B$ where A and B are zeroth-degree formulae, then at any point w in any model, if $w \Vdash^+ A$ then $w \Vdash^+ B$. Likewise, that $\mathbf{At}(B) \subseteq \mathbf{At}(A)$ entails that at any such point, $\gamma_w(B) \leq_w \gamma_w(A)$, whence $\models_{\mathbf{S}} A \rightarrow B$. $\square$

Given that Deutsch’s system is the natural result of modifying PAI to yield a paraconsistent system, the Deutsch-Oller system plays a central role in the theory of $\mathbf{S}$. It also provides insight into further means of generating Parry systems. For example, the system $\mathbf{S}_{\text{fde}}^*$ —the **PP**-fragment of $\mathbf{E}_{\text{fde}}$ introduced by Priest in [41]—would play a central role in a paraconsistent and paracomplete logic similar to $\mathbf{S}$.

2.5 Conclusions

As Parry frequently referenced, Kurt Gödel conjectured in [23] that **AI** might enjoy a “double-barrelled” analysis, *i.e.*, $A \rightarrow B$ is an **AI** theorem *iff*

1. $A \Rightarrow B$ is a theorem in a “carrier logic” for some connective $\Rightarrow$, and
2. $\mathbf{At}(B) \subseteq \mathbf{At}(A)$

In providing his semantics for PAI in [21], Fine essentially confirms Gödel’s conjecture and remarks on the wide variety of logics that can be generated by altering the “carrier logic,” *e.g.*, by interpreting $\Rightarrow$ as intuitionistic or relevant implication. (The “carrier logic” in the case of PAI itself is **S4**.)

We have observed that the same can be said for the Deutsch-Oller system S_{fde} and similar conclusions may be drawn about Johnson's **RC** and Deutsch's **S**. Does this suggest that, as Sylvan suggests in [43], containment logics are merely a syntactic gimmick?

If anything, the opposite conclusion should be drawn from the structure of **RC** and S_{fde} . Rather than resting, as Sylvan suggested, 'on a narrow and arbitrary assumption as to what counts as a concept' [43, p. 101], aligning containment logics with logics of nonsense provides an alternative foundation. That an isomorphism exists between concepts and atomic formulae is not necessary; merely making the claim that meaningfulness must be established in order to ensure the validity of an inference already starts one down the path towards Parry systems.

Moreover, however syntactical the flavor of the **PP** may be, this does not entail that Parry calculi deal in gimmickry. The **VSP**—an equally syntactical criterion—is, after all, a *symptom* of relevance rather than its *explication*. That the **PP** is suggested by a number of distinct and relatively natural positions on inference demonstrates that the Proscriptive Principle may emerge without overt appeal to syntax.

In the coming two chapters, we are going to examine two distinct contexts in which this relationship between nonsense and containment is apparent. In Chap. 3, we will examine Kit Fine's analysis of Richard Angell's containment logic **AC** in which nonsense will correspond to the absence of any truthmaker *or* falsemaker for a sentence at a world. In Chap. 4, we will consider containment logics through the lens of computation, in which nonsense will correspond to the failure of a procedure to terminate its computation.

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