

## Chapter 2

# Static Electric Field in Vacuum

**Abstract** This chapter introduces forces between charges at rest, which are not supposed to be inside any media (Coulomb's law). Concepts such as electric field or electric potential are introduced, as well as its calculus when produced by different charge distributions, including conductive materials. Gauss' law and its use to calculate electric field caused by certain charge distributions is also seen.

### 2.1 Electric Charge

Charge is a basic and characteristic property of the elementary particles which make up matter. There are two kinds of charges: positive and negative. Every portion of matter contains approximately equal amounts of each type. When speaking about charge, we are referring to the net sum of positive and negative. Therefore, when something is positively charged it is because the amount of positive charges (usually protons) is higher than the negative ones (usually electrons). The electric charge is found in multiples of the elementary charge  $e$  (electron or proton charge). It is an experimental fact that charge can be neither created nor destroyed. This is known as **the principle of conservation of charge**: for any process performed in an *isolated system*, net or total charge does not change or in a *non-isolated system* the charge introduced into a system is equal to its increase of charge. Charge is represented by  $q$  and its unit in International System (SI) is *coulomb* (C).<sup>1</sup> For continuous distributions of charge, and given the smallness of the elementary charge  $e$ , any small element of volume that we consider will be constituted by a large number of electrons and protons. Hence we can consider a charge density function as the limit of the charge per unit volume as the volume becomes infinitesimal, and the corresponding integration will allow us to obtain the overall charge of the object. It is defined as **volume charge density**  $\rho$  by

$$\rho = \lim_{\Delta V \rightarrow 0} \frac{\Delta q}{\Delta V} \equiv \frac{dq}{dV}, \quad (2.1)$$

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<sup>1</sup>Coulomb can be defined as a function of the elementary charge  $e$  as  $1 \text{ C} = 6.25 \times 10^{18} e$ . In later chapters, when magnetic experiments are discussed, it will be possible to define it as an ampere's derivative.

which represents the charge per unit volume at each point of a surface. The SI unit is  $\text{Cm}^{-3}$ . The overall charge  $q_V$  in the volume  $V$  is obtained as

$$q_V = \int_V \rho dV. \quad (2.2)$$

If the charge is distributed on one surface  $S$ , **surface charge density**  $\sigma$  can be defined as

$$\sigma = \lim_{\Delta S \rightarrow 0} \frac{\Delta q}{\Delta S} \equiv \frac{dq}{dS}, \quad (2.3)$$

which represents charge per unit area at each point of a surface. Its unit in SI is  $\text{Cm}^{-2}$ . Total charge  $q_S$  on  $S$  is obtained as

$$q_S = \int_S \sigma dS. \quad (2.4)$$

When charge is distributed on a material line  $L$ , **line charge density**  $\lambda$  can be defined as

$$\lambda = \lim_{\Delta l \rightarrow 0} \frac{\Delta q}{\Delta l} \equiv \frac{dq}{dl}, \quad (2.5)$$

that represents charge per unit of length at each point of the line. Its unit in SI is  $\text{Cm}^{-1}$ . Total charge  $q_L$  on  $L$  is obtained as:

$$q_L = \int_L \lambda dl. \quad (2.6)$$

## 2.2 Coulomb's Law

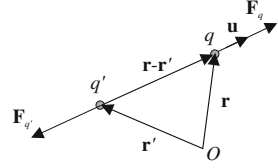
From several observations that took place in 18th century by Coulomb and others, it can be established that force between two electric charges at rest can be mathematically expressed by **Coulomb's law**:

$$\vec{F}_q = k \frac{qq'}{|\vec{r} - \vec{r}'|^2} \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|} = k \frac{qq'}{d^2} \mathbf{u}, \quad (2.7)$$

which states that forces between two point charges  $q$  and  $q'$  act along the line joining them, and are directly proportional to the product of these charges and inversely proportional to the square of the distance  $d = |\mathbf{r} - \mathbf{r}'|$  between them (Fig. 2.1).

Equation (2.7) express the force  $F_q$  which acts on  $q$  due to  $q'$ 's action.  $F_q$ 's direction is determined by the unitary vector  $\mathbf{u} = (\mathbf{r} - \mathbf{r}')/|\mathbf{r} - \mathbf{r}'|$ , oriented from  $q'$  to  $q$ , as well as charges' sign. Forces are repulsive if charges have the same sign, and attractive if they have opposite sign.  $\mathbf{F}_{q'}$  over  $q'$  due to  $q$  is the vector  $-\mathbf{F}_q$ . If  $(\mathbf{r} - \mathbf{r}')$  is replaced in (2.7) by  $(\mathbf{r}' - \mathbf{r})$ , Newton's third Law is obtained.

**Fig. 2.1** Forces between point charges



The value of the proportional constant  $k$  depends of the units system. In SI, it is

$$k = 10^{-7} c^2, \quad (2.8)$$

where  $c$  is the velocity of light in vacuum. It can also be written as

$$k = \frac{1}{4\pi\epsilon_0}, \quad (2.9)$$

where  $\epsilon_0$ , is the *permittivity of free space*. Therefore

$$k \approx 8.9875 \times 10^9 \text{ Nm}^2/\text{C}^2 \approx 9 \times 10^9 \text{ Nm}^2/\text{C}^2, \quad (2.10)$$

and

$$\epsilon_0 \approx 8.8542 \times 10^{-12} \text{ C}^2/(\text{Nm}^2). \quad (2.11)$$

When several  $n$  point charges  $q_j$  act on  $q$ , it's been experimentally established that the total force acting on a charge is the vector sum of the individual forces which act on it. This is known as **the superposition principle for electrostatic forces**. Therefore, the force is determined by the repeated application of (2.7):

$$\mathbf{F}_q = q \sum_{j=1}^n \frac{q_j}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}_j|^2} \frac{\mathbf{r} - \mathbf{r}_j}{|\mathbf{r} - \mathbf{r}_j|} = q \sum_{j=1}^n \frac{q_j}{4\pi\epsilon_0 d_j^2} \mathbf{u}_j, \quad (2.12)$$

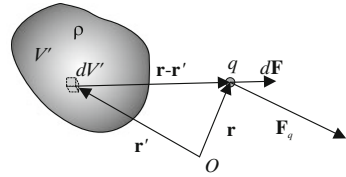
where  $d_j = |\mathbf{r} - \mathbf{r}_j|$  is the distance between the  $j$ -ith charge and  $q$ , and  $\mathbf{u}_j = (\mathbf{r} - \mathbf{r}_j)/|\mathbf{r} - \mathbf{r}_j|$  is the unitary vector in the direction from  $q_j$  to  $q$ .

The same principle can be applied in a continuous charge distribution case. If a very small volume  $dV'$  is considered at a point in the charge distribution, where the density is  $\rho$ , charge inside  $dV'$  is, according (2.1),  $dq' = \rho dV'$  (Fig. 2.2). If these values are substituted in (2.12) and the sum is substituted by an extended integral to the whole charge, it results:

$$\mathbf{F}_q = \frac{q}{4\pi\epsilon_0} \int_{V'} \frac{\rho}{|\mathbf{r} - \mathbf{r}'|^2} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} dV', \quad (2.13)$$

where  $\mathbf{r}$  is the point charge position and  $\mathbf{r}'$  is the position of each of the volume differentials. Figure 2.2 represents the force  $d\mathbf{F}$  of point element  $dq' = \rho dV'$  over

**Fig. 2.2** Force due to a continuous charge distribution



point charge  $q$ ;  $\mathbf{F}_q$ 's value comes from adding every element  $d\mathbf{F}$ . The same expression can be applied when a charge is distributed on a surface or on a line substituting  $dq' = \rho dV'$  by  $dq' = \sigma dS'$  or  $dq' = \lambda dl'$ , respectively. The integral in (2.13) is well behaved even in case  $q$  falls inside the charge distribution.

## 2.3 Electric Field

In (2.7), (2.12) and (2.13) it is observed that the force that acts on  $q$  is proportional to  $q$ . Therefore a vectorial field which is independent from  $q$  is introduced. Its dimensions are force per unit of charge. Hence **electric** or **electrostatic field**<sup>2</sup> can be defined as

$$\mathbf{E} = \lim_{q \rightarrow 0} \frac{\mathbf{F}}{q}, \quad (2.14)$$

where the test charge placed at the point goes to zero, so it can be assured that it does not affect the charge distribution which produces  $\mathbf{E}$ . The electric field unit in SI is  $\text{NC}^{-1}$ .

For a point charge, the expression of electric field is directly obtained from dividing by  $q$  in (2.7):

$$\mathbf{E}(\mathbf{r}) = \frac{q'}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^2} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} = \frac{q'}{4\pi\epsilon_0 d^2} \mathbf{u}. \quad (2.15)$$

When electric field's definition is applied to (2.12) and (2.13) a general equation can be obtained for the electric field due to a given distribution of charge at rest,

$$\begin{aligned} \mathbf{E}(\mathbf{r}) = & \frac{1}{4\pi\epsilon_0} \sum_{j=1}^n \frac{q_j}{|\mathbf{r} - \mathbf{r}_j|^2} \frac{\mathbf{r} - \mathbf{r}_j}{|\mathbf{r} - \mathbf{r}_j|} + \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} dV' + \\ & + \frac{1}{4\pi\epsilon_0} \int_{S'} \frac{\sigma(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} dS' + \frac{1}{4\pi\epsilon_0} \int_{L'} \frac{\lambda(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} dl'. \end{aligned} \quad (2.16)$$

<sup>2</sup>'Electrostatic field' is usually used when phenomena are time independent.

This is the mathematical expression of **superposition principle for electric field**: the electric field created by a number of charges is equal to the sum of the fields produced independently by each of them,<sup>3</sup> where the symbol  $\mathbf{r}$  represents the position vector of the point where the field is calculated (field point), and  $\mathbf{r}_j$  or  $\mathbf{r}'$  is the vector position of any of the charges or the charge differentials (source point).  $\mathbf{r} - \mathbf{r}'$  or  $\mathbf{r} - \mathbf{r}_j$  is the vector that goes from each of the source points to the field point, and its magnitude represents distance between them. The sum or integration is calculated over total charge: therefore the variable is not  $\mathbf{r}$ , but  $\mathbf{r}_j$  or  $\mathbf{r}'$ ; quantities  $\rho$ ,  $\sigma$  and  $\lambda$  can also be dependent on variables of position  $\mathbf{r}'$ .

It is not necessary to apply (2.2)'s formulas to calculate the force that acts on a charged particle  $q$  when introduced in a region where exists an electric field  $\mathbf{E}$ , but once  $\mathbf{E}$  is determined it is simple to see from (2.14) that:

$$\mathbf{F} = q\mathbf{E}. \quad (2.17)$$

## 2.4 Electrostatic Potential

It has been seen in Chap. 1 that if the field's curl is zero, then the vector can be expressed as the gradient of a scalar field. It's easy to demonstrate that  $\nabla \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = 0$ , and as every term of (2.16) corresponds to this form, we establish that the electrostatic field is irrotational, and therefore it is derived from a potential:

$$\nabla \times \mathbf{E}(\mathbf{r}) = 0 \Rightarrow \mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r}), \quad (2.18)$$

where  $V$  is called the **electrostatic potential**. It is important to note that if the phenomena were time-dependent, the electric field's curl would not be zero.

If we bear in mind the gradient's property  $dV = \nabla V \cdot d\mathbf{r}$  given by (1.39), and we apply it to (2.18) it results that:

$$V(\mathbf{r}) = - \int \mathbf{E}(\mathbf{r}) \cdot d\mathbf{r}. \quad (2.19)$$

It can be observed that potential represents potential energy per unit charge. So, if we remember potential energy of a conservative force is:

$$E_p(\mathbf{r}) = - \int \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}, \quad (2.20)$$

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<sup>3</sup>The superposition principle has been experimentally checked also for very high field intensities: in engineering practices with fields which reach several millions of volts per meter (accelerators, high voltage discharges), when calculating fields in electron orbits ( $E \approx 10^{11} \dots 10^{17}$  V/m) or when calculating the field of highly weight nucleus ( $E \approx 10^{22}$  V/m). For fields over  $10^{20}$  V/m vacuum polarization is introduced and makes the problem non-linear.

and, taking into account from (2.17) that  $\mathbf{F} = q\mathbf{E}$ , it results that

$$E_p(\mathbf{r})/q = - \int \mathbf{E}(\mathbf{r}) \cdot d\mathbf{r} = V(\mathbf{r}). \quad (2.21)$$

If two points  $A$  and  $B$  are taken, it is observed from (2.19) that the *potential difference* between two points is the circulation of the electrostatic field  $\mathbf{E}$  between these two points, along any path between them<sup>4</sup>:

$$V_A - V_B = \int_{r_A}^{r_B} \mathbf{E} \cdot d\mathbf{r}. \quad (2.22)$$

Any convenient point  $\mathbf{r}_B = \mathbf{r}_{ref}$  can be chosen as the potential reference, at which  $V_B = V_{ref} = 0$  in (2.22). It is common to choose infinity as potential reference. The expression we reach is

$$V_A = \int_{r_A}^{r_{ref}} \mathbf{E} \cdot d\mathbf{r} = \int_{r_A}^{\infty} \mathbf{E} \cdot d\mathbf{r}, \quad (2.23)$$

which represents the work done by an external agent in transferring the unit of positive charge from infinity to a considered point.

If the  $\mathbf{E}$  field is due to a point charge  $q'$  (2.15) and (2.19) is integrated, the electrostatic potential is obtained,

$$V(\mathbf{r}) = \frac{q'}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|} = \frac{q'}{4\pi\epsilon_0 d}. \quad (2.24)$$

The integration constant does not appear because infinity has been chosen as potential reference ( $V_\infty = 0 \Rightarrow C = 0$ ). If the same calculus is applied to (2.16) a general expression for the electrostatic potential is obtained:

$$\begin{aligned} V(\mathbf{r}) = & \frac{1}{4\pi\epsilon_0} \sum_{j=1}^n \frac{q_j}{|\mathbf{r} - \mathbf{r}_j|} + \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho}{|\mathbf{r} - \mathbf{r}'|} dV' + \frac{1}{4\pi\epsilon_0} \int_{S'} \frac{\sigma}{|\mathbf{r} - \mathbf{r}'|} dS' \\ & + \frac{1}{4\pi\epsilon_0} \int_{L'} \frac{\lambda}{|\mathbf{r} - \mathbf{r}'|} dl'. \end{aligned} \quad (2.25)$$

This is the mathematical expression of **the superposition principle for electrostatic potential**: the electrostatic potential created by a number of charges is equal to the sum of potentials caused by each of them independently.

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<sup>4</sup>It must be observed in expression (2.22) that the order of integral limits have been changed. This is due to the negative sign removal.

## 2.5 Flux of Electric Field. Gauss' Law

Let  $V$  be a region in space, bordered by  $\partial V$  and let  $\mathbf{n}$  be the outward unit normal to  $\partial V$  on every point of a surface. The flux  $\Phi_E$  through  $\partial V$  (1.28) of the electric field  $\mathbf{E}$  produced by a point charge  $q$  located at the origin is

$$\Phi_E = \oint_{\partial V} \frac{q}{4\pi\epsilon_0 r^3} \mathbf{r} \cdot \mathbf{n} dS. \quad (2.26)$$

where  $\mathbf{r}$  is a vector that goes from  $q$  to a point on  $\partial V$ .

Gauss' law states that if  $V$  is smooth enough and if  $q \notin \partial V$ , it is verified

$$\Phi_E = \oint_{\partial V} \frac{q}{4\pi\epsilon_0} \frac{\mathbf{r} \cdot d\mathbf{S}}{r^3} = \begin{cases} 0 & \text{if } q \notin V, \\ \frac{q}{\epsilon_0} & \text{if } q \in V. \end{cases} \quad (2.27)$$

If we have any charge distribution, and we apply the principle of superposition for electrostatic fields, the previous theorem can be generalized as:

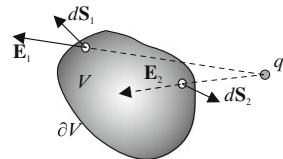
$$\Phi_E = \oint_{\partial V} \mathbf{E} \cdot d\mathbf{S} = \frac{q_{\text{in}}}{\epsilon_0}, \quad (2.28)$$

where  $q_{\text{in}}$  represents charge inside surface  $\partial V$ , which is usually called a *Gaussian surface*. This is known as **Gauss' law** or Gauss' theorem.

This theorem shows that the flux of the electric field through a closed surface only depends on the charge  $q_{\text{in}}$  inside the surface. It must be noted that the flux can be zero even though the field is not, as in Fig. 2.3, where the flux is positive in  $d\mathbf{S}_1$  and negative in  $d\mathbf{S}_2$ , and where the direction of field  $\mathbf{E}_2$  is towards  $V$ 's inner part. It must be also observed that the flux (not the field) through one surface is the same as another surface if the charge is in the volume bounded by the surface is the same.

Gauss' law allows calculating an electrostatic field created by charge distributions with different geometric and electric symmetries. This calculus is usually possible if a Gaussian surface with the same electric field (same magnitude and same angle with a normal vector to the surface) can be taken at each of its points.

**Fig. 2.3** Flux of an electric field through a closed surface due to a point charge



## 2.6 Electrostatic Equations

The integral expression of Gauss' law makes results dependent on the region we integrate. To avoid this problem, proper application of the divergence theorem results in a differential expression for the law:

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\varepsilon_0}, \quad (2.29)$$

where  $\rho = \rho(\mathbf{r})$  is the charge density on the considered point. This expression represents the *differential form of Gauss' law*,<sup>5</sup> which together with (2.18), compose electrostatics fundamental equations. If we remember the concept of divergence from Chap. 1, it can be observed that electric field sources are positive charge points, and sink ones are points with negative charge.

If we combine the two electrostatic equations, the result is

$$\Delta V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}, \quad (2.30)$$

which is known as the *Poisson equation*. In regions where charge density is null, we have

$$\Delta V(\mathbf{r}) = 0, \quad (2.31)$$

which is known as the *Laplace equation*.

Equations (2.30) and (2.31) are second order partial differential equations for a scalar field (electrostatic potential). If these equations are integrated and boundary conditions are given by a known charge distribution, the potential can be obtained for particular problems. The problem is simplified if an appropriate coordinate system is chosen. A Poisson (or Laplace) Equation solution is a unique one (*unity theorem*). This property allows us to establish methods to obtain the differential equation solution without specifically solving it (as occurs with the *method of images*, Sect. 7.4). This is because once a solution is obtained, it is unique, independent of the way it is obtained.

## 2.7 Electric Dipole

A special case of electric charge distribution can be studied: two equal and opposite charges separated by a small distance. This is known as an **electric dipole**. This can occur not only with two charges, but due to more complex charge distributions where

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<sup>5</sup>This equation is valid even when conditions are not static.



the effective centers of negative and positive charges satisfy dipole characteristics, as it will be seen in Chap. 3. The electric dipole is characterized by its **dipole moment**, expressed by

$$\mathbf{p} \equiv q\mathbf{d}, \quad (2.32)$$

whose SI unit is Cm. The magnitude  $\mathbf{d}$  is equal to the distance between charges, and  $\mathbf{d}$  has the direction from the negative charge to the positive charge. This is especially interesting for the case when the distance  $d$  goes to zero (it's very small compared to the other dimensions of the problem): a *point dipole* is formed. It has neither net charge nor space extension, but it is completely characterized by its dipole moment. Polar molecules are an example of a point dipole.<sup>6</sup> The electric field and the potential distribution produced by a point dipole can be calculated with the aid of the formulas of Sects. 2.3 and 2.4. The electric field is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{3(\mathbf{r} - \mathbf{r}') \cdot \mathbf{p}}{|\mathbf{r} - \mathbf{r}'|^5} (\mathbf{r} - \mathbf{r}') - \frac{\mathbf{p}}{|\mathbf{r} - \mathbf{r}'|^3} \right\}, \quad (2.33)$$

where the point dipole is located at point  $\mathbf{r}'$ . The potential distribution produced by a point dipole is given by

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}. \quad (2.34)$$

## 2.8 Conductors and Insulators

Materials have charged particles inside them which can move through-out the material under the influence of an outside electric field. These charged particles are called *charge carriers*. Charge carriers are electrons and ions in gases and liquids, electrons in crystalline solids (semiconductors and metals) and pairs of electrons in superconductors. The physical property used to measure the ease of charge movement is the conductivity.<sup>7</sup> According to electric behavior, materials can be divided into conductors, semiconductors and insulators (or dielectric).

**Conductors** are substances in which charges are free to move throughout the material under the influence of an outside electric field. Metallic conductors are the most characteristic example. The conductivity of metals generally increases with a decrease in temperature. At temperatures near absolute zero ( $T \approx 0^\circ\text{K}$ ), some conductors exhibit infinite conductivity and are called *superconductors*.

**Dielectrics** are substances in which charged particles are not free to move (low conductivity). These charges (nucleus and electrons) are strongly linked forming a material's atoms or molecules. In fact, they change position very little.

<sup>6</sup>[104] can be seen for more detailed information about the electric dipole.

<sup>7</sup>This concept will be studied on Chap. 4.

**Semiconductors** have electrical properties intermediate between conductors and dielectrics, though in electrostatic fields they behave as conductors.

A material's ability to conduct electricity can be understood with band theory: electrons in solid materials are distributed in bands, each one with a grade of energy. Electrons can change from one to another by absorbing or giving energy. Between bands there can be gaps or forbidden regions where an electron's presence is not possible. In the case of conducting materials, superior bands (conducting ones) are partially full, so the electron can move along them. For the insulating materials, the gap is large; they need a large energy to allow electrons to jump from the highest full band (valence's band) to the next one. In semiconductors, the necessary energy to go from the valence band to the conducting one is small.

Behavior of dielectric materials undergone to electric fields will be studied in next chapter. Charge carriers in conducting materials move until they reach positions where no net force is exerted, so they will have electrostatic balance. Hence, in electrostatic conditions:

- Electric field is null ( $\mathbf{E} = 0$ ) inside conducting materials, because equilibrium implies a null force, and  $\mathbf{E}$  is perpendicular on the surface.
- From (2.29), charge density  $\rho$  in the interior of the conductor is zero.
- From (2.18), each conductor forms an equipotential region of space.
- If (2.28) is applied, it can be deduced that the field in a very close point from conducting's surface is  $E = \sigma/\epsilon_0$ , where  $\sigma$  is the surface charge density of the conductor.

It must be observed that the Laplace equation (2.31) can be applied for conductors problems, because in almost every point the charge density is zero. You can solve the Laplace equation for every point outside the conductor if you know the boundary conditions, which involve the electrostatic potential  $V$ . Solution to the problem is already completed, because for the rest of the points (inside the conductor), the solution is the one that follows: as conducting materials are equipotential volumes, the potential inside them is the same as the one on its surface.

## 2.9 Biot–Savart-like Law in Electrostatics

The Biot–Savart law in one of the most basic relations in electromagnetism. We will study it in later chapters this law which allows us to calculate the total magnetic field  $\mathbf{B}$  at a point in space as superposition of  $d\mathbf{B}$  produced by the flow of current  $I$  through an infinitesimal path segment  $d\mathbf{l}$ . In [79], we can see an application of Biot–Savart law to obtain the electric field  $\mathbf{E}$  produced by a plane charge distribution, bounded by a curve  $C$  and kept at a fixed potential  $V$  while the rest of the plane is held at zero potential. If  $\mathbf{r}'$  locates the source point and  $\mathbf{r}$  refers to the field point, it follows that:

$$\mathbf{E}(\mathbf{r}) = \frac{V}{2\pi} \oint_C \frac{(\mathbf{r} - \mathbf{r}') \times d\mathbf{l}'}{|\mathbf{r} - \mathbf{r}'|^3}, \quad (2.35)$$

where  $d\mathbf{l}'$  is a length element of the integration path  $C$ . The direction of the integration around  $C$  is determined by the direction of the outward unit normal via the right-hand rule. Notice that to calculate the electric field we just need to take into account the contributions coming from the boundary contour  $C$ .

## Solved Problems

### Problems A

**2.1** In a cartesian coordinate system, with the axis in meters, two point charges are considered, one of them positive of 1 nC, located at the origin of coordinates, and the other one, negative of  $-20$  nC, located at  $A(0, 1)$ . Determine the resulting field at  $B(2, 0)$  and the necessary work to take a positive charge of  $3 \mu\text{C}$  from  $B(2, 0)$  to  $C(4, 2)$ .

### Solution

Applying the superposition principle for electrostatic field, field at  $B$  will be the vectorial addition of the fields due to each charge (Fig. 2.4), this is:

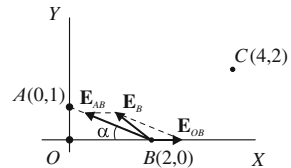
$$\mathbf{E}_B = \mathbf{E}_{OB} + \mathbf{E}_{AB}.$$

To calculate the field due to each charge, (2.15) is applied. Point  $B$ 's coordinates are  $\mathbf{r} = 2\mathbf{u}_x$ . In the case for the charge at  $O$ , position vector is  $\mathbf{r}' = 0$ , and we obtain  $\mathbf{r} - \mathbf{r}' = \mathbf{OB} = 2\mathbf{u}_x$  and  $|\mathbf{r} - \mathbf{r}'| = 2$ . Position vector of charge at  $A$  is  $\mathbf{r}' = \mathbf{u}_y$ , so  $\mathbf{r} - \mathbf{r}' = \mathbf{AB} = 2\mathbf{u}_x - \mathbf{u}_y$  and  $|\mathbf{r} - \mathbf{r}'| = \sqrt{5}$ . If (2.15) is applied to the charge located at  $O$ , the result is,

$$\mathbf{E}_{OB} = \frac{1}{4\pi\epsilon_0} \frac{q'}{|\mathbf{r} - \mathbf{r}'|^2} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} = 9 \cdot 10^9 \frac{10^{-9}}{4} \mathbf{u}_x = \frac{9}{4} \mathbf{u}_x.$$

And for the charge located at  $A$ ,

**Fig. 2.4** Field produced by point charges



$$\begin{aligned}\mathbf{E}_{AB} &= \frac{1}{4\pi\epsilon_0} \frac{q'}{|\mathbf{r} - \mathbf{r}'|^2} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} = 9 \cdot 10^9 \frac{(-20 \cdot 10^{-9})}{5} \left( \frac{2}{\sqrt{5}}\mathbf{u}_x - \frac{1}{\sqrt{5}}\mathbf{u}_y \right) \\ &= \frac{36}{\sqrt{5}} (-2\mathbf{u}_x + \mathbf{u}_y).\end{aligned}$$

The field could be obtained without using position vectors, and bearing in mind Fig. 2.4. Since there is a negative charge at  $A$ , field  $E_{AB}$  is pointed to  $A$ . If we use (2.15) to calculate the electric field magnitude and if we project it, we obtain the field expressed as,

$$\mathbf{E}_{AB} = 9 \cdot 10^9 \frac{20 \cdot 10^{-9}}{5} (-\cos \alpha \mathbf{u}_x + \sin \alpha \mathbf{u}_y) = 36 \left( \frac{-2}{\sqrt{5}}\mathbf{u}_x + \frac{1}{\sqrt{5}}\mathbf{u}_y \right).$$

The resulting field will be

$$\mathbf{E}_B = \mathbf{E}_{OB} + \mathbf{E}_{AB} = \left( \frac{9}{4} - \frac{72}{\sqrt{5}} \right) \mathbf{u}_x + \frac{36}{\sqrt{5}} \mathbf{u}_y = -29.95\mathbf{u}_x + 16.10\mathbf{u}_y.$$

Since the electrostatic field is conservative, the work necessary to take a charge from  $B$  to  $C$  is equal to the variation of potential energy between  $B$  and  $C$  (2.20), with negative sign. From (2.21) which relates electrostatic potential to the potential energy,

$$W_{BC} = -(E p_B - E p_C) = -q(V_B - V_C),$$

and potentials at points  $B$  and  $C$  due to the charge system of the problem must be calculated. To obtain the potential on each point, the superposition principle is applied, this is, the potential to be the added due to each charge,  $V_B = V_{OB} + V_{AB}$  and  $V_C = V_{OC} + V_{AC}$ . If (2.24) (potential produced by a point charge) is applied

$$V = \frac{1}{4\pi\epsilon_0} \frac{q'}{|\mathbf{r} - \mathbf{r}'|},$$

potentials due to each charge on each point  $V_{OB}$ ,  $V_{AB}$ ,  $V_{OC}$  and  $V_{AC}$  can be determined. It is necessary to define the terms  $(|\mathbf{r} - \mathbf{r}'|)$ , which are the distances from each charge to points  $B$  and  $C$ . On the case of point  $B$  the distances have already been determined for the calculus of the electric field, so potential in  $B$  will be

$$V_B = V_{OB} + V_{AB} = 9 \cdot 10^9 \left( \frac{10^{-9}}{2} + \frac{(-20 \cdot 10^{-9})}{\sqrt{5}} \right) = -76 \text{ V}.$$

For point  $C$ , its position vector is  $\mathbf{r} = 4\mathbf{u}_x + 2\mathbf{u}_y$ , and it becomes that  $\mathbf{r} - \mathbf{r}'$  values are  $\mathbf{OC} = 4\mathbf{u}_x + 2\mathbf{u}_y$  for the charge at  $O$  and  $\mathbf{AC} = (4\mathbf{u}_x + 2\mathbf{u}_y) - \mathbf{u}_y = 4\mathbf{u}_x + \mathbf{u}_y$  for the charge at  $A$ . Potential is

$$V_C = V_{OC} + V_{AC} = 9 \cdot 10^9 \left( \frac{10^{-9}}{\sqrt{20}} + \frac{(-20 \cdot 10^{-9})}{\sqrt{17}} \right) = -41.6 \text{ V}.$$

And circulation from point  $B$  to  $C$  is

$$W_{BC} = -q(V_B - V_C) = -3 \cdot 10^{-6}(-76 + 41.6) = 103.2 \cdot 10^{-6} \text{ J} = 103.2 \mu\text{J}.$$

- 2.2** In the space region defined by  $y > 0$ , a charge density  $\rho = cy$  exists, with  $c = 2 \mu\text{C}/\text{m}^4$  and  $y$  the distance (in meters) from any point to plane  $XOZ$ . Calculate:
- The flux of the electrostatic field through the prism's surface in Fig. 2.5.
  - Divergence of electrostatic field in the prism's faces which are parallel to plane  $XOZ$ .

### Solution

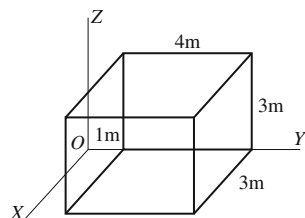
(a) Gauss' law (2.28) states the flux of the electric field due to a charge distribution,

$$\Phi_E = \oint_{\partial V} \mathbf{E} \cdot d\mathbf{S} = \frac{q_{\text{in}}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_V \rho dV.$$

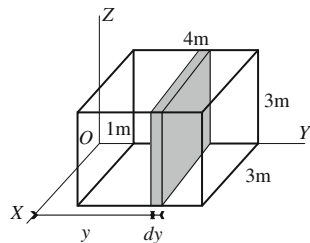
Therefore we must calculate the charge  $q_{\text{in}}$  inside the prism in Fig. 2.5. Since charge density  $\rho$  only depends on coordinate  $y$ , every point located at the same distance  $y$  have the same density. Let's consider the infinitesimal volume drawn in Fig. 2.6,  $dV = 3 \cdot 3 dy = 9 dy$ . At every point in it the density is  $cy$ , so the volume integral becomes a simple integral:

$$q_{\text{in}} = \int_V \rho dV = \int_1^5 cy \cdot 9 dy = \frac{9cy^2}{2} \Big|_1^5 = 216 \mu\text{Cm}^{-3}.$$

**Fig. 2.5** Prism of Problem 2.2



**Fig. 2.6** Differential element to calculate internal charge



And therefore the flux is

$$\Phi_E = \frac{q_{\text{in}}}{\varepsilon_0} = \frac{216 \cdot 10^{-6}}{8.85 \cdot 10^{-12}} = 24.4 \cdot 10^6 \text{ NC}^{-1}\text{m}^2.$$

(b) We obtain the divergence of the electric field from (2.29). If we particularize for each of the prism faces, we obtain:

$$\text{div } \mathbf{E} = \frac{\rho}{\varepsilon_0} = \frac{cy}{\varepsilon_0} \begin{cases} y = 1, \text{ div } \mathbf{E} = \frac{2 \cdot 10^{-6} \cdot 1}{8.85 \cdot 10^{-12}} = 2.26 \cdot 10^5 \text{ NC}^{-1}\text{m}^{-1} \\ y = 5, \text{ div } \mathbf{E} = \frac{2 \cdot 10^{-6} \cdot 5}{8.85 \cdot 10^{-12}} = 1.13 \cdot 10^6 \text{ NC}^{-1}\text{m}^{-1} \end{cases}$$

**2.3** Determine the electric field produced on any point in space by a very long line (infinite) charged with a uniform density  $\lambda$ .

### Solution<sup>8</sup>

This problem has cylindrical symmetry. Hence all the points at the same distance to the line have the same electric field magnitude, and  $\mathbf{E}$  is perpendicular to the line.<sup>9</sup> The problem can be solved by applying Gauss' law (2.28). We take as a Gaussian surface  $\partial V$  a closed cylindrical surface, with any length  $L$ , with the axis on the line and with radius  $r$  making the surface to pass through point  $P$ , the field desired to be calculated (discontinuous line in Fig. 2.7). The electric field at any point on the Gaussian surface is radial (perpendicular to the cylinder lateral surface), outward pointed, if we suppose the line as positively charged,<sup>10</sup> and has the same magnitude at every point on the lateral surface. It should be observed that the Gaussian surface is a closed surface and therefore to calculate the flux the cylinder lateral surface where the point  $P$  is and the cylinder bases, where the magnitude of the field is not constant and is different from the one at the lateral surface, should be considered. However, this is not a problem, since the flux through the bases is null due to the fact that  $d\mathbf{S}$  and  $\mathbf{E}$  are perpendicular at any point. Calculating the flux through the lateral surface, where  $d\mathbf{S}$  and  $\mathbf{E}$  are parallel:

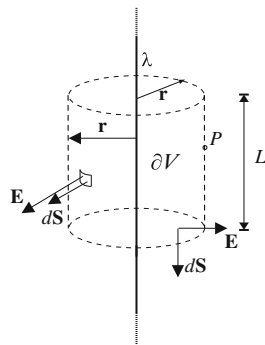
$$\Phi_E = \oint_{\partial V} \mathbf{E} \cdot d\mathbf{S} = \int_{S_{\text{lat}}} E dS = E \int_{S_{\text{lat}}} dS = E 2\pi r L,$$

<sup>8</sup>Infinite is usually used to express that the element is much longer than the distance  $r$  to point  $P$ , so the symmetry reasonings can be used for the calculus.

<sup>9</sup>To check this, the field can be considered to be produced at a point by an element  $dq$  and its symmetric regarding to the normal to the line by the considered point. Tangential components from one and another have the same magnitude, since they are at the same distance, and opposite directions. Then the result is a radial field.

<sup>10</sup>If the charge were negative, the charge sign on the solution indicates that the field vector has the opposite direction.

**Fig. 2.7** Gaussian surface and field vectors to apply Gauss' law to an infinite line



where  $S_{\text{lat}}$  is the cylinder lateral surface. On another side, if Gauss' law is applied, it is obtained

$$\Phi_E = \frac{q_{\text{in}}}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0}.$$

Equating both expressions,

$$E = \frac{\lambda}{2\pi\varepsilon_0 r},$$

shows that the field varies inversely with the distance to the line. This expression is the same as the one calculated by integration in Problem 2.4. Using vectors:

$$\mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0 r} \frac{\mathbf{r}}{r} = \frac{\lambda}{2\pi\varepsilon_0 r} \mathbf{u}_\rho, \quad (2.36)$$

where  $\mathbf{u}_\rho$  is the radial unit vector for cylindrical coordinates.

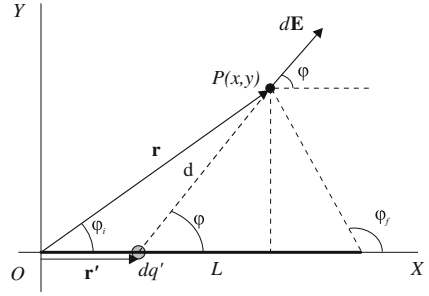
## Problems B

**2.4** Determine the electric field, at any point in space, produced by a line with length  $L$ , which has been uniformly charged by a total charge  $Q$ : (a) directly; (b) from electrostatic potential. Apply the result for the particular cases of the field produced by an infinitely long line with density  $\lambda$  and to the field produced by a semi-infinite line with density  $\lambda$  at a point located on the perpendicular to its extreme.

### Solution

(a) The problem above is drawn in Fig. 2.8, where a cartesian coordinate system has been defined, just to simplify the calculus. To achieve this, we choose a point  $P$  where we want to calculate the field (*field point*), and we define a plane  $XY$  as drawn. The  $X$  axis is on the charged line and the origin is at one extreme of the line.

**Fig. 2.8** Field produced by an finite line at any point



The coordinates of point  $P$  will be  $(x, y)$  and the electric field  $\mathbf{E}$  will be on plane  $XY$ .

To solve the problem we calculate the field due to an element  $dq'$  from the line, and the superposition principle is applied. The field produced by an element  $dq'$  is given by (2.15)

$$d\mathbf{E}(\mathbf{r}) = \frac{dq'}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|^2} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|},$$

where  $\mathbf{r} = x\mathbf{u}_x + y\mathbf{u}_y$  is the position vector of point  $P$  and  $\mathbf{r}' = x'\mathbf{u}_x$  is the position of each charge element (*source point*)  $dq'$ . Charge  $dq'$  is obtained from lineal charge density definition  $\lambda$  (2.5) as

$$dq' = \lambda dl = \lambda dx'.$$

$\lambda$  is obtained from (2.6), bearing in mind that, since it is a uniform charge,  $\lambda$  is constant:

$$\lambda = \frac{Q}{L}.$$

With  $\mathbf{r}$  and  $\mathbf{r}'$  values we obtain vector  $\mathbf{r} - \mathbf{r}'$  and the distance from  $dq'$  to point  $P$ :

$$\mathbf{r} - \mathbf{r}' = (x - x')\mathbf{u}_x + y\mathbf{u}_y, \quad d = |\mathbf{r} - \mathbf{r}'| = \sqrt{(x - x')^2 + y^2}.$$

If we substitute the electric field produced by  $dq'$  at point  $P$ , we have

$$d\mathbf{E}(\mathbf{r}) = \frac{dq'}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|^2} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{4\pi\epsilon_0} \frac{\lambda dx'}{[(x - x')^2 + y^2]} \frac{(x - x')\mathbf{u}_x + y\mathbf{u}_y}{\sqrt{(x - x')^2 + y^2}}.$$

The total field is obtained by applying the superposition principle (2.16):

$$\mathbf{E} = \frac{\lambda}{4\pi\epsilon_0} \int_L \frac{dx'}{[(x - x')^2 + y^2]} \frac{(x - x')\mathbf{u}_x + y\mathbf{u}_y}{\sqrt{(x - x')^2 + y^2}}.$$



Solving using integrals, where the only variable is  $x'$ , allows us to obtain the total field's value. A way of solving this integral is by introducing a variable change, depending on the angle  $\varphi$  in Fig. 2.8,

$$\cos \varphi = \frac{x - x'}{d}, \quad \sin \varphi = \frac{y}{d}, \quad \cot \varphi = \frac{\cos \varphi}{\sin \varphi} = \frac{x - x'}{y}.$$

If we take the derivative of the last expression, the result is

$$\frac{d\varphi}{\sin^2 \varphi} = \frac{dx'}{y}.$$

If we obtain  $dx'$  and  $d$  values,

$$dx' = \frac{y d\varphi}{\sin^2 \varphi}, \quad d = \frac{y}{\sin \varphi},$$

and we substitute into the integral, and substitute  $\sqrt{(x - x')^2 + y^2}$  for  $d$ , it follows

$$\begin{aligned} \mathbf{E} &= \frac{\lambda}{4\pi\epsilon_0} \int_L \frac{dx' (x - x')\mathbf{u}_x + y\mathbf{u}_y}{d^2} \\ &= \frac{\lambda}{4\pi\epsilon_0} \left( \int_{\varphi_i}^{\varphi_f} \frac{\sin^2 \varphi}{y^2} \frac{y d\varphi}{\sin^2 \varphi} \cos \varphi \mathbf{u}_x + \int_{\varphi_i}^{\varphi_f} \frac{\sin^2 \varphi}{y^2} \frac{y d\varphi}{\sin^2 \varphi} \sin \varphi \mathbf{u}_y \right) \\ &= \frac{\lambda}{4\pi\epsilon_0 y} \left( \int_{\varphi_i}^{\varphi_f} \cos \varphi d\varphi \mathbf{u}_x + \int_{\varphi_i}^{\varphi_f} \sin \varphi d\varphi \mathbf{u}_y \right). \end{aligned}$$

where  $\varphi_i$  and  $\varphi_f$  are the angles measured from the initial and final positions of the line, as shown in Fig. 2.8. It should be observed that  $y$ , the generic ordinate of a point field, is not a variable in the integral. If we solve it, it results

$$\mathbf{E} = \frac{\lambda}{4\pi\epsilon_0 y} ((\sin \varphi_f - \sin \varphi_i)\mathbf{u}_x + (\cos \varphi_i - \cos \varphi_f)\mathbf{u}_y). \quad (2.37)$$

The result can be written in Cartesian coordinates:

$$\begin{aligned} \mathbf{E} &= \frac{Q}{4\pi\epsilon_0 y L} \left[ \left( \frac{y}{\sqrt{(L-x)^2 + y^2}} - \frac{y}{\sqrt{x^2 + y^2}} \right) \mathbf{u}_x \right. \\ &\quad \left. + \left( \frac{x}{\sqrt{x^2 + y^2}} + \frac{L-x}{\sqrt{(L-x)^2 + y^2}} \right) \mathbf{u}_y \right], \end{aligned} \quad (2.38)$$

where  $\lambda$ 's value has been replaced by  $Q/L$ .

(b) To calculate the electric field at  $P$  from the potential  $V$ , we must obtain a potential generic expression at any point (for example  $P(x, y)$ ) and then obtain the field from

(2.18). It is important to notice that this procedure can be done because the potential at all the points is known, and therefore its gradient can be calculated. To obtain the potential at point  $P$ , the potential produced by a charge element  $dq'$  is expressed, taking the potential reference at infinity (2.24),

$$dV = \frac{1}{4\pi\epsilon_0} \frac{dq'}{|\mathbf{r} - \mathbf{r}'|} = \frac{\lambda}{4\pi\epsilon_0} \frac{dx'}{\sqrt{(x - x')^2 + y^2}},$$

and the superposition principle for electrostatic potential is applied (2.25):

$$V = \frac{\lambda}{4\pi\epsilon_0} \int_0^L \frac{dx'}{\sqrt{(x - x')^2 + y^2}}.$$

Solving this integral,

$$V = -\frac{\lambda}{4\pi\epsilon_0} \ln \left( x - x' + \sqrt{(x - x')^2 + y^2} \right) \Big|_0^L = \frac{\lambda}{4\pi\epsilon_0} \ln \frac{x + \sqrt{x^2 + y^2}}{x - L + \sqrt{(x - L)^2 + y^2}}.$$

If we apply now (2.18),

$$\mathbf{E} = -\nabla V = -(\partial V / \partial x) \mathbf{u}_x - (\partial V / \partial y) \mathbf{u}_y,$$

and if we solve the indicated partial derivative, the total field  $\mathbf{E}$  at  $P$  is obtained (2.38).

Let's consider the case of an *infinite line* (very long line). Since the line has an infinite length, (2.38) is not obvious. It is easier to use (2.37). It should be observed in Fig. 2.8 that if the line is infinite, initial and final angles are  $\varphi_i = 0$  and  $\varphi_f = \pi$ . If we substitute in (2.37), it results

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 y} \mathbf{u}_y.$$

Let's consider that the line begins at  $O$  and is very long (*semi-infinite line*) and  $P$  is over the perpendicular to the line at  $O$ . Point  $P$  coordinates are  $P(0, y)$ . If we consider (2.37), we observe in Fig. 2.8 that, since  $P(0, y)$ ,  $\varphi_i = \pi/2$  and  $\varphi_f = \pi$ . Then,

$$\mathbf{E} = \frac{\lambda}{4\pi\epsilon_0 y} (-\mathbf{u}_x + \mathbf{u}_y),$$

We should be cautious when applying the used procedure in section (b) to obtain the field when the line is infinitely long. We observe the potential for this charge distribution as

$$V = \lim_{L \rightarrow \infty} \frac{\lambda}{4\pi\epsilon_0} \ln \frac{x + \sqrt{x^2 + y^2}}{x - L + \sqrt{(x - L)^2 + y^2}} \rightarrow \infty.$$

So it is not possible to obtain the field from this potential. Infinite potential has been obtained because for charge distributions that spread in an infinite region, it can never be certain that this potential converges. However, the field from the potential can be obtained as follows: firstly we calculate the finite line potential, then its gradient, and then we make the line's length to infinity. This difficulty will also appear in the magnetostatic chapter.

**2.5** Determine the electric field and the potential at any point in space produced by a spherical crown where the internal radius is  $R_1$  and the external one  $R_2$ , with a total charge  $Q$ , for the following cases: (a) non conducting and an uniform charge distributed throughout the volume; and (b) metallic and on electrostatic equilibrium. Particularize the results for a solid sphere with radius  $R$ .

### Solution

(a) In this case the spherical crown has a uniform volume charge density  $\rho$  at every point between  $R_1$  and  $R_2$ , that can be calculated by applying (2.2):

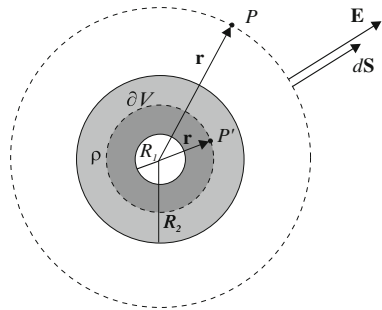
$$Q = \int_V \rho dV = \rho \int_V dV = \rho V = \rho \frac{4}{3} \pi (R_2^3 - R_1^3) \Rightarrow$$

$$\rho = \frac{Q}{V} = \frac{3Q}{4\pi(R_2^3 - R_1^3)}.$$

Because of its symmetry, Gauss' law (2.28) is applied, considering a spherical Gaussian surface  $\partial V$  (Fig. 2.9), concentric with the charge distribution, and passing through the point where we want to calculate the field. At every point inside a sphere whose radius is  $r \leq R_1$ , field is null since internal charge is zero:

$$E_{(r \leq R_1)} = 0.$$

**Fig. 2.9** Gaussian surfaces and field vectors to apply Gauss' law to a spherical crown



Due to the charge distribution symmetry, the electric field at any other point on the Gaussian surface will be radial,<sup>11</sup> outward<sup>12</sup> and with the same magnitude at every point on the surface. Two different expressions are obtained for the electric field, depending on the point to be studied, if it is outside or inside the spherical crown ( $P$  and  $P'$  in Fig. 2.9), since internal charge to the Gaussian surface has a different expression.

Let's firstly consider point  $P$ , outside the crown. The flux through the Gaussian surface passing through  $P$  is

$$\Phi_E = \oint_{\partial V} \mathbf{E} \cdot d\mathbf{S} = \oint_{\partial V} E dS = E \oint_{\partial V} dS = E4\pi r^2,$$

where  $4\pi r^2$  is the spherical surface's area, with radius  $r$ . We observe that the total charge within the Gaussian surface is all the charge of the spherical crown (grey in figure). If Gauss' law is applied, the flux is

$$\Phi_E = \frac{q_{\text{in}}}{\varepsilon_0} = \frac{Q}{\varepsilon_0} = \frac{\rho 4\pi(R_2^3 - R_1^3)}{3\varepsilon_0}.$$

If we equate the two previous expressions for the flux and we solve it, we obtain the field at an external point  $P$ ,

$$E_{(r \geq R_2)} = \frac{\rho(R_2^3 - R_1^3)}{3\varepsilon_0 r^2} = \frac{Q}{4\pi\varepsilon_0 r^2},$$

expression that coincides with the field produced by a point charge  $Q$  located at the spherical crown centre. In fact, the field produced by a point charge, known from Coulomb's law, can be calculated by applying Gauss' law to a random spherical surface whose centre is on the charge.

To know the field at a point  $P'$  (Fig. 2.9), inside the spherical crown, the same procedure is followed. We set up a spherical concentric surface, with radius  $r$ , passing through  $P'$ . The expression of the flux through a surface of radius  $r$  is the same, but since  $R_1 \leq r \leq R_2$ , internal charge to the Gaussian surface is now not the total charge of the crown charge, but only the dark grey region in the figure. If we calculate

$$\Phi_E = \frac{q_{\text{in}}}{\varepsilon_0} = \frac{\rho 4\pi(r^3 - R_1^3)}{3\varepsilon_0},$$

and then, from  $\Phi_E = E4\pi r^2$ , the field at a point  $P'$  inside the spherical crown is obtained,

<sup>11</sup> It can be checked by taking the field produced by a random element  $dq$  and its symmetrical with regard to the diameter that passes through the considered point. Tangential components from one to the other have the same magnitude and opposite directions, and the result is a radial field.

<sup>12</sup> It will be supposed, unless it is stated otherwise, these bodies are positively charged. If the charge is negative, the vector field has the opposite direction.

$$E_{(R_1 \leq r \leq R_2)} = \frac{\rho(r^3 - R_1^3)}{3\epsilon_0 r^2} = \frac{Q}{4\pi\epsilon_0 r^2} \frac{(r^3 - R_1^3)}{(R_2^3 - R_1^3)}.$$

Joining both results and expressing the field as a vector, it results

$$\mathbf{E} = \begin{cases} 0 & r \leq R_1, \\ \frac{Q}{4\pi\epsilon_0 r^2} \frac{(r^3 - R_1^3)}{(R_2^3 - R_1^3)} \mathbf{u}_r = \frac{\rho(r^3 - R_1^3)}{3\epsilon_0 r^2} \mathbf{u}_r & R_1 \leq r \leq R_2, \\ \frac{Q}{4\pi\epsilon_0 r^2} \mathbf{u}_r = \frac{\rho(R_2^3 - R_1^3)}{3\epsilon_0 r^2} \mathbf{u}_r & r \geq R_2, \end{cases} \quad (2.39)$$

where  $\mathbf{u}_r$  is the radial unit vector for spherical coordinates. The field at a point on the outside spherical surface can be calculated by using either one of the expressions, making  $r = R_2$ :

$$\mathbf{E}_{(r=R_2)} = \frac{Q}{4\pi\epsilon_0 R_2^2} \mathbf{u}_r = \frac{\rho(R_2^3 - R_1^3)}{3\epsilon_0 R_2^2} \mathbf{u}_r.$$

If the sphere were solid, with radius  $R$ , results can be obtained by replacing  $R_1 = 0$  and  $R_2 = R$ :

$$\mathbf{E} = \begin{cases} \frac{Q}{4\pi\epsilon_0} \frac{r}{R^3} \mathbf{u}_r = \frac{\rho r}{3\epsilon_0} \mathbf{u}_r & r \leq R, \\ \frac{Q}{4\pi\epsilon_0 r^2} \mathbf{u}_r = \frac{\rho R^3}{3\epsilon_0 r^2} \mathbf{u}_r & r \geq R, \end{cases} \quad (2.40)$$

And the field at a point on the surface would be:

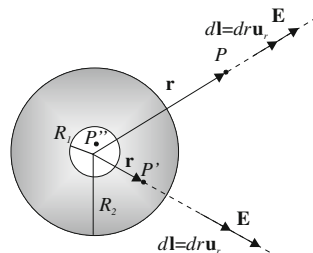
$$\mathbf{E}_{(r=R)} = \frac{Q}{4\pi\epsilon_0 R^2} \mathbf{u}_r = \frac{\rho R}{3\epsilon_0} \mathbf{u}_r.$$

To determine the potential at any point, we calculate the potential difference between that point and infinity, with null potential. Since the circulation of the electrostatic field is path-independent, we take the radial direction from the point, where  $\mathbf{E}$  and  $d\mathbf{l}$  are parallel,<sup>13</sup> as shown in Fig. 2.10.

For an external point  $P$  we have, if we circulate the field  $\mathbf{E}$  between  $P$  and infinity (Fig. 2.10), that

<sup>13</sup>It is not necessary to take a circulation line where  $\mathbf{E}$  and  $d\mathbf{l}$  are parallel, if we bear in mind the property of any vector  $\mathbf{r}$ , for which  $\mathbf{r} \cdot d\mathbf{r} = |\mathbf{r}|d|\mathbf{r}|$ . If we express  $\mathbf{E}$  depending on  $\mathbf{u}_r$  we would achieve the same calculus expression, but it is explained like this to make the circulation concept comprehension easier.

**Fig. 2.10** Potential calculation of a charged spherical crown



$$\begin{aligned}
 V_P - V_\infty = V_P &= \int_P^\infty \mathbf{E} \cdot d\mathbf{l} = \int_r^\infty E(r \geq R_2) dr \\
 &= \int_r^\infty \frac{\rho(R_2^3 - R_1^3)}{3\epsilon_0 r^2} dr = \\
 &= - \left. \frac{\rho(R_2^3 - R_1^3)}{3\epsilon_0 r} \right|_r^\infty.
 \end{aligned}$$

Then,

$$V_P = \frac{\rho(R_2^3 - R_1^3)}{3\epsilon_0 r} = \frac{Q}{4\pi\epsilon_0 r}.$$

It can be observed how the potential given by the charged spherical crown at an external point is the same as a point charge, located at the centre, with the same charge, would create.

If the point  $P'$  is inside the spherical crown, from which we need to circulate from  $P'$  to infinity, field expressions are different depending on where we circulate, inside or outside the spherical crown. Potential difference is obtained from

$$\begin{aligned}
 V_{P'} - V_\infty = V_{P'} &= \int_{P'}^\infty \mathbf{E} \cdot d\mathbf{l} = \int_r^{R_2} E(r \leq R_2) dr + \int_{R_2}^\infty E(r \geq R_2) dr \\
 &= \int_r^{R_2} \frac{\rho(r^3 - R_1^3)}{3\epsilon_0 r^2} dr + \int_{R_2}^\infty \frac{\rho(R_2^3 - R_1^3)}{3\epsilon_0 r^2} dr = \\
 &= \left. \frac{\rho r^2}{6\epsilon_0} \right|_r^{R_2} + \left. \frac{\rho R_1^3}{3\epsilon_0 r} \right|_r^{R_2} - \left. \frac{\rho(R_2^3 - R_1^3)}{3\epsilon_0 r} \right|_{R_2}^\infty.
 \end{aligned}$$

The potential will be

$$V_{P'} = \frac{\rho(R_2^2 - r^2)}{6\epsilon_0} + \frac{\rho R_1^3}{3\epsilon_0} \left( \frac{1}{R_2} - \frac{1}{r} \right) + \frac{\rho(R_2^3 - R_1^3)}{3\epsilon_0 R_2}.$$

If the point ( $P''$ ) is in the hole (Fig. 2.10), so  $r \leq R_1$ , given the field at the hole is null, circulation from point  $P''$  to the inner radius of the crown  $R_1$  is also null. Point  $P''$  potential is the same as the one at point  $P'$  located on the inner spherical surface, with  $r = R_1$ . If the previous result is specified for  $r = R_1$  the result is:

$$V_{P''} = \frac{\rho(R_2^2 - R_1^2)}{6\varepsilon_0} + \frac{\rho R_1^3}{3\varepsilon_0} \left( \frac{1}{R_2} - \frac{1}{R_1} \right) + \frac{\rho(R_2^3 - R_1^3)}{3\varepsilon_0 R_2}.$$

If all the results are joined,

$$V = \begin{cases} \frac{\rho(R_2^2 - R_1^2)}{6\varepsilon_0} + \frac{\rho R_1^3}{3\varepsilon_0} \left( \frac{1}{R_2} - \frac{1}{R_1} \right) + \frac{\rho(R_2^3 - R_1^3)}{3\varepsilon_0 R_2} & r \leq R_1, \\ \frac{\rho(R_2^2 - r^2)}{6\varepsilon_0} + \frac{\rho R_1^3}{3\varepsilon_0} \left( \frac{1}{R_2} - \frac{1}{r} \right) + \frac{\rho(R_2^3 - R_1^3)}{3\varepsilon_0 R_2} & R_1 < r \leq R_2, \\ \frac{\rho(R_2^3 - R_1^3)}{3\varepsilon_0 r} = \frac{Q}{4\pi\varepsilon_0 r} & r \geq R_2. \end{cases} \quad (2.41)$$

(b) If the spherical crown is metallic and has electrostatic balance, charge is only distributed on its external surface, as it can be deduced from conductor properties seen in Sect. 2.8. To obtain null electric field inside the conductor, the crown must be charged on its external surface with a uniform superficial density. This density can be obtained from (2.4):

$$Q = \int_S \sigma dS = \sigma \int_S dS = \sigma 4\pi R_2^2 \Rightarrow \sigma = \frac{Q}{4\pi R_2^2}.$$

The field at internal points ( $r < R_2$ ) is null, due to the charge distribution symmetry. To obtain the electric field produced by the crown at external points to it, and due to the fact that the problem's symmetry is analogous to the one in section (a), we proceed as it was done in that section. We use the same Gaussian surfaces (Fig. 2.9) but just changing that the charge  $Q$  is only on the surface. The flux through the spherical surface of radius  $r$  is, as it happened with the previous case,

$$\Phi_E = E 4\pi r^2.$$

If Gauss' law is applied to an external point  $P$  to the spherical crown,<sup>14</sup> ( $r > R_2$ ), results in

$$\Phi_E = \frac{q_{\text{in}}}{\varepsilon_0} = \frac{Q}{\varepsilon_0} = \frac{4\pi R_2^2 \sigma}{\varepsilon_0}.$$

In solving it, the result for an external point  $P$  is

$$E_{(r>R_2)} = \frac{Q}{4\pi\varepsilon_0 r^2} = \frac{\sigma R_2^2}{\varepsilon_0 r^2},$$

<sup>14</sup>It should be observed that Gauss' theorem cannot be applied to the points exactly located on the sphere's surface, due to the fact that in this case charges would be on the Gaussian surface, breaking the theorem's condition  $q \notin \partial V$ .

the same expression depending on the total charge  $Q$  that in section (a). If it is expressed with vector notation, the results are:

$$\mathbf{E} = \begin{cases} 0 & r < R_2, \\ \frac{Q}{4\pi\epsilon_0 r^2} \mathbf{u}_r = \frac{\sigma R_2^2}{\epsilon_0 r^2} \mathbf{u}_r & r > R_2, \end{cases} \quad (2.42)$$

where  $\mathbf{u}_r$  is the radial unit vector for spherical coordinates.

Since the field at external points to the metallic crown is the same as the one in section (a), if the same reference is taken (infinity), the potential is also the same for points outside the crown. For points inside it, given the field is null at these points, potential does not change and has the same value that on the crown's surface. Then,

$$V = \begin{cases} \frac{Q}{4\pi\epsilon_0 R_2} = \frac{\sigma R_2}{\epsilon_0} & r \leq R_2, \\ \frac{Q}{4\pi\epsilon_0 r} = \frac{\sigma R_2^2}{\epsilon_0 r} & r \geq R_2. \end{cases} \quad (2.43)$$

**2.6** Determine the electric field produced by a very long charged cylinder at any point, with inner radius  $R_1$  and external radius  $R_2$ , with a charge per unit of length  $q$ , for the following cases: (a) non conductor and uniformly charged; and (b) metallic and in electrostatic balance. Specify the results for a solid cylinder with radius  $R$ .

### Solution

(a) In this case the cylinder has a uniform volumetric charge density  $\rho$ , at every point between  $R_1$  and  $R_2$ , that can be calculated by applying (2.2), and considering the finite cylinder length  $L$ ,

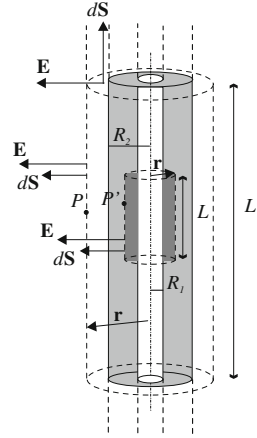
$$\begin{aligned} Q &= \int_V \rho dV = \rho \int_V dV = \rho V = \rho \pi (R_2^2 - R_1^2) L \Rightarrow \\ \Rightarrow \rho &= \frac{Q}{V} = \frac{Q/L}{\pi (R_2^2 - R_1^2)} = \frac{q}{\pi (R_2^2 - R_1^2)}. \end{aligned}$$

The problem has cylindrical symmetry: at points inside the cylinder ( $r < R_1$ ), the field is null, due to that symmetry and, for the other zones, all the points at the same distance to the cylinder axis have the same electric field magnitude, with direction perpendicular to that axis.<sup>15</sup> The problem can be solved by applying Gauss' law (2.28), the same as it was done for Problem 2.3. For this we take as a Gaussian surface

<sup>15</sup>To check this, consider the field produced at a point by an element  $dq$  and its symmetric pair with respect to a perpendicular to the cylinder axis at the considered point. The components parallel to the cylinder axis have the same magnitude, since they are at the same distance, and opposite direction, and we obtain a radial field.



**Fig. 2.11** Gaussian surface and field vectors to apply Gauss' law to an infinite cylinder



$\partial V$  a cylindrical surface, with any length  $L$ , with the same axis as the cylinder and radius  $r$ , making the surface pass through point  $P$  (or  $P'$ ) where we want to calculate the field (discontinuous line in Fig. 2.11).

The flux through the bases of the Gaussian surface, is null, since  $d\mathbf{S}$  and  $\mathbf{E}$  are perpendicular at any given point. Calculating the flux through the lateral surface, where  $d\mathbf{S}$  and  $\mathbf{E}$  are parallel:

$$\Phi_E = \oint_{\partial V} \mathbf{E} \cdot d\mathbf{S} = \int_{S_{\text{lat}}} E dS = E \int_{S_{\text{lat}}} dS = E 2\pi r L, \quad (2.44)$$

where  $S_{\text{lat}}$  is the cylinder lateral surface. Applying Gauss' law we have:

$$\Phi_E = \frac{q_{\text{in}}}{\varepsilon_0},$$

where  $q_{\text{in}}$ 's value depends on the point where the field is calculated, whether it's inside ( $P'$ ) or outside ( $P$ ) the charged cylinder. If we calculate the field at an external point  $P$ , the entire charged cylinder (with height  $L$ , light grey coloured in Fig. 2.11) remains inside the Gaussian surface, and it results

$$\Phi_{E(r > R_2)} = \frac{q_{\text{in}}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_{R_1}^{R_2} \rho dV = \frac{\rho}{\varepsilon_0} \pi (R_2^2 - R_1^2) L = \frac{qL}{\varepsilon_0},$$

If the point  $P'$  is inside the charged cylinder, there is a part of the charge outside the Gaussian surface, which produces no flux. Then the only inner charge that remains is the dark grey coloured one in Fig. 2.11, and it results

$$\Phi_{E(R_1 < r < R_2)} = \frac{q_{\text{in}}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_{R_1}^r \rho dV = \frac{\rho}{\varepsilon_0} \pi (r^2 - R_1^2) L.$$

Equating these expressions with (2.44) it results, at external points as  $P$ ,

$$E(r > R_2) = \frac{\rho\pi(R_2^2 - R_1^2)L}{\varepsilon_0 2\pi r L} = \frac{\rho(R_2^2 - R_1^2)}{2\varepsilon_0 r},$$

And at internal points as  $P'$ ,

$$E(R_1 < r < R_2) = \frac{\rho\pi(r^2 - R_1^2)L}{\varepsilon_0 2\pi r L} = \frac{\rho(r^2 - R_1^2)}{2\varepsilon_0 r}.$$

If we combine both results and we express them using vectors, it results

$$\mathbf{E} = \begin{cases} 0 & r \leq R_1, \\ \frac{\rho(r^2 - R_1^2)}{2\varepsilon_0 r} \mathbf{u}_\rho = \frac{q}{2\pi\varepsilon_0 r} \frac{(r^2 - R_1^2)}{(R_2^2 - R_1^2)} \mathbf{u}_\rho & R_1 \leq r \leq R_2, \\ \frac{\rho(R_2^2 - R_1^2)}{2\varepsilon_0 r} \mathbf{u}_\rho = \frac{q}{2\pi\varepsilon_0 r} \mathbf{u}_\rho & r \geq R_2, \end{cases}$$

where  $\mathbf{u}_\rho$  is the radial unit vector for cylindrical coordinates. The field at any point on the cylinder outside surface can be calculated by using either expressions, and making  $r = R_2$ .

$$\mathbf{E}_{(r=R_2)} = \frac{\rho(R_2^2 - R_1^2)}{2\varepsilon_0 R_2} \mathbf{u}_\rho = \frac{q}{2\pi\varepsilon_0 R_2} \mathbf{u}_\rho.$$

If the cylinder were solid, with radius  $R$ , the field can be obtained by replacing  $R_1 = 0$  and  $R_2 = R$ :

$$\mathbf{E} = \begin{cases} \frac{\rho r}{2\varepsilon_0} \mathbf{u}_\rho = \frac{q}{2\pi\varepsilon_0 r} \frac{r^2}{R^2} \mathbf{u}_\rho & r \leq R, \\ \frac{\rho R^2}{2\varepsilon_0 r} \mathbf{u}_\rho = \frac{q}{2\pi\varepsilon_0 r} \mathbf{u}_\rho & r \geq R. \end{cases}$$

And at any point on the surface

$$\mathbf{E}_{(r=R)} = \frac{\rho R}{2\varepsilon_0} \mathbf{u}_\rho = \frac{q}{2\pi\varepsilon_0 R} \mathbf{u}_\rho.$$

(b) If the cylinder is metallic (conductor) and in electrostatic balance, charge is only distributed on the external surface and its distribution is uniform.<sup>16</sup> Charge superficial density  $\sigma$  can be calculated from (2.4), considering a cylinder with length  $L$ :

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<sup>16</sup>If the superficial charge density were not uniform, and due to the cylindrical symmetry, the inner field in the conductor would not be null, against the electrostatic balance hypothesis.

$$Q = \int_S \sigma dS = \sigma \int_S dS = \sigma 2\pi R_2 L \Rightarrow \sigma = \frac{Q/L}{2\pi R_2} = \frac{q}{2\pi R_2}.$$

The field at points inside this surface ( $r < R_2$ ) is null, due to the symmetry of the charge distribution. To calculate the field at external points ( $r \geq R_2$ ) we do the same as in section (a). Calculating the flux through the Gaussian surface, the same (2.44) is obtained. Applying Gauss' law, it results:

$$\Phi_{E(r > R_2)} = \frac{q_{in}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_S \sigma dS = \frac{\sigma}{\varepsilon_0} S_{lat} = \frac{\sigma}{\varepsilon_0} 2\pi R_2 L = \frac{qL}{\varepsilon_0},$$

where  $S_{lat}$  is the lateral surface area of the external cylindrical surface, where the charge is distributed. If we calculate with (2.44) it results

$$E(r > R_2) = \frac{\sigma 2\pi R_2 L}{\varepsilon_0 2\pi r L} = \frac{\sigma R_2}{\varepsilon_0 r} = \frac{q}{\varepsilon_0 2\pi r}.$$

Combining all results,

$$\mathbf{E} = \begin{cases} 0 & r < R_2, \\ \frac{\sigma R_2}{\varepsilon_0 r} \mathbf{u}_\rho = \frac{q}{\varepsilon_0 2\pi r} \mathbf{u}_\rho & r > R_2, \end{cases}$$

where  $\mathbf{u}_\rho$  is the radial unit vector for cylindrical coordinates. The field at a closed point to the external surface of the cylinder can be calculated with  $r = R_2$ :

$$\mathbf{E}_{(r=R_2)} = \frac{\sigma}{\varepsilon_0} \mathbf{u}_\rho,$$

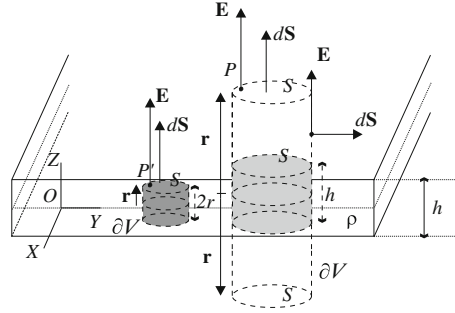
which is the known value for the field at points near the conductor surface.

**2.7** Determine the electric field and the potential produced at any point by a very large plate, with thickness  $d$ , on the following cases: (a) non conductor and uniformly charged with density  $\rho$ ; and (b) metallic and with electrostatic balance, with charge density  $\sigma$ . Particularize these results to the case in which the plate thickness is null (infinite plane). Note: take as the potential reference its central plane.

### Solution

(a) Let's firstly consider the case in which charge is uniformly distributed on the plate. Due to symmetry of the charge distribution, every point at the same distance from the central plane of the plate and far away from the ends has the same electric field value,

**Fig. 2.12** Gaussian surfaces and field vectors to apply Gauss' law to a uniformly charged plate



which is also perpendicular to the plate.<sup>17</sup> The problem can be solved by applying Gauss' law (2.28). As a Gaussian surface  $\partial V$ , and due to the symmetry, it can be chosen a straight cylinder (any parallelepiped surface would also be valid), with its axis perpendicular to the plate, with one of its bases passing through point where the field is calculated, and the other base symmetric to the previous one, referring to the central plane of the plate (Fig. 2.12). Two different expressions are obtained for the electric field, depending on the point to be studied, if it is outside or inside the plate ( $P$  and  $P'$  in Fig. 2.12), since charge inside the Gaussian surface has a different expression.

Electric field at any point on the Gaussian surface is perpendicular to the plate, outward if we suppose the plate positively charged, and with the same magnitude at every point of the two cylinder bases. The flux through the lateral surface of either gaussian cylinder  $\partial V$  is null, since  $\mathbf{E}$  and  $d\mathbf{S}$  are perpendicular at any point of this surface. There is only flux through the bases, it is

$$\Phi_E = \oint_{\partial V} \mathbf{E} \cdot d\mathbf{S} = \int_{B_{\text{upp}}} E dS + \int_{B_{\text{low}}} E dS = E 2S. \quad (2.45)$$

It should be noticed that  $S$  is the area of the upper base  $B_{\text{upp}}$  and that of the lower base  $B_{\text{low}}$ .

If we apply Gauss' law, it is obtained for an external point  $P$ ,

$$\Phi_E = \frac{q_{\text{in}}}{\varepsilon_0} = \frac{\rho Sh}{\varepsilon_0}.$$

Charge inside the Gaussian surface is only inside the cylinder of height  $h$  (marked in light grey in Fig. 2.12). That's the reason that the charged volume is just  $Sh$ . Equating this expression with (2.45), and solving, it results

<sup>17</sup>To check this, consider the field produced at a point by an element and its symmetric pair with respect to a perpendicular to the plate at the considered point. The components parallel to the plate have the same magnitude, since they are at the same distance, and opposite direction, so we need only to add the two normal components of the electric field.

$$E_{\text{ext}} = \frac{\rho h}{2\varepsilon_0}.$$

If we apply Gauss' law for an internal point  $P'$ , inside the plate, the flux is

$$\Phi_E = \frac{q_{\text{in}}}{\varepsilon_0} = \frac{\rho 2r S}{\varepsilon_0}$$

The cylinder intersection with the plate is the entire cylinder with height  $2r$  (marked in dark grey in Fig. 2.12). Equating this expression with (2.45) and solving, it results

$$E_{\text{in}} = \frac{\rho r}{\varepsilon_0}.$$

If it is expressed with vector notation, using the distance  $r$  to the central plane of the plate or Cartesian coordinates, it results

$$\mathbf{E} = \begin{cases} \frac{\rho r}{\varepsilon_0} \frac{\mathbf{r}}{r} = \frac{\rho r}{\varepsilon_0} \text{sgn}(z) \mathbf{u}_z & r \leq h/2 \quad (|z| \leq h/2), \\ \frac{\rho h}{2\varepsilon_0} \frac{\mathbf{r}}{r} = \frac{\rho h}{2\varepsilon_0} \text{sgn}(z) \mathbf{u}_z & r \geq h/2 \quad (|z| \geq h/2), \end{cases} \quad (2.46)$$

where  $(x, y, z)$  are the coordinates of field point  $P$  and  $\text{sgn}(z)$  the signum function of  $z$ , which indicates that the direction of electric field is downward at points under the central plane of the plate.

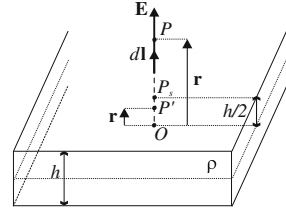
To determine the potential at any point  $P$ , and due to the fact that the potential reference is on the central plane of the plate, the circulation of the electric field must be calculated from the point  $P$  to any point of the central plane of the plate. Since circulation is independent of the chosen path, and any point of the central plane has null potential, we take as the circulation line the perpendicular one from the point to the plane, for which  $\mathbf{E}$  and  $d\mathbf{l} = dr \frac{\mathbf{r}}{r}$  are parallel,<sup>18</sup> as shown in Fig. 2.13. If we consider an internal point  $P'$  in Fig. 2.13 at a distance  $r$  from the central plane of the plate and if  $O$  is the point of the central plane perpendicular to  $P'$ , which has null potential, it results,

$$V_{P'} = V_{P'} - V_O = \int_{P'}^O \mathbf{E}_{\text{in}} \cdot d\mathbf{l} = \int_r^0 \frac{\rho}{\varepsilon_0} r dr = \left. \frac{\rho}{2\varepsilon_0} r^2 \right|_r^0 = -\frac{\rho r^2}{2\varepsilon_0}.$$

To determine the potential at an external point  $P$ , since the electric field expression is different depending on whether the point is inside or outside the plate, it is necessary to circulate  $\mathbf{E}$  from  $P$  to a point  $P_s$  on the surface using the expression for an external

<sup>18</sup>The same result is achieved without taking a circulation line in which  $\mathbf{E}$  and  $d\mathbf{l}$  are parallel, as it was already said in Problem 2.5.

**Fig. 2.13** Calculation of the potential of a charged plate



field, and then from this point  $P_s$  to the centre of the plate, using the expression for the field at internal points. The result is

$$\begin{aligned}
 V_P &= V_P - V_O = \int_P^O \mathbf{E} \cdot d\mathbf{l} = \int_P^{P_s} \mathbf{E}_{\text{ext}} \cdot d\mathbf{l} + \int_{P_s}^O \mathbf{E}_{\text{in}} \cdot d\mathbf{l} \\
 &= \int_r^{h/2} \frac{\rho h}{2\varepsilon_0} dr + \int_{h/2}^0 \frac{\rho r}{\varepsilon_0} dr = \frac{\rho h}{2\varepsilon_0} r \Big|_r^{h/2} + \frac{\rho}{2\varepsilon_0} r^2 \Big|_{h/2}^0 \\
 &= \frac{\rho}{2\varepsilon_0} \left( \frac{h^2}{2} - hr - \frac{h^2}{4} \right) = \frac{\rho h}{2\varepsilon_0} \left( \frac{h}{4} - r \right).
 \end{aligned}$$

It can be observed that calculated potentials for point  $P$  and for point  $P'$  are negative, which coincides with the fact that field  $\mathbf{E}$  has the direction of decreasing potentials. (b) If the plate is a conductor, charge is distributed over the lower and upper surfaces, since interior charge in conductors is null. Charge density  $\sigma$  on the surfaces must be homogeneous, because if not, field inside the plate wouldn't be null, as it has to be a balanced conductor. The field at internal points of the plate is therefore null. To calculate the field at external points, symmetry reasonings are the same as the ones in section (a). The Gaussian surface is the same as before, but now the charge is only in the intersection of the Gaussian surface with the plate surfaces (Fig. 2.14). The flux through the surface is obtained as in (2.45),

$$\Phi_E = E2S.$$

If Gauss' law is applied the result is

$$\Phi_E = \frac{q_{\text{in}}}{\varepsilon_0} = \frac{\sigma 2S}{\varepsilon_0}.$$

If both expressions are equated, the result is

$$E_{\text{ext}} = \frac{\sigma}{\varepsilon_0},$$



**2.8** We have a wire  $AB$  with length  $l$  and its line charge density is  $\lambda_1 = \lambda(1 + kx)$ , where  $x$  is the distance of a point of the wire to the central point  $M$  of segment  $OA$  (Fig. 2.15), and  $\lambda$  and  $k$  are two known constants. Perpendicularly to this wire at a distance  $a$  of its extreme  $A$ , an infinite wire with line charge density  $\lambda_2 = \lambda$  is placed. Determine the electric field at point  $M$ .

### Solution

To solve the problem, the superposition principle for electric fields is applied: fields at point  $M$  is the addition of the fields produced by both wires at that point. To calculate the field of wire  $AB$  we consider Fig. 2.16, where distance from any element of charge  $dq' = \lambda_1 dx$  to point  $M$  is expressed by variable  $x$ . Field  $d\mathbf{E}_1$  produced at point  $M$  by the differential element of charge  $dq'$  is given by (2.15):

$$\begin{aligned} d\mathbf{E}_1 &= \frac{dq'}{4\pi\epsilon_0 d^2} \mathbf{u} = \frac{1}{4\pi\epsilon_0} \frac{\lambda_1 dx}{x^2} (-\mathbf{u}_x) = \\ &= \frac{1}{4\pi\epsilon_0} \frac{\lambda(1 + kx)dx}{x^2} (-\mathbf{u}_x), \end{aligned}$$

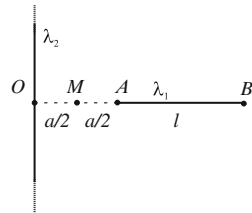
To calculate total field due to wire  $AB$ , the superposition principle (2.16) is applied:

$$\begin{aligned} \mathbf{E}_1 &= \int_{a/2}^{l+a/2} \frac{\lambda}{4\pi\epsilon_0} \frac{(1 + kx)dx}{x^2} (-\mathbf{u}_x) = \frac{\lambda}{4\pi\epsilon_0} \left( \int_{a/2}^{l+a/2} \frac{1}{x^2} dx + \int_{a/2}^{l+a/2} \frac{k}{x} dx \right) (-\mathbf{u}_x) = \\ &= \frac{\lambda}{4\pi\epsilon_0} \left( \frac{-1}{x} + k \ln x \right) \Big|_{a/2}^{l+a/2} (-\mathbf{u}_x) = \frac{\lambda}{4\pi\epsilon_0} \left[ \frac{2}{a} - \frac{2}{a+2l} + k \ln \frac{l+a/2}{a/2} \right] (-\mathbf{u}_x) = \\ &= \frac{\lambda}{2\pi\epsilon_0} \left[ -\frac{1}{a} + \frac{1}{a+2l} - \frac{k}{2} \ln \frac{a+2l}{a} \right] \mathbf{u}_x. \end{aligned}$$

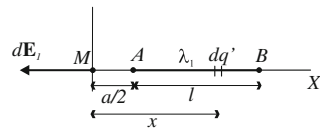
It should be observed that the limits of the integral are the ends of the charged wire.

To calculate the field produced by the infinite wire with density  $\lambda_2$  the problem's 2.3 result is applied,

**Fig. 2.15** Figure of Problem 2.8



**Fig. 2.16** Field produced by the finite wire





$$\mathbf{E}_2 = \frac{\lambda_2}{2\pi\epsilon_0 r} \mathbf{u}_\rho = \frac{\lambda}{2\pi\epsilon_0 a/2} \mathbf{u}_x = \frac{\lambda}{\pi\epsilon_0 a} \mathbf{u}_x,$$

where distance from the wire to the point field is  $a/2$  and the radial unit vector is, in this case,  $\mathbf{u}_x$ .

The electric field at point  $M$ , applying superposition principle, is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \frac{\lambda}{2\pi\epsilon_0} \left( \frac{1}{a} + \frac{1}{a+2l} - \frac{k}{2} \ln \frac{a+2l}{a} \right) \mathbf{u}_x.$$

- 2.9** Two straight conductors, parallel and infinite, with respective density charge  $\lambda_1 = \lambda$  and  $\lambda_2 = -\lambda$  are separated by a distance  $d$ . Calculate the potential difference between points  $A$  and  $B$  in Fig. 2.17.

### Solution

To calculate the potential difference between  $A$  and  $B$ ,  $V_A - V_B$ , it is necessary to know the electrostatic field at every point in a line between these points. For this, the superposition principle is applied, and the resulting field at each point is the addition of the fields produced by each wire independently. From Problem 2.3 it is known that electric field produced by an infinite line is perpendicular to this line, and with the same magnitude at every point of a cylindrical surface whose axis is the line. The electric field at  $P$  (Fig. 2.18) is given by (2.36), which for the line of density  $\lambda_1$  is

$$\mathbf{E}_1 = \frac{\lambda_1}{2\pi\epsilon_0 r_1} \frac{\mathbf{r}_1}{r_1} = \frac{\lambda}{2\pi\epsilon_0 r_1} \frac{\mathbf{r}_1}{r_1}.$$

The field produced by the line of density  $\lambda_2$  is

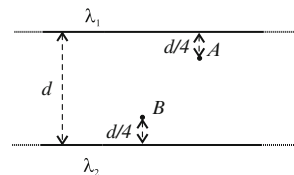
$$\mathbf{E}_2 = \frac{\lambda_2}{2\pi\epsilon_0 r_2} \frac{\mathbf{r}_2}{r_2} = \frac{-2\lambda}{2\pi\epsilon_0 r_2} \frac{\mathbf{r}_2}{r_2} = \frac{\lambda}{\pi\epsilon_0(d-r_1)} \frac{\mathbf{r}_1}{r_1}.$$

Adding both fields, the total electric field at any point  $P$  is obtained

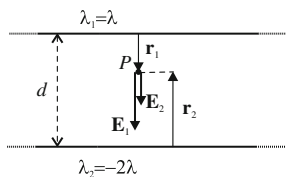
$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \frac{\lambda}{2\pi\epsilon_0} \left( \frac{1}{r_1} + \frac{2}{d-r_1} \right) \frac{\mathbf{r}_1}{r_1}.$$

To calculate the potential difference between  $A$  and  $B$ , a circulation of  $\mathbf{E}$  from  $A$  to  $B$  must be done. A line from  $A$  to  $B'$  has been taken (Fig. 2.19), where  $\mathbf{E}$  and

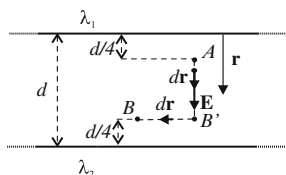
**Fig. 2.17** Figure of Problem 2.9



**Fig. 2.18** Fields produced by the two infinite wires



**Fig. 2.19** Scheme to calculate the potential difference between A and B



$d\mathbf{r}$  are parallel, and then from  $B'$  to  $B$  where they are perpendicular, and circulation null.<sup>19</sup> Applying (2.22),

$$\begin{aligned} V_A - V_B &= \int_{r_A}^{r_B} \mathbf{E} \cdot d\mathbf{r} = \\ &= \int_{d/4}^{3d/4} \frac{\lambda}{2\pi\epsilon_0} \left( \frac{1}{r} + \frac{2}{d-r} \right) dr = \\ &= \frac{\lambda}{2\pi\epsilon_0} \left( \ln r \Big|_{d/4}^{3d/4} - 2 \ln(d-r) \Big|_{d/4}^{3d/4} \right) = \frac{\lambda}{2\pi\epsilon_0} 3 \ln 3. \end{aligned}$$

**2.10** We have an isolated spherical conductor whose radius is  $R_1 = 4$  cm, and whose potential is 9000 V referring to ground. After, it is surrounded with a concentric spherical conducting layer, with inner radius  $R_2 = 8$  cm and exterior one  $R_3 = 10$  cm, isolated and with null total charge. Determine charges and potentials on the inner conductor, as well as the conducting layer, for the following cases: (a) Inner conductor and conducting layer isolated. (b) If the conducting layer is connected to ground. (c) If the layer is once again isolated and the conductor is connected to ground by a conducting wire that goes through a small hole in the layer.

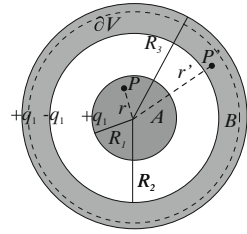
### Solution

(a) Note the charge distribution is not known. The isolated spherical conductor will have certain charge  $q_1$ , since its potential is not zero. If the expression of potential for a spherical conductor is applied ((2.43) of Problem 2.5), it results

$$V = \frac{q_1}{4\pi\epsilon_0 R_1}.$$

<sup>19</sup>As it was already indicated in Problem 2.5, it is not necessary to consider a specific circulation line, since  $\mathbf{r} \cdot d\mathbf{r} = |\mathbf{r}|d|\mathbf{r}|$ .

**Fig. 2.20** Isolated conducting sphere and isolated conducting layer



From where conductor's charge  $q_1$  is obtained,

$$q_1 = 4\pi\epsilon_0 R_1 V = 40 \text{ nC}.$$

If the conductor is surrounded by the conducting layer, Fig. 2.20, charges for both conductors are reorganized, until the balance is reached, all properties in Sect. 2.8 are verified. Since conductor A is isolated, charge  $q_1$  remains, and this charge can only be on its outsider surface. Due to the spherical symmetry of the figure, it will be uniformly distributed over the surface, and thus the electric field inside the conductor is null. On the conducting layer B, charges are distributed so that the field inside it is null. If Gauss' law (2.28) is applied to a Gaussian surface  $\partial V$  totally inside the conductor (discontinuous line in the figure),

$$\Phi_E = \oint_{\partial V} \mathbf{E} \cdot d\mathbf{S} = 0 = \frac{q_{\text{in}}}{\epsilon_0} \Rightarrow q_{\text{in}} = 0 = q_1 + q_{B,\text{in}},$$

where  $q_{B,\text{in}}$  is the charge of the inner surface of the conducting layer B. It should be observed that the flux is null because the field at every point of  $\partial V$  is zero. Therefore

$$q_{B,\text{in}} = -q_1.$$

If the principle of conservation of charge is applied to conductor B, the external surface charge of B is

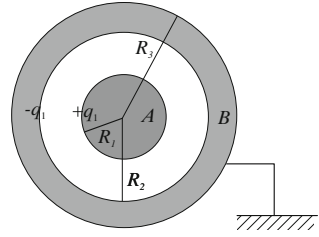
$$q_B = 0 = -q_1 + q_{B,\text{ex}} \Rightarrow q_{B,\text{ex}} = +q_1.$$

The charge distribution is the one shown in Fig. 2.20.

Potential at any point is derived from the superposition of potentials created by each of the charge distributions. For every distribution, (2.43) obtained in Problem 2.5 is applied. For conductor A, the distance  $r$  from any interior point  $P$  to the centre is lower (or equal) than the distance from the centre to the charge distributions ( $r \leq R_1, r < R_2, r < R_3$ ), so it results

$$V_A = \frac{1}{4\pi\epsilon_0} \left( \frac{q_1}{R_1} - \frac{q_1}{R_2} + \frac{q_1}{R_3} \right) = 9 \cdot 10^9 \cdot 40 \cdot 10^{-9} \left( \frac{1}{0.04} - \frac{1}{0.08} + \frac{1}{0.1} \right) = 8100 \text{ V}.$$

**Fig. 2.21** Isolated conducting sphere and grounded conducting layer



For conductor  $B$ , the distance  $r'$  from any interior point  $P'$  to the centre results in  $r' > R_1$ ,  $r' \geq R_2$ ,  $r' \leq R_3$ , so if (2.43) is applied,

$$V_B = \frac{1}{4\pi\epsilon_0} \left( \frac{q_1}{r'} - \frac{q_1}{r'} + \frac{q_1}{R_3} \right) = 9 \cdot 10^9 \cdot \frac{40 \cdot 10^{-9}}{0.1} = 3600 \text{ V.}$$

(b) Connecting the conducting layer to ground, Fig. 2.21, we equalize the potentials of conductor  $B$  and the ground. Ground potential is taken as a reference (0 V) and, therefore,

$$V_B = 0.$$

Since there is no potential difference between  $B$  and ground, there cannot exist any electric field between them (due to (2.18),  $\mathbf{E} = -\nabla V$ ). The field is null in  $B$ , as well as outside the conductor  $B$ . From (2.29) ( $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ ), it is obtained that on the outside surface of conductor  $B$  there cannot be charges. The other charge distributions remain the same, as seen by reapplying the reasoning from section (a). It can be observed how grounded conductor  $B$  does not have a null charge (it is not an isolated system), but it remains negatively charged. If (2.43) is applied, the result for conductor  $A$  is,

$$V_A = \frac{1}{4\pi\epsilon_0} \left( \frac{q_1}{R_1} - \frac{q_1}{R_2} \right) = 9 \cdot 10^9 \cdot 40 \cdot 10^{-9} \left( \frac{1}{0.04} - \frac{1}{0.08} \right) = 4500 \text{ V.}$$

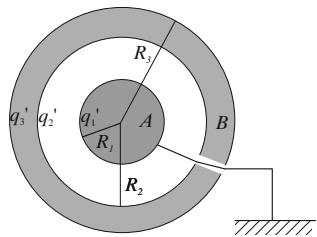
A device like this, a grounded conductor which surrounds another one, is the base of *electrostatic shields* and it is called a **Faraday cage**. Even though charge or potential inside it is changed, the electric field and potential outside it is always zero. Also, any external electric field would affect neither the electric field nor the potential of the conductors.

(c) If conducting layer  $B$  is disconnected from ground, its charge,  $-q_1 = -40 \text{ nC}$ , remains and distributes between the inner and outside surface of  $B$ ,

$$-q_1 = q'_2 + q'_3,$$

as seen in Fig. 2.22.

**Fig. 2.22** Grounded conducting sphere and isolated conducting layer



Conductor  $A$  does not keep its charge anymore, since it is grounded. New charge is called  $q'_1$ . Applying the reasoning from section (a), the charge on the inner surface of conductor  $B$  will be

$$q'_2 = -q'_1.$$

It is also known that conductor  $A$  potential is zero, since it is grounded. If (2.43) is applied to conductor  $A$  the result is

$$V_A = 0 = \frac{1}{4\pi\epsilon_0} \left( \frac{q'_1}{R_1} + \frac{q'_2}{R_2} + \frac{q'_3}{R_3} \right) \Rightarrow \frac{q'_1}{0.04} + \frac{q'_2}{0.08} + \frac{q'_3}{0.1} = 0.$$

With these three equations the new values of the charges are obtained,

$$q'_1 = \frac{4}{9} q_1 = 17.8 \text{ nC}, \quad q'_2 = -\frac{4}{9} q_1 = -17.8 \text{ nC}, \quad q'_3 = -\frac{5}{9} q_1 = -22.2 \text{ nC}.$$

If (2.43) is applied we obtain conductor  $B$  potential,

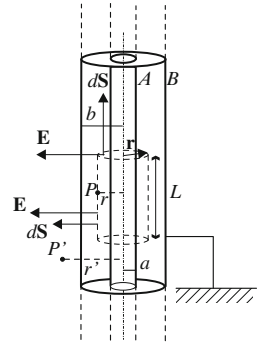
$$V_B = \frac{1}{4\pi\epsilon_0} \left( \frac{q'_1}{r'} + \frac{q'_2}{r'} + \frac{q'_3}{R_3} \right) = 9 \cdot 10^9 \cdot \frac{-22.2 \cdot 10^{-9}}{0.1} = -2000 \text{ V}.$$

**2.11** Consider two coaxial conductor cylindrical surfaces,  $A$  and  $B$ , with infinite length, whose radii are  $a$  and  $b$ . Outer conductor  $B$  is grounded and potential of inner conductor  $A$  is  $V_a$ . The space between both conductors and outside of conductor  $B$  is a vacuum. (a) Calculate surface charge densities on both conductors. (b) If  $P$  is a point between  $A$  and  $B$ , and  $P'$  is outside of conductor  $B$ , calculate potential difference  $V_P - V_{P'}$ .

### Solution

(a) Figure 2.23 shows both cylinders. Firstly, since charges can freely move inside conductors, we must study how charges distribute inside both conductors. As it was seen in Sect. 2.8, charges distribute on conductor surfaces, so the field inside them is zero. Due to the symmetry, charge has to be distributed uniformly on the surface. The

**Fig. 2.23** Set of two coaxial hollow cylindrical conductors, the exterior one connected to ground



entire charge of conductor  $A$  is distributed on its external surface.<sup>20</sup> Let's suppose that  $q_a$  is the charge on conductor  $A$  for a finite length  $L$ . If Gauss' law for a coaxial cylindrical surface  $\partial V$  totally inside conductor  $B$ , with length  $L$  is applied, the result is, following the same reasoning as Problem 2.10,

$$\Phi_E = \oint_{\partial V} \mathbf{E} \cdot d\mathbf{S} = 0 = \frac{q_{\text{in}}}{\varepsilon_0} \Rightarrow q_{\text{in}} = 0 = q_a + q_{B,\text{in}},$$

where  $q_{B,\text{in}}$  is the charge on inner surface of conductor  $B$ . It should be observed that the flux is null since the field on every point of the considered Gaussian surface is zero. Therefore

$$q_{B,\text{in}} = -q_a.$$

As conductor  $B$  is grounded, its potential is zero. The electric field outside  $B$  is zero and charge on the outside surface of the cylinder is also zero, as it was previously explained on Problem 2.10.

To obtain charge densities, we find the electric field's expression at any point  $P$  between both conductors. Due to the cylindrical symmetry, the problem can be solved by applying Gauss' law (2.28) in a similar way as it was done in Problem 2.6. To do it, we take as a cylindrical Gaussian surface  $\partial V$ , with any length  $L$ , with the same axis as the cylinder, and with radius  $r$  so that the surfaces passes through point  $P$  where we want to calculate the field (discontinuous line in Fig. 2.23). The flux through the bases of the Gaussian surface is zero, since  $d\mathbf{S}$  and  $\mathbf{E}$  are perpendicular at any point of the bases. So it only remains to calculate the flux through the lateral surface, where  $d\mathbf{S}$  and  $\mathbf{E}$  are parallel:

$$\Phi_E = \oint_{\partial V} \mathbf{E} \cdot d\mathbf{S} = \int_{S_{\text{lat}}} E dS = E \int_{S_{\text{lat}}} dS = E 2\pi r L,$$

<sup>20</sup>If there were a charge on the internal surface of conductor  $A$ , an electric field inside the conductor should exist, but this would contradict electrostatic equations.

where  $S_{\text{lat}}$  is the cylinder lateral surface. On the other side, if we apply Gauss' law, we obtain

$$\Phi_E = \frac{q_{\text{in}}}{\varepsilon_0} = \frac{\sigma_a 2\pi a L}{\varepsilon_0},$$

where  $\sigma_a$  is charge density of cylinder  $A$ . It should be observed that  $q_{\text{in}}$  is the charge inside the Gaussian surface, and that there's only charge on the lateral surface of cylinder  $A$ , whose radius is  $a$ . If both expressions are equalized, electric field is obtained,

$$E = \frac{\sigma_a a}{\varepsilon_0 r} \Rightarrow \mathbf{E} = \frac{\sigma_a a}{\varepsilon_0 r} \mathbf{u}_\rho,$$

where  $\mathbf{u}_\rho$  is the radial unit vector for cylindrical coordinates. As additional information, the potential of both conductors is known. The potential difference  $V_A - V_B$  between them is  $V_a - 0 = V_a$ , since  $B$  is grounded. If we circulate the electric field by following a line perpendicular to the cylinder axis, so that  $\mathbf{E}$  and  $d\mathbf{l}$  are parallel, the result is

$$V_A - V_B = V_a = \int_a^b \mathbf{E} \cdot d\mathbf{l} = \int_a^b \frac{\sigma_a a}{\varepsilon_0 r} dr = \frac{\sigma_a a}{\varepsilon_0} \ln \frac{b}{a},$$

from which charge density  $\sigma_a$  of conductor  $A$  can be calculated,

$$\sigma_a = \frac{V_a \varepsilon_0}{a \ln \frac{b}{a}}.$$

On a piece of cylinder with length  $L$ ,  $A$ 's charge will be  $q_a = \sigma_a 2\pi a L$ . As charge at inner surface of conductor  $B$  is  $-q_a$ , it results that its density  $\sigma_b$  can be obtained from

$$\sigma_a 2\pi a L = -\sigma_b 2\pi b L \Rightarrow \sigma_b = -\frac{a}{b} \sigma_a = -\frac{V_a \varepsilon_0}{b \ln \frac{b}{a}}.$$

The outer surface of conductor  $B$  mentioned previously is not charged due to the fact that it is grounded.

(b) To calculate the potential between points  $P$  and  $P'$  in Fig. 2.23, at a distance  $r$  and  $r'$  from the axis, we circulate the electric field between both points,<sup>21</sup>

$$V_P - V_{P'} = V_P = \int_r^{r'} \mathbf{E} \cdot d\mathbf{l} = \int_r^{r'} \frac{\sigma_a a}{\varepsilon_0 r} dr = \frac{\sigma_a a}{\varepsilon_0} \ln \frac{b}{r} = \frac{\ln \frac{b}{r}}{\ln \frac{b}{a}} V_a.$$

---

<sup>21</sup> We will bear on mind the property of every vector  $\mathbf{r}$ :  $\mathbf{r} \cdot d\mathbf{r} = |\mathbf{r}|d|\mathbf{r}|$ . It can be reasoned in a similar way by taking the following circulating line: first the perpendicular to the axis from  $P$ , where  $\mathbf{E}$  and  $d\mathbf{l}$  are parallel, until we reach the distance of  $r'$  from the center of the cylinders, and then by circulating parallel to the cylinder's axis, whose circulation's value is null, since  $\mathbf{E}$  and  $d\mathbf{l}$  are perpendicular.

It should be observed that  $P'$ 's potential is null: it has ground potential since there is no electric field outside of conductor  $B$ . This is also the reason to use  $b$  as the superior limit of the integral, and not  $r'$ : between  $b$  and  $r'$ , the electric field is zero.

## Problems C

**2.12** Determine the field produced at the coordinate's origin by a circular-shaped arc wire with radius  $R$  in Fig. 2.24, symmetrically placed with respect to the  $X$  axis, and charged with positive charge density  $\lambda = k|\sin \theta|$ , where  $\theta$  is the angle with the horizontal of the position vector of any differential element in the wire, and  $k$  is a constant.

### Solution

The wire in the problem has a symmetric charge with respect to the  $X$  axis. If any charge element  $dq$  is taken (Fig. 2.25), field  $d\mathbf{E}$  produced by this element of charge has the same magnitude, and forms the same angle  $\theta$  with  $X$  axis as the field produced by its symmetric  $dq'$  in reference to this axis. Components parallel to the  $Y$  axis of these fields cancel; components along the  $X$  axis will be added. The total field produced by a wire has, therefore, a null vertical component, while a horizontal component is twice the one produced by the piece of wire placed in the first quadrant.

Considering the field produced by element  $dq = \lambda dl$ , which according to (2.15) is

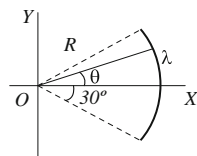
$$d\mathbf{E} = \frac{\lambda dl}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^2} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} = \frac{k|\sin \theta| dl}{4\pi\epsilon_0 R^2} (-\cos \theta \mathbf{u}_x - \sin \theta \mathbf{u}_y),$$

where  $\mathbf{r} = 0$  is field point  $O$  position and  $\mathbf{r}' = R \cos \theta \mathbf{u}_x + R \sin \theta \mathbf{u}_y$  is the source point  $dq$  position.

The total field produced by the wire is

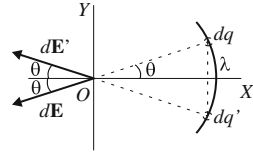
$$\begin{aligned} \mathbf{E} &= \int d\mathbf{E} = \int_L \frac{k|\sin \theta| dl}{4\pi\epsilon_0 R^2} (-\cos \theta \mathbf{u}_x - \sin \theta \mathbf{u}_y) = \\ &= \int_{L+} \frac{k \sin \theta dl}{4\pi\epsilon_0 R^2} (-\cos \theta \mathbf{u}_x - \sin \theta \mathbf{u}_y) + \int_{L-} \frac{k(-\sin \theta) dl}{4\pi\epsilon_0 R^2} (-\cos \theta \mathbf{u}_x - \sin \theta \mathbf{u}_y), \end{aligned}$$

**Fig. 2.24** Figure of Problem 2.12





**Fig. 2.25** Field produced at coordinate's origin by the circular-shaped wire



where  $L+$  indicates the piece of wire in the first quadrant and  $L-$  indicates the piece of wire in the fourth quadrant. Length element  $dl$ , arch of circle, can be expressed as a function of angle  $\theta$ ,  $dl = R d\theta$ , and the integral results in<sup>22</sup>

$$\begin{aligned}
 \mathbf{E} &= \int_0^{\pi/6} \frac{k \sin \theta R d\theta}{4\pi\epsilon_0 R^2} (-\cos \theta \mathbf{u}_x - \sin \theta \mathbf{u}_y) + \int_{-\pi/6}^0 \frac{k(-\sin \theta) R d\theta}{4\pi\epsilon_0 R^2} (-\cos \theta \mathbf{u}_x - \sin \theta \mathbf{u}_y) = \\
 &= -\frac{k}{4\pi\epsilon_0 R} \left[ \left( \int_0^{\pi/6} \sin \theta \cos \theta d\theta - \int_{-\pi/6}^0 \sin \theta \cos \theta d\theta \right) \mathbf{u}_x \right. \\
 &\quad \left. + \left( \int_0^{\pi/6} \sin^2 \theta d\theta - \int_{-\pi/6}^0 \sin^2 \theta d\theta \right) \mathbf{u}_y \right] = \\
 &= -\frac{k}{4\pi\epsilon_0 R} \left[ \left( \left. \frac{\sin^2 \theta}{2} \right|_0^{\pi/6} - \left. \frac{\sin^2 \theta}{2} \right|_{-\pi/6}^0 \right) \mathbf{u}_x \right. \\
 &\quad \left. + \left( \left. \left( \frac{2\theta - \sin 2\theta}{4} \right) \right|_0^{\pi/6} - \left. \left( \frac{2\theta - \sin 2\theta}{4} \right) \right|_{-\pi/6}^0 \right) \mathbf{u}_y \right] \\
 &= -\frac{k}{4\pi\epsilon_0 R} \left[ \left( \frac{1}{8} - 0 - 0 + \frac{1}{8} \right) \mathbf{u}_x + \frac{1}{4} (\pi/3 - \sin(\pi/3) - \pi/3 - \sin(-\pi/3)) \mathbf{u}_y \right] \\
 &= -\frac{k}{16\pi\epsilon_0 R} \mathbf{u}_x.
 \end{aligned}$$

The same results can be reached bearing in mind the previous symmetry considerations,

$$\begin{aligned}
 \mathbf{E} &= \int_L \frac{k |\sin \theta| dl}{4\pi\epsilon_0 R^2} (-\cos \theta \mathbf{u}_x - \sin \theta \mathbf{u}_y) = 2 \int_{L+} \frac{k \sin \theta dl}{4\pi\epsilon_0 R^2} (-\cos \theta \mathbf{u}_x) \\
 &= -2 \int_0^{\pi/6} \frac{k \sin \theta \cos \theta R d\theta}{4\pi\epsilon_0 R^2} \mathbf{u}_x = \\
 &= -\frac{k}{2\pi\epsilon_0 R} \int_0^{\pi/6} \sin \theta \cos \theta d\theta \mathbf{u}_x = -\frac{k}{2\pi\epsilon_0 R} \left. \frac{\sin^2 \theta}{2} \right|_0^{\pi/6} \mathbf{u}_x \\
 &= -\frac{k}{2\pi\epsilon_0 R} \left( \frac{1}{8} - 0 \right) \mathbf{u}_x = -\frac{k}{16\pi\epsilon_0 R} \mathbf{u}_x.
 \end{aligned}$$

<sup>22</sup>To solve the integral, remember that  $\sin^2 \theta = (1 - \cos 2\theta)/2$ .

**2.13** The spherical crown in Fig. 2.26 (sectioned by plane  $XY$ ) is shown, whose centre is at point  $C(1\text{ m}, 0, 0)$ , with inner radius  $R_i = 20\text{ cm}$  and exterior one  $R_e = 50\text{ cm}$ , has a non homogeneous charge density  $\rho = k/r$  with  $k = 2\text{ }\mu\text{C/m}^2$  and  $r$  the distance to the crown's centre measured in meters. The wire in the figure, in  $XY$  plane, is infinitely long, makes  $45^\circ$  with  $X$  axis, and has a charge per unit of length  $\lambda = 30\text{ nC/m}$ . Determine the electric field produced at point  $A(1\text{ m}, -1\text{ m}, 0)$ .

### Solution

To calculate the field at point  $A$ , the superposition principle is applied, adding the fields produced by the spherical crown and the wire at that point. The field produced by each of the distributions can be obtained by applying Gauss' law (2.28).

For the spherical crown, the same procedure as Problem 2.5 is applied: we take a spherical Gaussian surface, concentric with the charged crown, that passes over point  $A$  on the field to be calculated (Fig. 2.27). Field  $\mathbf{E}_\rho$  is radial and to determine its magnitude, the flux through this Gaussian surface is calculated

$$\Phi_{E_\rho} = \oint_{\partial V} \mathbf{E}_\rho \cdot d\mathbf{S} = E_\rho 4\pi r^2,$$

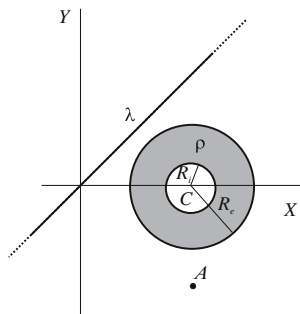
If Gauss' theorem is applied,

$$\Phi_{E_\rho} = \frac{q_{\text{in}}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_{V_{\text{in}}} \rho dV = \frac{1}{\varepsilon_0} \int_{V_{\text{in}}} \frac{k}{r} 4\pi r^2 dr = \frac{4\pi k}{\varepsilon_0} \int_{R_i}^{R_e} r dr = \frac{4\pi k}{\varepsilon_0} \frac{R_e^2 - R_i^2}{2},$$

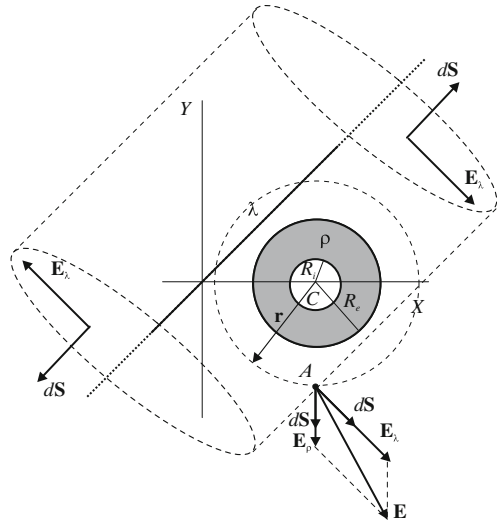
since the volume element in a sphere can be written as  $dV = 4\pi r^2 dr$ . Equating both flux calculations and taking the data, results are

$$E_\rho = \frac{0.105k}{\varepsilon_0} = 23729\text{ N/C}.$$

**Fig. 2.26** Figure of Problem 2.13



**Fig. 2.27** Fields produced by the charged distributions and Gaussian surfaces



And using vector notation, if Fig. 2.27 is observed:

$$\mathbf{E}_\rho = -23729 \mathbf{u}_y \text{ N/C.}$$

For the wire, the procedure of Problem 2.4 is applied: a cylindrical Gaussian surface is drawn in Fig. 2.27, whose axis is the wire, with any length  $L$  and passing over point  $A$ . Field  $\mathbf{E}_\lambda$  is perpendicular to the wire, and to obtain its magnitude, the flux through the surface is calculated,

$$\Phi_{E_\lambda} = \oint_{\partial V} \mathbf{E}_\lambda \cdot d\mathbf{S} = \int_{S_{\text{lat}}} E_\lambda dS = E_\lambda 2\pi r L|_{r=\sqrt{2} \text{ m}},$$

since the distance from the wire to  $A$ , the radius of the cylindrical surface, is  $\sqrt{2}$  meters long. And applying Gauss' theorem,

$$\Phi_{E_\lambda} = \frac{q_{\text{in}}}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0}.$$

Equating the two calculations of the flux, results are

$$E_\lambda = \frac{\lambda}{2\pi\varepsilon_0 r} = \frac{30 \cdot 10^{-9}}{2\pi\varepsilon_0 \sqrt{2}} = 381.5 \text{ N/C.}$$

And using vector notation

$$\mathbf{E}_\lambda = 381.5 \frac{\mathbf{u}_x - \mathbf{u}_y}{\sqrt{2}} = 270 (\mathbf{u}_x - \mathbf{u}_y) \text{ N/C.}$$

The total field is obtained by adding both fields (vectorially):

$$\mathbf{E} = \mathbf{E}_\rho + \mathbf{E}_\lambda = (270 \mathbf{u}_x - 23999 \mathbf{u}_y) \text{ N/C.}$$

**2.14** The space region defined by equation  $0 < z < 2$  (with Cartesian coordinates in meters) has a charge density  $\rho = k|z - 1|$  with  $k = 8 \mu\text{C}/\text{m}^4$ . (a) Determine the electric field value for the points in the region defined by the sphere with its centre at the coordinates' origin and radius 2 m. (b) Determine the divergence value of the electric field at the previous points. (c) Determine the electric field flux through the previous sphere surface.

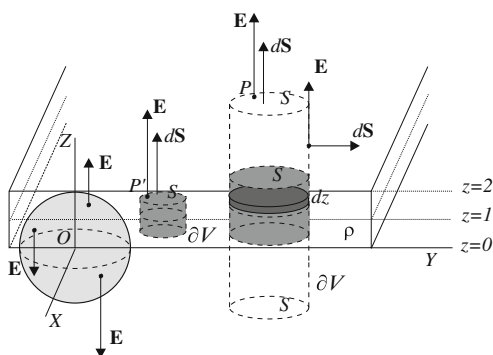
### Solution

(a) Figure 2.28 represents the region of the problem. Charge distribution is symmetric in reference to the center plane  $z = 1$ , so Gauss' law (2.28) can be easily applied. A similar procedure as the one used for the infinite plate on Problem 2.7 is followed. Now the points at the field to be calculated are a few specific points defined by the sphere indicated in the statement and light grey coloured in the figure. Points in the higher hemisphere are all the inside points of the charged plate, while points in the lower hemisphere are outside the plate. If Gauss' law is applied as in Problem 2.7, with cylindrical Gaussian surfaces indicated in Fig. 2.28 (which are the ones used before)

$$\Phi_E = \oint_{\partial V} \mathbf{E} \cdot d\mathbf{S} = \int_{B_{\text{upp}}} E dS + \int_{B_{\text{low}}} E dS = E 2S.$$

To calculate the charge inside the Gaussian surface, it should be taken into account that the charge density is variable. As the charge is symmetric in reference to the

**Fig. 2.28** Figure of Problem 2.14



medium plane, we will determine the higher half charge, where  $|z - 1| = z - 1$ , and we will double the resulting charge. For points outside of the charged region, as point  $P$ , it results

$$\Phi_E = \frac{q_{\text{in}}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_{V_{\text{in}}} \rho dV = \frac{1}{\varepsilon_0} 2 \int_1^z k(z-1) S dz = \frac{2kS}{\varepsilon_0} \left. \frac{(z-1)^2}{2} \right|_1^z = \frac{kS}{\varepsilon_0}.$$

To calculate the integral we have taken a volume element  $dV = Sdz$  (in dark grey in the figure), that represents a differential cylinder whose area is  $S$  and its height is  $dz$  and it is located at any distance from the medium plane. If we compare both flux expressions, the results are

$$E_{\text{ext}} = \frac{k}{2\varepsilon_0} = \frac{8 \cdot 10^{-6}}{2\varepsilon_0} = 4.52 \cdot 10^5 \text{ NC}^{-1},$$

If Gauss' law is applied to an interior point  $P'$ , it is obtained

$$\Phi_E = \frac{q_{\text{in}}}{\varepsilon_0} = \frac{1}{\varepsilon_0} 2 \int_1^z k(z-1) S dz = \frac{2kS}{\varepsilon_0} \left. \frac{(z-1)^2}{2} \right|_1^z = \frac{kS}{\varepsilon_0} (z-1)^2.$$

where  $z$  is the coordinate in meters of point  $P'$ . Comparing both flux expressions, results are

$$E_{\text{in}} = \frac{k}{2\varepsilon_0} (z-1)^2 = 4.52 \cdot 10^5 (z-1)^2 \text{ NC}^{-1}.$$

Vectorially, at points over the center plane ( $z = 1$ ) the sense of field vector is  $\mathbf{u}_z$  and for the ones below it is  $-\mathbf{u}_z$ .

Every point in the lower hemisphere ( $z < 0$ ) is exterior to the charged region, and since they are below the center plane, the sense of the vector field  $\mathbf{E}$  is the negative  $Z$ -axis, as can be observed in Fig. 2.28. Points in the higher hemisphere are all inside the plate. The vector field has the orientation parallel to the  $Z$ -axis in the positive direction at points above the center plane, and it has the opposite orientation at points below the medium plane. If we express the result using vector notation and Cartesian coordinates, it results, for the points in the sphere  $x^2 + y^2 + z^2 \leq 4$ :

$$\mathbf{E} = \begin{cases} -4.52 \cdot 10^5 \mathbf{u}_z \text{ NC}^{-1} & z < 0, \\ -4.52 \cdot 10^5 (z-1)^2 \mathbf{u}_z \text{ NC}^{-1} & 0 \leq z \leq 1, \\ 4.52 \cdot 10^5 (z-1)^2 \mathbf{u}_z \text{ NC}^{-1} & 1 \leq z \leq 2, \end{cases}$$

with  $z$  expressed in meters.

(b) From (2.29) the result is

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\varepsilon_0} = \begin{cases} 0 & x^2 + y^2 + z^2 \leq 4 \text{ and } z < 0, \\ \frac{k|z-1|}{\varepsilon_0} = 9.04 \cdot 10^5 |z-1| \text{ NC}^{-1} \text{ m}^{-1} & x^2 + y^2 + z^2 \leq 4 \text{ and } z \geq 0. \end{cases}$$

It can be observed that the lower hemisphere points are outside of the charged zone ( $\rho = 0$ ) and the electric field's divergence at these points is null.

(c) If Gauss' law (2.28) is applied to the surface defined by the sphere it results

$$\Phi_E = \oint_{\partial V} \mathbf{E} \cdot d\mathbf{S} = \frac{q_{\text{in}}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_{V_{\text{sph}}} \rho dV = \frac{1}{\varepsilon_0} \int_{V_{\text{upsph}}} k|z-1| dV,$$

where  $V_{\text{upsph}}$  is the upper hemisphere's volume. To calculate the integral we use spherical coordinates. It can be observed that  $r$  varies from 0 to 2 (sphere's radius), coordinate  $\phi$  goes from 0 to  $2\pi$  because the whole circumference is described, and coordinate  $\theta$  goes from 0 to  $\pi/2$ , since only the upper hemisphere is charged:

$$\Phi_E = \frac{1}{\varepsilon_0} \int_{V_{\text{upsph}}} k|z-1| dV = \frac{1}{\varepsilon_0} \int_0^R \int_0^{\pi/2} \int_0^{2\pi} k|r \cos \theta - 1| r^2 \sin \theta dr d\theta d\phi,$$

where we have taken into account that  $z = r \cos \theta$ . It should be observed that function  $|z-1| = |r \cos \theta - 1|$  has a different value depending on  $z$ , whether  $z > 1$  or  $z < 1$ . Where  $z = 1$  and the sphere's radius  $R = 2$ , results are  $\cos \theta = 1/2 \rightarrow \theta = \pi/3$ . Using two integrals to substitute the absolute value expression and solving them, results are

$$\begin{aligned} \Phi_E &= \frac{k}{\varepsilon_0} \left( \int_0^2 \int_0^{\pi/3} \int_0^{2\pi} (r \cos \theta - 1) r^2 \sin \theta dr d\theta d\phi \right. \\ &\quad \left. - \int_0^2 \int_{\pi/3}^{\pi/2} \int_0^{2\pi} (r \cos \theta - 1) r^2 \sin \theta dr d\theta d\phi \right) = \\ &= \frac{2\pi k}{\varepsilon_0} \left( \int_0^2 \int_0^{\pi/3} (r \cos \theta - 1) r^2 \sin \theta dr d\theta - \int_0^2 \int_{\pi/3}^{\pi/2} (r \cos \theta - 1) r^2 \sin \theta dr d\theta \right) = \\ &= \frac{2\pi k}{\varepsilon_0} \left( \int_0^{\pi/3} \left( 4 \cos \theta \sin \theta - \frac{8}{3} \sin \theta \right) d\theta - \int_{\pi/3}^{\pi/2} \left( 4 \cos \theta \sin \theta - \frac{8}{3} \sin \theta \right) d\theta \right) = \\ &= \frac{2\pi k}{\varepsilon_0} \left( 4 \frac{\sin^2 \theta}{2} \Big|_0^{\pi/3} + \frac{8}{3} \cos \theta \Big|_0^{\pi/3} - 4 \frac{\sin^2 \theta}{2} \Big|_{\pi/3}^{\pi/2} - \frac{8}{3} \cos \theta \Big|_{\pi/3}^{\pi/2} \right) \\ &= \frac{2\pi k}{\varepsilon_0} \left( \frac{3}{2} - \frac{4}{3} - \frac{1}{2} + \frac{4}{3} \right). \end{aligned}$$

The flux obtained is

$$\Phi_E = \frac{2\pi k}{\varepsilon_0} = 5.68 \cdot 10^6 \text{ Nm}^2 \text{ C}^{-1}.$$

**2.15** Calculate the electric field produced at point  $P$  in Fig. 2.29 by the cylinder with volumetric uniform density  $\rho$ , whose height is  $H$ , inner radius  $R_i$  and exterior one  $R_e$ .

### Solution

The problem can be solved by direct integration using cylindrical coordinates. Any charge element  $dq$  (Fig. 2.30), at  $\mathbf{r}' = r \cos \phi \mathbf{u}_x + r \sin \phi \mathbf{u}_y + z \mathbf{u}_z$  is expressed by using cylindrical coordinates as follows

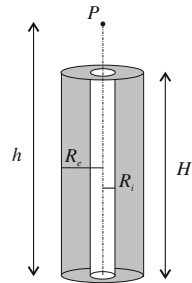
$$dq = \rho dV = \rho r dr d\phi dz.$$

The field produced by this element at point  $P$ , located at  $\mathbf{r} = h \mathbf{u}_z$ , is

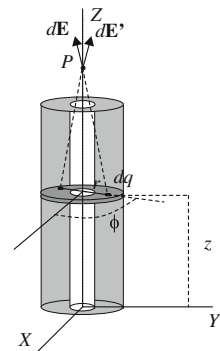
$$\begin{aligned} d\mathbf{E} &= \frac{dq}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = \\ &= \frac{\rho}{4\pi\epsilon_0} \frac{-r \cos \phi \mathbf{u}_x - r \sin \phi \mathbf{u}_y + (h - z) \mathbf{u}_z}{[r^2 + (h - z)^2]^{3/2}} r dr d\phi dz. \end{aligned}$$

If the superposition principle is applied

**Fig. 2.29** Figure of Problem 2.15



**Fig. 2.30** Charge differentials for a finite cylinder



$$\mathbf{E} = \int_{R_i}^{R_e} \int_0^{2\pi} \int_0^H \frac{\rho}{4\pi\epsilon_0} \frac{-r^2 \cos \phi \mathbf{u}_x - r^2 \sin \phi \mathbf{u}_y + (h-z)r \mathbf{u}_z}{(r^2 + (h-z)^2)^{3/2}} dr d\phi dz.$$

The sine integral and the cosine integral between 0 and  $2\pi$  are null, and therefore, components in  $\mathbf{u}_x$  and  $\mathbf{u}_y$  are null. This can be deduced from the figure's symmetry: the field can only have a component in  $\mathbf{u}_z$  as each  $dq$  has its symmetric  $dq'$  in a horizontal plane whose  $d\mathbf{E}'$  makes the same angle with the vertical, and it gives opposite components in  $\mathbf{u}_x$  and  $\mathbf{u}_y$ , and equal components in  $\mathbf{u}_z$ . The integral to solve is

$$\begin{aligned} \mathbf{E} &= \frac{\rho}{4\pi\epsilon_0} \int_{R_i}^{R_e} \int_0^{2\pi} \int_0^H \frac{(h-z)r}{(r^2 + (h-z)^2)^{3/2}} dr d\phi dz \mathbf{u}_z \\ &= \frac{\rho}{2\epsilon_0} \int_{R_i}^{R_e} \int_0^H \frac{(h-z)r}{(r^2 + (h-z)^2)^{3/2}} dr dz \mathbf{u}_z, \end{aligned}$$

because  $\int_0^{2\pi} d\phi = 2\pi$ . Solving in  $r$ , bearing in mind that the derivative of the function  $[r^2 + (h-z)^2]$  is  $2r$ , the result is

$$\begin{aligned} \mathbf{E} &= \frac{\rho}{2\epsilon_0} \int_0^H \left. \frac{-(h-z)}{(r^2 + (h-z)^2)^{1/2}} \right|_{R_i}^{R_e} dz \mathbf{u}_z = \\ &= -\frac{\rho}{2\epsilon_0} \int_0^H \left( \frac{(h-z)}{\sqrt{R_e^2 + (h-z)^2}} - \frac{(h-z)}{\sqrt{R_i^2 + (h-z)^2}} \right) dz \mathbf{u}_z. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbf{E} &= \frac{\rho}{2\epsilon_0} \left( \left. \sqrt{R_e^2 + (h-z)^2} \right|_0^H - \left. \sqrt{R_i^2 + (h-z)^2} \right|_0^H \right) \mathbf{u}_z \\ &= \frac{\rho}{2\epsilon_0} \left( \sqrt{R_e^2 + (h-H)^2} - \sqrt{R_i^2 + (h-H)^2} - \sqrt{R_e^2 + h^2} + \sqrt{R_i^2 + h^2} \right) \mathbf{u}_z. \end{aligned}$$

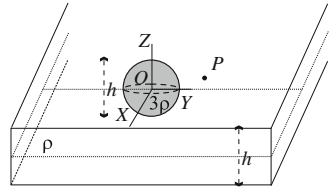
**2.16** Consider an infinite plate with thickness  $h = 1$  m in Fig. 2.31, with uniform charge density  $\rho = 20 \mu\text{C}/\text{m}^3$ , except inside a spherical cavity with charge density three times that of the plate, with the centre the medium plane, and diameter  $h$ . Determine the electric field produced at point  $P(1, 2, 2)$ , with the coordinates measured in meters referring to the coordinate axis located at the centre of the sphere, as shown in Fig. 2.31.

### Solution

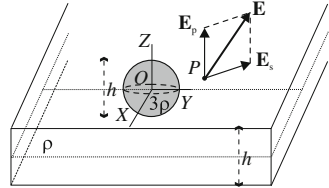
To calculate the electric field produced by the charge distribution, it can be observed that the superposition principle can be applied, and therefore it can be calculated as the addition of the fields produced by the two distributions with high symmetry:



**Fig. 2.31** Infinite plate with a spherical cavity and higher density



**Fig. 2.32** Superposition principle for fields produced by the plate and the sphere



an infinite plate with uniform charge density  $\rho$  and a uniformly charged sphere with density  $2\rho$ . In this way the spherical area would have a charge distribution, by adding of the previous ones, of  $3\rho$  indicated on the problem's statement. Fields produced by these distributions (Fig. 2.32) can be obtained from exercises previously worked in this book.

To calculate the field  $\mathbf{E}_p$  produced by the whole plate, without the cavity, we apply the obtained results in Problem 2.7, bearing in mind that point  $P$  is outside the plate. According to (2.46), the electric field results

$$\mathbf{E}_p = \frac{\rho h}{2\epsilon_0} \text{sgn}(z) \mathbf{u}_z = \frac{20 \cdot 10^{-6} \cdot 1}{2 \cdot 8.85 \cdot 10^{-12}} \mathbf{u}_z = 1.130 \cdot 10^6 \mathbf{u}_z \text{ N/C.}$$

To calculate the field produced by the sphere  $\mathbf{E}_s$ , we apply the result of Problem 2.5, and bear in mind that the sphere is solid (2.40), with density  $2\rho$  and radius  $h/2$  and that point  $P$  is outside the sphere. The distance  $r$  from the sphere's centre to point  $P$  is calculated from the vector's position at point  $P$ :

$$\mathbf{r} = \mathbf{u}_x + 2\mathbf{u}_y + 2\mathbf{u}_z, \quad r = 3, \quad \mathbf{u}_r = \frac{\mathbf{u}_x + 2\mathbf{u}_y + 2\mathbf{u}_z}{3}.$$

The field results

$$\begin{aligned} \mathbf{E}_s &= \frac{2\rho \left(\frac{h}{2}\right)^3}{3\epsilon_0 r^2} \mathbf{u}_r = \frac{\rho h^3}{12\epsilon_0 r^2} \mathbf{u}_r = \frac{20 \cdot 10^{-6} \cdot 1^3}{12 \cdot 8.85 \cdot 10^{-12} \cdot 3^2} \frac{\mathbf{u}_x + 2\mathbf{u}_y + 2\mathbf{u}_z}{3} \\ &= 6975(\mathbf{u}_x + 2\mathbf{u}_y + 2\mathbf{u}_z) \text{ N/C.} \end{aligned}$$

The field  $\mathbf{E}$  produced by the plate with the cavity is the addition of the fields  $\mathbf{E}_p$  and  $\mathbf{E}_s$ ,

$$\mathbf{E} = \mathbf{E}_p + \mathbf{E}_s = (6975\mathbf{u}_x + 13950\mathbf{u}_y + 1.144 \cdot 10^6 \mathbf{u}_z) \text{ N/C.}$$

**2.17** Consider the bent wire in Fig. 2.33, where coordinates are expressed in meters, with uniform charge density  $\lambda = 8 \mu\text{C/m}$ . The horizontal piece is very long (semi-infinite) and the other one,  $L = 2 \text{ m}$  long, making an angle of  $60^\circ$  with respect to the horizontal axis. Determine the electric field and the electric potential at point  $P(2, 1)$ .

### Solution

To solve the problem, the superposition principle can be applied if the problem is considered as the addition of the fields and potentials produced by a semi-infinite horizontal wire and a finite one ( $60^\circ$ ) in respect to the horizontal axis. For the case of the semi-infinite wire, we consider Fig. 2.34 and (2.37) of Problem 2.4,

$$\mathbf{E}_1 = \frac{\lambda}{4\pi\epsilon_0 y} ((\sin \varphi_f - \sin \varphi_i)\mathbf{u}_x + (\cos \varphi_i - \cos \varphi_f)\mathbf{u}_y).$$

Angle  $\varphi_i$  can be obtained from its tangent:  $\tan \varphi_i = y/x = 1/2 \Rightarrow \varphi_i = 30^\circ$ . Angle  $\varphi_f$  is  $180^\circ$ , since the wire is infinite along the  $+X$  direction. Therefore,

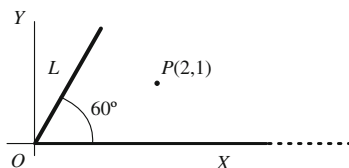
$$\begin{aligned} \mathbf{E}_1 &= \frac{8 \cdot 10^{-6}}{4\pi\epsilon_0 \cdot 1} ((\sin 180^\circ - \sin 30^\circ)\mathbf{u}_x + (\cos 30^\circ - \cos 180^\circ)\mathbf{u}_y) \\ &= 71.9 \cdot 10^3 (-0.5\mathbf{u}_x + 1.87\mathbf{u}_y), \end{aligned}$$

$$\mathbf{E}_1 = (-36\mathbf{u}_x + 134.2\mathbf{u}_y) \cdot 10^3 \text{N/C}.$$

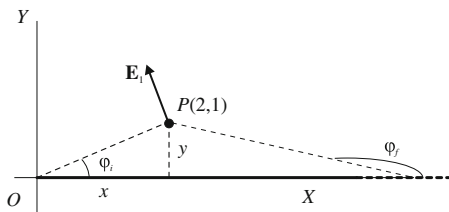
Considering the finite wire of length  $L = 2 \text{ m}$  (Fig. 2.35) and applying (2.37) of Problem 2.4 again, the result is

$$\mathbf{E}_2 = \frac{\lambda}{4\pi\epsilon_0 h} ((\sin \varphi_f - \sin \varphi_i)\mathbf{u}_x + (\cos \varphi_i - \cos \varphi_f)\mathbf{u}_y).$$

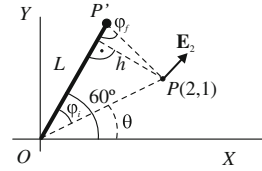
**Fig. 2.33** Charged wire of Problem 2.17



**Fig. 2.34** Field produced by the semi-infinite wire



**Fig. 2.35** Field produced by the finite wire



Let's determine the coordinates of the wire's extreme  $P'(x', y')$ :

$$x' = L \cos 60^\circ = 1, \quad y' = L \sin 60^\circ = \sqrt{3},$$

so

$$\mathbf{PP}' = (1 - 2)\mathbf{u}_x + (\sqrt{3} - 1)\mathbf{u}_y = -\mathbf{u}_x + 0.732\mathbf{u}_y,$$

with  $|PP'| = 1.239$ .

The vector magnitude  $\mathbf{r} = \mathbf{OP}$  is  $|OP| = \sqrt{5}$ , and angle  $\theta$  is obtained from

$$\tan \theta = 1/2 \rightarrow \theta = 26.6^\circ.$$

Angle  $\varphi_i$  is, therefore,

$$\varphi_i = 60^\circ - \theta = 33.4^\circ.$$

Distance  $h$  from point  $P$  to the wire is obtained from

$$h = |OP| \sin \varphi_i = 1.232.$$

The value of  $\varphi_f$  is obtained from

$$\sin \varphi_f = \frac{h}{|PP'|} = 0.994 \rightarrow \varphi_f = 83.9^\circ.$$

If we substitute in  $\mathbf{E}_2$  expression

$$\begin{aligned} \mathbf{E}_2 &= \frac{8 \cdot 10^{-6}}{4\pi\epsilon_0 \cdot 1.232} ((\sin 83.9^\circ - \sin 33.4^\circ)\mathbf{u}_x + (\cos 33.4^\circ - \cos 83.9^\circ)\mathbf{u}_y) = \\ &= 58.44 \cdot 10^3 (0.44\mathbf{u}_x + 0.73\mathbf{u}_y) = (25.9\mathbf{u}_x + 42.6\mathbf{u}_y) \cdot 10^3 \text{N/C}. \end{aligned}$$

The total field at point  $P$ , if the superposition principle is applied, is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = (-10.1\mathbf{u}_x + 176.8\mathbf{u}_y) \cdot 10^3 \text{N/C}.$$

**2.18** A thin, flat plate, which has the shape of a regular  $n$ -sided polygon, inscribed in a circle of radius  $a$  in plane  $XY$ , is considered. This plate is maintained at a fixed potential  $V$ , while the rest of the plane  $XY$  is held at zero potential. Apply Biot–Savart-like law in electrostatics to calculate the electric field produced by this plate at a point  $P$  located over the perpendicular to the plate by its centre.

### Solution

Figure 2.36 shows the schematic view of the regular  $n$ -sided polygon plate, located in the  $XY$ -plane, with its centre at the origin, and kept at a potential  $V$  ( $n = 6$  in Fig. 2.36). In this plane ( $z = 0$ ), the potential is set to zero in the region outside the plate. To solve the problem we apply the (2.35),

$$\mathbf{E}(\mathbf{r}) = \frac{V}{2\pi} \oint_C \frac{(\mathbf{r} - \mathbf{r}') \times d\mathbf{l}'}{|\mathbf{r} - \mathbf{r}'|^3}.$$

$\mathbf{r}$  refers to the field point and  $\mathbf{r}'$  locates the source point.  $d\mathbf{l}'$  is an element of length of the integration path  $C$ . As discussed in Sect. 2.9, we just need to calculate the contributions coming from the boundary contour  $C$ .

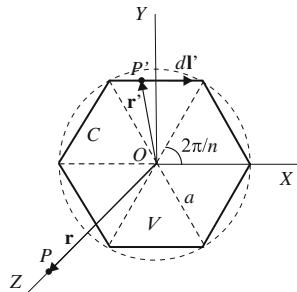
Looking at Fig. 2.36, it can be seen that electric field at point  $P$  can be calculated as the superposition of the field due to  $n$  triangles obtained by joining the centre of circle  $O$  to the vertices of the polygon: the sense of integration along their common sides is opposite for two adjacent triangles and adding all the contributions from these triangles, we obtain the integration around the contour  $C$ . Therefore, the only non-zero contributions come from the  $n$  sides that define the contour  $C$ . By symmetry, when the  $n$  fields are added vectorially, only the components located along  $Z$ -axis remains; the  $XY$ -plane components add to zero.

Then, considering the point  $P$  at  $Z$ -axis in the Fig. 2.36 and a point  $P'$  and  $d\mathbf{l}'$  at the upper straight side,

$$\mathbf{r} = z\mathbf{u}_z, \quad \mathbf{r}' = x'\mathbf{u}_x + a \cos \frac{\pi}{n} \mathbf{u}_y,$$

$$\mathbf{r} - \mathbf{r}' = -x'\mathbf{u}_x - a \cos \frac{\pi}{n} \mathbf{u}_y + z\mathbf{u}_z \quad |\mathbf{r} - \mathbf{r}'| = (x'^2 + a^2 \cos^2(\pi/n) + z^2)^{1/2},$$

**Fig. 2.36** Schematic view of the regular  $n$ -sided polygon plate



$$d\mathbf{l}' = dx' \mathbf{u}_x \quad (\mathbf{r} - \mathbf{r}') \times d\mathbf{l}' = z dx' \mathbf{u}_y + a \cos \frac{\pi}{n} dx' \mathbf{u}_z.$$

Calling  $b^2 = a^2 \cos^2(\pi/n) + z^2$  and applying the (2.35) to calculate the contribution  $E_{iz}$  along  $\mathbf{u}_z$ ,

$$\begin{aligned} E_{iz} &= \frac{V}{2\pi} \int_{-a \sin(\pi/n)}^{a \sin(\pi/n)} \frac{a \cos(\pi/n) dx'}{(x'^2 + b^2)^{3/2}} = \frac{Va \cos(\pi/n)}{2\pi b^2} \left. \frac{x'}{\sqrt{x'^2 + b^2}} \right|_{-a \sin(\pi/n)}^{a \sin(\pi/n)} = \\ &= \frac{Va \cos(\pi/n)}{2\pi b^2} \frac{2a \sin(\pi/n)}{\sqrt{a^2 \sin^2(\pi/n) + b^2}} = \frac{V}{2\pi} \frac{a^2 \sin(2\pi/n)}{[a^2 \cos^2(\pi/n) + z^2] \sqrt{a^2 + z^2}}. \end{aligned}$$

Adding the  $n$  contributions from the  $n$  sides that define the contour  $C$ , the total electric field at  $P$  is obtained,

$$\mathbf{E} = nE_{iz} = \frac{V}{2\pi} \frac{na^2 \sin(2\pi/n)}{[a^2 \cos^2(\pi/n) + z^2] \sqrt{a^2 + z^2}} \mathbf{u}_z.$$

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