2 Auxiliary Material and Notation

This chapter introduces our notation conventions and gives a short overview over some results that will be used without references throughout the book.

Complex analysis. There are many books covering elementary complex analysis. Our standard reference is the book of Ahlfors [1].

A *domain* is an open connected set in $\mathbb{C}$. The domain we are dealing with most of the time is the *unit disc* $\{z \in \mathbb{C} |\, |z| < 1\}$ denoted by $\mathbb{D}$. Its boundary is the *unit circle* $\{z \in \mathbb{C} |\, |z| = 1\}$ and will be denoted by $\mathbb{T}$. Other discs will be denoted by $B_r(z^*) := \{z \in \mathbb{C} |\, |z - z^*| < r\}$. Given a circle $C \subset \mathbb{C}$, the disc bounded by C will be denoted by $\operatorname{int} C$, while $\operatorname{ext} C = \mathbb{C} \setminus (C \cup \operatorname{int} C)$.

The boundary of a domain U is denoted by ∂U and the closure $U \cup \partial U$ is denoted by $\overline{U}$. Some caution is required here since the complex conjugate of a complex number z is denoted by $\overline{z}$.

By Log we denote the principal branch of the complex logarithm defined on the slit domain $\mathbb{C}^- := \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. By Arg we denote the principal branch of the complex argument function, i.e. $\operatorname{Arg}(z) = \operatorname{Im} \operatorname{Log}(z)$. log resp. arg denotes any branch of these functions if it is not necessary (or possible) to specify the branch explicitly.

To keep the overview, we have the following convention for variables: z always refers to a variable ranging on $\mathbb{D}$ or sometimes $\mathbb{C}$, while t is a variable ranging on $\mathbb{T}$. Arguments of complex numbers are denoted by Greek letters, mostly τ, sometimes also ϕ or θ.

Möbius transformations. A Möbius transformation is a holomorphic function of the form

$$T(z) = \frac{az + b}{cz + d}$$

where a, b, c, d are complex constants such that $ad - bc \neq 0$. These functions play a special role in complex analysis due to their geometric properties. It is natural to understand them as mappings on the extended complex plane $\mathbb{C} \cup \{\infty\}$ by writing $T(\infty) = \frac{a}{c}$ and $T(-\frac{d}{c}) = \infty$ if $c \neq 0$ (or $T(\infty) = \infty$ if $c = 0$).

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Möbius transformations have the following properties (for proofs, see Chapter 3, Section 3 of [1]):

(i) T is a bijective map on the extended complex plane $\mathbb{C} \cup \{\infty\}$.

(ii) T is holomorphic on $\mathbb{C} \setminus \{T^{-1}(\infty)\}$ and its derivative vanishes nowhere.

(iii) If $C \subset \mathbb{C}$ is a *Möbius circle*, i.e. a circle or a line, $T(C)$ is a Möbius circle, too.[1]

(iv) T can be written as a composition of linear functions and the *inversion* $z \mapsto \frac{1}{z}$.

(v) Let $C \subset \mathbb{C}$ be a circle and $T^{-1}(\infty) \notin C$. Let $z_1, z_2, z_3 \in \mathbb{C}$ lie in counter-clockwise order on C. Then

- if $T^{-1}(\infty) \in \operatorname{ext} C$, then $T(\operatorname{int} C) = \operatorname{int} T(C)$, $T(\operatorname{ext} C) = \operatorname{ext} T(C)$, and $T(z_1), T(z_2), T(z_3)$ lie in counter-clockwise order on C again.
- if $T^{-1}(\infty) \in \operatorname{int} C$, then $T(\operatorname{int} C) = \operatorname{ext} T(C)$, $T(\operatorname{ext} C) = \operatorname{int} T(C)$, and $T(z_1), T(z_2), T(z_3)$ lie in clockwise order on C.

If $c = 0$, let $d = 1$ w.l.o.g. Then $T(z) = az + b$ is a linear function. If in addition $a \in \mathbb{T}$, then T is an isometry of $\mathbb{C}$ and is called a *plane rigid motion*. Hence, there are 3 real degrees of freedom to construct a plane rigid motion.

Conformal automorphisms. A function $f\colon \mathbb{D} \to \mathbb{D}$ is bijective and conformal iff it is a composition of a rotation $z \mapsto \lambda z$ for $\lambda \in \mathbb{T}$ and a Möbius transformation of the form

$$z \mapsto \frac{z - z_0}{1 - z\overline{z_0}}$$

for some $z_0 \in \mathbb{D}$. In particular, there are exactly three real degrees of freedom for constructing such a map. For a proof, see [1] (Chapter 4, 3.4).

Blaschke products. A (finite) Blaschke product of degree n is a function of the form

$$B(z) = c \prod_{j=1}^{n} \frac{z - z_j}{1 - \overline{z_j} z}$$

[1] In this context, a line should be thought of as a circle containing the point ∞. From this, it is clear that T maps a circle C to a circle again iff $T^{-1}(\infty) \notin C$.

for $c \in \mathbb{T}, z_1, \ldots, z_n \in \mathbb{D}$. The set of all finite Blaschke products is denoted by $\mathcal{B}$.

Note that $z_1, \ldots, z_n$ are precisely the zeros of B counted with multiplicity. The perhaps most important property of Blaschke products is that each $B \in \mathcal{B}$ satisfies

$$|B(t)| = 1 \; \forall t \in \mathbb{T}$$

Since all poles of B lie outside $\mathbb{D}$, B is bounded and continuous on $\overline{\mathbb{D}}$.

Blaschke products can be understood as "polynomials in the disc": A Blaschke product of degree n is a conformal self-map of $\mathbb{D}$ that attains each value in $\mathbb{D}$ exactly n times. For details, see [23].

As the term "finite" suggests, one can also consider *infinite Blaschke products* with an infinite number of zeros in $\mathbb{D}$. These functions are not relevant for this book, though.

Function spaces. For further details on the spaces introduced here, any textbook on functional analysis or measure theory can be consulted, e.g. [7]. A function $f\colon \mathbb{T} \to \mathbb{C}$ is said to be in the *Lebesgue space* $L^p(\mathbb{T})$ for $1 \leq p < \infty$ if its normalized L^p-norm

$$\|f\|_p := \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(e^{i\tau}) \right|^p \mathrm{d}\tau \right)^{\frac{1}{p}}$$

is smaller than ∞. f is said to be in $L^\infty(\mathbb{T})$ if it is bounded on $\mathbb{T} \setminus E$ for a set E with Lebesgue measure zero and the L^∞-norm is defined by

$$\|f\|_\infty := \inf\{C \in \mathbb{R} | \{t \in \mathbb{T} | f(t) \geq C\} \text{ has Lebesgue measure zero}\}.$$

For $1 \leq p \leq \infty$, $L^p(\mathbb{T})$ is a Banach space w.r.t. the L^p-norm. The space $L^2(\mathbb{T})$ is a Hilbert space endowed with the scalar product

$$(f, g)_2 := \left(\frac{1}{2\pi} \int_0^{2\pi} f(e^{i\tau}) \overline{g(e^{i\tau})} \mathrm{d}\tau \right)^{\frac{1}{2}}.$$

For a continuous function $f\colon \mathbb{T} \to \mathbb{C}$, we define its *modulus of continuity*

$$\omega_f(\theta) := \sup_{|\phi - \tau| < \theta} \left| f(e^{i\phi}) - f(e^{i\tau}) \right|.$$

f is said to be Hölder continuous with Hölder exponent α if its $\mathcal{C}^\alpha$-seminorm

$$|f|_{\mathcal{C}^\alpha} := \sup_{\theta\in[0,\pi]} \frac{\omega_f(\theta)}{\theta^\alpha}$$

is finite. For $k \in \mathbb{N}_0$ and $\alpha \in [0,1]$, we denote by $\mathcal{C}^{k+\alpha}(\mathbb{T})$ the set of all functions whose k-th derivative exists and is Hölder continuous with Hölder exponent α. Endowed with the $\mathcal{C}^{k+\alpha}$-norm

$$\|f\|_{\mathcal{C}^{k+\alpha}} := \sum_{j=0}^{k} \left\|f^{(j)}\right\|_\infty + \left|f^{(k)}\right|_{\mathcal{C}^\alpha},$$

these spaces are Banach spaces, too.

Matrices and vectors. Let $v \in \mathbb{R}^n$. If not explicitly stated otherwise, v_j always denotes the j-th component of v such that $v = (v_1, \ldots, v_n)^T$. We will frequently drop the transposition for vectors and just write $v = (v_1, \ldots, v_n)$.

Given a differentiable function $f\colon U \to \mathbb{R}^k$ for a domain $U \subset \mathbb{R}^n$, we denote its Jacobian evaluated at $x \in U$ by $\mathrm{D}f|_x \in \mathbb{R}^{k\times n}$. In context of matrices and vectors, the dot $\cdot$ always denotes the usual matrix product. For example, the chain rule of differentiation reads

$$\mathrm{D}(f \circ g)|_x = \mathrm{D}f|_{g(x)} \cdot \mathrm{D}g|_x$$

in our notation.

A diagonal matrix with diagonal entries $a_1, \ldots, a_n$ will be denoted by $\mathrm{diag}(a_1, \ldots, a_n)$. $I_n = \mathrm{diag}(1, \ldots, 1)$ denotes the $n \times n$ unit matrix. Furthermore, $\mathbb{1}_n = (1, \ldots, 1)^T \in \mathbb{R}^n$ denotes the vector where all entries are equal to 1.

Implicit Function Theorem. We will make use of the well-known Implicit Function Theorem. For convenience, we state the theorem here. The formulation we are using is a combination of the formulations in [17] (Theorem 5.4 in Chapter XVIII) and in [6] (Theorem 7.43).

Theorem (Implicit Function Theorem)**.** *Let $U \subset \mathbb{R}^p \times \mathbb{R}^n$ be open and $F\colon U \to \mathbb{R}^n$ continuously differentiable. Let $(x_0, y_0) \in \mathbb{R}^p \times \mathbb{R}^n$ such that $F(x_0, y_0) = 0$ and the matrix*

$$\mathrm{D}_y F|_{(x_0,y_0)}$$

is regular. Then there are open sets $V_1 \subset \mathbb{R}^p$ *and* $V_2 \subset \mathbb{R}^n$ *with* $x_0 \in V_1$ *and* $y_0 \in V_2$ *and a continuously differentiable function*

$$g\colon V_1 \to V_2$$

with $g(x_0) = y_0$ *such that the following holds:* $(x, y) \in V_1 \times V_2$ *satisfies* $F(x, y) = 0$ *if and only if* $g(x) = y$*. Moreover, the Jacobian of* g *at* x_0 *is given by*

$$\mathrm{D}g|_{x_0} = -\left(\mathrm{D}_y F|_{(x_0,y_0)}\right)^{-1} \cdot \mathrm{D}_x F|_{(x_0,y_0)}.$$

Manifolds. To avoid the technical machinery of general abstract differentiable manifolds, we confine ourselves with introducing submanifolds of $\mathbb{R}^n$. For details on manifolds and rigorous definitions of the termini introduced here, we refer to [16].

A *k-dimensional submanifold of* $\mathbb{R}^n$ is a set $M \subset \mathbb{R}^n$ which is locally given as the zero set of a function $F\colon \mathbb{R}^n \to \mathbb{R}^{n-k}$ with maximal rank. This means, given $x \in M$, there is an open set $U \subset \mathbb{R}^n$ with $x \in U$ and a function $F\colon U \to \mathbb{R}^{n-k}$ such that $M \cap U = F^{-1}(\{0\})$. DF has maximal rank at each point on $M \cap U$. The kernel of $DF|_x$ is the *tangent space* at x. This space is denoted by $T_x M$ and consists of all vectors which are tangent to the manifold at M.

By the Implicit Function Theorem, M can be parametrized locally: Given $x \in M$, there are $V \subset \mathbb{R}^k$ and $U \subset \mathbb{R}^n$ open with $x \in M$ and a function $g\colon V \to \mathbb{R}^n$ such that $g(V) = M \cap U$. The function g is called a *parametrization* of M.

Given $U \subset \mathbb{R}^n$ open and $M' := M \cap U$, a diffeomorphism $f\colon M' \to \mathbb{R}^k$ with full rank is called a *chart* on M.

We will mostly work with manifolds consisting of circle packings. In this context, an element of the manifold is usually denoted by $\mathcal{P}$ and a tangential vector is denoted by p.

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