Chapter 2
Technical Preparation

This chapter's focus is on providing some technical background which is needed for the subsequent analysis to be carried out on the surplus models themselves in later chapters. This background is not meant to be comprehensive in the sense that well-known mathematical topics are normally assumed to be known and are not discussed in much detail. Rather, less commonly known topics specific to this monograph are discussed. A brief review is provided of Lagrange polynomials due to their importance in numerous subsequent places in the monograph. The so-called 'Dickson–Hipp' operator, which generalizes both Laplace transforms and distribution tails, is of central importance in many of the models, and is then discussed. As much of the monograph utilizes defective renewal equation methodology, this topic and the closely related compound geometric and compound geometric convolution methodology is also reviewed. Finally, the important classes of mixed Erlang and Coxian distributions, which have attracted much attention in recent years in the applied probability and actuarial literature due to their mathematical tractability, are briefly summarized.

2.1 Lagrange Polynomials

Suppose that $x_1, x_2, \ldots, x_n$ are distinct numbers, and that $h(x)$ is any polynomial of degree at most $n-1$. Then $h(x)$ may be expressed in the form

$$h(x) = \sum_{i=1}^{n} h(x_i) \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}, \tag{2.1}$$

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so that $h(x)$ may be re-expressed as a linear combination of its functional values $h(x_1), h(x_2), \ldots, h(x_n)$.

Example 2.1 The choice $h(x) = 1$ yields interesting and useful identities involving (arbitrarily chosen) numbers $x_1, x_2, \ldots, x_n$. In this case (2.1) becomes

$$1 = \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}, \tag{2.2}$$

and with $x = 0$ it follows that for $n \geq 2$

$$\sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x_j}{x_j - x_i} = 1,$$

which is also true for $n = 1$ when the empty product is assumed to be 1. Also, the right-hand side of (2.2) is a polynomial of degree $n - 1$ because it is the sum of n such polynomials, one for each i. The coefficient of x^{n-1} in the i-th polynomial is $1/\prod_{j=1, j\neq i}^{n} (x_i - x_j)$, and since the coefficient of x^{n-1} must be 0 for $n \geq 2$, it follows that (for $n \geq 2$)

$$\sum_{i=1}^{n} \left\{ \prod_{\substack{j=1 \\ j \neq i}}^{n} (x_i - x_j) \right\}^{-1} = \sum_{i=1}^{n} \left\{ \prod_{\substack{j=1 \\ j \neq i}}^{n} (x_j - x_i) \right\}^{-1} = 0, \tag{2.3}$$

where the equality on the right follows by multiplying both sides of the outer equality by $(-1)^{n-1}$. □

2.2 Dickson–Hipp Operators and Equilibrium Distributions

In this section, we introduce the Dickson–Hipp operator (including the Laplace transform as a special case) as well as some useful properties of this operator. The results for this transform and the related equilibrium distributions will be extensively used throughout this monograph.

Let r be a number, and $h(x)$ an integrable function. Then define

$$T_r h(x) = e^{rx} \int_x^{\infty} e^{-ry} h(y) dy, \qquad \mathrm{Re}(r) \geq 0, \quad x \geq 0, \tag{2.4}$$

called the Dickson–Hipp transform of the function $h(x)$ (e.g. Dickson and Hipp (2001), Li and Garrido (2004)). A change in the variable of integration results in the alternative representation

$$T_r h(x) = \int_0^\infty e^{-ry} h(x+y)dy. \tag{2.5}$$

Clearly, T_r is a linear operator in that

$$T_r \left\{ \sum_{i=1}^{n} a_i h_i(x) \right\} = \sum_{i=1}^{n} a_i T_r h_i(x), \tag{2.6}$$

as is obvious from (2.4) or (2.5). Also, the Laplace transform is a special case, i.e.

$$\widetilde{h}(s) = \int_0^\infty e^{-sy} h(y)dy = T_s h(0). \tag{2.7}$$

Furthermore, the integrated tail may be obtained from

$$\int_x^\infty h(y)dy = T_0 h(x).$$

Example 2.2 **Mixture of exponentials**
Suppose that $h(x) = \sum_{i=1}^{k} q_i h_i(x)$ for $x > 0$ where $h_i(x) = \beta_i e^{-\beta_i x}$ for $\beta_i > 0$ and $0 \le q_i < 1$ with $\sum_{i=1}^{k} q_i = 1$. Then, from (2.6) with (2.5), one finds

$$T_r h(x) = \sum_{i=1}^{k} q_i T_r h_i(x) = \sum_{i=1}^{k} q_i \beta_i e^{-\beta_i x} \int_0^\infty e^{-(\beta_i + r)y} dy = \sum_{i=1}^{k} \frac{q_i \beta_i}{\beta_i + r} e^{-\beta_i x}.$$

□

It is of interest to consider repeated application of the operator. Thus, define for $n = 1, 2, \ldots,$

$$T_{r_1, r_2, \ldots, r_n} h(x) = T_{r_1} T_{r_2} \cdots T_{r_n} h(x). \tag{2.8}$$

For $n = 2$, a change in the order of integration yields

$$\begin{aligned} T_{r_1, r_2} h(x) &= e^{r_1 x} \int_x^\infty e^{-r_1 y} \left\{ T_{r_2} h(y) \right\} dy \\ &= e^{r_1 x} \int_x^\infty e^{-(r_1 - r_2)y} \int_y^\infty e^{-r_2 t} h(t) dt dy \\ &= e^{r_1 x} \int_x^\infty e^{-r_2 t} \left\{ \int_x^t e^{-(r_1 - r_2)y} dy \right\} h(t) dt. \end{aligned} \tag{2.9}$$

Therefore, if $r_1 \neq r_2$, it follows that

$$T_{r_1,r_2}h(x) = e^{r_1 x}\int_x^{\infty} e^{-r_2 t}\left\{\frac{e^{-(r_1-r_2)x} - e^{-(r_1-r_2)t}}{r_1 - r_2}\right\}h(t)dt$$
$$= \frac{e^{r_2 x}\int_x^{\infty} e^{-r_2 t}h(t)dt - e^{r_1 x}\int_x^{\infty} e^{-r_1 t}h(t)dt}{r_1 - r_2},$$

that is,

$$T_{r_1,r_2}h(x) = \frac{T_{r_2}h(x) - T_{r_1}h(x)}{r_1 - r_2}, \qquad r_1 \neq r_2. \tag{2.10}$$

Thus, the Laplace transform of the Dickson–Hipp transform is a special case of (2.10), i.e.

$$\int_0^{\infty} e^{-sy}\left\{T_r h(y)\right\}dy = T_{s,r}h(0) = \frac{T_r h(0) - T_s h(0)}{s - r},$$

and using (2.7),

$$\int_0^{\infty} e^{-sy}\left\{T_r h(y)\right\}dy = \frac{\widetilde{h}(r) - \widetilde{h}(s)}{s - r}. \tag{2.11}$$

It is clear from (2.10) that

$$T_{r_1,r_2}h(x) = T_{r_2,r_1}h(x), \tag{2.12}$$

and so the order of application of the operator is unimportant. Clearly, the same is true for (2.8) by repeated application of (2.12) if $r_i \neq r_j$ for $i \neq j$. In fact, we have the following generalization of (2.10).

Theorem 2.1 *If $r_i \neq r_j$ for $i \neq j$, then for $n \geq 1$,*

$$T_{r_1,r_2,\ldots,r_n}h(x) = \sum_{i=1}^{n} a_i T_{r_i}h(x), \tag{2.13}$$

where

$$a_i = \frac{1}{\prod_{\substack{j=1\\ j\neq i}}^{n}\left(r_j - r_i\right)}, \qquad i = 1, 2, \ldots, n.$$

Proof Clearly, (2.13) holds for $n = 2$ as (2.13) reduces to (2.10) in this case. We will prove that the result holds for all n by induction on n, and thus we assume that (2.13) holds for n. Then using (2.12) repeatedly, it follows that

$$T_{r_1,r_2,\ldots,r_n,r_{n+1}}h(x) = T_{r_{n+1},r_1,r_2,\ldots,r_n}h(x) = T_{r_{n+1}}T_{r_1,r_2,\ldots,r_n}h(x),$$

and by the inductive hypothesis together with (2.6) and (2.10),

$$T_{r_1,r_2,\ldots,r_n,r_{n+1}}h(x) = T_{r_{n+1}}\left\{\sum_{i=1}^{n}\frac{T_{r_i}h(x)}{\prod\limits_{\substack{j=1\\j\neq i}}^{n}\left(r_j-r_i\right)}\right\}$$

$$=\sum_{i=1}^{n}\frac{T_{r_{n+1},r_i}h(x)}{\prod\limits_{\substack{j=1\\j\neq i}}^{n}\left(r_j-r_i\right)}=\sum_{i=1}^{n}\frac{T_{r_i}h(x)-T_{r_{n+1}}h(x)}{(r_{n+1}-r_i)\prod\limits_{\substack{j=1\\j\neq i}}^{n}\left(r_j-r_i\right)}.$$

But using (2.3), it follows that

$$T_{r_1,r_2,\ldots,r_{n+1}}h(x) = \left\{\sum_{i=1}^{n}\frac{T_{r_i}h(x)}{\prod\limits_{\substack{j=1\\j\neq i}}^{n+1}\left(r_j-r_i\right)}\right\} - \{T_{r_{n+1}}h(x)\}\left(\sum_{i=1}^{n}\left\{\prod_{\substack{j=1\\j\neq i}}^{n+1}\left(r_j-r_i\right)\right\}^{-1}\right)$$

$$=\left\{\sum_{i=1}^{n}\frac{T_{r_i}h(x)}{\prod\limits_{\substack{j=1\\j\neq i}}^{n+1}\left(r_j-r_i\right)}\right\} - \left\{T_{r_{n+1}}h(x)\right\}\left(-\left\{\prod_{\substack{j=1\\j\neq n+1}}^{n+1}\left(r_j-r_{n+1}\right)\right\}^{-1}\right),$$

and (2.13) also holds for $n+1$. □

Although less important in what follows than the case with distinct r_i, the case with identical r_i is straightforward. It is clear from (2.9) that

$$T_{r,r}h(x) = e^{rx}\int_x^{\infty}(t-x)e^{-rt}h(t)dt,$$

and by induction on n that

$$T_{\underbrace{r,r,\ldots,r}_{n\ \text{terms}}}h(x) = \frac{e^{rx}}{(n-1)!}\int_x^{\infty}(t-x)^{n-1}e^{-rt}h(t)dt.$$

In connection with probability distributions, it is often convenient to allow for distributions which have discrete or both discrete and continuous components, rather than strictly continuous densities. In particular, if $F(y) = 1-\overline{F}(y) = \Pr(Y \leq y)$, for $y \geq 0$, is a distribution function (df), it is often useful to replace the right-hand side of (2.4) by $e^{rx}\int_x^{\infty}e^{-ry}dF(y)$, which essentially involves the replacement of $h(y)dy$ by the more general $dF(y)$. In a similar manner to the derivation of (2.11), it follows that

$$\int_0^\infty e^{-sy}\left\{e^{ry}\int_y^\infty e^{-rt}dF(t)\right\}dy = \frac{\widetilde{f}(r)-\widetilde{f}(s)}{s-r}, \tag{2.14}$$

where

$$\widetilde{f}(s) = E\left(e^{-sY}\right) = \int_0^\infty e^{-sy}dF(y). \tag{2.15}$$

Replacement of r by 0 in (2.14) yields

$$\int_0^\infty e^{-sy}\overline{F}(y)dy = \frac{1-E(e^{-sY})}{s}, \tag{2.16}$$

and letting $s \to 0$ yields, by L'Hopital's rule,

$$E(Y) = \int_0^\infty \overline{F}(y)dy. \tag{2.17}$$

Each of (2.14), (2.16) and (2.17) hold for any nonnegative random variable Y, even if Y has discrete mass points.

Let $F_{1,r}(y) = 1 - \overline{F}_{1,r}(y)$, for $y \geq 0$, be defined by

$$\overline{F}_{1,r}(y) = \frac{e^{ry}\int_y^\infty e^{-rx}\overline{F}(x)dx}{\int_0^\infty e^{-rx}\overline{F}(x)dx} = \frac{\int_0^\infty e^{-rx}\overline{F}(x+y)dx}{\int_0^\infty e^{-rx}\overline{F}(x)dx}, \qquad y \geq 0, \tag{2.18}$$

and $F_{1,r}(y)$ in (2.18) is a df, as it is a mixture of those of the form $1-\overline{F}(x+y)/\overline{F}(x)$. In Dickson–Hipp notation, $\overline{F}_{1,r}(y) = \{T_r\overline{F}(y)\}/\{T_r\overline{F}(0)\}$. Thus, $F_{1,r}(y)$ is differentiable (even if $F(y)$ has discrete mass points), with derivative $f_{1,r}(y) = -\overline{F}'_{1,r}(y)$ from (2.18), namely

$$f_{1,r}(y) = \frac{\overline{F}(y) - re^{ry}\int_y^\infty e^{-rx}\overline{F}(x)dx}{\int_0^\infty e^{-rx}\overline{F}(x)dx}.$$

But integration by parts yields

$$e^{ry}\int_y^\infty e^{-rx}dF(x) = \overline{F}(y) - re^{ry}\int_y^\infty e^{-rx}\overline{F}(x)dx,$$

and therefore

$$f_{1,r}(y) = \frac{e^{ry}\int_y^\infty e^{-rx}dF(x)}{\int_0^\infty e^{-rx}\overline{F}(x)dx}, \qquad y > 0. \tag{2.19}$$

The Laplace transform of (2.19) is, using (2.14), (2.15) and (2.16), given by

$$\widetilde{f}_{1,r}(s) = \int_0^\infty e^{-sy} f_{1,r}(y)dy = \left(\frac{r}{s-r}\right) \frac{\widetilde{f}(r) - \widetilde{f}(s)}{1 - \widetilde{f}(r)}. \tag{2.20}$$

We remark that when $r = 0$, (2.19) reduces to

$$f_{1,0}(y) = \frac{\overline{F}(y)}{E(Y)} \tag{2.21}$$

using (2.17). The probability density function (pdf) (2.21) is often referred to as an equilibrium pdf, and consequently (2.19) as a generalized equilibrium pdf (e.g. Willmot and Lin 2001, Sect. 9.2).

Example 2.3 **Mixture of exponentials**
Suppose that $\overline{F}(x) = \sum_{i=1}^k q_i \overline{F}_i(x)$ for $x \geq 0$ where $\overline{F}_i(x) = e^{-\beta_i x}$ for $\beta_i > 0$ and $0 \leq q_i < 1$ with $\sum_{i=1}^k q_i = 1$. Then, using the result in Example 2.2, the generalized equilibrium pdf (2.19) is given by

$$f_{1,r}(y) = \frac{T_r f(y)}{\int_0^\infty e^{-rx}\overline{F}(x)dx} = \frac{\sum_{i=1}^k \frac{q_i \beta_i}{\beta_i + r} e^{-\beta_i y}}{\sum_{j=1}^k \frac{q_j}{\beta_j + r}} = \sum_{i=1}^k q_i(r) f_i(y),$$

where

$$q_i(r) = \frac{\frac{q_i}{\beta_i + r}}{\sum_{j=1}^k \frac{q_j}{\beta_j + r}}, \qquad i = 1, 2, \ldots, k,$$

and $f_i(y) = \beta_i e^{-\beta_i y}$ for $y > 0$. □

Moments of $F_{1,r}(y)$ are easily obtainable. One has from (2.19) that

$$\begin{aligned}\int_0^\infty y^n f_{1,r}(y)dy &= \frac{\int_0^\infty y^n e^{ry} \int_y^\infty e^{-rx} dF(x)dy}{\int_0^\infty e^{-rx}\overline{F}(x)dx} \\ &= \frac{\int_0^\infty e^{-rx} \left\{\int_0^x y^n e^{ry} dy\right\} dF(x)}{\int_0^\infty e^{-rx}\overline{F}(x)dx}.\end{aligned} \tag{2.22}$$

For $r \neq 0$, one has the identity (easily proved by induction on n)

$$\int_0^x y^n e^{ry} dy = \frac{n!}{(-r)^{n+1}} \left\{1 - e^{rx} \sum_{j=0}^n \frac{(-rx)^j}{j!}\right\}. \tag{2.23}$$

Substitution of (2.23) into the numerator of (2.22) yields

$$\int_0^\infty e^{-rx}\left\{\int_0^x y^n e^{ry}dy\right\}dF(x) = \frac{n!}{(-r)^{n+1}}\int_0^\infty e^{-rx}\left\{1-e^{rx}\sum_{j=0}^n \frac{(-rx)^j}{j!}\right\}dF(x)$$
$$= \frac{n!}{(-r)^{n+1}}\left\{\int_0^\infty e^{-rx}dF(x) - \sum_{j=0}^n \frac{(-r)^j}{j!}\int_0^\infty x^j dF(x)\right\}$$
$$= \frac{n!}{(-r)^{n+1}}\left\{\widetilde{f}(r) - \sum_{j=0}^n \frac{(-r)^j}{j!}E\left(Y^j\right)\right\}.$$

Thus, for $r \neq 0$, it follows that for $n = 1, 2, \ldots,$

$$\int_0^\infty y^n f_{1,r}(y)dy = \frac{n!}{(-r)^n}\left\{1+\sum_{j=1}^n \frac{(-r)^j}{j!}\frac{E\left(Y^j\right)}{1-\widetilde{f}(r)}\right\}. \tag{2.24}$$

For $r = 0$, it follows easily from (2.22) that

$$\int_0^\infty y^n f_{1,0}(y)dy = \frac{E\left(Y^{n+1}\right)}{(n+1)E(Y)}. \tag{2.25}$$

For a detailed discussion of higher order equilibrium distributions in connection with higher stop-loss moments, see Willmot (2002b) or Willmot et al. (2005) for example.

2.3 Defective Renewal Equations

Suppose that $m(x)$ satisfies the integral equation

$$m(x) = \phi\int_0^x m(x-y)dF(y) + v(x), \qquad x \geq 0, \tag{2.26}$$

where $0 < \phi < 1$, $F(y) = 1 - \overline{F}(y)$ is a df with $F(0) = 0$, and $v(x) \geq 0$ is locally bounded (i.e. $v(x) < \infty$ for $x < \infty$). Then (2.26) is called a defective renewal equation.

In order to discuss the solution to (2.26), we let the Laplace–Stieltjes transform of F be $\widetilde{f}(s) = \int_0^\infty e^{-sy}dF(y)$. Then define $\overline{F}^{*n}(y) = 1 - F^{*n}(y)$ to be the tail of the distribution of the n-fold convolution of F with itself, i.e. the associated Laplace transform is $\int_0^\infty e^{-sy}\overline{F}^{*n}(y)dy = \{1-[\widetilde{f}(s)]^n\}/s$. It is convenient to introduce the compound geometric df $G(y) = 1 - \overline{G}(y) = \Pr(L \leq y)$ associated with (2.26) by

$$\overline{G}(y) = \sum_{n=1}^\infty (1-\phi)\phi^n\overline{F}^{*n}(y), \qquad y \geq 0. \tag{2.27}$$

Clearly,

$$\overline{G}(0) = \phi,$$

so that $G(y)$ has a discrete mass point of $1-\phi$ at 0. One has

$$E\left(e^{-sL}\right) = \int_0^\infty e^{-sx} dG(x) = \sum_{n=0}^\infty (1-\phi)\phi^n \left\{\widetilde{f}(s)\right\}^n = \frac{1-\phi}{1-\phi\widetilde{f}(s)}, \tag{2.28}$$

where $\widetilde{f}(s)$ is given by (2.15). Taking Laplace transforms of (2.26) yields $\widetilde{m}(s) = \phi\widetilde{m}(s)\widetilde{f}(s) + \widetilde{v}(s)$, and solving for $\widetilde{m}(s)$ yields with (2.28),

$$\widetilde{m}(s) = \frac{\widetilde{v}(s)E\left(e^{-sL}\right)}{1-\phi}. \tag{2.29}$$

The solution (2.29) may be expressed in a more convenient form with additional assumptions about $G(y)$ or $v(x)$. First, if $F(y)$ has density $f(y) = F'(y)$, then $G(y) = 1-\phi+\int_0^y g(x)dx$, where

$$g(y) = \sum_{n=1}^\infty (1-\phi)\phi^n f^{*n}(y), \qquad y > 0, \tag{2.30}$$

is a compound geometric density (and $f^{*n}(y) = dF^{*n}(y)/dy$). Thus, using (2.28), the Laplace transform of (2.30) is

$$\widetilde{g}(s) = \int_0^\infty e^{-sy} g(y)dy = \frac{1-\phi}{1-\phi\widetilde{f}(s)} - (1-\phi). \tag{2.31}$$

Therefore, (2.29) may be expressed as

$$\widetilde{m}(s) = \frac{1}{1-\phi}\,\widetilde{g}(s)\widetilde{v}(s) + \widetilde{v}(s),$$

which yields upon inversion (e.g. Resnick (1992), Sect. 3.5)

$$m(x) = \frac{1}{1-\phi}\int_0^x v(y)g(x-y)dy + v(x). \tag{2.32}$$

Next, we consider assumptions about $v(x)$ rather than $G(x)$. First note that (2.16) yields

$$\widetilde{\overline{G}}(s) = \int_0^\infty e^{-sy}\overline{G}(y)dy = \frac{1-E\left(e^{-sL}\right)}{s}. \tag{2.33}$$

Then assuming that one may write

$$s\widetilde{v}(s) = C - \widetilde{v}_*(s), \tag{2.34}$$

it follows that (2.29) may be expressed as

$$\widetilde{m}(s) = \frac{\widetilde{v}(s)\left\{1 - \left[1 - E\left(e^{-sL}\right)\right]\right\}}{1-\phi} = \frac{\widetilde{v}(s) - s\widetilde{v}(s)\widetilde{\overline{G}}(s)}{1-\phi},$$

i.e.

$$\widetilde{m}(s) = \frac{\widetilde{v}(s) - C\widetilde{\overline{G}}(s) + \widetilde{v}_*(s)\widetilde{\overline{G}}(s)}{1-\phi}. \tag{2.35}$$

For example, if $v(x)$ is differentiable, (2.34) holds with $\widetilde{v}_*(s) = \int_0^\infty e^{-sx}\{-v'(x)\}dx$ and $C = v(0)$, and (2.35) yields

$$m(x) = \frac{v(x) - v(0)\overline{G}(x) - \int_0^x v'(x-y)\overline{G}(y)dy}{1-\phi}. \tag{2.36}$$

Similarly, if $\widetilde{v}(s) = \widetilde{v}_r(s)$ where

$$\widetilde{v}_r(s) = k\frac{\widetilde{h}(r) - \widetilde{h}(s)}{s-r}, \tag{2.37}$$

then

$$s\widetilde{v}_r(s) = k\frac{s}{s-r}\left\{\widetilde{h}(r) - \widetilde{h}(s)\right\} = k\left(1 + \frac{r}{s-r}\right)\left\{\widetilde{h}(r) - \widetilde{h}(s)\right\},$$

i.e. $s\widetilde{v}_r(s) = k\widetilde{h}(r) - \left\{k\widetilde{h}(s) - r\widetilde{v}_r(s)\right\}$ and (2.34) holds with $C = k\widetilde{h}(r)$ and $\widetilde{v}_*(s) = k\widetilde{h}(s) - r\widetilde{v}_r(s)$. Thus, if $\widetilde{v}(s)$ is given by (2.37) and (2.35) becomes

$$\widetilde{m}(s) = \frac{\widetilde{v}_r(s) - k\widetilde{h}(r)\widetilde{\overline{G}}(s) + \left\{k\widetilde{h}(s) - r\widetilde{v}_r(s)\right\}\widetilde{\overline{G}}(s)}{1-\phi}. \tag{2.38}$$

Hence if $\widetilde{v}(s) = \widetilde{v}_r(s) = k\int_0^\infty e^{-sx}\{T_r h(x)\}dx$, then (2.38) yields

$$m(x) = \left(\frac{k}{1-\phi}\right)\left[\{T_r h(x)\} - \widetilde{h}(r)\overline{G}(x) + \int_0^x \{h(y) - rT_r h(y)\}\,\overline{G}(x-y)dy\right].$$

A similar result holds if $\widetilde{v}(s) = \widetilde{v}_r(s) = \int_0^\infty e^{-sx}v_r(x)dx$, where $v_r(x) = ke^{rx}\int_x^\infty e^{-rt}dH(t)$, with $H(t) = 1 - \overline{H}(t)$ is a (possibly discrete) df, as is clear from (2.14) and (2.37) with $\widetilde{h}(s) = \int_0^\infty e^{-sx}dH(x)$. In particular, with $r = 0$, if $v(x) = k\overline{H}(x)$, it follows from (2.38) that

$$m(x) = \frac{k}{1-\phi}\left\{\int_0^x \overline{G}(x-y)dH(y) + \overline{H}(x) - \overline{G}(x)\right\}.$$

There are special cases of (2.26) that deserves mention.

Example 2.4 **Compound geometric tail**
From (2.28) and (2.33), we find that

$$\tilde{\overline{G}}(s) = \frac{1}{s}\left\{1 - \frac{1-\phi}{1-\phi\tilde{f}(s)}\right\} = \left\{\frac{\phi}{1-\phi\tilde{f}(s)}\right\}\frac{1-\tilde{f}(s)}{s}, \tag{2.39}$$

which may be rearranged as

$$\tilde{\overline{G}}(s) = \phi\tilde{f}(s)\tilde{\overline{G}}(s) + \phi\frac{1-\tilde{f}(s)}{s}.$$

Inversion of this Laplace transform relationship yields

$$\overline{G}(x) = \phi\int_0^x \overline{G}(x-y)dF(y) + \phi\overline{F}(x). \tag{2.40}$$

Comparison of (2.40) with (2.26) yields the conclusion that the solution to (2.26) when $v(x) = \phi\overline{F}(x)$ is $m(x) = \overline{G}(x)$ given by (2.27). □

Example 2.5 **Compound geometric density**
It follows from (2.31) that

$$\tilde{g}(s) = (1-\phi)\left\{\frac{1}{1-\phi\tilde{f}(s)} - 1\right\} = \frac{\phi(1-\phi)\tilde{f}(s)}{1-\phi\tilde{f}(s)}.$$

Thus, $\tilde{g}(s) = \phi\tilde{f}(s)\tilde{g}(s) + \phi(1-\phi)\tilde{f}(s)$, yielding

$$g(x) = \phi\int_0^x g(x-y)f(y)dy + \phi(1-\phi)f(x), \tag{2.41}$$

and the compound geometric density (2.30) also satisfies a defective renewal equation. □

While the solution to (2.26) is complicated in general, there is some asymptotic help available. Suppose that $F(y)$ is nonarithmetic (i.e. has a continuous component) and there exists an $R > 0$ satisfying

$$\int_0^\infty e^{Ry}dF(y) = \frac{1}{\phi}. \tag{2.42}$$

If $e^{Rx}v(x)$ is "directly Riemann integrable"(to be discussed momentarily), then

$$m(x) \sim Ce^{-Rx}, \qquad x \to \infty, \tag{2.43}$$

where

$$C = \frac{\int_0^\infty e^{Ry} v(y)dy}{\phi \int_0^\infty ye^{Ry} dF(y)}, \tag{2.44}$$

and $a(x) \sim b(x)$ as $x \to \infty$ means that $\lim_{x\to\infty} a(x)/b(x) = 1$. A sufficient condition for $e^{Rx}v(x)$ to be directly Riemann integrable is that $e^{Rx}v(x) \le h(x)$ where $h(x)$ is nonnegative, nonincreasing, and Riemann integrable (i.e. $\int_0^\infty h(x)dx < \infty$). This condition is in turn satisfied if $\int_0^\infty e^{(R+\epsilon)x} v(x)dx < \infty$ for some $\epsilon > 0$. To see this, note that $e^{(R+\epsilon)x}v(x)$ is locally bounded on $(0, \infty)$ because the same is true of $v(x)$. Also, because $\lim_{x\to\infty} e^{(R+\epsilon)x}v(x) = 0$, there exists a $K < \infty$ such that $e^{(R+\epsilon)x}v(x) \le K$, i.e. $e^{Rx}v(x) \le h(x)$ with $h(x) = Ke^{-\epsilon x}$. The asymptotic result (2.43) is sometimes called a Cramer–Lundberg result. Finally, if (2.42) holds then

$$C_L e^{-Rx} \le m(x) \le C_U e^{-Rx}, \qquad x \ge 0, \tag{2.45}$$

where $C_L = \inf_{z\ge 0} \alpha(z)$, $C_U = \sup_{z\ge 0} \alpha(z)$, and

$$\alpha(z) = \frac{e^{Rz}v(z)}{\phi \int_z^\infty e^{Ry} dF(y)}.$$

In particular, (2.27) satisfies

$$\overline{G}(x) \le e^{-Rx}, \qquad x \ge 0. \tag{2.46}$$

The Lundberg bounds (2.45) are derived in Willmot et al. (2001). A more detailed discussion is provided in Sect. 8.3.

We introduce the function $\overline{G}(x, y)$, for $x \ge 0$ and $y \ge 0$, satisfying the defective renewal equation

$$\overline{G}(x, y) = \phi \int_0^x \overline{G}(x - t, y)dF(t) + \phi \overline{F}(x + y), \tag{2.47}$$

so that $\overline{G}(x, 0) = \overline{G}(x)$, from (2.40). It will be shown that $\overline{G}(x, y)$ is useful when analyzing the deficit at ruin in the renewal risk model (in Sect. 4.5). Alternatively, (2.47) has an expression as follows. Taking the Laplace transform of (2.47) with the aid of (2.28) yields

$$\int_0^\infty e^{-sx}\overline{G}(x, y)dx = \frac{\phi \int_0^\infty e^{-sx}\overline{F}(x + y)dx}{1 - \phi \tilde{f}(s)} = \frac{\phi}{1-\phi} \left\{\int_0^\infty e^{-sx} dG(x)\right\} \left\{\int_0^\infty e^{-sx}\overline{F}(x + y)dx\right\}. \tag{2.48}$$

Therefore, inversion of the Laplace transform results in

$$\overline{G}(x, y) = \frac{\phi}{1-\phi} \int_0^x \overline{F}(x+y-t)dG(t). \tag{2.49}$$

Next, we consider the "excess loss" or "residual lifetime distribution" with df $F_x(y) = 1 - \overline{F}_x(y)$, where

$$\overline{F}_x(y) = \frac{\overline{F}(x+y)}{\overline{F}(x)}, \qquad y \geq 0, \tag{2.50}$$

and

$$f_x(y) = F'_x(y) = \frac{f(x+y)}{\overline{F}(x)}. \tag{2.51}$$

It is convenient to define

$$\overline{A}_x(y) = 1 - A_x(y) = \frac{\overline{G}(x, y)}{\overline{G}(x)} \tag{2.52}$$

satisfying

$$\overline{A}_x(y) = \frac{\int_0^x \overline{F}_{x-t}(y)\overline{F}(x-t)dG(t)}{\int_0^x \overline{F}(x-t)dG(t)}, \qquad y \geq 0, \tag{2.53}$$

due to (2.49) with $y = 0$ and (2.50). Clearly, (2.53) is a proper tail distribution as it is a mixture of (2.50), mixed over x. In fact (2.52) is the tail distribution of the deficit at ruin given that ruin occurs, which will be discussed in Sect. 4.5. Then the residual lifetime tail of the compound geometric distribution

$$\overline{G}_x(y) = \frac{\overline{G}(x+y)}{\overline{G}(x)}, \qquad y \geq 0, \tag{2.54}$$

also satisfies the defective renewal equation

$$\overline{G}_x(y) = \phi \int_0^y \overline{G}_x(y-t)dF(t) + \phi\overline{F}(y) + (1-\phi)\overline{A}_x(y). \tag{2.55}$$

To see this, introduce Θ_y with df $F_y(x)$ independent of L. Then

$$\Pr(L + \Theta_y > x) = \overline{G}(x) + \int_0^x \overline{F}_y(x-t)dG(t) = \overline{F}_y(x) + \int_0^x \overline{G}(x-t)dF_y(t),$$

and thus, with the help of (2.50) and (2.49) is given by

$$\begin{aligned}\overline{G}(x,y) &= \frac{\phi}{1-\phi}\overline{F}(y)\left\{\overline{F}_y(x)+\int_0^x \overline{G}(x-t)dF_y(t)-\overline{G}(x)\right\}\\ &= \frac{\phi}{1-\phi}\left\{\overline{F}(x+y)+\int_y^{x+y}\overline{G}(x+y-t)dF(t)-\overline{G}(x)\overline{F}(y)\right\}.\end{aligned} \tag{2.56}$$

Then, from (2.40), it may be expressed as

$$\overline{G}(x,y) = \frac{\phi}{1-\phi}\left\{\frac{\overline{G}(x+y)}{\phi}-\int_0^y \overline{G}(x+y-t)dF(t)-\overline{G}(x)\overline{F}(y)\right\}.$$

Dividing the above equation by $\overline{G}(x)$ followed by rearranging terms yields (2.55).

Interestingly, it can be demonstrated that (2.54) is the tail of $L+V_x$ where V_x is independent of L with df $A_x(y)$ in (2.52), namely

$$\overline{G}_x(y) = \frac{\overline{G}(x+y)}{\overline{G}(x)} = \Pr(L+V_x > y), \qquad y \geq 0, \tag{2.57}$$

or equivalently

$$\overline{G}_x(y) = \overline{G}(y)+\int_0^y \overline{A}_x(y-t)dG(t) = \overline{A}_x(y)+\int_0^y \overline{G}(y-t)dA_x(t). \tag{2.58}$$

It follows that the residual lifetime distribution of the compound geometric distribution is actually the convolution of the compound geometric distribution itself and the distribution $A_x(y)$. To prove (2.57), taking Laplace transforms of (2.55) and using (2.16) results in

$$\tilde{\overline{G}}_x(s) = \phi\tilde{\overline{G}}_x(s)\tilde{f}(s)+\phi\frac{1-\tilde{f}(s)}{s}+(1-\phi)\frac{1-E(e^{-sV_x})}{s},$$

where $\tilde{\overline{G}}_x(s) = \int_0^\infty e^{-sy}\overline{G}_x(y)dy$. Then rearranging the above equation and using (2.28) yields

$$\begin{aligned}\tilde{\overline{G}}_x(s) &= \frac{\phi\{1-\tilde{f}(s)\}+(1-\phi)\{1-E(e^{-sV_x})\}}{s\{1-\phi\tilde{f}(s)\}} = \frac{1-\phi\tilde{f}(s)-(1-\phi)E(e^{-sV_x})}{s\{1-\phi\tilde{f}(s)\}}\\ &= \frac{1-E(e^{-sL})E(e^{-sV_x})}{s},\end{aligned}$$

and the inversion gives (2.57). We remark that the expression for $\overline{G}_x(y)$ in (2.58) appears to be very useful in the study of the reliability properties of the compound geometric distribution. The df $F(x)$ is said to be new worse (better) than used or NWU (NBU) if $\overline{F}(x+y) \geq (\leq) \overline{F}(x)\overline{F}(y)$ for all $x \geq 0$ and $y \geq 0$ (see e.g. Barlow and Proschan (1975)). It is well known that the compound geometric distribution is NWU (e.g. Brown (1990)). From (2.58), this result is found immediately since the integral term on the right-hand side of (2.58) is non-negative. See Willmot (2002a) for analytic results on the compound geometric residual lifetime distributions in connection with the distribution of the deficit ruin as well as some reliability-based properties of the compound geometric distribution.

Example 2.6 **Compound geometric convolution**
Motivated by the previous discussion, we now consider more generally the df of the compound geometric convolution $\mathscr{K}(x) = 1 - \overline{\mathscr{K}}(x) = G * C(x)$, where $C(x) = 1 - \overline{C}(x)$ is the df of a positive random variable independent of L. So, the tail of the compound geometric convolution is given by

$$\overline{\mathscr{K}}(x) = \overline{G}(x) + \int_0^x \overline{C}(x-t)dG(t) = \overline{C}(x) + \int_0^x \overline{G}(x-t)dC(t). \tag{2.59}$$

From (2.40), it is obvious that $\overline{\mathscr{K}}(x) = \overline{G}(x)/\phi = \overline{G}(x)/\overline{G}(0)$ if $C(x) = F(x)$. Then it is known that the tail df of the compound geometric convolution satisfies the defective renewal equation (e.g. Willmot and Lin (2001), p. 174),

$$\overline{\mathscr{K}}(x) = \phi \int_0^x \overline{\mathscr{K}}(x-t)dF(t) + \phi\overline{F}(x) + (1-\phi)\overline{C}(x). \tag{2.60}$$

Similar to (2.47), we introduce the function

$$\overline{\mathscr{G}}(x, y) = \int_0^x \overline{G}(x-t, y)dC(t) + \overline{C}(x+y). \tag{2.61}$$

From (2.59), we know $\overline{\mathscr{G}}(x, 0) = \overline{\mathscr{K}}(x)$ due to $\overline{G}(x, 0) = \overline{G}(x)$ with $\overline{G}(x, y)$ given by (2.47). Then using (2.48) one finds the Laplace transform of the integral on the right-hand side of (2.61) as

$$\begin{aligned} &\int_0^\infty e^{-sx}\left\{\int_0^x \overline{G}(x-t, y)dC(t)\right\}dx \\ &= \left\{\int_0^\infty e^{-sx}\overline{G}(x, y)dx\right\}\left\{\int_0^\infty e^{-sx}dC(x)\right\} \\ &= \frac{\phi}{1-\phi}\left\{\int_0^\infty e^{-sx}dG(x)\right\}\left\{\int_0^\infty e^{-sx}\overline{F}(x+y)dx\right\}\left\{\int_0^\infty e^{-sx}dC(x)\right\}. \end{aligned}$$

Since $\int_0^\infty e^{-sx}d\mathscr{K}(x) = E(e^{-sL})\int_0^\infty e^{-sx}dC(x)$, we get

$$\int_0^\infty e^{-sx}\left\{\int_0^x \overline{G}(x-t, y)dC(t)\right\}dx = \frac{\phi}{1-\phi}\int_0^\infty e^{-sx}\left\{\int_0^x \overline{F}(x+y-t)d\mathscr{K}(t)\right\}dx.$$

Therefore, by the uniqueness of the Laplace transform, (2.61) satisfies

$$\overline{\mathscr{G}}(x, y) = \overline{C}(x + y) + \frac{\phi}{1 - \phi} \int_0^x \overline{F}(x + y - t) d\mathscr{K}(t). \tag{2.62}$$

Next, we define

$$\overline{\mathscr{A}}_x(y) = 1 - \mathscr{A}_x(y) = \frac{\overline{\mathscr{G}}(x, y)}{\overline{\mathscr{K}}(x)}, \tag{2.63}$$

and the residual lifetime tail of the df $\mathscr{K}(y)$ as

$$\overline{\mathscr{K}}_x(y) = \frac{\overline{\mathscr{K}}(x + y)}{\overline{\mathscr{K}}(x)}, \qquad y \geq 0.$$

But, the second term on the right-hand side of (2.62) has the same form as (2.49) with G replaced by $\mathscr{K}$, and using (2.56) with $\overline{G} = \overline{\mathscr{K}}$ results in

$$\begin{aligned}
&\overline{\mathscr{G}}(x, y) \\
&= \overline{C}(x + y) + \frac{\phi}{1 - \phi} \left\{ \overline{F}(x + y) + \int_y^{x+y} \overline{\mathscr{K}}(x + y - t) dF(t) - \overline{\mathscr{K}}(x)\overline{F}(y) \right\} \\
&= \overline{C}(x + y) + \frac{\phi}{1 - \phi} \left\{ \overline{F}(x + y) + \int_0^{x+y} \overline{\mathscr{K}}(x + y - t) dF(t) - \int_0^y \overline{\mathscr{K}}(x + y - t) dF(t) - \overline{\mathscr{K}}(x)\overline{F}(y) \right\} \\
&= \overline{C}(x + y) + \frac{\phi}{1 - \phi} \left\{ \frac{\overline{\mathscr{K}}(x + y) - (1 - \phi)\overline{C}(x + y)}{\phi} - \int_0^y \overline{\mathscr{K}}(x + y - t) dF(t) - \overline{\mathscr{K}}(x)\overline{F}(y) \right\},
\end{aligned}$$

where the last equality is due to (2.60). Dividing by $\overline{\mathscr{K}}(x)$ and rearranging terms yields the defective renewal equation for $\overline{\mathscr{K}}_x(y)$ given by

$$\overline{\mathscr{K}}_x(y) = \phi \int_0^y \overline{\mathscr{K}}_x(y - t) dF(t) + \phi \overline{F}(y) + (1 - \phi)\overline{\mathscr{A}}_x(y). \tag{2.64}$$

Then similar to (2.57) and (2.58), the stochastic composition result for the residual lifetime of the compound geometric convolution $\mathscr{K}(y)$ is also available as follows. Let $\mathscr{V}_x$ be independent of L with df $\mathscr{A}_x(y)$. Then taking Laplace transforms of (2.64) and using (2.28) it follows that

$$\int_0^\infty e^{-sx} \overline{\mathscr{K}}_x(y) dy = \frac{\phi\{1 - \widetilde{f}(s)\} + (1 - \phi)\{1 - E(e^{-s\mathscr{V}_x})\}}{s\{1 - \phi \widetilde{f}(s)\}} = \frac{1 - E(e^{-sL})E(e^{-s\mathscr{V}_x})}{s},$$

and inverting the above equation identifies the residual lifetime tail $\overline{\mathscr{K}}_x(y)$ as

$$\overline{\mathscr{K}}_x(y) = \frac{\overline{\mathscr{K}}(x + y)}{\overline{\mathscr{K}}(x)} = \Pr(L + \mathscr{V}_x > y), \tag{2.65}$$

or equivalently

$$\overline{\mathscr{K}}_x(y) = \overline{G}(y) + \int_0^y \overline{\mathscr{A}}_x(y-t)dG(t) = \overline{\mathscr{A}}_x(y) + \int_0^y \overline{G}(y-t)d\mathscr{A}_x(t).$$

□

Example 2.7 **Classical Poisson risk model with diffusion**
The classical Poisson risk model with diffusion is defined by $U_t = u + ct - S_t + W_t$ for $t \geq 0$, where W_t is a Wiener process with drift 0 and variance $2D$, and U_t without W_t is given in (3.1). The details of the model are described in Sect. 3.1. From Dufresne and Gerber (1991), it is shown that the survival probability $\overline{\psi}(u)$, namely $\overline{\psi}(u) = \Pr\{U_t \geq 0 \text{ for all } t \geq 0 | U_0 = u\}$, is the df of a compound geometric convolution. More precisely, let us assume the Poisson rate λ and the claim amount distribution $P(y) = 1 - \overline{P}(y)$ with mean $E(Y) = \int_0^\infty y dP(y)$, and let $P_1(y) = \int_0^y \overline{P}(x)dx/E(Y)$ be the equilibrium df of $P(y)$,

$$C(x) = 1 - e^{-\frac{c}{D}x}, \qquad x \geq 0,$$

and $F(x) = C * P_1(x)$ be the convolution df. Then, if $\phi = \lambda E(Y)/c$ in (2.30), the survival probability $\overline{\psi}(u)$ is a df of the compound geometric convoluted with $C(x)$ (i.e. $\overline{\psi}(u) = G * C(x)$). Therefore, the results for the compound geometric convolution obtained previously are applicable to $\overline{\psi}(u)$. □

Lastly, we remark that if $C(x) = F(x)$, then $\overline{\mathscr{K}}(x) = \overline{G}(x)/\overline{G}(0)$, and thus, all results for the compound geometric convolution are reduced to those for the compound geometric tail $\overline{G}(x)$. See Willmot and Cai (2004) for further details related to the residual lifetime of compound geometric convolution and its risk and queueing-theoretic applications.

2.4 Mixed Erlang Distributions

The mixed Erlang class of distributions is dense in the class of positive continuous probability distributions (e.g. Tijms (1994), pp. 163–164), and is extremely well suited for analytic evaluation of risk-theoretic quantities. It is also a very large class of distributions, and includes many distributions whose membership in the class is not immediately obvious, such as phase-type distributions (e.g. Shanthikumar (1985)).

Concerning parameter estimation of the mixed Erlang distribution, Lee and Lin (2010) studied numerical experiments to fit Erlang mixtures to data using maximum likelihood estimation using the EM algorithm. As discussed in Lee and Lin (2010), the EM algorithm for a finite mixture of Erlangs provides an effective iterative scheme and has fast convergence. However, there is an issue of overfitting with many Erlang terms in the mixed model. A detailed discussion regarding estimation can be founded in Lee and Lin (2010). See also Verbelen et al. (2015) for fitting a finite mixture of Erlangs to censored and truncated data using the EM algorithm.

For $\beta > 0$ and $j = 1, 2, 3, \ldots$, define the Erlang-j pdf to be

$$\mathscr{E}_{\beta,j}(y) = \frac{\beta\,(\beta y)^{j-1}\,e^{-\beta y}}{(j-1)!}, \qquad y > 0, \tag{2.66}$$

with Laplace transform

$$\widetilde{\mathscr{E}}_{\beta,j}(s) = \int_0^\infty e^{-sy}\mathscr{E}_{\beta,j}(y)dy = \left(\frac{\beta}{\beta+s}\right)^j. \tag{2.67}$$

For $j = 1$, $\mathscr{E}_{\beta,1}(y) = \beta e^{-\beta y}$ is the exponential pdf, and $\widetilde{\mathscr{E}}_{\beta,1}(s) = \beta/(\beta+s)$. For risk-theoretic calculations, it is of interest to consider $\mathscr{E}_{\beta,j}(x+y)$ where $x \geq 0$ and $y \geq 0$. Clearly,

$$\begin{aligned}
\mathscr{E}_{\beta,j}(x+y) &= \frac{\beta^j e^{-\beta(x+y)}}{(j-1)!}\sum_{k=0}^{j-1}\binom{j-1}{k}x^k y^{j-1-k} \\
&= \frac{1}{\beta}\sum_{k=0}^{j-1}\left\{\frac{\beta^{k+1}x^k e^{-\beta x}}{k!}\right\}\left\{\frac{\beta^{j-k}y^{j-k-1}e^{-\beta y}}{(j-k-1)!}\right\} \\
&= \frac{1}{\beta}\sum_{k=0}^{j-1}\mathscr{E}_{\beta,k+1}(x)\mathscr{E}_{\beta,j-k}(y).
\end{aligned}$$

That is,

$$\mathscr{E}_{\beta,j}(x+y) = \frac{1}{\beta}\sum_{k=1}^{j}\mathscr{E}_{\beta,k}(x)\mathscr{E}_{\beta,j+1-k}(y). \tag{2.68}$$

Next, let $\{q_1, q_2, \ldots\}$ be a discrete counting distribution with probability generating function (pgf)

$$Q(z) = \sum_{j=1}^{\infty} q_j z^j. \tag{2.69}$$

Then for $y > 0$

$$f(y) = \sum_{j=1}^{\infty} q_j\mathscr{E}_{\beta,j}(y) = \sum_{j=1}^{\infty} q_j\frac{\beta\,(\beta y)^{j-1}\,e^{-\beta y}}{(j-1)!} \tag{2.70}$$

is said to be a mixed Erlang pdf.

It follows from (2.67) and (2.69) that the Laplace transform of (2.70) is

$$\tilde{f}(s) = \int_0^\infty e^{-sy}f(y)dy = \sum_{j=1}^{\infty} q_j\left(\frac{\beta}{\beta+s}\right)^j = Q\left(\frac{\beta}{\beta+s}\right), \tag{2.71}$$

so that the mixed Erlang class of distributions may also be viewed as the class of compound distributions with exponential secondary distribution.

If $q_j = 1$ then $f(y)$ reduces to the Erlang-j pdf. It also includes distributions such as sums and mixtures of Erlang distributions with different scale parameters (Willmot and Woo (2007)), as is now described.

For $\beta_1 < \beta$, the algebraic identity

$$\frac{\beta_1}{\beta_1 + s} = \frac{\beta}{\beta + s}\left\{\frac{\frac{\beta_1}{\beta}}{1 - \left(1 - \frac{\beta_1}{\beta}\right)\left(\frac{\beta}{\beta+s}\right)}\right\}, \tag{2.72}$$

expresses (in Laplace transform form) the fact that a zero-truncated geometric sum of exponential random variables has an exponential distribution. Thus, if a distribution has Laplace transform $\widetilde{f}(s)$ which depends on s via the function $\beta_1/(\beta_1 + s)$ for different values of β_1, (2.72) may sometimes be used to express $\widetilde{f}(s)$ in the mixed Erlang form (2.71).

Example 2.8 **Mixture of two exponentials**
Suppose that $f(y) = p\beta_1 e^{-\beta_1 y} + (1-p)\beta_2 e^{-\beta_2 y}$ where $0 < \beta_1 < \beta_2$ and $0 < p < 1$. Then

$$\widetilde{f}(s) = p\frac{\beta_1}{\beta_1 + s} + (1 - p)\frac{\beta_2}{\beta_2 + s},$$

and again using (2.72), it may be expressed in the form (2.71), i.e. $\widetilde{f}(s) = Q(\frac{\beta_2}{\beta_2+s})$ with

$$Q(z) = z\left\{1 - p + p\left\{\frac{\frac{\beta_1}{\beta_2}}{1 - \left(1 - \frac{\beta_1}{\beta_2}\right)z}\right\}\right\},$$

which may be expressed as

$$Q(z) = \left\{1 - p + p\left(\frac{\beta_1}{\beta_2}\right)\right\}z + p\sum_{j=2}^{\infty}\left(\frac{\beta_1}{\beta_2}\right)\left(1 - \frac{\beta_1}{\beta_2}\right)^j z^j.$$

Thus, the coefficients q_j of z^j are obtained as $q_1 = 1 - p + p(\beta_1/\beta_2)$ and

$$q_j = p\left(\frac{\beta_1}{\beta_2}\right)\left(1 - \frac{\beta_1}{\beta_2}\right)^j, \qquad j = 2, 3, \ldots.$$

□

Example 2.9 **Sum of independent gammas**
Suppose that

$$\widetilde{f}(s) = \prod_{i=1}^{n}\left(\frac{\beta_i}{\beta_i + s}\right)^{\alpha_i}, \tag{2.73}$$

corresponding to the sum of independent gamma random variables. Let $\beta = \sup_i \beta_i$, and using (2.72) it follows that (2.73) may be expressed formally in the form (2.71), i.e. $\widetilde{f}(s) = Q\{\beta/(\beta+s)\}$, where

$$Q(z) = z^m \prod_{i=1}^{n} \left\{ \frac{\frac{\beta_i}{\beta}}{1 - \left(1 - \frac{\beta_i}{\beta}\right) z} \right\}^{\alpha_i} \tag{2.74}$$

with $m = \sum_{i=1}^{n} \alpha_i$. Thus, if m is a positive integer, (2.74) is a pgf, corresponding to the convolution of negative binomial distributions, shifted to the right by m. In particular, if $\alpha_i = 1$ for $i = 1, 2, \ldots, n$, then (2.73) is the Laplace transform of the generalized Erlang distribution (e.g. Gerber and Shiu (2005)), which is thus in the mixed Erlang class. The coefficients q_j of z^j in (2.74) may be evaluated recursively, and for some choices of n and the α_is, also analytically (e.g. Willmot and Woo (2007)). □

Example 2.10 **Mixture of Erlangs with different scale parameters**
Suppose that

$$f(y) = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} p_{ik} \mathscr{E}_{\beta_i,k}(y),$$

where $p_{ik} \geq 0$, $\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} p_{ik} = 1$, and $\beta = \sup_i \beta_i < \infty$. Then

$$\widetilde{f}(s) = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} p_{ik} \left(\frac{\beta_i}{\beta_i + s} \right)^k, \tag{2.75}$$

and using (2.72) and (2.75) may be expressed in the form (2.71) with

$$Q(z) = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} p_{ik} z^k \left\{ \frac{\frac{\beta_i}{\beta}}{1 - \left(1 - \frac{\beta_i}{\beta}\right) z} \right\}^k. \tag{2.76}$$

A negative binomial expansion in (2.76) yields

$$\begin{aligned}
Q(z) &= \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} p_{ik} \sum_{m=0}^{\infty} \binom{k+m-1}{k-1} \left(\frac{\beta_i}{\beta}\right)^k \left(1 - \frac{\beta_i}{\beta}\right)^m z^{m+k} \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} p_{ik} \sum_{j=k}^{\infty} \binom{j-1}{k-1} \left(\frac{\beta_i}{\beta}\right)^k \left(1 - \frac{\beta_i}{\beta}\right)^{j-k} z^j \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{j} p_{ik} \binom{j-1}{k-1} \left(\frac{\beta_i}{\beta}\right)^k \left(1 - \frac{\beta_i}{\beta}\right)^{j-k} \right\} z^j.
\end{aligned}$$

Thus, interchanging the order of the first two summations implies that the coefficients q_j of z^j in (2.76) are given by

$$q_j = \sum_{i=1}^{\infty} \sum_{k=1}^{j} p_{ik} \binom{j-1}{k-1} \left(\frac{\beta_i}{\beta}\right)^k \left(1 - \frac{\beta_i}{\beta}\right)^{j-k}, \qquad j = 1, 2, \ldots. \tag{2.77}$$

We remark that (2.77) holds even if $\beta_i = \beta$ for some i (with the usual notational convention that $0^0 = 1$). □

The mixed Erlang class defined by (2.70) or (2.71) is thus quite large, and is extremely tractable mathematically, as will become evident.

It follows from (2.16) and (2.71) that the tail of the mixed Erlang distribution has Laplace transform

$$\int_0^\infty e^{-sy} \overline{F}(y) dy = \frac{1 - Q\left(\frac{\beta}{\beta+s}\right)}{s} = \sum_{j=1}^{\infty} q_j \left\{ \frac{1 - \left(\frac{\beta}{\beta+s}\right)^j}{s} \right\}.$$

But one has the geometric series

$$\sum_{k=1}^{j} \left(\frac{\beta}{\beta+s}\right)^k = \left(\frac{\beta}{\beta+s}\right) \frac{1 - \left(\frac{\beta}{\beta+s}\right)^j}{1 - \frac{\beta}{\beta+s}} = \beta \, \frac{1 - \left(\frac{\beta}{\beta+s}\right)^j}{s},$$

and thus

$$\int_0^\infty e^{-sy} \overline{F}(y) dy = \frac{1}{\beta} \sum_{j=1}^{\infty} q_j \sum_{k=1}^{j} \left(\frac{\beta}{\beta+s}\right)^k = \frac{1}{\beta} \sum_{k=1}^{\infty} \left(\frac{\beta}{\beta+s}\right)^k \sum_{j=k}^{\infty} q_j,$$

i.e.

$$\int_0^\infty e^{-sy} \overline{F}(y) dy = \frac{1}{\beta} \sum_{k=0}^{\infty} \overline{Q}_k \left(\frac{\beta}{\beta+s}\right)^{k+1}, \tag{2.78}$$

where

$$\overline{Q}_k = \sum_{j=k+1}^{\infty} q_j, \qquad k = 0, 1, 2, \ldots. \tag{2.79}$$

Therefore, the mixed Erlang tail is given by

$$\overline{F}(y) = \frac{1}{\beta} \sum_{k=0}^{\infty} \overline{Q}_k \mathscr{E}_{\beta,k+1}(y) = e^{-\beta y} \sum_{k=0}^{\infty} \overline{Q}_k \frac{(\beta y)^k}{k!}. \tag{2.80}$$

Example 2.11 **Residual lifetime distribution of mixed Erlang distributions**
It follows from (2.68) that

$$\begin{aligned} f(x+y) &= \sum_{j=1}^{\infty} q_j \mathscr{E}_{\beta,j}(x+y) \\ &= \frac{1}{\beta} \sum_{j=1}^{\infty} q_j \sum_{k=1}^{j} \mathscr{E}_{\beta,k}(y) \mathscr{E}_{\beta,j+1-k}(x) \\ &= \frac{1}{\beta} \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} q_j \mathscr{E}_{\beta,k}(y) \mathscr{E}_{\beta,j+1-k}(x). \end{aligned}$$

Let $n = k - 1$ and $m = j - k$ to obtain

$$f(x+y) = \frac{1}{\beta} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_{m+n+1} \mathscr{E}_{\beta,m+1}(x) \mathscr{E}_{\beta,n+1}(y). \tag{2.81}$$

Substitution of (2.81) into (2.51) yields, using (2.80),

$$f_x(y) = \frac{\frac{1}{\beta} \sum_{j=1}^{\infty} \mathscr{E}_{\beta,j}(y) \sum_{k=0}^{\infty} q_{j+k} \mathscr{E}_{\beta,k+1}(x)}{\frac{1}{\beta} \sum_{k=0}^{\infty} \overline{Q}_k \mathscr{E}_{\beta,k+1}(x)},$$

i.e.

$$f_x(y) = \sum_{j=1}^{\infty} q_{j,x} \mathscr{E}_{\beta,j}(y), \qquad y \geq 0, \tag{2.82}$$

where

$$q_{j,x} = \frac{\sum_{k=0}^{\infty} q_{j+k} \mathscr{E}_{\beta,k+1}(x)}{\sum_{k=0}^{\infty} \overline{Q}_k \mathscr{E}_{\beta,k+1}(x)}, \qquad j = 1, 2, \ldots. \tag{2.83}$$

Clearly, (2.79) implies that $\sum_{j=1}^{\infty} q_{j,x} = 1$, and thus $f_x(y)$ in (2.82) is again a mixture of Erlangs, but with different weights. □

Example 2.12 **Generalized equilibrium distribution of mixed Erlang distributions**
Consider the distribution defined by (2.18). Differentiating (2.18) implies that (2.19) may also be expressed as

$$f_{1,r}(y) = \frac{\int_0^\infty e^{-rx} f(x+y)dx}{\int_0^\infty e^{-rx}\overline{F}(x)dx} = \frac{\int_0^\infty e^{-rx}\overline{F}(x) f_x(y)dx}{\int_0^\infty e^{-rx}\overline{F}(x)dx}, \tag{2.84}$$

and (2.84) is a mixture over x of pdf of the form (2.51). In the mixed Erlang case, substitution of (2.82) into (2.84) yields

$$f_{1,r}(y) = \sum_{j=1}^{\infty} q_j(r)\mathscr{E}_{\beta,j}(y), \tag{2.85}$$

where

$$q_j(r) = \frac{\int_0^\infty q_{j,x} e^{-rx}\overline{F}(x)dx}{\int_0^\infty e^{-rx}\overline{F}(x)dx}, \qquad j = 1, 2, \ldots. \tag{2.86}$$

Clearly, (2.85) is again a mixture of Erlang pdfs, and we will now simplify (2.86). First, consider the denominator of (2.86). It follows from (2.67) and (2.80) that

$$\int_0^\infty e^{-rx}\overline{F}(x)dx = \frac{1}{\beta}\sum_{k=0}^{\infty}\overline{Q}_k\widetilde{\mathscr{E}}_{\beta,k+1}(r) = \frac{1}{\beta+r}\sum_{k=0}^{\infty}\overline{Q}_k\left(\frac{\beta}{\beta+r}\right)^k.$$

For the numerator of (2.86), it follows from (2.83) and (2.80) that

$$q_{j,x}e^{-rx}\overline{F}(x) = \frac{e^{-rx}}{\beta}\sum_{k=0}^{\infty} q_{j+k}\mathscr{E}_{\beta,k+1}(x),$$

and, again using (2.67),

$$\int_0^\infty q_{j,x}e^{-rx}\overline{F}(x)dx = \frac{1}{\beta}\sum_{k=0}^{\infty} q_{j+k}\widetilde{\mathscr{E}}_{\beta,k+1}(r) = \frac{1}{\beta+r}\sum_{k=0}^{\infty} q_{j+k}\left(\frac{\beta}{\beta+r}\right)^k.$$

Thus, (2.86) becomes

$$q_j(r) = \frac{\sum_{k=0}^{\infty} q_{j+k}\left(\frac{\beta}{\beta+r}\right)^k}{\sum_{k=0}^{\infty}\overline{Q}_k\left(\frac{\beta}{\beta+r}\right)^k}, \qquad j = 1, 2, \ldots. \tag{2.87}$$

To obtain more insight into the discrete distribution (2.87), we consider the pgf

$$Q_{1,\frac{\beta}{\beta+r}}(z) = \frac{1-\frac{\beta}{\beta+r}}{1-Q\left(\frac{\beta}{\beta+r}\right)}\frac{Q(z)-Q\left(\frac{\beta}{\beta+r}\right)}{z-\frac{\beta}{\beta+r}},$$

e.g. Klugman et al. (2013) (pp. 129–131) or Willmot and Woo (2013) (pp. 189–190). Then,

$$Q_{1,\frac{\beta}{\beta+r}}(z) = \sum_{n=0}^{\infty} q_{n,1}\left(\frac{\beta}{\beta+r}\right) z^n,$$

where $\overline{Q}_j = \sum_{i=j+1}^{\infty} q_i$ and

$$q_{n,1}\left(\frac{\beta}{\beta+r}\right) = \frac{\sum_{j=n+1}^{\infty} q_j \left(\frac{\beta}{\beta+r}\right)^{j-n-1}}{\sum_{j=0}^{\infty} \overline{Q}_j \left(\frac{\beta}{\beta+r}\right)^{j}} = \frac{\sum_{j=0}^{\infty} q_{j+n+1} \left(\frac{\beta}{\beta+r}\right)^{j}}{\sum_{j=0}^{\infty} \overline{Q}_j \left(\frac{\beta}{\beta+r}\right)^{j}},$$

and so (2.87) is $q_j(r) = q_{j-1,1}(\frac{\beta}{\beta+r})$. Thus,

$$\sum_{j=1}^{\infty} q_j(r) z^j = \sum_{j=1}^{\infty} q_{j-1,1}\left(\frac{\beta}{\beta+r}\right) z^j = z Q_{1,\frac{\beta}{\beta+r}}(z) = Q_r^*(z)$$

with $Q_r^*(z) = z Q_{1,\frac{\beta}{\beta+r}}(z)$. Directly, we may write

$$\begin{aligned} Q_r^*\left(\frac{\beta}{\beta+s}\right) &= \left(\frac{\beta}{\beta+s}\right) \frac{1-\frac{\beta}{\beta+r}}{1-Q\left(\frac{\beta}{\beta+r}\right)} \frac{Q\left(\frac{\beta}{\beta+s}\right) - Q\left(\frac{\beta}{\beta+r}\right)}{\frac{\beta}{\beta+s} - \frac{\beta}{\beta+r}} \\ &= \frac{\frac{\beta}{\beta+s}\frac{r}{\beta+r}}{\frac{\beta}{\beta+s} - \frac{\beta}{\beta+r}} \frac{Q\left(\frac{\beta}{\beta+s}\right) - Q\left(\frac{\beta}{\beta+r}\right)}{1-Q\left(\frac{\beta}{\beta+r}\right)} = \frac{r}{\beta+r-\beta-s} \frac{Q\left(\frac{\beta}{\beta+s}\right) - Q\left(\frac{\beta}{\beta+r}\right)}{1-Q\left(\frac{\beta}{\beta+r}\right)} \\ &= \frac{r}{s-r} \frac{Q\left(\frac{\beta}{\beta+r}\right) - Q\left(\frac{\beta}{\beta+s}\right)}{1-Q\left(\frac{\beta}{\beta+r}\right)}, \end{aligned}$$

which is (2.20) with $\widetilde{f}(s) = Q\left(\frac{\beta}{\beta+s}\right)$, as expected.

Note that $zQ_{1,t}(z)$ is of the same form as the discrete ladder height pgf in the compound binomial model, which will be discussed later in Example 7.4. We remark that from Feller (1968) (p. 265),

$$\sum_{k=0}^{\infty} \overline{Q}_k z^k = \frac{1-Q(z)}{1-z}, \tag{2.88}$$

and thus if $r > 0$, (2.87) may be expressed as

$$q_j(r) = \left(\frac{r}{\beta + r}\right) \frac{\sum_{k=0}^{\infty} q_{j+k} \left(\frac{\beta}{\beta+r}\right)^k}{1 - Q\left(\frac{\beta}{\beta+r}\right)},$$

whereas if $r = 0$,

$$q_j(0) = \frac{\overline{Q}_{j-1}}{\sum_{k=0}^{\infty} \overline{Q}_k}, \qquad j = 1, 2, \ldots. \tag{2.89}$$

That is, from (2.21), the equilibrium pdf of the mixed Erlang distribution is

$$f_{1,0}(y) = \frac{\overline{F}(y)}{E(Y)} = \sum_{j=1}^{\infty} q_j(0)\mathscr{E}_{\beta,j}(y),$$

where $q_j(0)$ is given by (2.89), again of mixed Erlang form. □

Example 2.13 **Esscher transform of mixed Erlang distributions**
As $\widetilde{f}(s) = Q(\frac{\beta}{\beta+s})$, we get

$$\frac{\widetilde{f}(\mu+s)}{\widetilde{f}(\mu)} = \frac{Q\left(\frac{\beta}{\beta+\mu+s}\right)}{Q\left(\frac{\beta}{\beta+\mu}\right)} = \frac{Q\left(\frac{\beta}{\beta+\mu} \cdot \frac{\beta+\mu}{\beta+\mu+s}\right)}{Q\left(\frac{\beta}{\beta+\mu}\right)} = Q^*\left(\frac{\beta+\mu}{\beta+\mu+s}\right),$$

where

$$Q^*(z) = \frac{Q\left(\frac{\beta}{\beta+\mu} z\right)}{Q\left(\frac{\beta}{\beta+\mu}\right)},$$

or equivalently

$$q_n^* = \frac{\left(\frac{\beta}{\beta+\mu}\right)^n q_n}{Q\left(\frac{\beta}{\beta+\mu}\right)}.$$

□

Example 2.14 **A compound geometric distribution**
For the compound geometric random variable L with Laplace–Stieltjes transform given by (2.28), substitution of the mixed Erlang Laplace transform (2.71) yields

$$E\left(e^{-sL}\right) = C\left(\frac{\beta}{\beta+s}\right),$$

where

$$C(z) = \sum_{j=0}^{\infty} c_j z^j = \frac{1-\phi}{1-\phi Q(z)} \tag{2.90}$$

is a discrete compound geometric pgf. Thus L has a pdf

$$g(y) = \sum_{j=1}^{\infty} c_j \mathscr{E}_{\beta,j}(y), \qquad y > 0,$$

and $\Pr(L = 0) = c_0 = 1 - \phi$. Rearrangement of (2.90) gives rise to

$$C(z) = \phi Q(z) C(z) + (1 - \phi),$$

which yields, upon equating coefficients of z^n, the identity

$$c_n = \phi \sum_{k=1}^{n} q_k c_{n-k}, \qquad n = 1, 2, \ldots, \tag{2.91}$$

and (2.91) may be used to evaluate $\{c_n;\ n = 0, 1, 2, \ldots\}$ numerically, beginning with $c_0 = 1 - \phi$.

For any $\alpha \geq 0$ it follows from (2.81) that

$$\begin{aligned}
\int_y^{\infty} (x-y)^{\alpha} g(x)dx &= \int_0^{\infty} x^{\alpha} g(x+y)dx \\
&= \frac{1}{\beta} \int_0^{\infty} x^{\alpha} \left\{ \sum_{n=0}^{\infty} \mathscr{E}_{\beta,n+1}(y) \sum_{m=0}^{\infty} c_{m+n+1} \mathscr{E}_{\beta,m+1}(x) \right\} dx \\
&= \frac{1}{\beta} \sum_{n=0}^{\infty} \mathscr{E}_{\beta,n+1}(y) \sum_{m=0}^{\infty} c_{m+n+1} \int_0^{\infty} x^{\alpha} \mathscr{E}_{\beta,m+1}(x)dx.
\end{aligned}$$

One has easily from (2.66) that

$$\int_0^{\infty} x^{\alpha} \mathscr{E}_{\beta,m+1}(x)dx = \frac{\Gamma(m+\alpha+1)}{m!\beta^{\alpha}},$$

and thus

$$\int_y^{\infty} (x-y)^{\alpha} g(x)dx = \sum_{n=0}^{\infty} \mathscr{E}_{\beta,n+1}(y) \sum_{m=0}^{\infty} \frac{c_{m+n+1}\Gamma(m+\alpha+1)}{m!\beta^{\alpha+1}},$$

i.e.

$$\int_y^{\infty} (x-y)^{\alpha} g(x)dx = e^{-\beta y} \sum_{n=0}^{\infty} \gamma_{n,\alpha} \frac{(\beta y)^n}{n!}, \tag{2.92}$$

where

$$\gamma_{n,\alpha} = \sum_{m=0}^{\infty} c_{m+n+1} \frac{\Gamma(m+\alpha+1)}{m!\beta^{\alpha}}, \qquad n = 0, 1, 2, \ldots.$$

With $\alpha = 0$, one has that $\gamma_{n,0} = \sum_{m=0}^{\infty} c_{m+n+1} = \overline{C}_n$ and (2.92) becomes

$$\overline{G}(y) = \Pr(L > y) = e^{-\beta y} \sum_{n=0}^{\infty} \overline{C}_n \frac{(\beta y)^n}{n!}, \qquad y \geq 0. \tag{2.93}$$

The coefficients $\{\overline{C}_n;\ n = 0, 1, 2, \ldots\}$ have generating function from (2.90) given by

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{C}_n z^n &= \frac{1}{1-z} \left\{ 1 - \frac{1-\phi}{1-\phi Q(z)} \right\} \\ &= \frac{\phi}{1-\phi Q(z)} \; \frac{1-Q(z)}{1-z}. \end{aligned} \tag{2.94}$$

Again (2.94) implies, upon equating coefficients of z^n,

$$\overline{C}_n = \frac{\phi}{1-\phi} \sum_{k=0}^{n} c_k \overline{Q}_{n-k}, \qquad n = 0, 1, 2 \ldots,$$

using (2.88) and (2.90). Alternatively, (2.94) may be rearranged as

$$\sum_{n=0}^{\infty} \overline{C}_n z^n = \phi Q(z) \left\{ \sum_{n=0}^{\infty} \overline{C}_n z^n \right\} + \phi \frac{1-Q(z)}{1-z},$$

which yields, upon equating coefficients of z^n, the discrete defective renewal equation

$$\overline{C}_n = \phi \sum_{k=1}^{n} q_k \overline{C}_{n-k} + \phi \overline{Q}_n, \qquad n = 1, 2 \ldots. \tag{2.95}$$

The coefficients $\{\overline{C}_n;\ n = 0, 1, 2, \ldots\}$ may be computed recursively from (2.95), beginning with $\overline{C}_0 = \phi$. $\square$

2.5 Coxian Distributions

Another useful class of distributions is the class of Coxian-n distributions with Laplace transform

$$\tilde{f}(s) = \int_0^\infty e^{-sy} dF(y) = \frac{a(s)}{\prod_{i=1}^{m} (\lambda_i + s)^{n_i}}, \tag{2.96}$$

where $\lambda_i > 0$ for $i = 1, 2, \ldots, m$, with $\lambda_i \neq \lambda_j$ for $i \neq j$. Also, n_i is a positive integer for $i = 1, 2, \ldots, m$, and $n = n_1 + n_2 + \ldots + n_m$. Thus the denominator of (2.96) is polynomial of degree n, while $a(s)$ is a polynomial of degree $n - 1$ or less.

As $\tilde{f}(0) = 1$, it follows that $a(0) = \prod_{i=1}^{m} \lambda_i^{n_i}$, and if $a(s) = a(0)$ for all $s > 0$ then (2.96) is the Laplace transform of the sum of m independent Erlangian distributed random variables. Of course, if $m = n = 1$, (2.96) is an exponential Laplace transform.

In fact, a partial fraction expansion of (2.96) results in

$$\tilde{f}(s) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} p_{ij} \left(\frac{\lambda_i}{\lambda_i + s} \right)^j \tag{2.97}$$

for some constants p_{ij}. While not particularly important in what follows, an explicit expression for p_{ij} is

$$p_{ij} = \frac{\lambda_i^{-j}}{(n_i - j)!} \frac{d^{n_i - j}}{ds^{n_i - j}} \left\{ \prod_{\substack{k=1 \\ k \neq i}}^{m} \frac{a(s)}{(\lambda_k + s)^{n_k}} \right\} \Bigg|_{s=-\lambda_i}.$$

It is clear from (2.97) that $\sum_{i=1}^{m} \sum_{j=1}^{n_i} p_{ij} = 1$, but it is not necessary that $p_{ij} \geq 0$ for all i and j. Thus (2.97) implies that the Coxian-n distribution is said to be that of a combination of Erlangs. In particular, when $n_i = 1$ for all i, the distribution is that of a combination of exponentials (e.g. Dufresne (2007)). Furthermore, if $p_{ij} \geq 0$ for all i and j, the distribution is of the mixed Erlang form of the type discussed in the previous section with a single scale parameter.

Example 2.15 **Coxian-2 distribution**
We now consider the Coxian-2 case with $n = 2$ in some detail. Then (2.96) may be expressed as

$$\tilde{f}(s) = \frac{a_1 s + a_0}{(s + \lambda_1)(s + \lambda_2)},$$

where $\lambda_1 = \lambda_2$ is not excluded. Clearly, $a_0 = \lambda_1\lambda_2$, and it is convenient notationally to reparameterize by letting $a_1 = \lambda_1(1-p)$. Thus the Coxian-2 Laplace transform may be written as

$$\widetilde{f}(s) = \int_0^\infty e^{-sy} f(y)dy = \frac{\lambda_1(1-p)s + \lambda_1\lambda_2}{(s+\lambda_1)(s+\lambda_2)}. \tag{2.98}$$

It is clear from (2.98) that if $p = 0$ then $\widetilde{f}(s)$ is the Laplace transform of an exponential distribution with mean $1/\lambda_1$, and if $\lambda_2 = \lambda_1(1-p)$ then $\widetilde{f}(s)$ is the Laplace transform of an exponential distribution with mean $1/\lambda_2$. We wish to exclude these cases from the ensuing analysis.

As (2.98) may be written as

$$\widetilde{f}(s) = (1-p)\frac{\lambda_1}{s+\lambda_1} + p\,\frac{\lambda_1\lambda_2}{(s+\lambda_1)\,(s+\lambda_2)}, \tag{2.99}$$

it follows that the Coxian-2 pdf may be expressed as

$$f(y) = \lambda_1(1-p)e^{-\lambda_1 y} + \lambda_1\lambda_2 p e^{-\lambda_1 y} h(y), \tag{2.100}$$

where

$$h(y) = \int_0^y e^{(\lambda_1-\lambda_2)x}dx. \tag{2.101}$$

Of course, $h(y)$ is easy to evaluate, but its form depends on whether λ_1 equals λ_2 or not. In any event, $f(0) = \lambda_1(1-p)$, which implies that $p \le 1$.

It follows from (2.100) that the tail may be expressed as

$$\overline{F}(y) = (1-p)e^{-\lambda_1 y} + p\left\{e^{-\lambda_1 y} + \lambda_1 e^{-\lambda_1 y} h(y)\right\},$$

i.e.

$$\overline{F}(y) = e^{-\lambda_1 y}\left\{1 + \lambda_1 p h(y)\right\}. \tag{2.102}$$

If $\lambda_1 \ge \lambda_2$ then from (2.101) $\lim_{y\to\infty} h(y) = \infty$, and from (2.102) one must have $p \ge 0$, because if $p < 0$ then $e^{\lambda_1 y}\overline{F}(y)$ would become negative for large y. But $p \ne 0$, and thus if $\lambda_1 \ge \lambda_2$, it follows that $0 < p \le 1$. Hence (2.99) is a mixture of an exponential Laplace transform with mean $1/\lambda_1$, and the Laplace transform of the sum of two exponentials with means $1/\lambda_1$ and $1/\lambda_2$ if $0 < p < 1$ and $\lambda_1 \ge \lambda_2$, and if $p = 1$ then (2.99) is the Laplace transform of the sum of two exponentials with means $1/\lambda_1$ and $1/\lambda_2$. If $\lambda_1 = \lambda_2$ then (2.99) is a mixed Erlang Laplace transform of the type discussed in the previous section.

On the other hand, if $\lambda_1 < \lambda_2$ then from (2.101)

$$h(y) = \frac{1 - e^{-(\lambda_2-\lambda_1)y}}{\lambda_2 - \lambda_1},$$

which implies from (2.102) that

$$\lim_{y\to\infty} e^{\lambda_1 y}\overline{F}(y) = 1 + \lambda_1 p \lim_{y\to\infty} h(y) = 1 + p\frac{\lambda_1}{\lambda_2 - \lambda_1}.$$

Since this limit cannot be negative, one must have $p \geq 1 - \lambda_2/\lambda_1$, or equivalently $\lambda_2 \geq \lambda_1(1-p)$. But $\lambda_2 \neq \lambda_1(1-p)$ and thus if $\lambda_1 < \lambda_2$ then $1 - \frac{\lambda_2}{\lambda_1} < p \leq 1$ but $p \neq 0$.

We now show that if $\lambda_1 < \lambda_2$ and $p < 0$ then (2.99) is the Laplace transform of the mixture of two exponentials with means $1/\lambda_1$ and $1/\lambda_2$. To see this, note that if $1 - \frac{\lambda_2}{\lambda_1} < p < 0$ then we may write $p = \alpha(1 - \frac{\lambda_2}{\lambda_1})$ where $0 < \alpha < 1$. Next, if $\lambda_1 \neq \lambda_2$, it follows that

$$\frac{\lambda_1\lambda_2}{(s+\lambda_1)(s+\lambda_2)} = \left(\frac{\lambda_2}{\lambda_2-\lambda_1}\right)\frac{\lambda_1}{s+\lambda_1} + \left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)\frac{\lambda_2}{s+\lambda_2}, \tag{2.103}$$

which expresses the fact that the sum of two independent exponential random variables with different means has pdf which is a combination of two exponential terms. Substitution of (2.103) into (2.99) yields the fact that if $\lambda_1 \neq \lambda_2$ then

$$\widetilde{f}(s) = \left(1 - p + p\frac{\lambda_2}{\lambda_2-\lambda_1}\right)\frac{\lambda_1}{s+\lambda_1} + \left(p\frac{\lambda_1}{\lambda_1-\lambda_2}\right)\frac{\lambda_2}{s+\lambda_2},$$

i.e.

$$\widetilde{f}(s) = \left(1 - p\frac{\lambda_1}{\lambda_1-\lambda_2}\right)\frac{\lambda_1}{s+\lambda_1} + \left(p\frac{\lambda_1}{\lambda_1-\lambda_2}\right)\frac{\lambda_2}{s+\lambda_2}, \tag{2.104}$$

which is again the Laplace transform of a combination of two exponentials. Thus, if $p = \alpha(1 - \frac{\lambda_2}{\lambda_1})$ then $\alpha = p\frac{\lambda_1}{\lambda_1-\lambda_2}$, and (2.104) becomes

$$\widetilde{f}(s) = (1-\alpha)\frac{\lambda_1}{s+\lambda_1} + \alpha\frac{\lambda_2}{s+\lambda_2},$$

which, for $0 < \alpha < 1$, is the Laplace transform of a mixture of two exponentials.

To summarize, the Coxian-2 distribution has Laplace transform (2.99) which for $p = 1$ is that of the sum of two independent exponential random variables (possibly with different means), for $0 < p < 1$ is that of a mixture of an exponential and the sum of two exponentials, and for $1 - \frac{\lambda_2}{\lambda_1} < p < 0$ (where $\lambda_1 < \lambda_2$) is that of the mixture of two exponentials.

If $\lambda_1 \neq \lambda_2$, it follows from (2.104) that

$$\overline{F}(y) = \left(1 - p\frac{\lambda_1}{\lambda_1-\lambda_2}\right)e^{-\lambda_1 y} + \left(p\frac{\lambda_1}{\lambda_1-\lambda_2}\right)e^{-\lambda_2 y},$$

and from (2.50), the excess loss tail is easily expressed as

$$\overline{F}_x(y) = \frac{\overline{F}(x+y)}{\overline{F}(x)} = \left(1 - p_x \frac{\lambda_1}{\lambda_1 - \lambda_2}\right) e^{-\lambda_1 y} + \left(p_x \frac{\lambda_1}{\lambda_1 - \lambda_2}\right) e^{-\lambda_2 y},$$

where

$$p_x = \frac{pe^{-\lambda_2 x}}{\overline{F}(x)}. \tag{2.105}$$

Similarly, if $\lambda_1 = \lambda_2$, then from (2.102),

$$\overline{F}(y) = e^{-\lambda_2 y}\left\{1 + p\lambda_2 y\right\},$$

and

$$\overline{F}_x(y) = \frac{\overline{F}(x+y)}{\overline{F}(x)} = e^{-\lambda_2 y}\left\{1 + p_x\lambda_2 y\right\},$$

again with p_x given by (2.105). Thus, the excess loss distribution with pdf $f_x(y) = f(x+y)/\overline{F}(x)$ is again of Coxian-2 form, but with p replaced by p_x in (2.105). That is, from (2.98),

$$\widetilde{f}_x(s) = \int_0^\infty e^{-sy} f_x(y)dy = \frac{\lambda_1(1-p_x)s + \lambda_1\lambda_2}{(s+\lambda_1)(s+\lambda_2)}. \tag{2.106}$$

For the generalized equilibrium distribution defined by (2.18), it follows from (2.84) and (2.106) that (2.20) becomes

$$\begin{aligned} \widetilde{f}_{1,r}(s) &= \frac{\int_0^\infty e^{-rx}\overline{F}(x)\widetilde{f}_x(s)dx}{\int_0^\infty e^{-rx}\overline{F}(x)dx} \\ &= \frac{\lambda_1(1-p_{1,r})s + \lambda_1\lambda_2}{(s+\lambda_1)(s+\lambda_2)}, \end{aligned} \tag{2.107}$$

where

$$p_{1,r} = \frac{\int_0^\infty p_x e^{-rx}\overline{F}(x)dx}{\int_0^\infty e^{-rx}\overline{F}(x)dx}. \tag{2.108}$$

Clearly, (2.107) implies that the generalized equilibrium distribution (2.18) is again of Coxian-2 form, but with p replaced by $p_{1,r}$ in (2.108), which will now be simplified.

Using (2.16) and (2.99), one finds that the Laplace transform of the tail $\overline{F}(y)$ satisfies

$$
\begin{aligned}
\frac{1-\widetilde{f}(s)}{s} &= \frac{1}{s}\left\{1-(1-p)\frac{\lambda_1}{\lambda_1+s}-p\frac{\lambda_1\lambda_2}{(\lambda_1+s)(\lambda_2+s)}\right\}\\
&= \frac{1}{s}\left\{(1-p)\left(1-\frac{\lambda_1}{\lambda_1+s}\right)+p\left(1-\frac{\lambda_1\lambda_2}{(\lambda_1+s)(\lambda_2+s)}\right)\right\}\\
&= \frac{1-p}{s+\lambda_1}+p\frac{s+\lambda_1+\lambda_2}{(\lambda_1+s)(\lambda_2+s)}\\
&= \frac{(1-p)(s+\lambda_2)+p(s+\lambda_1+\lambda_2)}{(s+\lambda_1)(s+\lambda_2)},
\end{aligned}
$$

i.e.

$$
\int_0^\infty e^{-sx}\overline{F}(x)dx = \frac{s+\lambda_2+\lambda_1 p}{(s+\lambda_1)(s+\lambda_2)}. \tag{2.109}
$$

Also, from (2.105),

$$
\int_0^\infty p_x e^{-rx}\overline{F}(x)dx = p\int_0^\infty e^{-(r+\lambda_2)x}dx = \frac{p}{r+\lambda_2},
$$

and using (2.109) and (2.108) simplifies to

$$
p_{1,r} = \frac{p\,(r+\lambda_1)}{r+\lambda_2+\lambda_1 p}. \tag{2.110}
$$

To summarize, the generalized equilibrium pdf $f_{1,r}(y)$ is again of Coxian-2 form, but with p replaced by $p_{1,r}$ given by (2.110). In particular, when $r=0$, the equilibrium pdf $f_{1,0}(y)=\overline{F}(y)/E(Y)$ is of Coxian-2 form with p replaced by $p_{1,0}$.

The compound geometric tail $\overline{G}(y)$ has Laplace transform, from (2.28) and (2.33), given by

$$
\int_0^\infty e^{-sy}\overline{G}(y)dy = \frac{1}{s}\left\{1-\frac{1-\phi}{1-\phi\widetilde{f}(s)}\right\} = \left\{\frac{\phi}{1-\phi\widetilde{f}(s)}\right\}\frac{1-\widetilde{f}(s)}{s}.
$$

In the case when $\widetilde{f}(s)$ has the Coxian-2 Laplace transform (2.98), then from (2.109), this yields

$$
\begin{aligned}
\int_0^\infty e^{-sy}\overline{G}(y)dy &= \frac{\phi\frac{s+\lambda_2+\lambda_1 p}{(s+\lambda_1)(s+\lambda_2)}}{1-\phi\frac{\lambda_1(1-p)s+\lambda_1\lambda_2}{(s+\lambda_1)(s+\lambda_2)}}\\
&= \frac{\phi(s+\lambda_2+\lambda_1 p)}{s^2+\{\lambda_1+\lambda_2-\phi\lambda_1(1-p)\}s+\lambda_1\lambda_2(1-\phi)}.
\end{aligned}
$$

Thus,

$$
\int_0^\infty e^{-sy}\overline{G}(y)dy = \frac{\phi(s+\lambda_2+\lambda_1 p)}{(s+R_1)(s+R_2)}, \tag{2.111}
$$

where

$$R_1, R_2 = \frac{1}{2}\Big[\{\lambda_1+\lambda_2-\phi\lambda_1(1-p)\}\pm\sqrt{\{\lambda_1+\lambda_2-\phi\lambda_1(1-p)\}^2-4\lambda_1\lambda_2(1-\phi)}\,\Big]. \tag{2.112}$$

The roots R_1 and R_2 given by (2.112) are real, distinct, and positive. To see this, assume that $\lambda_1 \geq \lambda_2$, implying that $0 < p \leq 1$. Thus,

$$\lambda_1+\lambda_2-\phi\lambda_1(1-p) = \lambda_2+\lambda_1\{1-\phi(1-p)\} > 0,$$

and

$$\begin{aligned}
&\{\lambda_1+\lambda_2-\phi\lambda_1(1-p)\}^2-4\lambda_1\lambda_2(1-\phi)\\
&= \{\lambda_1[1-\phi(1-p)]-\lambda_2\}^2+4\lambda_1\lambda_2\{[1-\phi(1-p)]-(1-\phi)\}\\
&= \{\lambda_1[1-\phi(1-p)]-\lambda_2\}^2+4\lambda_1\lambda_2\phi p,
\end{aligned}$$

which is clearly positive. If $\lambda_1 < \lambda_2$, then $\lambda_2-\lambda_1(1-p) > 0$, and therefore

$$\lambda_1+\lambda_2-\phi\lambda_1(1-p) = \lambda_1+\lambda_2(1-\phi)+\phi\{\lambda_2-\lambda_1(1-p)\} > 0,$$

and also

$$\begin{aligned}
&\{\lambda_1+\lambda_2-\phi\lambda_1(1-p)\}^2-4\lambda_1\lambda_2(1-\phi)\\
&= \{\lambda_1-[\lambda_2-\phi\lambda_1(1-p)]\}^2+4\lambda_1\{[\lambda_2-\phi\lambda_1(1-p)]-\lambda_2(1-\phi)\}\\
&= \{\lambda_1-[\lambda_2-\phi\lambda_1(1-p)]\}^2+4\phi\lambda_1\{\lambda_2-\lambda_1(1-p)\},
\end{aligned}$$

again clearly positive.

Clearly, (2.111) may be expressed as

$$\int_0^\infty e^{-sy}\overline{G}(y)dy = \frac{\phi}{R_2-R_1}\left\{\frac{\lambda_2+\lambda_1 p-R_1}{s+R_1}+\frac{R_2-\lambda_2-\lambda_1 p}{s+R_2}\right\},$$

resulting in

$$\overline{G}(y) = \frac{\phi}{R_2-R_1}\left\{(\lambda_2+\lambda_1 p-R_1)\,e^{-R_1 y}+(R_2-\lambda_2-\lambda_1 p)\,e^{-R_2 y}\right\}, \qquad y \geq 0,$$

a combination of exponentials. □

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