

# Semantics of the Probabilistic Typed Lambda Calculus

Dirk Draheim

# Semantics of the Probabilistic Typed Lambda Calculus

Markov Chain Semantics, Termination  
Behavior, and Denotational Semantics

 Springer

Dirk Draheim  
Large-Scale Systems Group  
Tallinn University of Technology  
Tallinn, Estonia

ISBN 978-3-642-55197-0      ISBN 978-3-642-55198-7 (eBook)  
DOI 10.1007/978-3-642-55198-7

Library of Congress Control Number: 2017932370

© Springer-Verlag Berlin Heidelberg 2017

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, express or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Printed on acid-free paper

This Springer imprint is published by Springer Nature  
The registered company is Springer-Verlag GmbH Germany  
The registered company address is: Heidelberger Platz 3, 14197 Berlin, Germany

---

## Preface

Today's information systems operate in probabilistic environments. Programs need to react to probabilistic events. Therefore, a rigorous understanding of probabilistic program behavior becomes ever more important. Probabilistic programming is relevant in its own right, as a means to implement randomized algorithms. This book takes a foundational approach to the semantics of probabilistic programming. It deals with the probabilistic typed lambda calculus, which is the typed lambda calculus with recursion plus probabilistic choice.

We elaborate a Markov chain semantics for the probabilistic lambda calculus. As part of this operational semantics, we define a reduction semantics and an evaluation semantics in terms of Markov chain hitting probabilities. The Markov chain semantics unlocks probability theory and Markov chain theory to be used in reasoning about probabilistic programs. Also, we introduce the notions of reduction graphs and reduction trees. Reduction graphs and reduction trees are not part of but rather accompany the Markov chain semantics. They unlock results from graph theory. These prove useful, e.g., in reasoning about termination behavior. On the basis of this, we investigate the termination behavior of probabilistic programs. We introduce the notions of termination degree, bounded termination and path stoppability and investigate their mutual relationships. Path stoppability characterizes a broadened class of termination and allows for the computation of program runs that are otherwise considered as non-terminating.

Furthermore, we elaborate a denotational semantics for the probabilistic lambda calculus. The domains of this denotational semantics are probabilistic pre-distributions as base domains and  $\omega$ -continuous function spaces as higher-type domains. We show the basic correspondence between the denotational semantics and the established Markov chain semantics.

---

# Contents

<b>1</b>	<b>Introduction</b>	1
1.1	Motivation	3
1.2	The Probabilistic Lambda Calculus	5
1.3	Termination Behavior of Probabilistic Programs	9
1.4	Denotational Semantics	11
1.5	Chapter Outline and Further Remarks	14
<b>2</b>	<b>Preliminary Mathematics</b>	17
2.1	Probability Theory	18
2.2	Markov Chains	23
2.3	Graph Theory	33
2.4	Inductive Definitions	39
2.5	Miscellaneous	59
<b>3</b>	<b>Syntax and Operational Semantics</b>	65
3.1	Syntax of the Probabilistic Lambda Calculus	66
3.2	Operational Semantics of the Typed $\lambda$ -Calculus	73
3.3	The Probabilistic Operational Semantics	76
3.4	Important Readings	91
<b>4</b>	<b>Termination Behavior</b>	93
4.1	Introductory Examples of Termination Behavior	96
4.2	Bounded and Unbounded Termination	101
4.3	Program Executions and Program Runs	103
4.4	The Reduction Graph	107
4.5	Central Graph Cover Lemmas	115
4.6	Path Stoppability	120
4.7	Program Reduction Trees	127
4.8	Characteristics of Bounded Termination	131

**5    Denotational Semantics** ..... 135

    5.1    Domains and Denotations ..... 136

    5.2    Well-Definedness of the Denotational Semantics ..... 145

    5.3    Semantic Correspondence..... 157

    5.4    Important Readings on Domains and Probabilism ..... 189

**References** ..... 193

**Index** ..... 211

Semantics of the Probabilistic Typed Lambda Calculus  
Markov Chain Semantics, Termination Behavior, and  
Denotational Semantics

Draheim, D.

2017, VIII, 218 p. 6 illus., Hardcover

ISBN: 978-3-642-55197-0