Chapter 2
Celerity Regulators on Networks

Knowing where and when one vehicle should arrive, we want to *compute* from where and when one should depart to reach timely their desired position. For traveling takes time, duration and uses positions on the network: we take these observations as our data before deriving geodesics starting from a departure position at a departure time for reaching an arrival position at an arrival time. Beforehand, we shall illustrate our results on a road system in the plane.

2.1 Traveling Takes Time

Time involves departure time, arrival time, waiting time, travel time, etc. Observe that *time* was used with two meanings in this sentence: dates in the two first cases, duration in the two next ones. Time is indeed particularly polysemous, and thus, dangerous, encapsulating many meanings,[1] among which the notions of

1. *instants*, which evolve. This connotation of the word has many synonyms, among which *dates*, such as departure and arrival time, *chronological* time, *calendar* time, *final* time, deadlines, etc.;
2. *durations*[2], such as in travel time, minimal time, waiting time, age in population dynamics, investment period in economics, etc.;

[1] In Chinese, the sinogram for time (shi2jian1) 时间, involves also the concept of duration. The first sinogram (shi2) 时, denotes the concept of time itself, since it involves the ideogram of "sun" in the left and the ideogram of unit of measure in the right. The second ideogram 间, conveys the concept of duration, since it means interval.

[2] Phylogenesis has "taken all its time" to synchronize the "perceptions" of "durations" by living beings in the sense that they adjust their biological clocks to those of celestial mechanics that made cyclical facetious ellipses turning around the earth. Chronological time requires memory to order these cycles, and may not be perceived in an operational way by most species.

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3. *evolutions* (of vehicles, for instance), which are functions of time (instants), *arriving at* a given position, *passing through* this position, or *starting* from this position;
4. and many other meanings which are not relevant in this study.

Retrospective and Prospective Temporal Windows

Given a chronological time and a position, we shall study separately and successively the problems of

1. *arriving* at this position;
2. *starting* from this position,

before "compounding" their solutions or "concatenating" their evolutions to link a departure position to an arrival position by a "geodesic". This leads us to differentiate what happened *before* and what will happen *later*, and thus, to introduce the concepts of retrospective and prospective *temporal windows* associated with a

1. *chronological time* $T \in \mathbb{R}$,
2. "closing" a *retrospective* temporal window $[T - \Omega^{in}, T]$ at departure date $T - \Omega^{in}$ or "opening" a *prospective* temporal windows[3] $[T, T + \Omega^{ou}]$ at date $T + \Omega^{ou}$;
3. with (prospective) *inward duration*[4] $\Omega^{in} \geq 0$ or (prospective) *outward duration* $\Omega^{ou} \geq 0$,

on which are defined *evolutions*.

Retrospective temporal windows describe the past or the history, which *may be known*, whereas *prospective* temporal windows represent the future, which *is not known, just forecast*.

2.1.1 Time, Duration and Temporal Windows

Temporal windows are thus parameterized by *two-dimensional* vectors $(T, \Omega) \in \mathbb{R} \times \mathbb{R}_+$, describing the "dual nature of time".[5] *Instants* $T := [T, T] \in \mathbb{R}$ are temporal windows of duration $\Omega = 0$, and thus, parameterized by *one-dimensional scalars*, easier to deal with. Instants or dates $t := [t, t] \in [T - \Omega, T]$ are usual called "current times", or, unfortunately, "times". The smallest duration measured so far is the yoctoseconde (10^{-24} s). Therefore, for the time, *instants do not exist yet physically*, and remain the privilege of mathematicians who do not hesitate to let durations converge to 0 in difference quotients for obtaining derivatives of all kinds and obediences.

[3]Most of the time, they are intervals $[0, H]$ where 0 is regarded as *"the" origin of time* and H is interpreted as the *horizon*, finite or infinite.

[4]Finite or infinite, in principle, but finite in this book, since there is no infinite travel.

[5]See [7, Andersen and Grush] and [115, 116, 117, Dobbs] among a myriad of references on time.

The intersection $[T - \Omega^{in}, T] \cap [T, T + \Omega^{ou}]$ is usually regarded as an (evolving) *present* T, separating past from future. It is an instant, which should be called an *"absent"*, since instants do not physically exist yet.[6]

Consequently, mathematically at least, *the "cone"* $\mathbb{R}_+$ *of durations has a natural origin*, 0. *This is not the case of the set* $\mathbb{R}$ *of instants, which has no physical origin*[7] $\mathbf{O} \in \mathbb{R}$ other than arbitrary, *conventional* and *consensual* for being real. Instants range over the real line $\mathbb{R}$, but *only its total order structure should be used*, by stating that an instant s is anterior or posterior to another instant t and using the concepts of infimum and supremum. However, the vector structure of $\mathbb{R}$ is underlying its lattice structure, let only because durations duration $s - t \geq 0$ are differences between two chronological times s posterior to t, where 0 is the origin of duration and not "the" origin of chronological time,[8] whereas addition allows us to translate temporal windows.

Chronological time plays the role of a *"numéraire of evolutions"* (see Sect. 8.2, p. 601, of [20, *La mort du devin, l'émergence du démiurge*]), for comparing evolutions between them by comparing each of them with the duration $t \mapsto \max(0, t - \mathbf{O})$, as in economics, where the numéraire is used for comparing commodities by comparing the value of each of them with the value of the numéraire.

The chronological time was measured by the ephemerides through gnomons and sundials, and now, by clocks, whereas the duration was obtained by clepsydra, and, since the Xth century, by hourglasses, or by the difference between two chronological times.

A *(calendar) duration is an evolution*[9] $t \mapsto \mathbb{R}_+$, the velocity of which is either equal to 1 on its domain, in which case it is called retrospective or inward, or -1, in which case it is called prospective or outward:

1. *retrospective durations* $o^{in} : t \in [T - \Omega^{in}, T] \mapsto o^{in}(t) := t - (T - \Omega^{in}) \in [0, \Omega^{in}]$, increasing from 0 at departure time $T - \Omega^{in}$ up to time T, when it is equal to Ω^{in};

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[6] See *Didier Nordon* in his book *Scientaisies. Chroniques narquoises d'un mathématicien*, [187, Nordon], p. 87, quoting the novel *Deux heures moins dix* by *Mikhail Shishkin*.

[7] *"What was God doing before He created the Heavens and the Earth?"* asked *Augustine of Hippo* in his confessions. What was the universe behaving before the *Big Bang*, ask some physicists? Mathematically, one can introduce *helicoidal durations* which are concatenations of durations $d_{(T_i, T_i - T_{i-1})}$ on successive temporal windows $[T_{i-1}, T_i]$ (see *La valeur n'existe pas. À moins que[...]*, [28, Aubin]). Introducing the concepts of temporal windows and duration function circumvents the question of origin of time.

[8] The max-plus algebra structure on $\mathbb{R}$, in particular, hyperspaces (of subsets of a given space) an another example, could provide an adequate framework to study evolutionary problems (see [3, Akian, Quadrat J.-P. and Viot], [4, Akian, Gaubert and Kolokoltsov], [5, Akian, Bapat and Gaubert], [6, Akian, David and Gaubert], [40, Aubin and Dordan], etc.).

[9] See Definition 5.3.1, p. 131, and Fig. 5.3.2, p. 132, for the definition of durations with variable velocities, the *average velocity* of which is equal to ± 1, instead of requiring that the instantaneous velocity are equal to ± 1 as for the calendar duration.

2. *prospective durations* $o^{ou} : t \in [T, T + \Omega^{ou}] \mapsto o^{ou}(t) := (T + \Omega^{ou}) - t \in [0, \Omega^{ou}]$, decreasing from Ω^{ou} at time T down to arrival time $T + \Omega^{ou}$ when it vanishes.

They can be extended to $\mathbb{R}$ by setting

$$\begin{cases} o^{in} : t \in \mathbb{R} \mapsto o^{in}(t) := \max(0, \min(\Omega^{in}, t - (T - \Omega^{in}))) \in [0, \Omega^{in}] \\ o^{ou} : t \in \mathbb{R} \mapsto o^{ou}(t) := \max(0, \min(\Omega^{ou}, (T + \Omega^{ou}) - t)) \in [0, \Omega^{ou}] \end{cases} \tag{2.1}$$

2.1.2 Temporal Window Dependent Evolutions

We denote by $\mathbb{R}^{|p|}$, where $|p| \in \mathbb{N}$, the $|p|$-dimensional vector space of *positions* $p = (p_1; \ldots, p_{|p|})$ described by $|p|$ components.

Definition 2.1.1 (*Inward and Outward Evolutions*) An *evolution* $p(\cdot) : t \in \mathbb{R} \mapsto \mathbb{R}^{|p|}$ is a time-dependent function. We associate with each them

1. the retrospective temporal window $[T - \Omega^{in}, T]$, the restriction $p^{in}(\cdot) := p^{in}_{(T,\Omega)^{in}}(\cdot) : t \in [T - \Omega, T] \mapsto p^{in}(t) := p(t) \in \mathbb{R}^{|p|}$, called an *incoming evolution* (or *inward evolution*), depending on chronological time $T \in \mathbb{R}$, inward duration $\Omega^{in} \geq 0$ and current time $t \in [T - \Omega^{in}, T]$;
2. the prospective temporal window $[T, T + \Omega^{ou}]$, the restriction $p^{ou}(\cdot) := p^{ou}_{(T,\Omega^{ou})}(\cdot) : t \in [T, T + \Omega^{ou}] \mapsto p^{ou}(t) := p(t) \in \mathbb{R}^{|p|}$, called an *outgoing evolution* (or *outward evolution*), depending on chronological time $T \in \mathbb{R}$, outward duration $\Omega^{ou} \geq 0$ and current time $t \in [T, T + \Omega^{ou}]$.

In this study, the chronological time T is fixed, the durations Ω^{in} or Ω^{ou} and the current time t evolve. For simplifying notations, we do not mention the reference to (T, Ω^{in}) or (T, Ω^{ou}) when there is no ambiguity.

Remark **Forward and Backward Evolutions** Inward and outward evolutions we defined are *forward* evolutions in time, implicitly, one way trips. If we want to study round-trips, we need to consider *backward* inward evolutions $\overleftarrow{p}^{in}(\cdot)$ and outward evolutions $\overleftarrow{p}^{ou}(\cdot)$ defined respectively by

$$\begin{cases} \forall\, t \in [T - \Omega^{ou}, T] \;\; \overleftarrow{p}^{in}(t) = p^{ou}(2T - t) \\ \forall\, t \in [T, T + \Omega^{in}] \;\; \overleftarrow{p}^{ou}(t) = p^{in}(2T - t) \end{cases} \tag{2.2}$$

In other words, if we consider the *concatenation* $p(\cdot)$ of $p^{in}(\cdot)$ and $p^{ou}(\cdot)$ defined on $[T - \Omega^{in}, T + \Omega^{ou}]$ by

$$p(t) := \begin{cases} p^{in}(t) \text{ if } t \in [T - \Omega^{in}, T] \\ p^{ou}(t) \text{ if } t \in [T, T + \Omega^{ou}] \end{cases} \tag{2.3}$$

the backward evolution $\overleftarrow{p}(\cdot)$ defined by $\overleftarrow{p}(t) := p(2T - t)$ is the concatenation of $\overleftarrow{p}^{in}(\cdot)$ and $\overleftarrow{p}^{ou}(\cdot)$ defined on $[T - \Omega^{ou}, T + \Omega^{in}]$ by

$$\overleftarrow{p}(t) := \begin{cases} \overleftarrow{p}^{in}(t) \text{ if } t \in [T - \Omega^{ou}, T] \\ \overleftarrow{p}^{ou}(t) \text{ if } t \in [T, T + \Omega^{in}] \end{cases} \tag{2.4}$$

This is how we can study forward and backward evolutions on a temporal window $[T - \Omega^{in}, T + \Omega^{ou}]$ around T (see Fig. 3.1.12, p. 58). ■

The retrospective point of view allows us to govern evolutions not only by differential inclusions $p'(t) \in M(t, p(t))$ depending on current time $t \in [T - \Omega^{in}, T]$ and position, but also, by *duration-chaperoned* (or duration-structured) differential inclusions $p'(t) \in M(t, t - (T - \Omega), p(t))$ depending on current time $t \in [T - \Omega, T]$, *duration* $o^{in}(t) = t - (T - \Omega)$ and position $p(t)$. It costs nothing mathematically to add those duration functions. The difficulties would be the same if we restricted our attention only to differential inclusions $p'(t) \in M(p(t))$. However, we need time and duration for defining travel. The same is true for outward evolutions in a prospective framework.

Remark **Population Dynamics**—In mathematical demography and population dynamics, McKendrick and Hamilton-Jacobi-McKendrick partial differential equations provide the state of a population depending on chronological time T and on age (they are "structured" by age). "Age" is an example of duration[10] and the evolutionary systems involving both time and age are examples of "duration-chaperoned" McKendrick partial differential equations.

Those partial differential equation are closely related to evolutions governed by differential inclusions of the form

$$\forall\, (T, \Omega) \in \mathbb{R} \times \mathbb{R}_+, \quad \forall\, t \in [T - \Omega, T], \quad p'(t) = F(t, t - (T - \Omega), p(t)) \tag{2.5}$$

The age-structured standard approach started with the establishment of the McKendrick partial differential equation relating the population and its partial derivatives with respect to time and age. Age-structured partial differential equations have been extensively studied (see, among an abundant literature, [8, Anita], [22, Aubin], [153, Iannelli], [160, Keyfitz N. and Keyfitz B.], [211, Von Foerster], [214, Webb], etc.). ■

Remark **Prediction and Retroprediction** Let us consider an evolutionary system governing evolutions and single-out a property that evolutions should enjoy on a given temporal window. For instance, staying on the road during travel (called viability) is the minimal requirement. Can we predict that this property would be satisfied by at

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[10] Age is defined on prospective temporal windows since we know only the date of birth.

least one evolution, some or all evolutions in the future? Can we "retropredict" that one, some or all of them satisfied this property in the past?

1. *Retroprediction* of a given "evolutionary property" by an evolutionary system requires *analysis* in the past. This means that at given *chronological time* T, this property should be satisfied on retrospective temporal windows $[T - \Omega^{in}, T]$ for a subset of inward durations Ω^{in}. This subset is a qualitative measure of the validity of the retroprediction (the larger this set, the more valid the retroprediction).
2. *Prediction* of a given "evolutionary property" of evolutions governed by an evolutionary system requires *experimentation*. This means that for a *given outward duration*, the property is satisfied on a window $[T, T + \Omega^{ou}]$ for some subset of *chronological times* T at which the experimentation is conducted. This subset is a qualitative measure of the validity of the prediction.

Predictive systems have been adopted in physics since the founding fathers: *Fermat*, *Newton*, *Leibnitz*, *Maupertuis*, etc., shared the belief that the laws of mechanics are deterministic and, consequently, predictive whenever the sensitivity to initial conditions is tamed. This is the reason why prospective temporal windows were privileged in mathematics motivated by physical sciences. This is not the case with the life sciences: they cannot be predicted, by lack of experimentation. However, they can be analyzed. ■

These new features have been motivated by several problems in various fields: traffic congestion (where the duration is the travel time), economic dynamics, population dynamics, collision problems (where the duration is time until collision), dynamical games for intercepting an evader by a pursuer (see [32, Aubin, Chen Luxi and Désilles] and Sect. 2.4.4, p. 35), and, above all, in life sciences: *The past may be studied and known, the future cannot be known and only predicted*.

Since we have chosen to privilege the retrospective temporal windows, evolutions and systems, and to alleviate the notations, *we shall drop the exponent* ^{in}, *setting for example* $\Omega := \Omega^{in}$, *and use the exponent*^{ou} *when we deal with prospective systems*.

2.2 Traveling Needs Space

We begin by distinguishing the network and the mobiles which circulate on a network.[11] For instance, vehicles on road networks are mobiles evolving along the roads

[11] In Chinese, *"Wang(3)"* 网, used for instance to designate the WEB, is an evolution of the sinogram  drawn on tortoise shells, meaning *net* to catch fish or capture of birds.

through the cross-roads, where they may be required to stop, ions Na^+, K^+, etc., are crossing ionic pumps of the membranes for propagating the nervous influx along neurons until they release or destroy neurotransmitters in the synapses[12] of a neural network, hormones[13] circulate in endocrine systems between endocrinal glands and receptors, money transfers propagate in economic-financial networks of banks and their exchanges are settled in the clearing houses, information is propagated in a network of computers devices until bits[14] are transferred in their computer nodes, etc.

Although these general networks motivated the mathematical results described in this book, this study is primarily motivated by road networks[15] or transportation networks,[16] and, in a less extent by neural networks[17] and economic-financial networks.[18] The following pages concentrate on road systems.

However, we propose in Sect. 3.5, p. 84, *Synaptic Networks of Ionic Networks*, we suggest that the links between ionic pumps of a neuron and between synapses can use the mathematical metaphors of junctions and intermodal traffic systems studied in Sect. 3.4.2, p. 77.

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[12]Synapses are "junction" in Greek, which join the axon of a "pre-synaptic" neuron and the dendrite of a "post-synaptic". They have been discovered in 1897 by *Charles Sherrington*.

[13]From the Greek *horme*, meaning *impulse*, harmones have been coined in 1905 by *Ernest Starling*, who isolated them in 1902 together with *William Bayliss*.

[14]A bit (contraction of "binary digit" proposed by *John W. Tukey* in 1947 and next popularized by *Claude E. Shannon* in 1948) is transmitted by serial or parallel transmission one at a time in computing devices. The encoding of data by discrete bits was used in Bacon's cipher (1626), in the punched cards invented by Basile Bouchon and Jean-Baptiste Falcon (1732) and developed by Joseph Marie Jacquard (1804), the beginning of a long history.

[15]See [30, Aubin, Bayen and Saint-Pierre], [21, 23, Aubin], [77, Chen Luxi], [32, 33, Aubin and Chen Luxi]. We also refer to [48, Aubin and Martin] for a "microscopic" analysis of traffic management studying the evolution of mobiles on a network in the framework of a direct approach without using celerities.

[16]The word "transport" is polysemous, particularly in mathematics: it is used in many different perspectives, such as the Fokker-Plank partial differential equations, involving drift and diffusion. This is not the meaning which we use in this study. Neither do we address the optimal network and transport problems, which have been the topic of an extensive literature. For instance, see *Network flows and monotropic optimization*, [202, Rockafellar], by *Terry Rockafellar*. In the Monge-Kantorovitch perspective, we refer to the monographs *Optimal Transport: Old and New*, [209, Villani] and *Optimal Transportation and Applications*, [210, Villani], by *Cédric Villani*. The mean field approach can also be used (see *Mean Field Games and Mean Field Type Control Theory Series*, [61, Bensoussan, Frehse and Phillip]). Therefore, we shall use only the word "traffic" in this book to avoid confusion.

[17]See for instance, among a myriad of references, *The Handbook of Brain Theory and Neural Networks*, [9, Arbib], by *Michael Arbib*, [68, Burnod], [193, Pakdaman], *Neural Networks and Qualitative Physics: a Viability Approach*, [16, Aubin], etc.

[18]See for instance *Rama Cont*, [86, Cont R., Moussa A. and Santos], analyzing the potential for contagion and systemic risk in a network of interlinked financial institutions, [175, Lehalle and Laruelle] and [17, 19, Aubin], in a connectionist perspective.

Road traffic information exists at least since the "bematists" (from the Greek *bema*, single pace) who were specialists in ancient Greece who measured distances by counting their steps. They were, so to speak, the first "odometers" (from the Greek *hodòs*, "path").

Those who accompanied *Alexander the Great* on his campaign in Asia measured the distances with an astonishing degree of precision. With milestones, travel duration could be estimated from the information on distances, a *spatial metaphor of time*,[19] actually, duration.[20] The first mechanical odometer known may be the Greek *Antikythera mechanism*, made around 100 BC, discovered in a shipwreck. However, their invention as crank mechanism counting pebbles at each turn of the wheel, as clepsydra for counting time, goes back to *Archimedes of Syracuse*. The milestones appeared on the Appian Way and were an important part of the Roman road network. Spatial origins were invented, such as the lost "golden milestone" at the center of Rome since all roads lead to Rome! They were used for deducing travel durations between two positions. At the time, only the position was indicated, the velocity not being an issue yet!

Nowadays, road traffic regulation, as rudimentary and frustrating as it is, is known to everyone: they are provided by speed limit signs (an upper bound of the velocity used by the mobiles passing by), or more coercive information, as traffic lights or signals, imposing a speed limit equal to 0 during a given time-interval, or other signs providing qualitative advices, etc.

2.3 Celerity Regulators for Advising Viable Velocities

2.3.1 The Double Nature of Traffic Networks

Definition 2.3.1 (*Traffic Networks*) A *traffic network* is defined in this book by

1. a subset $K := \bigcup_{i \in \mathbb{I}} K_i \subset \mathbb{R}^{|p|}$ of *positions* $p \in K$, which is the union of "sections" K_i labeled by $i \in \mathbb{I}$;

[19] In his 1960's study of the *Yamonamö*, still stone age tribes at this time, *Napoleon Chagnon* reports in [76, *Noble Savages*] that the distance between two places is measured by durations (number of "sleeps" during a trip), a *time metaphor of space!* The Yanomamö had no word for numbers beyond 2 and had to sit down to estimate a long distance using their toes.

[20] Distance and time where later used together to derive the concept of velocity.

2. *"junctions"* $\partial_{\mathbb{J}} K := \bigcap_{j \in \mathbb{J}} K_j$ separating sections, where $j \in \mathbb{J} \subset \mathbb{I}$ are labels of some roads defining a junctions. Junctions are unions $\partial_{\mathbb{J}} K := \partial_{\overleftarrow{\mathbb{J}}} K \cup \partial_{\overrightarrow{\mathbb{J}}} K$ of two disjoint *prejunctions* and *postjunctions* (see Sect. 3.4.2, p. 77).

Some junctions $\partial_{\mathbb{J}} K$ may be empty, and, for each nonempty junction $\partial_{\mathbb{J}} K$, the prejunction $\partial_{\overleftarrow{\mathbb{J}}} K$ or postjunction $\partial_{\overrightarrow{\mathbb{J}}} K$ may be empty (arrival and departure junctions respectively). More general types of junctions will be studied in Sect. 3.4, p. 75.

Definition 2.3.2 (*Viable Evolutions*) We regard a subset $K \subset \mathbb{R}^{|p|}$ of positions as a network. An evolution $p(\cdot) : \mathbb{R} \mapsto \mathbb{R}^{|p|}$ is said to be *viable* in K on a temporal window if $p(t)$ belongs to K for all instants of this temporal window.

This book addresses two types of questions:

1. ***When and from where*** a mobile can depart to arrive at an arrival time T at an arrival position $p \in K$ while duration $\Omega \geq 0$?
   We shall build and compute for this purpose *Cournot maps* associating with each arrival time, duration and position the (possibly empty) subset of departing time and positions from which mobiles can evolve on the network to reach the prescribed arrival time and position for the prescribed duration.
2. ***How*** the evolution of any mobile can be governed to arrive at time T at a position $p \in K$ from an initial position $p(T - \Omega) \in K$ at initial time $T - \Omega$?
   We shall build and compute for this purpose *celerity regulators* advising celerities at each time and position to whatever mobile passing at that time and that position to reach the arrival requirements while staying on the network.

The celerity is a form of "macroscopic velocity" *attached to the network* at each time and each position advising the viable velocity that vehicles should follow for staying on the network and satisfy other traffic requirements, such as congestion, etc. It is not a "microscopic velocity" attached to an arbitrary vehicle. For instance, for road networks, only some hints are provided by speed limit signs[21] or traffic lights at junctions, or other signs informing the driver with adequate messages. The objective of this investigation is to provide this information *at each time and each position*.

[21] The first maximum speed limit was the 10 mph (16 km/h) limit introduced in 1861 in the first "locomotive act" by Great Britain.

Remark **Tracking Nominal Paths** Using celerity regulators providing at each time and position the velocity for at least one evolution viable on the road, *there is no need to plan a* nominal trajectory *linking the initial position to the arrival position*, because it will be "discovered" by solving its own differential inclusion (2.6), p. 22 (see [177, Liniger and Lygeros]).

It can then be used as another tool of *trajectory planning* by providing the trajectory of a viable evolution as a byproduct. The designing of nominal trajectory have been the topic of a vast literature and many operational types of software have been built to find them, using for instance sampling algorithms requiring collision detection tests of the network.

Once nominal paths[22] have been obtained, the next problem is to track them more or less precisely. Many *different trajectory tracking devices* (or *motion planning solver*) have been proposed.

It could then be tempting to use another trajectory tracking device for following an evolution provided by the regulation map $p'(t) \in R_K(t, p(t))$. However, the velocities provided may, when they are explicitly known, be *different* than the one provided by the celerity regulator, even though they track the same nominal trajectory $t \mapsto \{p(t)\}$, *so that the information on velocities provided by the celerity regulator may be lost* and will no longer respond to upstream requirements for which they have been designed. Indeed, by following the trajectory provided by a *different trajectory tracking device* $p'(t) \in T_{\{p\}}(t, p(t))$, its velocity $p'(t)$ *is not necessarily contained in the regulation map* $R_K(t, p(t)) \subset T_{\{p\}}(t, p(t))$ taking into account *other* information than staying on the road (see Theorem 4.3.2, p. 108). ■

The next problem is to provide a *single-valued celerity feedback* $r_K(t, x) \in R_K(t, x)$, so that the evolution of the vehicle is governed by the differential equation

$$p''(t) = g(t, p(t), r_K(t, p(t))) \tag{2.6}$$

in such a way that, automatically, $p(t) \in K$. In some sense, the celerity regulator R_K is "the mother of all single-valued celerity feedbacks" $r_K(t, p) \in R_K(t, p)$ regular enough for solutions to differential equation (2.6), p. 22, to exist (for instance, continuity or even less demanding regularity requirements).

> **1** *(**The Double Nature of a Traffic Network**) The question asked at each chapter of this book is twofold:*
>
> 1. *For whom and for which purposes (viability, congestion, fuel efficiency, environment damage control, road traffic safety indicators or many other "traffic specifications") are the celerity regulators designed and how are they computed?*

[22] A path, an orbit, whenever they mean a trajectory, is the *set of positions* $\{p(t)\}$ visited by an*evolution* $p(\cdot) : t \mapsto p(t)$ (its image $p([T - \Omega, T])$). The abuse of language identifying evolution and trajectory is often made, the context allowing to erase the polysemy.

2. *How are the celerity regulators used by the mobiles circulating in the network to regulate their evolution?*

The approach recommended in this book regards a traffic network both as the information system *providing the celerities $p'(t)$ and as the set of individual mobiles using the celerities to circulate on the network.*

Remark **Direct and Inverse Approaches**—The way we pose the first questions is an *inverse approach*. Once the Cournot map and the celerity regulator are built, the second question is a *direct approach* studying how individual mobiles must adapt to the traffic network.

In a nutshell, *the purpose of this study is to investigate both an "inverse approach" for providing celerities and a"direct approach" for using them.*

An abundant literature on traffic networks deals with the direct approach. The inverse approach constructs regulators (feedbacks, retroaction, etc.) piloting evolutions viable in an environment, here, a network.

2.3.2 The Viability Approach

However, on the mathematical side, one may observe that for a century, all first order partial differential equations using positions p among their state variables, conservation laws as well Hamilton-Jacobi equations, use in their characteristic system the differential equation $p'(t) = \textit{something}$. At first glance, this does not make sense. *Except if one regards p'(t) as an advised velocity, which is the point of this study.* We thus propose a *mathematical* framework for describing a traffic network as an *information provider*. For that purpose, we suggest a mathematical framework no longer based on partial differential equations, but on their characteristic systems of differential equations. Knowing it, one can define "viable capture basins" of targets viable in environments, some of them being graphs of (set-valued) maps which are solutions to conservation laws or epigraphs of functions which are solutions to Hamilton-Jacobi-Bellman equations. These "viability solutions" are taken in a weak or generalized sense of set-valued analysis.

For each example, the only task which remains to be done is to *translate* them in the framework and the language of traffic networks and to design the viability algorithms for computing them. This is the choice we made in this book: the traffic problem is set in Definition 3.2.2, p. 63, and its solution is described in Quotation 4, p. 64.

Hence, there are two parts in this study:

1. the construction of these capture basins for translating such or such problem and derive their properties valid for any kind of evolutionary systems[23] (see Sect. 4.2, p. 97);
2. the construction of traffic regulators, which is specific to each class of evolutionary systems. Here, we consider only evolutionary systems generated by *differential inclusions* and we use the tangential conditions provided by viability theorems and the viability algorithm for computing them (see Sect. 4.3, p. 108).

Intuitively, there is no problem to understand the concepts and the results concerning velocities and celerity by assuming that the evolutions are differentiable, knowing that we can overcome the lack of classical differentiability: derivatives can be extended to any set-valued maps (see Sect. 8.1, p. 211) and/or extended functions or tubes in a variety of ways, if the reader wants to know how this is done.

We do not need partial differential equations for constructing celerity regulators, although the “viability solutions” provided by the viable capture basins under the characteristic systems are generalized solutions of their partial differential equations. This is explained in *Viability Theory. New Directions*, [31, Aubin, Bayen and Saint-Pierre], as well as in many books and articles published since the beginning of the 1980’s on the links between viability theory and first-order partial differential equations (see [130, Frankowska] for a review.). Adaptations of this study at a high abstract level providing a unified point of view to potential specific problems (traffic engineering, neural network, economic and financial networks, etc.) is postponed to future investigations, since they require the contributions of their specialists. Their validation to questions in other fields remains to be done in deep collaboration with the experts. ■

Viability algorithms, a topic of Set-Valued Numerical Analysis, provide approximations of celerity regulators[24] we are looking for. Numerical results are available for road traffic-based models.[25]

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[23] Such as historical differential inclusions, or mutational equations, operational differential inclusions, etc.

[24] They are defined by their graphs (graphical approach to maps), which are the actual subsets delivered as “control boards” embedded in the vehicles or read by drivers on a (future) dedicated velocity regulator. The viability algorithms compute subsets, and thus control boards.

[25] We refer to [56, 57, Bayen, Claudel, Saint-Pierre], [81, 82, Claudel, Bayen], [112, Désilles], Chap. 14, p. 563, of *Viability Theory. New Directions*, [31, Aubin, Bayen and Saint-Pierre] for available numerical results. A software for computing Cournot maps is presently under investigation and implementation by *Anya Désilles*.

All maps and relations studied in this book are *"viable capture basins"* of an auxiliary target viable in an auxiliary environment under an auxiliary evolutionary system, *incoming maps and Cournot maps inherit all properties of viable capture basins*. The only task which remains to be done is to *translate* them in the framework and the language of traffic networks and to design the viability algorithms for computing them. This is the choice we made in this book.

2.3.3 New Specifications: Advised Accelerations and Jerks

Even though the velocities are perceived or advised by the celerity regulators, accelerations are needed for achieving them. Hence the celerity may or must depend not only on time, duration and position, but also on other dynamical features, such as the acceleration, the jerks, etc. For example, if we wish not only that the evolution of the vehicle remains viable on the network, but also satisfies speed limits, stops at traffic lights, etc., we need to add to the evolutions of the position $p(t)$ at least its celerity $p'(t)$, its (advised) acceleration $p''(t)$ and (advised) jerk $p'''(t)$.

Evolutions of positions $p(t)$, of their celerities $p'(t)$, their (advised) acceleration $p''(t)$ and their (advised) jerk $p'''(t)$ range over finite dimensional vector spaces of dimension $|p|$. They are respectively called (by an ancient abuse of language) spaces of traffic positions (or *displacements*) p, velocities p', accelerations p'' and jerks p''', although these vectors are no longer depending on time. They are four different vector spaces isomorphic to $\mathbb{R}^{|p|}$.

Since we deal with *advised* velocities, accelerations and jerks, we do not take into consideration the mass $m = 1$. Otherwise, in rational mechanics,[26] forces are equal to mp'' and their derivatives mp''' are called *yanks*.

However, *we can identify the space of accelerations (forces) to the dual of the space of velocities and the space of jerks (yanks) to the dual of the space of positions*. The bilinear forms $\langle p'', p' \rangle$ and $\langle p''', p \rangle$ are interpreted as *mechanical power* (for vehicles of mass one). The bilinear form $\langle p'', p \rangle$ has the dimension of a *mechanical work*.

---

[26]The concepts of velocity and acceleration have been defined through Leibniz differential calculus in 1698 and 1700 by *Pierre Varignon*, who coined the word acceleration. In mechanics, the third derivative $p'''(t)$ of the evolution $p(\cdot)$ is nicknamed the *jerk* (or *jolt*, *lurche*), the fourth the *jouce* or *snap*, the fifth the *crackle* and the sixth the *pop* (as least for one-dimensional systems). The first four derivatives of a force $f(t) := m(t)p''(t)$ have been called *yanks*, *tugs*, *snatches* and *shakes*. The jerk tames in some sense the speed for comfort or even safety of the driver and its passenger by avoiding brutal accelerations and decelerations. They need time to adapt to acceleration and deceleration, but to their variations. Acceleration and jerk thresholds are needed to maintain their physiological viability of the passengers by protecting them from "jerky motion".

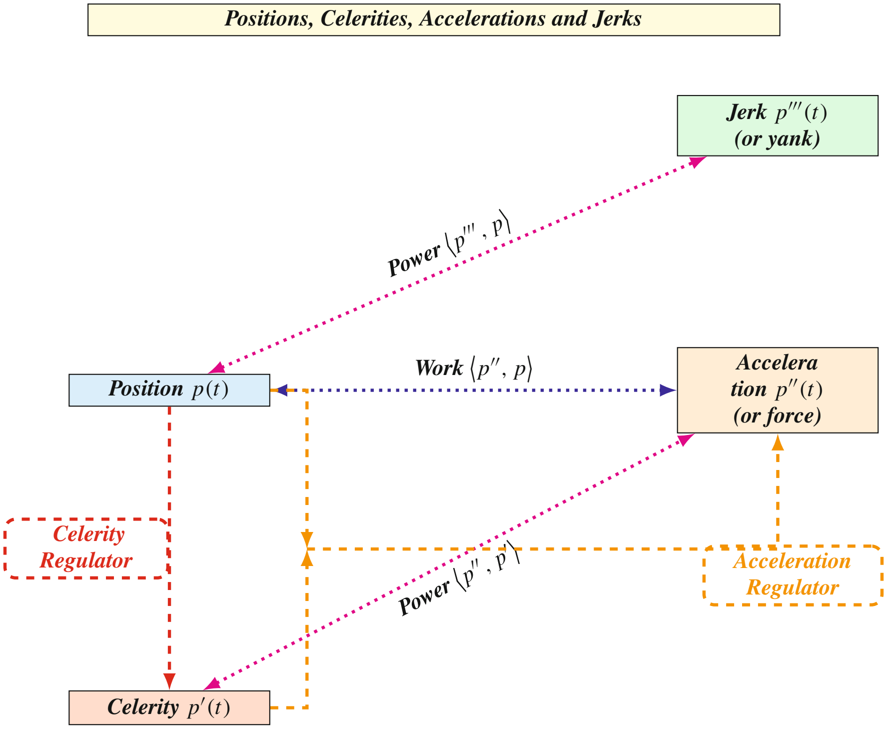


Figure 2.3.3 *[**Positions, Celerities, Accelerations and Jerks**] This diagram displays the four finite dimensional vector spaces isomorphic to $\mathbb{R}^{|p|}$ on which evolve the traffic position, the celerity, the acceleration and the jerk respectively. This is the reason why we regard them as spaces of positions, celerities, acceleration and jerks.*
Dotted lines symbolize the power between accelerations and celerities and the power between jerks and positions one hand, and the work between celerities and positions on the other one

Vectors of $\mathbb{R}^{|p|}$ (for $p > 1$) are more difficult to perceive by human brains than numbers, so that it may be beneficial to use numerical functions on vector spaces to summarize and grasp these vectors. They can be any function $V : p \mapsto V(p)$, and, and among them, linear forms. For our problems dealing with transport, two of these indicators are quite useful:

1. the *speed* $u(t) := \|p'(t)\|$, the norm of the velocity;
2. the *kinetic energy*[27] $e(t) = \dfrac{1}{2}\|p'(t)\|^2$.

[27]Without going back to Aristotle, kinetic energy (from the Greek *energeia* "force in action" and *kinesis*, "movement"), was introduced by *Gottfried Leibniz* and *Johann Bernoulli*, under the

Speed limits $u(t) = \|p'(t)\| \leq c^\sharp$ are equivalent to energy limits $e(t) = \frac{1}{2}\|p'(t)\|^2 \leq e^\sharp := \frac{1}{2}\|c^\sharp\|^2$ (for unit mass).

We prefer to use kinetic energy since its first derivative $e'(t) = \pi(t) := \langle p'(t), p''(t)\rangle$ is an instantaneous power (for unit mass) and its second derivative is $e''(t) = \langle p'(t), p'''(t)\rangle + \|p''(t)\|^2$. They are simpler formulas than

1. the first derivative of the speed

$$u'(t) = \frac{\langle p'(t), p''(t)\rangle}{\|p'(t)\|} \tag{2.7}$$

2. the second derivative of the speed

$$u''(t) = \frac{\langle p'(t), p'''(t)\rangle + \|p''(t)\|^2}{\|p'(t)\|} - \frac{\langle p'(t), p''(t)\rangle^2}{\|p'(t)\|^3} \tag{2.8}$$

The first derivatives of speed[28] and energy involve celerities and accelerations and their second derivatives the jerks.

2.4 Reaching Unexpected Position in Two-Dimensional Networks

This section provides examples of Cournot maps displaying the initial positions for reaching an arrival position signaled by an alarm as well as the celerity provided by the celerity regulator to *unmanned automated guided vehicles*.

The real difficulty is to satisfy the first requirement that the evolution of vehicles is viable in the sense that it remains on the road.

2.4.1 Reaching a Target at Alarm Signals

The inputs are:

(Footnote 27 continued)
name of the living force, *"vis viva"* or *"potentia motrix"*. *Émilie du Châtelet* published in 1741 a small book entitled *Réponse de Madame la Marquise du Chastelet, a la lettre que M. de Mairan lui a écrite sur la question des forces vives* in which she explained *Gravesande*'s experiment, by dropping brass marbles in soft clay from different heights to settle the *vis viva dispute*. Together with *Christine de Pisan* and *Olympe de Gouges*, *Émilie du Châtelet* figures among the exceptional learned women of pre-revolutionary French History (see *La Dame d'Esprit: a Biography of the Marquise Du Châtelet*, [218, Zinsser] and *Emilie du Chatelet: Daring Genius of the Enlightenment*, [219, Zinsser], among recent biographies in English.).

[28] We could use the words *haste* for the first derivative $u'(t)$ of the speed and *surge* for its second derivative.

1. The network, which is the one described by Fig. 7.3, p. 180;
2. The *evolutionary system* providing the *celerities* $p'(t) \in M(t, p(t))$, the *velocities advised to* any *vehicle circulating on the network*, using their own dynamics depending on *time, position and velocity* describing celerity constraints

$$\forall\, t \in \mathbb{R}, \quad p'(t) \in M(t, p(t)) \tag{2.9}$$

where

$$M(t, p) := \{(u\cos(\varphi), u\sin(\varphi))\}_{u \in [0, c^{\sharp}(\varphi(t))] \text{ and } \varphi \in [-\varphi^{\flat}(t), \varphi^{\sharp}(t)]} \tag{2.10}$$

is controlled by

a. the norm $u := \|p'\|$ of the celerity (the advised "speed") satisfying the constraints $u(t) \in [0, c^{\sharp}(\varphi(t))]$ where $c^{\sharp}(\varphi)$ is an a priori speed limit depending on the direction (for simplicity of the notation, we do not assume that it depends on time);
b. the direction φ satisfying the constraints $\varphi(t) \in [-\varphi^{\flat}(t), \varphi^{\sharp}(t)]$ where $-\pi \le -\varphi^{\flat}(t) \le 0 \le \varphi^{\sharp}(t) \le +\pi$.

The Cournot software designed by *Anya Désilles* (see [112, 113, Désilles]) computes a celerity regulator $r_{(M,K)} : (t, m) \rightsquigarrow r_{(M,K)}(t, m)$. Recall that any vehicle governed by differential equation $m''(t) = g_m(t, m(t), m'(t))$ is regulated by

$$m''(t) = g_m(t, m(t), r_{(M,K)}(t, m(t))) \tag{2.11}$$

3. The departure tube:

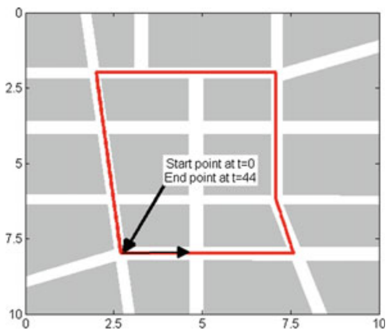


Figure 2.4.1 *[Departure Tube] The departure tube $D : t \mapsto D(t) := \{d(t)\} \in K$ is the trajectory of the monitoring round patrolled by any security vehicle (in red). The derivative of the departure map $d(\cdot)$ is denoted by $d'(t) = (v(t)\cos(\delta(t)), v(t)\sin(\delta(t)))$ where $v(t) := \|d'(t)\|$ is the "speed" and $\delta(t)$ the direction of the vehicle. The initial departure position is displayed, as well as the direction of circulation indicated by an arrow*

Once an alarm signal is received, any security vehicle must leave its round at an appropriate initial time to reach with minimal duration the position from where the alarm has been emitted (see Fig. 2.4.1).

The first task is to compute for all $T \geq t^\star$, the minimal duration $\Omega^\flat(T, p^\star)$ for reaching position $p^\star$ at time T by viable evolutions $t \mapsto p(t)$ governed by the system (2.9), p. 28.

The second task is to find the capture time $T^\star$ maximizing over T the initial date $T - \Omega^\flat(T, p^\star)$ and check whether the latest initial date $T^\star - \Omega^\flat(T^\star, p^\star) \geq t^\star$ when the patrolling vehicle must deviate from its round is posterior to the alarm date $t^\star$.

The software of the Cournot algorithm computes, first, the *celerity regulator* map $(t, p) \mapsto (u, \varphi) \in R_{(M,K)}(t, p)$ which controls the evolution of the vehicle by providing the controls $(u(t), \varphi(t)) \in R_{(M,K)}(t, p(t))$ describing the velocity and the direction allowing the car to remain on the roads of the network until it reaches at time $T^\star$ at the arrival position $p^\star$, and, second, computes the minimal Cournot duration $\Omega^\flat(T^\star, p^\star)$, the departure time $T - \Omega$ and the initial position $p(T - \Omega) = d(T - \Omega)$.

The two figures below illustrate how the Cournot map determines *when*, the departure time $T^\star - \Omega^\flat(T^\star, p^\star) \geq t^\star$ and *where*, position $d(T^\star - \Omega^\flat(T^\star, p^\star))$, the patrolling vehicle must leave its monitoring round to reach the position $p^\star$ signaled when the alarm was emitted at time $t^\star$.

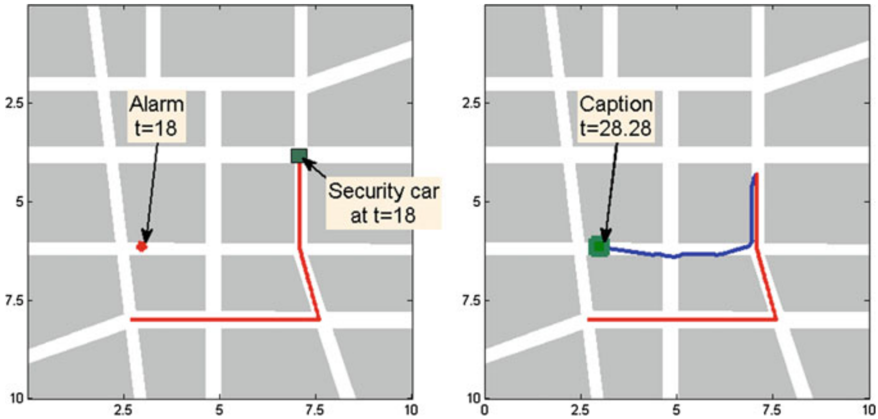
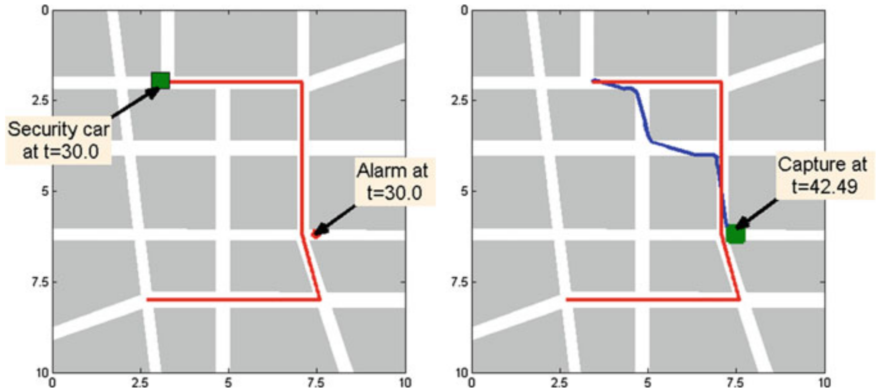


Figure 2.4.2 ***[Reachability of a Position]*** *The figures from the* left *describe the position* $p^\star$ (red bullet) *and the time* $t^\star$ *at which the alarm is emitted and the position* (green square) *of the security vehicle at this time* $t^\star$*. Only the trajectory of the patrolling vehicle* (in red) *after alarm time is displayed.*
The software can thus simulate when, where *and* how *a security vehicle must* deviate *from the monitoring round when at some time* $t^\star$ *an alarm signal is received from a position* $p^\star$ *to reach this position with the smallest duration: the figures from the right display the departure time* $T - \Omega^\flat(T, p^\star) \geq t^\star$*, and thus, the initial position* $d(T - \Omega^\flat(T, p^\star))$ *on the monitoring path, to follow the trajectory* (in blue) *to reach the position signaled by the alarm.*



2.4.2 Reaching a Target and Stopping at Alarm Signals

This first example when the only requirement asked to any vehicle passing by is to remain on the road may be insufficient. We may also prescribe, for instance, that the vehicles *stop at* position $p^\star$ at capture time $T^\star$ (for example, at stop signs and traffic lights). This requires to add to the position a new variable, regarded as a *specification*, to enrich the mathematical description of the problem (see Chap. 5, p. 123, devoted to the study of traffic networks with general specifications). In the second example, we choose for specification the speed $u(t) = \|p'(t)\|$ and demand that it vanishes at position $p^\star$ at time $T^\star$. This is done by "adding" one part of the control, u, to the position for transforming it as a new state, (p, u), while keeping the direction φ as the new control.

The new set of constraints is

$$\forall\, t \in \mathbb{R}, \quad (p(t), u(t)) \ \in \ K \times [0, c^\sharp(\varphi(t))] \tag{2.12}$$

and the new *control system*

$$\forall\, t \in \mathbb{R}, \quad (p'(t), u'(t)) \ \in \ M_1(t, p(t), u(t)) \tag{2.13}$$

where

$$\begin{cases} M_1(t, p, u) \ := \ M(t, p) \times [-\gamma^\flat(t), \gamma^\sharp(t)] \\ = \ \{((u\cos(\varphi), u\sin(\varphi)), \gamma)\}_{\gamma \in [-\gamma^\flat(t), \gamma^\sharp(t)]} \text{ and } _{\varphi \in [-\varphi^\flat(t), \varphi^\sharp(t)]} \end{cases} \tag{2.14}$$

is controlled by the derivative of the speed $\gamma \in [-\gamma^\flat(t), \gamma^\sharp(t)]$ and by the directions $\varphi \in [-\varphi^\flat(t), \varphi^\sharp(t)]$.

The Cournot algorithm provides a new celerity regulator $r^K_{M_1} : (T, \Omega, p, u) \mapsto (\gamma, \varphi) \in R^K_{M_1}(T, \Omega, p, u)$.

The new departure map is described by the function $t \mapsto d_1(t) = (d(t), \upsilon(t)) \in \mathbb{R}^2 \times [0, c^\sharp(\delta(t))]$ where $\upsilon(t) := \|d'(t)\|$.

In order to arrive *and stop* at $p^\star$, the new alarm is described by the pair $(p^\star, 0)$. Therefore, the Cournot map associate with any $(T, \Omega, p, 0)$ the subset $\mathbb{C}_1[D](T, \Omega, p, 0)$ of initial position-velocity pair $(p(T - \Omega), u(T - \Omega)) = (d(T - \Omega), \upsilon(T - \Omega))$ from which an evolution $t \mapsto (p(t), u(t))$ starts from $d(T - \Omega)$ with the velocity $\upsilon(T - \Omega) = \|d'(T - \Omega)\|$ arriving at $(p(^\star), 0)$ at time T.

Once such an alarm signal is received, any security vehicle leaves its round at an appropriate initial time to reach with minimal duration the position *and stop*. We next associate with any T the minimal duration $\Omega^\flat(T, p^\star, 0)$ needed for an evolution $t \mapsto (p(t), u(t))$ governed by the system (2.21), p. 34, to arrive at $p^\star$ at time T with a velocity 0 and maximize over the times $T \geq t^\star$ the departure dates $T - \Omega^\flat(T, p^\star, 0)$.

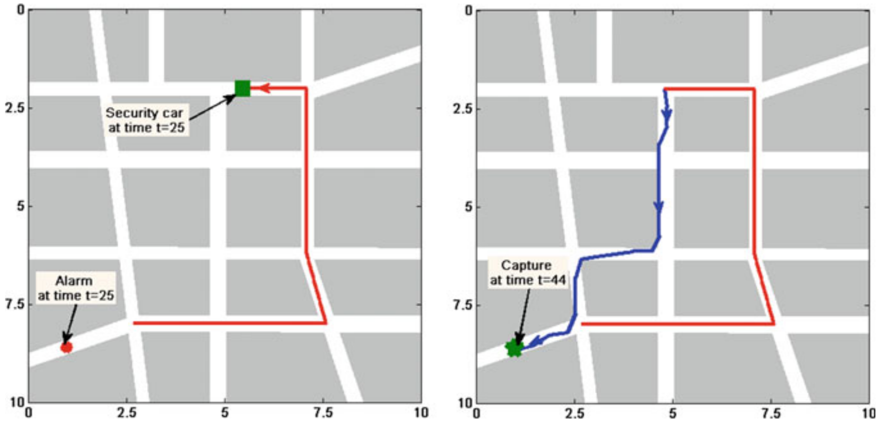
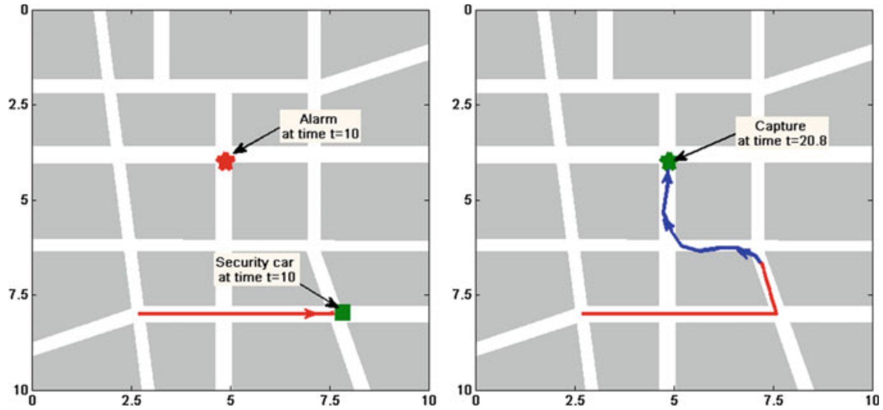


Figure 2.4.3 ***[Reaching and Stopping at a Position]*** *The figures on the left describe the position $p^\star$ (red bullet) and indicates the time $t^\star$ at which the alarm is emitted and the position of the security vehicle (green square) at this time. Only the trajectory of the patrolling vehicle (in red) after alarm time is displayed.*
The figures on the right display the initial time and thus, the initial position on the monitoring path, whenever a security vehicle must deviate to reach the target following the trajectory (in blue) arriving and stopping at a time $T^\star$ at the alarm position $p^\star$ (the green square locating the vehicle coincides with and hides the red bullet of the position to be reached).



The Cournot algorithm provides also the evolution of the speed $u(t)$ advised to the vehicles circulating on the network:

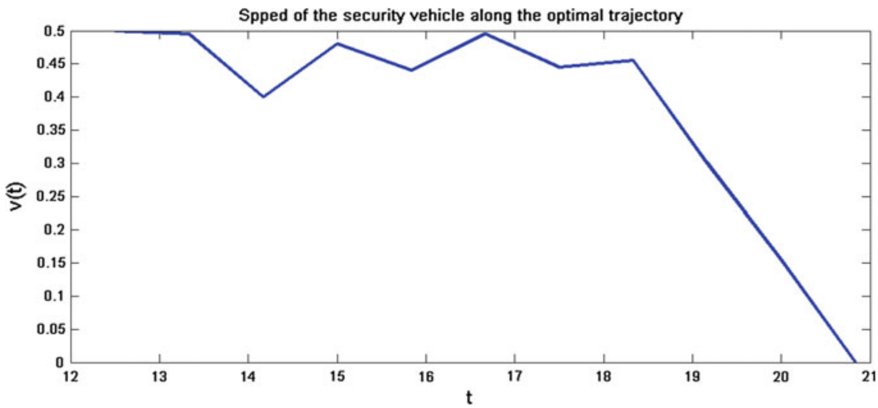


Figure 2.4.4 ***[Evolution of the Celerity]*** *This figure displays the graph of the evolution of the advised speed to the vehicle controlled by the derivative of the speed which vanishes at arrival time at arrival position of Fig. 2.4.3, p. 31*

We can add as well another specification in a third example, the direction φ, to exchange its role of control to the one of a second specification added to (p, u) for introducing a third state, (p, u, φ), controlled by the derivative of the speed and the angular velocity, as well as other specifications.

The new set of constraints is

$$\forall\, t \in \mathbb{R}, \quad (p(t), u(t), \varphi(t)) \;\in\; K \times [0, c^{\sharp}(\varphi(t))] \times [-\varphi^{\flat}(t), \varphi^{\sharp}(t)] \qquad (2.15)$$

and the new *control system*

$$\forall\, t \in \mathbb{R}, \quad (p'(t), u'(t), \varphi'(t)) \;\in\; M_2(t, p(t), u(t), \varphi(t)) \tag{2.16}$$

where[29]

$$\begin{cases} M_2(t, p, u, \varphi) \;:=\; M_1(t, p, u, \varphi) \times [-\omega^\flat(t), \omega^\sharp(t)] \\ = \;\{((u\cos(\varphi), u\sin(\varphi)), \gamma, \omega)\}_{\gamma \,\in\, [-\gamma^\flat(t), \gamma^\sharp(t)] \text{ and } \omega \,\in\, [-\omega^\flat(t), \omega^\sharp(t)]} \end{cases} \tag{2.18}$$

is controlled by the derivative $\gamma \in [-\gamma^\flat(t), \gamma^\sharp(t)]$ of the speed and by the angular velocity $\omega \in [-\omega^\flat(t), \omega^\sharp(t)]$.

Since $d'(t) = (v(t)\cos(\delta(t)); v(t)\sin(\delta(t)))$, the new departure map is defined by $t \mapsto (p, u, \varphi) = d_2(t) := (d(t), v(t), \delta(t))$.

In order to arrive and stop at $p^\star$ at time T with a prescribed direction $\varphi^\star$, the new alarm is described by the tripe $(p^\star, 0, \varphi^\star)$. Once such an alarm signal is received, any security vehicle leaves its round at an appropriate initial time to reach with minimal duration the position and stop, this time, with the specific direction. We associate with any T the minimal duration $\Omega^\flat(T, p^\star, 0, \varphi^\star)$ for a viable evolution $t \mapsto (p(t), u(t), \varphi(t))$ governed by the system (2.18), p. 33, to arrive at $p^\star$ with a velocity 0 and prescribed direction $\varphi^\star$ at time T. Next, we maximize over T the initial date $T - \Omega^\flat(T, p^\star, 0, \varphi^\star)$ and check wether it is later than the alarm date $t^\star$ (see Fig. 2.4.4).

Naturally, adding specifications[30] increases the state space and the dimension of the grids, requiring more computation time, more memory and dynamic memory management and thus, more powerful computers. Still, the mathematics and the algorithms remain the same.

---

[29] This dynamical system has one equation more than the one of the *Dubins' car* introduced in 1957 by *Lester Dubins*. Its system assumes that the speed u is constant, so that he was not preoccupied by decelerating or accelerating issues:

$$\forall\, t \in \mathbb{R}, \quad (p'(t), \varphi'(t)) \;\in\; \{((u\cos(\varphi), u\sin(\varphi)), \omega)\}_{\omega \,\in\, [-\omega^\flat(t), \omega^\sharp(t)]} \tag{2.17}$$

In this pioneering work, this probabilist studied the shortest trajectory connecting two positions with a constraint on the curvature of the path and with prescribed initial and terminal tangents to the path, but without road constraints. He proved that this trajectory is a concatenation of three strategies: denoting by S going straight, R, turning right with maximum curvature and L turning left, and thus, six combinations of them. See Sect. 3.1, p. 105, of *Viability Theory. New Directions*, [31, Aubin, Bayen and Saint-Pierre], for examples in the case of road constraints for reaching a known target.

[30] See for instance [212, Vandanjon, Coiret and Lorino] for governing viable cornering manoeuvres.

2.4.3 Starting from Junctions

We may assume that each junction p_j of the network (cross-road, in this example) is equipped with traffic lights for regulating the circulation. With each cross road p_j, we associate $h^j > 0$ and the increasing sequence of times $\cdots < t_j^n < t_j^{n+1} < \cdots$ where $t_j^{n+1} =: t_j^n + h_j$.

We denote by $\alpha_j \in]0,1[$ the duration of the red light of the traffic light at junction p_j and we associate with any junction p_j the set-valued map $t \mapsto D_j(t) \subset K \times [0, c_j^\sharp(t)]$ where

$$\Delta_j(t) = \begin{cases} \{p_j\} \times \{0\} & \text{if } t \in [t_j^n, t_j^{n+1}[\\ \{p_j\} \times [0, c_j^\sharp(t)] & \text{if } t \in [t_j^n + \alpha_j h_j, t_j^{n+1}[\end{cases} \tag{2.19}$$

describing the operating system of the traffic lights, red when $t_j^n \le T - \Omega < t_j^n + \alpha_j h_j$, and green when $t_j^n + \alpha_j h_j \le T - \Omega < t_j^{n+1}$.

The *control system*

$$\forall\, t \in \mathbb{R}, \quad (p'(t), u'(t)) \in M_1(t, p(t), u(t)) \tag{2.20}$$

where

$$M_1(t, p, u) := \{(u\cos(\varphi), u\sin(\varphi)), \gamma\}_{\gamma \in [-\gamma^\flat(t), \gamma^\sharp(t)] \text{ and } \varphi \in [-\varphi^\flat(t), \varphi^\sharp(t)]} \tag{2.21}$$

is controlled by the derivative of the speed $\gamma \in [-\gamma^\flat(t), \gamma^\sharp(t)]$ and by the directions $\varphi \in [-\varphi^\flat(t), \varphi^\sharp(t)]$.

The departure map $t \rightsquigarrow D_1(t) \subset K \times \mathbb{R}_+$ is defined by

$$D_1(t) := \begin{cases} \Delta_j(t) \text{ if } \exists\, j \text{ such that } p = p_j \\ \emptyset \qquad\qquad \text{otherwise} \end{cases} \tag{2.22}$$

Then the Cournot map associates with any (T, Ω, p, u) the subset $\mathbb{C}_1[D](T, \Omega, p, u)$ of initial position-velocity pair $(p(T - \Omega), u(T - \Omega)) \in D_1(T - \Omega)$ from which the vehicle starts from $p(T - \Omega) = p_j$ for some j, stopping at p_j if for some n, $t_j^n \le T - \Omega < t_j^n + \alpha_j h_j$, and passing by without stopping if $t_j^n + \alpha_j h_j \le T - \Omega < t_j^{n+1}$.

This Cournot map generates a time-duration dependent oriented graph on the network with junctions, where the vertices are the junctions of the network and the edges are the Cournot maps

$$N(T, \Omega, p_j, p_k) := \mathbb{C}_1[\Delta_j](T, \Omega, p_k, \upsilon_k) \tag{2.23}$$

linking positions p_j to p_k between departure time $T - \Omega$ and T when the arrival velocity is υ_k (most often taken equal to 0).[31]

2.4.4 Application: Pursuer-Invader Dynamical Games

These examples of arrival and Cournot maps could be of some use in the context of pursuer-invader dynamical games[32] (see [32, Aubin, Chen Luxi and Désilles] for more details).

Here, *we consider only the behavior of the pursuer*, *who knows* his viability constraints, the control system governing the evolutions *he can pilot* and his departure tube. The pursuer thus computes off-line its Cournot map and the celerity regulator.

The pursuer *does know none of these data about the invader*. He can only *detect* or *forecast* when and where the invader can arrive at a future time and position. Then, he computes his Cournot map for finding an initial position from which he can pilot an evolution capturing the invader and the celerity regulator.

Therefore, the pursuer has to devise a forecast mechanism for predicting *when, where and whether* the state x of the invader will be at time T in the domain of the Cournot map. For instance, this evolution can be extrapolated from the knowledge of the evolution on an adequate interval. Or, knowing the constraints, departure state and the dynamics of the invader, computes the evolutions governed by the invader's dynamics specifying whether the evolution of the invader will enter the arrival states.

1. This point of view allows the pursuer to overcome *some uncertainty*, by assuming that the invader may enter several forecast arrival time-position pairs in the detection set. Therefore, he can trigger as many Cournot evolutions capturing them;
2. The pursuer may also correct later the evolution whenever the pursuer
    a. has already governed an evolution for intercepting the previous forecast;
    b. makes a new forecast of the time-position of the invader;
    c. uses the trajectory he is following as a new departure map from which he may deviate to reach the newly forecast time-position state of the invader to correct it for taking into account this new information for capturing new state of the invader.

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[31] See Proposition 3.3.2, p. 74, for general concatenations of Cournot maps and Sect. 4.4, p. 112, for other example of departure maps.

[32] The literature on differential games is so abundant that it is impossible to quote all the contributions, which figure, for instance, in the recent proceedings, *Advances in Dynamic Games: Theory, Applications, and Numerical Methods for Differential and Stochastic Games*, [71, Cardaliaguet and Cressman], *Games and dynamic games*, [147, Haurie, Krawczyk and Zaccour]. However, viability techniques have been introduced in Chap. 14 of *Viability Theory*, [13, Aubin], [72, Cardaliaguet and Plaskacz], [73, 74, Cardaliaguet, Quincampoix and Saint-Pierre] among many other articles.

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