On the Group of Transformations of Classical Types of Seventh Chords

Sonia Cannas[1,2](✉), Samuele Antonini[1](✉), and Ludovico Pernazza[1](✉)

[1] Dipartimento di Matematica "Felice Casorati", Università di Pavia, Pavia, Italy
sonia.cannas01@universitadipavia.it,
{samuele.antonini,ludovico.pernazza}@unipv.it
[2] IRMA, Université de Strasbourg, Strasbourg, France

Abstract. This paper presents a generalization of the well-known neo-Riemannian group PLR to the classical five types of seventh chord (dominant, minor, half-diminished, major, diminished) considered as tetrachords with a marked root and proving that it is isomorphic to the abstract group $S_5 \ltimes \mathbb{Z}_{12}^4$. This group includes as subgroups the PLR group and several other groups already appeared in the literature.

Keywords: Transformational theory · neo-Riemannian group
Semi-direct product · Seventh chord

1 Introduction

Since the pioneering works by David Lewin [8,9] and Guerino Mazzola [10,11], the main idea of *transformational theory* is to model musical transformations using algebraic structures. The most famous example is probably the neo-Riemannian group PLR, that acts on the set of all 24 minor and major triads of twelve-tone equal temperament and is abstractly isomorphic to the dihedral group of order 24. It is generated by the P, L and R operations that transform major triads in minor triads (and vice versa) shifting a single note by a semitone or a whole tone. They were introduced by 19th-century music theorist Hugo Riemann [12] for pure intervals. Lewin rediscovered the PLR operations, defined them considering the equal temperament, and gave birth to a branch of the transformational theory called *neo-Riemannian theory*.

Neo-Riemannian transformations can be modelled with several geometric structures, of which the most important is the Tonnetz, first introduced by Euler [4] and later studied by the several musicologists of the 19th century, such as Wilhelm Moritz Drobisch, Carl Ernst Naumann, Arthur von Oettingen and the same Hugo Riemann. From a mathematical point of view the Tonnetz is an infinite two-dimensional simplicial complex which tiles the plane with triangles where 0-simplices represent pitch classes, and 2-simplices identify major and minor triads: the relative position of 2-simplices makes it also a natural tool in the theory of parsimonious voice leading.

O. A. Agustín-Aquino et al. (Eds.): MCM 2017, LNAI 10527, pp. 13–25, 2017.
https://doi.org/10.1007/978-3-319-71827-9_2

In addition to triads, seventh chords are often used in the music literature. A natural question arises: can we define a group similar to the neo-Riemannian group PLR acting on the set of seventh chords (of the twelve-tone equal temperament)? More precisely: can we define a group of transformations between seventh chords to describe parsimonious voice leading, so that the generators fix three notes and move a single note by a semitone or a whole tone? Problems on relationships between seventh chords were studied by Childs [2], Gollin [6], by Fiore and Satyendra [5], by Arnett and Barth [1] and by Kerkez [7] for some of the types of seventh chords. In this paper we will extend their studies considering all five "classical" types of seventh chords: dominant, minor, half-diminished, major, diminished.

In Sect. 2 we provide some preliminaries about the neo-Riemannian group. Section 3 presents briefly the known results about the generalization of the PLR-group to seventh chords, and a classification of all transformations between seventh chords shifting a single note by a semitone or a whole tone. In the fourth and final section we will define the $PLRQ$ group, generalizing the PLR group, and we identify its abstract algebraic structure.

2 The neo-Riemannian Group *PLR*

The neo-Riemannian group PLR is generated by the following P, L and R operations.

– P ("Parallel"): if the triad is major, P moves the third down a semitone, while if the triad is minor P moves the third up a semitone.
– L ("Leading-Tone"): if the triad is major L moves the root down a semitone, while if the triad is minor L moves the fifth up a semitone.
– R ("Relative"): if the triad is major, R moves the fifth up a whole tone, while if the triad is minor R moves the root down a whole tone.

There exist many ways to represent algebraically or geometrically such transformations. We will denote pitch classes by elements of the cyclic group of 12 elements $\mathbb{Z}/12\mathbb{Z}$ (or, more briefly, $\mathbb{Z}_{12}$) and n-chords by n-ples of pitch classes in brackets, ordered in the ascending direction (as induced by the linear order of pitches) and starting from some pitch class of reference.

In this notation, Crans, Fiore and Satyendra [3] use twelve equally-spaced points on a circle to represent pitch classes and relate the above operations to an inversion operation I_{k+h} as follows. Let S be the set of all 24 minor and major triads $\{[x_1, x_2, x_3] \mid x_1, x_2, x_3 \in \mathbb{Z}_{12}, x_2 = x_1 + 3 \text{ or } x_2 = x_1 + 4, x_3 = x_1 + 7\}$; then

$$P([x_1, x_2, x_3]) = I_{x_1+x_3}([x_1, x_2, x_3]) \tag{1}$$

$$R([x_1, x_2, x_3]) = I_{x_1+x_2}([x_1, x_2, x_3]) \tag{2}$$

$$L([x_1, x_2, x_3]) = I_{x_2+x_3}([x_1, x_2, x_3]) \tag{3}$$

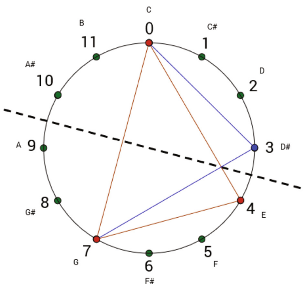

Fig. 1. $P(C) = c$

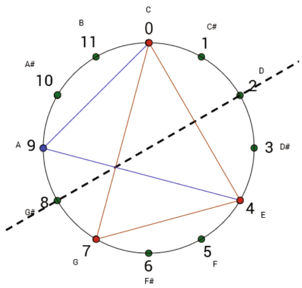

Fig. 2. $R(C) = a$

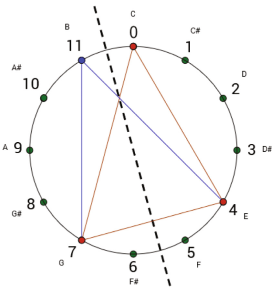

Fig. 3. $L(C) = e$

where I_{k+h} is the reflection of the circle across the axis of the line passing through k and h. As depicted in Figs. 1, 2 and 3, when applied to the triad of C major P gives c (C-minor), R gives a (A-minor) and L gives e (E-minor).

Another way to define the P, L and R operations is proposed by Arnett and Barth [1]:

$$P\colon M \leftrightarrow m \qquad P\colon [x, x+4, x+7] \leftrightarrow [x, x+3, x+7] \tag{4}$$

$$R\colon M \leftrightarrow m-3 \qquad R\colon [x, x+4, x+7] \leftrightarrow [x, x+4, x+9] \tag{5}$$

$$L\colon M \leftrightarrow m+4 \qquad L\colon [x, x+4, x+7] \leftrightarrow [x-1, x+4, x+7] \tag{6}$$

where M represents a major triad, m a minor one, -3 and 4 are the numbers of semitones to be added to each component of the parallel triad (where here, as in the whole paper, the sum is made $\mod 12$, i.e. in the group $\mathbb{Z}_{12}$).

It is an easy calculation to verify that P is obtained as $RLRLRLR$, therefore the PLR group is in fact generated by R and L. The isomorphism to the dihedral group of order 24 becomes apparent noting that the element $RPLP$ is a translation up a semitone and therefore has order 12. We can visualize these operations in the neo-Riemannian Tonnetz (see Fig. 4), a simplicial complex where

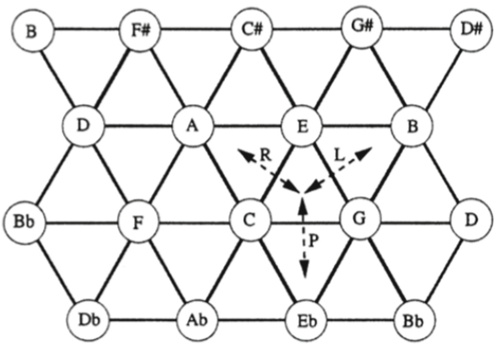

Fig. 4. Reflections preserving a triangle's edge in the Tonnetz represent the P, L and R operations

the vertices represent pitch classes and in which notes connected by a horizontal segment have intervals of a perfect fifth, while the other two directions represent major and minor thirds. Triangles sharing an edge represent triads that share two notes, while the third one differs only by a semitone or a whole tone. We can observe that reflections preserving a triangle's edge represent P, L and R operations (and realize a parsimonious voice leading).

3 Transformations Between Seventh Chords

Childs investigated transformational parsimonious voice leading between dominant and half-diminished sevenths in [2]. In particular he studied transformations that fix two notes and move the other two notes by a semitone or a whole tone.

Gollin also studied the relationships between the same types of sevenths chords [6]. He introduced a possible three-dimensional expansion of the Tonnetz in which horizontal planes contain copies of the traditional Tonnetz, while segments in a chosen direction outside the plane represent intervals of minor seventh. While the Tonnetz tiles the plane with triangles, its three-dimensional expansion tiles the three-dimensional Euclidean space with tetrahedra, representing dominant and half-diminished seventh chords, and triangular prisms (not representing chords). There are six transformations between tetrahedra sharing a common edge: they are represented spatially as a "flip" of the two tetrahedra about their common edge. Each "edge-flip" maintains at least the two notes represented by the two vertices of the common edge, and in one case the two tetrahedra share three notes (see Fig. 5).

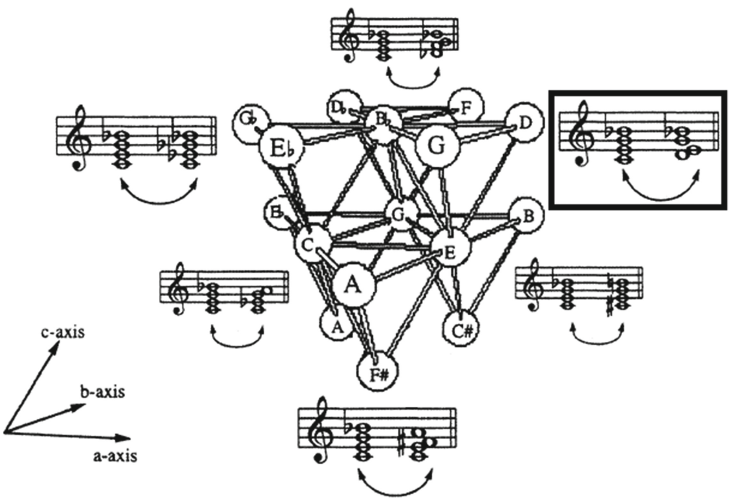

Fig. 5. The six edge-flips between tetrahedra. In the upper right the only flip in which the tetrahedra represent seventh chords sharing three common notes

Arnett and Barth [1] start from the three-dimensional expansion of the Tonnetz introduced by Gollin and observe that Gollin's study does not include the minor seventh chords, very common in the music literature. Therefore they propose to consider a set of 36 chords consisting of all dominant, half-diminished and minor seventh chords and to find the transformations between them that maintain three common notes. They define the following five operations:

$$\begin{aligned}
P1 &\colon D \leftrightarrow m & P1&\colon [x, x+4, x+7, x+10] \leftrightarrow [x, x+3, x+7, x+10]\\
P2 &\colon m \leftrightarrow hd & P1&\colon [x, x+3, x+7, x+10] \leftrightarrow [x, x+3, x+6, x+10]\\
R1 &\colon D \leftrightarrow m-3 & P1&\colon [x, x+4, x+7, x+10] \leftrightarrow [x, x+4, x+7, x+9]\\
R2 &\colon m \leftrightarrow hd-3 & R2&\colon [x, x+3, x+7, x+10] \leftrightarrow [x, x+3, x+7, x+9]\\
L &\colon D \leftrightarrow hd+4 & P1&\colon [x, x+4, x+7, x+10] \leftrightarrow [x+2, x+4, x+7, x+10]
\end{aligned}$$

The first four transformations move a single note by a semitone, whereas L shifts a note by a whole tone. In fact, L is the algebraic formalization of the edge-flip between tetrahedra representing seventh chords with three common notes described in Gollin's three-dimensional Tonnetz.

Although this study includes more types of seventh chords than Childs' and Gollin's ones, other important types of seventh chords are not considered and the algebraic structure of these transformations is not analyzed.

Kerkez gives an idea to extend the PLR group to seventh chords in [7].

Let H be the set of major and minor seventh chords, that is,

$$\begin{aligned}
H = &\{[x_1, x_2, x_3, x_4] | x_1, x_2, x_3, x_4 \in \mathbb{Z}_{12}, x_2 = x_1 + 4, x_3 = x_1 + 7, x_4 = x_1 + 11\} \cup\\
&\{[x_1, x_2, x_3, x_4] \,|\, x_1, x_2, x_3, x_4 \in \mathbb{Z}_{12}, x_2 = x_1 + 3, x_3 = x_1 + 7, x_4 = x_1 + 10\}
\end{aligned}$$

Kerkez defines the following two maps $P, S\colon H \to H$:

$$\begin{aligned}
P[a, b, c, d] &= [(type[a, b, c, d]) \cdot 2 + d, a, b, c]\\
S[a, b, c, d] &= [b, c, d, (-1) \cdot (type[a, b, c, d]) \cdot 2 + a]
\end{aligned}$$

where

$$type(t) = \begin{cases} 1 & \text{if } t \text{ is a minor seventh}\\ -1 & \text{if } t \text{ is a major seventh} \end{cases}$$

P maps each major seventh to its relative minor seventh moving the seventh down a whole tone. Vice versa, it maps each minor seventh to its relative major seventh moving the root up a whole tone.

S maps each major seventh to the minor seventh having root 4 semitones up, moving its root up a whole tone. Vice versa, it maps each minor seventh to the major seventh having root 4 semitones down, moving its seventh down a whole tone.

Kerkez proves that transformations P and S act on H generating a group again isomorphic to the dihedral group D_{12} of order 24.

In his work, Kerkez considers only major and minor seventh chord. But, as he noted in his conclusions, these transformations are just two of the possible operations between seventh chords.

3.1 Transformations Between Seventh Chords

We want to find all transformations between seventh chords describing parsimonious voice leading, i.e. those that fix three notes and move only one note by a semitone or a whole tone. We consider the following types of seventh chords: dominant (D), minor (m), half-diminished (hd), major (M) and diminished (d), and let $\tilde{H}$ be the set of all seventh chords of these 5 types. We first analyze transformations moving just one note by one semitone: if it exists, let us call Q_{i+} the map that sends each type of seventh chord to another type moving the i-th member up a semitone, where $i = R, T, F, S$ depending on whether the member is considered to be the root (R), the third (T), fifth (F) or seventh (S), respectively. Likewise, let Q_{i-} be the map that moves the i-th member down a semitone. We have the following:

$$
\begin{array}{lllll}
Q_{R+}(D) = d & Q_{R+}(m) = D & Q_{R+}(hd) = m & Q_{R+}(M) = hd & Q_{R+}(d) = hd \\
\cancel{Q_{R-}(D)} & \cancel{Q_{R-}(m)} & Q_{R-}(hd) = M & \cancel{Q_{R-}(M)} & Q_{R-}(d) = D \\
\cancel{Q_{T+}(D)} & Q_{T+}(m) = D & \cancel{Q_{T+}(hd)} & \cancel{Q_{T+}(M)} & Q_{T+}(d) = hd \\
Q_{T-}(D) = m & \cancel{Q_{T-}(m)} & \cancel{Q_{T-}(hd)} & \cancel{Q_{T-}(M)} & Q_{T-}(d) = D \\
\cancel{Q_{F+}(D)} & \cancel{Q_{F+}(m)} & Q_{F+}(hd) = m & \cancel{Q_{F+}(M)} & Q_{F+}(d) = hd \\
\cancel{Q_{F-}(D)} & Q_{F-}(m) = hd & \cancel{Q_{F-}(hd)} & \cancel{Q_{F-}(M)} & Q_{F-}(d) = D \\
Q_{S+}(D) = M & \cancel{Q_{S+}(m)} & \cancel{Q_{S+}(hd)} & \cancel{Q_{S+}(M)} & Q_{S+}(d) = hd \\
Q_{S-}(D) = m & Q_{S-}(m) = hd & Q_{S-}(hd) = d & Q_{S-}(M) = D & Q_{S-}(d) = D
\end{array}
$$

The maps that do not produce any of the classical types of seventh chords have been overstruck. We observe that some transformations are inverse to each other:

$$
\begin{array}{llll}
Q_{R+}(M) = hd & Q_{R-}(hd) = M & \Rightarrow & Q_R \colon M \leftrightarrow hd \\
Q_{R+}(m) = D & Q_{S-}(D) = m & \Rightarrow & Q_R, Q_S \colon D \leftrightarrow m \\
Q_{R+}(hd) = m & Q_{S-}(m) = hd & \Rightarrow & Q_R, Q_S \colon hd \leftrightarrow m \\
Q_{S+}(D) = M & Q_{S-}(M) = D & \Rightarrow & Q_S \colon D \leftrightarrow M \\
Q_{T+}(m) = D & Q_{T-}(D) = m & \Rightarrow & Q_T \colon m \leftrightarrow D \\
Q_{F+}(hd) = m & Q_{F-}(m) = hd & \Rightarrow & Q_F \colon hd \leftrightarrow m
\end{array}
$$

It remains to consider the following operations:

$$
\begin{array}{lllll}
Q_{R+}(D) = d & Q_{R-}(d) = D & Q_{T-}(d) = D & Q_{F-}(d) = D & Q_{S-}(d) = D \\
Q_{S-}(hd) = d & Q_{S+}(d) = hd & Q_{R+}(d) = hd & Q_{T+}(d) = hd & Q_{F+}(d) = sd
\end{array}
$$

$Q_{R+}(D) = d$ is the inverse of $Q_{R-}(d) = D, Q_{T-}(d) = D, Q_{F-}(d) = D$ and $Q_{S-}(d) = D$. This is due to the particular symmetry of the interval structure of diminished sevenths, in which the members of the chord play an identical role: for example the diminished seventh $C\sharp^{o7} = [C\sharp, E, G, Bb]$ acoustically coincides to the diminished seventh $E\sharp^{o7} = [E, G, B\flat, D\flat]$ because they are enharmonically equivalent. Unlike the other four types, the diminished sevenths would be only three (and not twelve), e.g. $C, C\sharp, D$, because the other nine chords are three by three enharmonic to them. This explains why we have four transformations that have the same inverse. To obtain a set of well-defined musical transformations, we will consider the diminished seventh as 12 distinct chords, using the marked root to distinguish them. Hence we have 4 transformations between diminished and half-diminished seventh chords and 4 transformations between diminished and dominant seventh chords

$$\begin{array}{llcl}
Q_{S-}(hd) = d & Q_{R-}(d) = hd & \Rightarrow & Q_R, Q_S\colon hd \leftrightarrow d \\
Q_{S-}(hd) = d & Q_{T-}(d) = hd & \Rightarrow & Q_T, Q_S\colon hd \leftrightarrow d \\
Q_{S-}(hd) = d & Q_{F-}(d) = hd & \Rightarrow & Q_F, Q_S\colon hd \leftrightarrow d \\
Q_{S-}(hd) = d & Q_{S-}(d) = hd & \Rightarrow & Q_S\colon hd \leftrightarrow d \\
Q_{R+}(D) = d & Q_{R-}(d) = D & \Rightarrow & Q_R\colon D \leftrightarrow d \\
Q_{R+}(D) = d & Q_{T-}(d) = D & \Rightarrow & Q_R, Q_T\colon D \leftrightarrow d \\
Q_{R+}(D) = d & Q_{F-}(d) = D & \Rightarrow & Q_R, Q_F\colon D \leftrightarrow d \\
Q_{R+}(D) = d & Q_{S-}(d) = D & \Rightarrow & Q_R, Q_S\colon D \leftrightarrow d
\end{array}$$

Now we consider the transformations that move a single note by a whole tone. Analogously to what was done above, if they exist let us call Q_{i++} the map which sends each type of seventh chord in another type moving the i-th member up a whole tone, and Q_{i-} the map which moves the i-th member down a whole tone. We obtain another classical type of seventh chords only moving the root up a whole tone and the seventh down a whole tone:

$$\begin{array}{lllll}
Q_{R++}(D)=hd & Q_{R++}(m)=M & \cancel{Q_{R++}(hd)} & Q_{R++}(M) = m & \cancel{Q_{R++}(d)} \\
\cancel{Q_{S--}(D)} & Q_{S--}(m)=M & Q_{S--}(hd) = D & Q_{S--}(M)=m & \cancel{Q_{S--}(d)}
\end{array}$$

Again, we find some transformations that are the inverse one another:

$$\begin{array}{llcl}
Q_{R++}(D) = hd & Q_{S--}(hd) = D & \Rightarrow & Q_R, Q_S\colon D \leftrightarrow hd \\
Q_{R++}(m) = M & Q_{S--}(M) = m & \Rightarrow & Q_R, Q_S\colon m \leftrightarrow M \\
Q_{R++}(M) = m & Q_{S--}(m) = M & \Rightarrow & Q_R, Q_S\colon M \leftrightarrow m
\end{array}$$

Overall we have 17 transformations corresponding to a parsimonious voice leading among our 5 types of seventh chords.

We want to define these transformations in a similar way to the neo-Riemannian operations. We will use the Arnett and Barth's notation, but we want to formalize it more precisely.

Definition 1. *We define a* cyclic marked chord $[\underline{x_1}, x_2, \ldots, x_n]$ *as a chord constituted by the* n *musical notes* $x_1, x_2, \ldots, x_n$*, so that acoustically* $[\underline{x_1}, x_2, \ldots, x_n] = [x_2, \ldots, x_n, \underline{x_1}] = \cdots = [x_n, \underline{x_1}, \ldots, x_2]$*, where* $x_i \in \mathbb{Z}_{12}$ *and the note corresponding to the root of the chord is underlined.*

As above, in cyclic marked chords all notes will be expressed in terms of a single note by adding or subtracting the appropriate number of semitones.

We start defining a parallel operation P for seventh chords. Let $P_{ij}\colon \tilde{H} \to \tilde{H}$ be the maps which send a i-th type of seventh chord to a j-th type of seventh chord, $1 \le i, j \le 5$ and $i \neq j$, and vice versa, and that fix the other types. 4 of the 17 transformations are parallel operations:

$$\begin{aligned}
Q_T\colon D \leftrightarrow m \quad &\Leftrightarrow \quad P_{12}\colon [\underline{x}, x+4, x+7, x+10] \leftrightarrow [\underline{x}, x+3, x+7, x+10] \\
Q_S\colon D \leftrightarrow M \quad &\Leftrightarrow \quad P_{14}\colon [\underline{x}, x+4, x+7, x+10] \leftrightarrow [\underline{x}, x+4, x+7, x+11] \\
Q_F\colon m \leftrightarrow hd \quad &\Leftrightarrow \quad P_{23}\colon [\underline{x}, x+3, x+7, x+10] \leftrightarrow [\underline{x}, x+3, x+6, x+10] \\
Q_S\colon hd \leftrightarrow d \quad &\Leftrightarrow \quad P_{35}\colon [\underline{x}, x+3, x+6, x+10] \leftrightarrow [\underline{x}, x+3, x+6, x+9]
\end{aligned}$$

Remark 1. P_{12} and P_{23} coincide with $P1$ and $P2$ defined by Arnett and Barth.

Now we consider a relative operation R. We observe that if the triad is major $R = P \circ T_{-3} = T_{-3} \circ P$, if it is minor $R = P \circ T_3 = T_3 \circ P$. Then let $R_{ij}\colon \tilde{H} \to \tilde{H}$ be the maps which send a i-th type of seventh chord to a j-th type of seventh chord transposed three semitones down, a j-th type of seventh to a i-th type of seventh transposed three semitones up, and fix the other types. Then:

$$R_{ij} = T_{\pm 3} \circ P_{ij} = P_{ij} \circ T_{\pm 3} \qquad \forall i, j \in \{1, 2, 3, 4, 5\} \tag{7}$$

Now, 5 of the 17 transformations are relative operations:

$$\begin{aligned}
Q_R, Q_S\colon D \leftrightarrow m-3 \;&\Leftrightarrow R_{12}\colon [\underline{x}, x+4, x+7, x+10] \leftrightarrow [x, x+4, x+7, \underline{x+9}] \\
Q_R, Q_S\colon m \leftrightarrow hd-3 \;&\Leftrightarrow R_{23}\colon [\underline{x}, x+3, x+7, x+10] \leftrightarrow [x, x+3, x+7, \underline{x+9}] \\
Q_R, Q_S\colon M \leftrightarrow m-3 \;&\Leftrightarrow R_{42}\colon [\underline{x}, x+4, x+7, x+11] \leftrightarrow [x, x+4, x+7, \underline{x+9}] \\
Q_R, Q_S\colon hd \leftrightarrow d-3 \;&\Leftrightarrow R_{35}\colon [\underline{x}, x+3, x+6, x+10] \leftrightarrow [x, x+3, x+6, \underline{x+9}] \\
Q_F, Q_S\colon d \leftrightarrow hd-3 \;&\Leftrightarrow R_{53}\colon [\underline{x}, x+3, x+6, x+9] \leftrightarrow [x, x+3, x+7, \underline{x+9}]
\end{aligned}$$

Remark 2. R_{12} and R_{23} coincide with $R1$ and $R2$ defined by Arnett and Barth. Moreover R_{42} coincide with the map P defined by Kerkez.

For the operation L we observe that if the triad is major $L = P \circ T_4 = T_4 \circ P$, if it is minor $L = P \circ T_{-4} = T_{-4} \circ P$. Then let $L_{ij}\colon H \to H$ be the maps which send a i-th type of seventh chord to a j-th type of seventh chord transposed four semitones up, a j-th type of seventh to a i-th type of seventh transposed four semitones down, and fix the other types. Then:

$$L_{ij} = T_{\pm 4} \circ P_{ij} = P_{ij} \circ T_{\pm 4} \qquad \forall i, j \in \{1, 2, 3, 4, 5\} \tag{8}$$

This time, 3 of the 17 transformations are L_{ij} operation:

$$\begin{aligned}
Q_{R++}\colon D \leftrightarrow hd+4 \quad &\Leftrightarrow \quad L_{13}\colon [\underline{x}, x+4, x+7, x+10] \leftrightarrow [x+2, \underline{x+4}, x+7, x+10] \\
Q_R\colon D \leftrightarrow d+4 \quad &\Leftrightarrow \quad L_{15}\colon [\underline{x}, x+4, x+7, x+10] \leftrightarrow [x+1, \underline{x+4}, x+7, x+10] \\
Q_{R++}\colon M \leftrightarrow m+4 \quad &\Leftrightarrow \quad L_{42}\colon [\underline{x}, x+4, x+7, x+11] \leftrightarrow [x+2, \underline{x+4}, x+7, x+11]
\end{aligned}$$

Remark 3. L_{13} coincides with L defined by Arnett and Barth end the "edge-flip" described by Gollin in his three-dimensional Tonnetz.
L_{42} coincides with S defined by Kerkez.

We have identified 12 of the 17 transformations between seventh chords as operations similar to P, L and R. We now see that the other transformations correspond to new operations obtained by the composition of a parallel transformation and a transposition (with a number of semitones different from 3 and 4).
We denote by:

- Q_{ij} the maps which send a i-th type of seventh chord to a j-th type of seventh chord transposed one semitone up, a j-th type of seventh to a i-th type of seventh transposed one semitone down, and fix the other types;
- RR_{ij} the maps which send a i-th type of seventh chord to a j-th type of seventh chord transposed six semitones, and fix the other types;
- QQ_{ij} the maps which send a i-th type of seventh chord to a j-th type of seventh chord transposed two semitones up, a j-th type of seventh to a i-th type of seventh transposed two semitones down, and fix the other types;
- N_{ij} the maps which send a i-th type of seventh chord to a j-th type of seventh chord transposed five semitones up, a j-th type of seventh to a i-th type of seventh transposed five semitones down, and fix the other types.

With these transformations we can define the missing operations in the following way:

$$\begin{aligned}
Q_R, Q_S\colon M \leftrightarrow hd+1 \quad &\Leftrightarrow \quad Q_{43}\colon [x, x+4, x+7, x+11] \leftrightarrow [x+1, x+4, x+7, x+11]\\
Q_R\colon D \leftrightarrow d+1 \quad &\Leftrightarrow \quad Q_{15}\colon [\underline{x}, x+4, x+7, x+10] \leftrightarrow [\underline{x+1}, x+4, x+7, x+10]\\
Q_T, Q_S\colon hd \leftrightarrow d-6 \quad &\Leftrightarrow \quad RR_{35}\colon [\underline{x}, x+3, x+6, x+10] \leftrightarrow [x, x+3, \underline{x+6}, x+9]\\
Q_R, Q_T\colon d \leftrightarrow D+2 \quad &\Leftrightarrow \quad QQ_{51}\colon [\underline{x}, x+3, x+6, x+9] \leftrightarrow [x, \underline{x+2}, x+6, x+9]\\
Q_R, Q_F\colon d \leftrightarrow D+5 \quad &\Leftrightarrow \quad N_{51}\colon [\underline{x}, x+3, x+6, x+9] \leftrightarrow [x, x+3, \underline{x+5}, x+9]
\end{aligned}$$

Remark 4. Crans, Fiore and Satyendra define P, L and R as inversions I_n; since inversions are isometries, they leave unchanged lengths and angles, and minor and major triads geometrically are represented by triangles which the edge lengths correspond to $3, 4$ and 5 semitones. This idea could in principle also be used to define transformations between seventh chords, but it can not be applied to all types since the lengths of the edges and the angles of the quadrilaterals that compose them are not equal. We have only 2 quadrilaterals that are isometric: the one representing the dominant sevenths and the one representing half-diminished sevenths. There exists a unique transformation between this types of seventh chords, L_{13}.

To visualize the 17 transformations just defined we can represent them in a graph whose vertices represent the types of seventh chord, and the edges represent the transformations between them. Therefore we have 5 vertices and 17 edges Fig. 6.

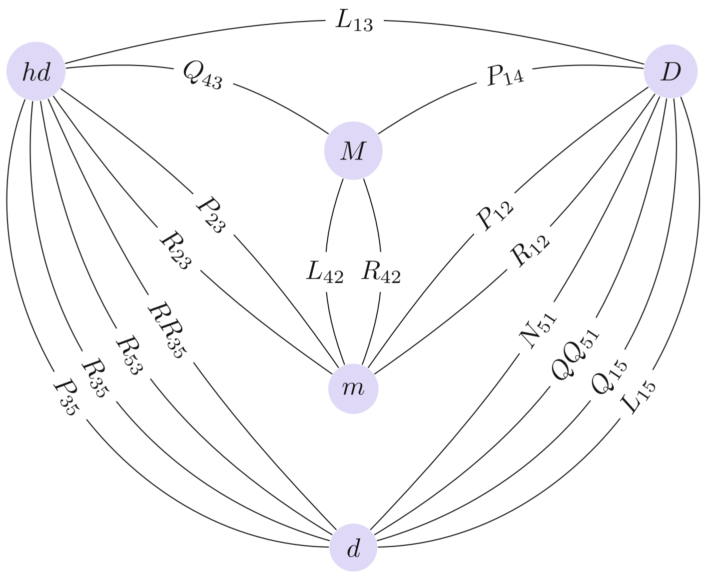

Fig. 6. The graph representing the 17 transformations between seventh chords.

4 The *PLRQ* Group

Let $PLRQ$ be the group generated by the 17 transformations among seventh chords. Each transformation $t \in PLRQ$ exchanges two types of sevenths and fixes the others, thus we can associate to it a permutation of S_5 (more precisely, a transposition). This information is not sufficient to identify the transformation: to identify it, we add a vector $v \in \mathbb{Z}_{12}^5$, in which the i-th component, $i \in \{1, \ldots, 5\}$, is the number of semitones of which the root of the chord of type i has to be shifted to become the root of the chord of type j. It is easy to see that in this way no ambiguity is possible.

We write the 17 transformations between seventh chords as pairs of elements $(\sigma, v) \in S_5 \times \mathbb{Z}_{12}^5$ explicitly:

$$P_{12}\colon [\underline{x}, x+4, x+7, x+10] \leftrightarrow [\underline{x}, x+3, x+7, x+10] \quad (\sigma, v) = ((12), (0,0,0,0,0))$$
$$P_{14}\colon [\underline{x}, x+4, x+7, x+10] \leftrightarrow [\underline{x}, x+4, x+7, x+11] \quad (\sigma, v) = ((14), (0,0,0,0,0))$$
$$P_{23}\colon [\underline{x}, x+3, x+7, x+10] \leftrightarrow [\underline{x}, x+3, x+6, x+10] \quad (\sigma, v) = ((23), (0,0,0,0,0))$$
$$P_{35}\colon [\underline{x}, x+3, x+6, x+10] \leftrightarrow [\underline{x}, x+3, x+6, x+9] \quad (\sigma, v) = ((35), (0,0,0,0,0))$$
$$R_{12}\colon [\underline{x}, x+4, x+7, x+10] \leftrightarrow [x, x+4, x+7, \underline{x+9}] \quad (\sigma, v) = ((12), (-3,3,0,0,0))$$
$$R_{23}\colon [\underline{x}, x+3, x+7, x+10] \leftrightarrow [x, x+3, x+7, \underline{x+9}] \quad (\sigma, v) = ((23), (0,-3,3,0,0))$$
$$R_{42}\colon [\underline{x}, x+4, x+7, x+11] \leftrightarrow [x, x+4, x+7, \underline{x+9}] \quad (\sigma, v) = ((42), (0,3,0,-3,0))$$
$$R_{35}\colon [\underline{x}, x+3, x+6, x+10] \leftrightarrow [x, x+3, x+6, \underline{x+9}] \quad (\sigma, v) = ((35), (0,0,-3,0,3))$$
$$R_{53}\colon [\underline{x}, x+3, x+6, x+9] \leftrightarrow [x, x+3, x+7, \underline{x+9}] \quad (\sigma, v) = ((53), (0,0,3,0,-3))$$

$L_{13}\colon [\underline{x}, x+4, x+7, x+10] \leftrightarrow [x+2, \underline{x+4}, x+7, x+10] \quad (\sigma, v) = ((13), (4, 0, -4, 0, 0))$

$L_{15}\colon [\underline{x}, x+4, x+7, x+10] \leftrightarrow [x+1, \underline{x+4}, x+7, x+10] \quad (\sigma, v) = ((15), (4, 0, 0, 0, -4))$

$L_{42}\colon [\underline{x}, x+4, x+7, x+11] \leftrightarrow [x+2, \underline{x+4}, x+7, x+11] \quad (\sigma, v) = ((42), (0, -4, 0, 4, 0))$

$Q_{43}\colon [\underline{x}, x+4, x+7, x+11] \leftrightarrow [\underline{x+1}, x+4, x+7, x+11] \quad (\sigma, v) = ((43), (0, 0, -1, 1, 0))$

$Q_{15}\colon [\underline{x}, x+4, x+7, x+10] \leftrightarrow [\underline{x+1}, x+4, x+7, x+10] \quad (\sigma, v) = ((15), (1, 0, 0, 0, -1))$

$RR_{35}\colon [\underline{x}, x+3, x+6, x+10] \leftrightarrow [x, x+3, \underline{x+6}, x+9] \quad (\sigma, v) = ((35), (0, 0, -6, 0, 6))$

$QQ_{51}\colon [\underline{x}, x+3, x+6, x+9] \leftrightarrow [x, \underline{x+2}, x+6, x+9] \quad (\sigma, v) = ((51), (-2, 0, 0, 0, 2))$

$N_{51}\colon [\underline{x}, x+3, x+6, x+9] \leftrightarrow [x, x+3, \underline{x+5}, x+9] \quad (\sigma, v) = ((51), (-5, 0, 0, 0, 5))$

More precisely, we can represent each transformation $t \in PQRL$ as an element of

$$S_5 \times Z \qquad \text{where } Z = \{v \in \mathbb{Z}_{12}^5 | \sum_{i=1}^{5} v_i = 0\},$$

since this is clearly true for all the 17 generators. The mapping thus defined becomes a group homomorphism if we define on this set the following operation:

$$\begin{aligned}
&(\sigma_k, v_k) \circ \cdots \circ (\sigma_1, v_1) \\
&= \big(\sigma_k \cdots \sigma_1, v_1 + \sigma_1^{-1}(v_2) + (\sigma_2\sigma_1)^{-1}(v_3) + \cdots + (\sigma_{k-1} \cdots \sigma_1)^{-1}(v_k)\big) \qquad (9) \\
&= \big(\sigma_k \cdots \sigma_1, v_1 + \sigma_1^{-1}(v_2) + \sigma_1^{-1}\sigma_2^{-1}(v_3) + \cdots + \sigma_1^{-1} \cdots \sigma_{k-1}^{-1}(v_k)\big)
\end{aligned}$$

We want to prove that $PLRQ$ is isomorphic to $S_5 \ltimes Z$. We remind the definition of semidirect product of two subgroups.
Let G be a group. If G contains two subgroups H and K such that

(i) $G = HK$;
(ii) $K \trianglelefteq G$;
(iii) $H \cap K = 1$;

G is the *semidirect product* of H and K. Conversely, given two groups H and K and a group homomorphism $\phi\colon H \to Aut(K)$, we can construct a new group $H \ltimes K$ defining in the cartesian product $H \times K$ the following operation:

$$(h_1, k_1)(h_2, k_2) = (h_1 h_2, \phi_{h_2}(k_1) \cdot k_2)$$

Theorem 1. *The group $PLRQ$ is isomorphic to $S_5 \ltimes \mathbb{Z}_{12}^4$.*

Proof. First of all we prove that $PLRQ$ is isomorphic to $S_5 \ltimes Z$.

We observe that the subgroup formed by the elements (Id, v) is normal. In fact, for all $(\sigma, v) \in S_5 \times Z, (Id, v') \in \{Id\} \times Z$, we have

$$(\sigma, v)(Id, v')(\sigma, v)^{-1} = (\sigma\sigma^{-1}, -v + \sigma(v') + \sigma(v)) = (Id, v'') \in \{Id\} \times Z$$

On the other hand, since S_5 is generated by transpositions, it is easy to see that, calling O the origin in $\mathbb{Z}_{12}^5$, $S_5 \times \{O\} < PLRQ$, since we already have in it $(P_{12}, O), (P_{14}, O), (P_{23}, O), (P_{35}, O)$.

To prove our thesis we are only left to see that there is a subgroup isomorphic to Z in $PLRQ$ having trivial intersection with $S_5 \times \{O\}$. But this is exactly the subgroup of the elements of type (Id, v). In fact we compute the permutations and vectors associated to $R_{42}L_{42}$, $P_{14}L_{42}P_{14}R_{12}$, $P_{12}L_{13}P_{12}R_{23}$:

$$\begin{aligned}
R_{42}L_{42} = (\sigma', v')& \\
\sigma' =&\sigma_2\sigma_1 = (42)(42) = Id \\
v' =&v_1 + \sigma_1^{-1}(v_2) \\
=&(0,-4,0,4,0) + (0,-3,0,3,0) \\
=&(0,-7,0,7,0) \\
P_{14}L_{42}P_{14}R_{12} = (\sigma'', v'')& \\
\sigma'' =&\sigma_4\sigma_3\sigma_2\sigma_1 = (14)(42)(14)(12) = Id \\
v'' =&v_1 + \sigma_1^{-1}(v_2) + \sigma_1^{-1}\sigma_2^{-1}(v_3) + \sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}(v_4) \\
=&(-3,3,0,0,0) + (0,0,0,0,0) + (-4,4,0,0,0) + (0,0,0,0,0) \\
=&(7,-7,0,0,0) \\
P_{12}L_{13}P_{12}R_{23} = (\sigma''', v''')& \\
\sigma''' =&\sigma_4\sigma_3\sigma_2\sigma_1 = (12)(13)(12)(23) = Id \\
v''' =&v_1 + \sigma_1^{-1}(v_2) + \sigma_1^{-1}\sigma_2^{-1}(v_3) + \sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}(v_4) \\
=&(0,-3,3,0,0) + (0,0,0,0,0) + (0,-4,4,0,0) + (0,0,0,0,0) \\
=&(0,7,-7,0,0)
\end{aligned}$$

With the following elements just computed

$$\begin{aligned}
R_{42}L_{42} &= (Id, (0,-7,0,7,0)) \\
P_{14}L_{42}P_{14}R_{12} &= (Id, (7,-7,0,0,0)) \\
P_{12}L_{13}P_{12}R_{23} &= (Id, (0,7,-7,0,0))
\end{aligned} \tag{10}$$

we can generate each element $(Id, (v_1, v_2, v_3, v_4, 0))$, with $(v_1, v_2, v_3, v_4, 0) \in \mathbb{Z}_{12}^5$ such that $\sum_1^4 v_i = 0$. To see this, taken $a, b, c \in \mathbb{Z}$, we have to solve

$$\begin{aligned}
a(0,-7,0,7,0) + b(7,-7,0,0,0) + c(0,7,-7,0,0) &\equiv (v_1, v_2, v_3, v_4, 0) \pmod{12} \\
(-7b, -7a + 7b - 7c, 7c, 7a) &\equiv (v_1, v_2, v_3, v_4, 0) \pmod{12}
\end{aligned}$$

$$\begin{cases} -7b \equiv v_1 \\ -7a + 7b - 7c \equiv v_2 \\ 7c \equiv v_3 \\ 7a \equiv v_4 \end{cases} \Rightarrow \begin{cases} -7b \equiv v_1 \\ 7b - 7c \equiv v_2 + 7a \\ 7c \equiv -v_3 \\ 7a \equiv v_4 \end{cases} \Rightarrow \begin{cases} 7b \equiv -v_1 \\ -v_1 - v_3 \equiv v_2 + v_4 \\ 7c \equiv -v_3 \\ 7a \equiv v_4 \end{cases}$$

which is solvable because 7 is coprime with 12.

To obtain all elements $(Id, (v_1, v_2, v_3, v_4, v_5))$, with $(v_1, v_2, v_3, v_4, v_5) \in \mathbb{Z}_{12}^5$ such that $\sum_1^5 v_i = 0$, it is sufficient add to the 3 generators listed in 10 the generator $P_{1235}R_{23}P_{12}L_{15}L_{13} = (Id, (7,0,0,0,-7))$.

But it is evident that $Z \simeq \mathbb{Z}_{12}^4$, hence $PLRQ \simeq S_5 \ltimes \mathbb{Z}_{12}^4$. $\square$

References

1. Arnett, J., Barth, E.: Generalizations of the Tonnetz: Tonality Revisited. http://personal.denison.edu/~lalla/MCURCSM2011/10.pdf
2. Childs, A.: Moving beyond Neo-Riemannian triads: exploring a transformational model for Seventh Chords. J. Music Theory **42**(2), 181–193 (1998)
3. Crans, A., Fiore, T.M., Satyendra, R.: Musical actions of Dihedral Groups. Am. Math. Monthly **116**(6), 479–495 (2009)
4. Euler, L.: Tentamen novae theoriae musicae, Opera Omnia I(III), Teubner, Stuttgart (1739)
5. Fiore, T.M., Satyendra, R.: Generalized contextual groups. Music Theory Online, **11**(3) (2005)
6. Gollin, E.: Aspects of Three-Dimensional Tonnetze. J. Music Theory **42**(2), 195–206 (1998)
7. Kerkez, B.: Extension of Neo-Riemannian PLR-group to Seventh Chords. Mathematics, Music, Art, Architecture, Culture (2012)
8. Lewin, D.: Transformational techniques in atonal and other music theories. Perspect. New Music **21**, 312–371 (1982)
9. Lewin, D.: Generalized Musical Intervals and Transformations. Yale University Press, New Haven (1987)
10. Mazzola, G.: Gruppen und Kategorien in der Musik: Entwurf einer mathematischen Musiktheorie. Heldermannr, Lemgo (1985)
11. Mazzola, G.: Geometrie der Tone. Birkhuser, Basel (1990)
12. Riemann, H.: Handbuch der Harmonielehre. Breitkopf und Hrtel, Leipzig (1887)
13. Rotman, J.J.: Advanced modern algebra. Am. Math. Soc. (2010)

http://www.springer.com/978-3-319-71826-2

Mathematics and Computation in Music
6th International Conference, MCM 2017, Mexico City,
Mexico, June 26-29, 2017, Proceedings
Agustín-Aquino, O.A.; Lluis-Puebla, E.; Montiel, M. (Eds.)
2017, XI, 373 p. 147 illus., Softcover
ISBN: 978-3-319-71826-2