Chapter 2
The Equation for the Stokesian Stream Function and Its Solutions

Abstract This chapter presents and discusses the equation for the *Stokes*ian stream function. The equation emerges as the one non-zero component of the curl of the two-dimensional momentum equation with the velocity components given as spatial derivatives of the stream function. The stream function is defined such that its derivatives yield a solenoidal velocity field. The analyses of the flows discussed in Part I of this book are based on this function. In view of our search for analytical solutions, we are restricted to laminar two-dimensional flow in simple geometries. The equations of change therefore need no turbulence modelling, the concept of the *Stokes*ian stream function can be applied for representing the flow velocity, and the boundary conditions are easy to formulate and implement analytically in the general solutions. The fluids are treated as incompressible and *Newton*ian or linear viscoelastic. The linear viscoelastic liquids exhibit a viscosity depending on frequency, but not on shear rate. Furthermore, we restrict this analysis to flows without heat and mass transfer, i.e. we solve the continuity and momentum equations and disregard the influence of viscous dissipation on the energy budget of the flow. We are therefore restricted to flow without viscous heating. Problems of heat and mass transfer are the subjects of Part II of this book.

In the following we present the equations for the stream function for two-dimensional flow problems, following the structure in [2]. We present the function in Cartesian, cylindrical and spherical coordinates, which are geometrically most relevant for many flow problems. The presentation is structured according to linearity, time dependence and dependence of pressure on the spatial coordinates.

2.1 The Equation for the Stream Function in Cartesian Coordinates

The equation for the *Stokes*ian stream function emerges from the momentum equation with the velocity formulated by means of the stream function. The pressure gradient, if applicable, is eliminated from the vectorial formulation of the momentum equation by taking the curl of the equation. The result is a scalar fourth-order partial differential equation for the stream function. In cases of constant pressure throughout the flow

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field, we rewrite the momentum equation component in the main flow direction using the stream function and obtain a third-order partial differential equation. The latter applies to flows such as the flow along submerged flat plates, free submerged jets, free shear layers, wakes, etc.

We take the main flow velocity component to be directed along the x axis of the *Cartesian (rectangular) coordinate system* (x, y, z), assuming that the flow field does not depend on the coordinate z and the z velocity component w is zero. Therefore, the velocity vector is $(u, v, 0)$. The stream function ψ_r in Cartesian coordinates is introduced by the definitions

$$u = \frac{\partial \psi_r}{\partial y}, \quad v = -\frac{\partial \psi_r}{\partial x} \tag{2.1}$$

of the x and y velocity components. This definition ensures a solenoidal, i.e. divergence-free, velocity field, so that the special form of the continuity equation (1.2) for constant fluid density is automatically satisfied. Introducing this two-dimensional velocity vector into the momentum equation (1.3), with the material law (1.8), and taking the curl of the resulting equation to make the gradient fields of pressure and body force potential disappear (all gradient fields are irrotational), we obtain the fourth-order PDE [2]

$$\frac{\partial}{\partial t}\left(\nabla^2 \psi_r\right) + \frac{\partial(\psi_r, \nabla^2 \psi_r)}{\partial(x, y)} = \nu \nabla^4 \psi_r \tag{2.2}$$

for the stream function ψ_r. In this and all the following corresponding equations formulated in different coordinate systems, the *Jacobi*an reads

$$\frac{\partial(f, g)}{\partial(x, y)} = \begin{vmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{vmatrix} . \tag{2.3}$$

The nabla operator to the second and fourth powers in Cartesian coordinates reads

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} , \tag{2.4}$$

which is the Laplace operator in Cartesian coordinates, and

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} , \tag{2.5}$$

respectively.

2.1.1 Linear, Unsteady Flow

Analytical solutions of Eq. (2.2) are found for linear flow fields, where the *Jacobi*an is either negligible or vanishes exactly. The linearisation leads to the equation

$$\left(-\frac{1}{\nu}\frac{\partial}{\partial t} + \nabla^2\right)\nabla^2\psi_r = 0\,. \tag{2.6}$$

In his 1935 paper on the instability of a liquid jet immersed in an immiscible viscous fluid, *Tomotika* used the following idea for determining the stream function in cylindrical coordinates, which we use here for the Cartesian case [3]: since the operators $-1/\nu\partial/\partial t + \nabla^2$ and ∇^2 in Eq. (2.6) are commutative with each other, the stream function may be composed of two parts as per $\psi_r = \psi_{r,1} + \psi_{r,2}$, where $\psi_{r,1}$ and $\psi_{r,2}$ are solutions of the two partial differential equations

$$\nabla^2\psi_{r,1} = 0 \tag{2.7}$$

and

$$-\frac{1}{\nu}\frac{\partial\psi_{r,2}}{\partial t} + \nabla^2\psi_{r,2} = 0\,, \tag{2.8}$$

respectively. The Eqs. (2.7) and (2.8) are solved by separation of variables, which reveals the functions $\psi_{r,1}$ and $\psi_{r,2}$ as products of eigenfunctions of the two operators in the Cartesian coordinates and in time. We assume a wave-like solution in the coordinate direction x of the main flow, i.e. a solution proportional to the function $exp(ikx)$, where k is the wavenumber $2\pi/\lambda$ and λ the wavelength of the spatially periodic process. The eigenfunction in time is an exponential function of a non-dimensional time $-\alpha t$, where we interpret α as a complex angular frequency with a damping rate as the real and an angular frequency as the imaginary parts. For $\psi_{r,1}$ we obtain

$$\psi_{r,1} = \left(C_1 e^{ky} + C_2 e^{-ky}\right) e^{ikx-\alpha t}\,. \tag{2.9}$$

For the function $\psi_{r,2}$ we obtain

$$\psi_{r,2} = \left(C_1' e^{k_1 y} + C_2' e^{-k_1 y}\right) e^{ikx-\alpha t}\,, \tag{2.10}$$

where $k_1^2 = k^2 - \alpha/\nu$. The stream function can therefore be written in a general form as

$$\psi_r - \psi_{r,0} = \left(C_1 e^{ky} + C_2 e^{-ky} + C_1' e^{k_1 y} + C_2' e^{-k_1 y}\right) e^{ikx-\alpha t}\,, \tag{2.11}$$

where $\psi_{r,0}$ is a constant.

The special case where linearity is maintained, but a constant transport velocity U_0 in one coordinate direction—in the x direction, say—occurs, the partial differential equation for the stream function (2.6) changes its form into

$$\left[-\frac{1}{\nu}\left(\frac{\partial}{\partial t}+U_0\frac{\partial}{\partial x}\right)+\nabla^2\right]\nabla^2\psi_r=0\,. \tag{2.12}$$

The solution of the equation is obtained by the same means as for the previous version. It reads

$$\psi_r-\psi_{r,0}=\left(C_1e^{ky}+C_2e^{-ky}+C_1'e^{ly}+C_2'e^{-ly}\right)e^{ikx-\alpha t}\,, \tag{2.13}$$

where $l^2=k^2+(-\alpha+ikU_0)/\nu$. We will discuss applications of this stream function to the stability analysis of plane liquid sheets in Chap. 6.

For a spatially two-dimensional linear unsteady flow, with the flow velocity varying with the coordinate y, but not with x, in contrast, the differential equation for the stream function reads

$$\left(-\frac{1}{\nu}\frac{\partial}{\partial t}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial^2\psi_r}{\partial y^2}=0\,. \tag{2.14}$$

The solution of this equation is obtained along the same lines as the previous ones and reads

$$\psi_r-\psi_{r,0}=\left(C_1y+C_2+C_1'e^{iqy}+C_2'e^{-iqy}\right)e^{-\alpha t}\,, \tag{2.15}$$

where we have defined $q=(\alpha/\nu)^{1/2}$.

For the special case of constant pressure throughout the flow field, such as in plane *Couette* flow, Eq. (2.14) reduces to

$$\left(-\frac{1}{\nu}\frac{\partial}{\partial t}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial\psi_r}{\partial y}=0\,, \tag{2.16}$$

since pressure then needs no elimination from the momentum equations. This reduces the order of the differential equation by one. For unsteady processes in such flow, such as the start-up or fade-out of the flow treated as hydraulically developed in the direction x of the motion, the stream function may be of the form

$$\psi_r=\psi_{rs}(y)+f(y)e^{-\alpha t}\,, \tag{2.17}$$

where α is a real rate of change of the flow in time. The steady part $\psi_{rs}(y)$ satisfies the ordinary differential equation

$$\frac{d^3\psi_{rs}}{dy^3}=0 \tag{2.18}$$

with the solution

$$\psi_{rs}=C_1y^2+C_2y+C_3\,. \tag{2.19}$$

Substituting the formulation (2.17) of the stream function into Eq. (2.16), and accounting for (2.18), we obtain the following ordinary differential equation for the unknown function $f(y)$:

$$\frac{\alpha}{\nu} f' + f''' = 0 \,. \tag{2.20}$$

Integration yields the form

$$\frac{\alpha}{\nu} f + f'' = C_1' \tag{2.21}$$

with the constant C_1'. The solution of this inhomogeneous ordinary differential equation is readily obtained as the sum of the general solution of the (harmonic) homogeneous form of the equation and the particular solution $f_p(y) = C = C_1' \nu/\alpha$ of the inhomogeneous equation. The final solution for the stream function reads

$$\begin{aligned} \psi_r - \psi_{r,0} = {} & C_1 y^2 + C_2 y + \\ & + [C_3 \cos qy + C_4 \sin qy + C]\, e^{-\alpha t} \,, \end{aligned} \tag{2.22}$$

where $\psi_{r,0}$ is a constant and q is defined as in the previous case.

For the special case of a constant pressure gradient in the flow field, such as in plane *Poiseuille* flow, Eq. (2.14) becomes

$$\left(-\frac{1}{\nu} \frac{\partial}{\partial t} + \frac{\partial^2}{\partial y^2} \right) \frac{\partial \psi_r}{\partial y} = C \,, \tag{2.23}$$

where C represents the constant pressure gradient. The stream function has the same structure as in (2.17). The steady part $\psi_{rs}(y)$ satisfies the ordinary differential equation

$$\frac{d^3 \psi_{rs}}{dy^3} = C \,. \tag{2.24}$$

The solution reads

$$\psi_{rs} = \frac{C}{3} y^3 + C_2 y^2 + C_3 y + C_4 \,. \tag{2.25}$$

Substituting the formulation (2.17) of the stream function into Eq. (2.23), and accounting for (2.24), we obtain the following ordinary differential equation for the unknown function $f(y)$:

$$\frac{\alpha}{\nu} f' + f''' = 0 \,. \tag{2.26}$$

Integration yields the form

$$\frac{\alpha}{\nu} f + f'' = C_1' \tag{2.27}$$

with the constant C_1'. The solution of this inhomogeneous ordinary differential equation is readily obtained as the sum of the general solution of the (harmonic) homogeneous form of the equation and the particular solution $f_p(y) = C = C_1'\nu/\alpha$ of the inhomogeneous equation. The final solution for the stream function reads

$$\begin{aligned}\psi_r - \psi_{r,0} = C_1 y^3 + C_2 y^2 + C_3 y + \\ + \left[C_4 \cos qy + C_5 \sin qy + C_6\right] e^{-\alpha t} \ ,\end{aligned} \tag{2.28}$$

where $\psi_{r,0}$ is a constant and q is defined as in the previous case.

The details of the solutions of these equations depend on the evolution of the flow with time, i.e. whether or not the flow is (periodically) time-dependent throughout or converges to a steady form of the motion. We will go into the details of this in Chap. 3.

Self-similar, unsteady flow

Another behaviour of the flow may arise in situations where nor a time neither a length scale exists in the flow field. This is the case in flow fields along infinite structures without any geometrical elements with length scales, e.g. along flat plates, without imprinted flow time scale. The flow then behaves as self-similar and produces a time-dependent length scale by diffusive propagation of momentum. The flow field may then be a function of one Cartesian spatial coordinate (e.g. y) and time t combined in one self-similar coordinate. Pressure may come out as constant throughout the flow field, since the absence of a length scale of the flow field may include that pressure does not vary along the contour. The stream function ψ_r of the flow and a ratio of the spatial coordinate y and a power of a diffusive length scale, $(\nu t)^\beta$, which forms a self-similar coordinate η_r, may then be set as

$$\psi_r = C_s\,(\nu t)^\alpha\, f_r(\eta_r), \quad \eta_r = D_s \frac{y}{(\nu t)^\beta} \ . \tag{2.29}$$

The coefficients C_s and D_s are introduced for dimensional reasons and for convenience. Substituting the velocity component u in the main flow direction into the momentum equation for this problem, which is a diffusion equation for the velocity component u, we obtain

$$\alpha\,(\nu t)^{\alpha-\beta-1}\, f_r' - \beta\,(\nu t)^{\alpha-\beta-1}\left(f_r''\eta_r + f_r'\right) = (\nu t)^{\alpha-3\beta}\, f_r''' D_s^2 \ . \tag{2.30}$$

For the concept of self-similarity to work, we require that this equation is an ODE for the function $f_r(\eta_r)$. This is the case if and only if the dependency on time disappears, i.e. if the exponents of (νt) on the two sides of the equation are the same. This requirement leads to the value $\beta = 1/2$. The ODE for the function $f_r(\eta_r)$ now reads

$$\alpha f_r' - \frac{1}{2}\left(f_r''\eta_r + f_r'\right) = f_r''' D_s^2 \ . \tag{2.31}$$

Further conditions determining the exponent α in the stream function and the coefficient D_s in the self-similar coordinate depend on the details of the flow situation. We will use Eq. (2.31) for analysing the flow of the *First Stokesian Problem* in Chap. 3.

This kind of flow behaviour occurs in cylindrical and spherical flows at large distances from the cylindrical or spherical surfaces, if applicable, where the radius of the surface has lost its influence on the flow field. Otherwise the radius of curvature acts as a length scale and prevents self-similar behaviour. This behaviour is seen already when transforming the equations of motion in cylindrical and spherical coordinates into the self-similar forms.

2.1.2 Linear, Steady Flow

For linear steady flow problems in Cartesian coordinates, the equation determining the stream function reads

$$\nabla^4 \psi_r = 0 \,. \tag{2.32}$$

Equation (2.32) is the biharmonic equation. It is solved with the aim to find the solution in terms of eigenfunctions of the differential operators involved. Separating the stream function into one function of the coordinate x and one of the coordinate y, and assuming the stream function to be wave-like in the x direction of the main flow, we obtain the solution

$$\psi_{r,sx} = \left(C_1 e^{ky} + C_2 e^{-ky}\right)\left(C_3 e^{ikx} + C_4 e^{-ikx}\right) \,. \tag{2.33}$$

An alternative is to assume the solution to be wave-like in the y direction. Then the roles of the coordinates are just interchanged, resulting in the solution

$$\psi_{r,sy} = \left(C_1 e^{kx} + C_2 e^{-kx}\right)\left(C_3 e^{iky} + C_4 e^{-iky}\right) \,. \tag{2.34}$$

A special case is a linear, steady, hydraulically developed flow field with the main motion in the x direction. The flow field then does not depend on the x coordinate. The stream function, therefore, also cannot depend on that coordinate. The reduced form of Eq. (2.32) is the statement that the fourth-order derivative of the stream function with respect to the coordinate y is zero. The solution of that equation is

$$\psi_r = C_0 + C_1 y + C_2 y^2 + C_3 y^3 \,. \tag{2.35}$$

Three of the four constants are determined by boundary conditions, while C_0 is a free reference value.

A special case is seen in flows with constant pressure in the flow field, such as steady plane *Couette* flow. This kind of problems is characterised by the equation

$$\nabla^2 \psi_r = D \,, \tag{2.36}$$

where D is a constant. The order of this PDE for the stream function is lower than (2.32), since elimination of the pressure from the momentum equation is not required for its derivation and the form (2.36) was obtained by one integration already. The form of the stream function corresponding to (2.35) in this case is therefore

$$\psi_{r,p} = D_0 + D_1 y + D_2 y^2 \ . \tag{2.37}$$

The values of all the integration constants in the above solutions depend on the particular flow problem. We will go into the details of their determination in Chaps. 3 and 4.

2.1.3 Nonlinear, Steady Flow with Constant Pressure

Another class of flows accessible with analytical methods are boundary-layer flows in simple geometries with constant pressure throughout, such as flows along flat plates, free submerged jets, free shear layers, wakes, etc. The pressure is constant in these flow fields, since, in the region far from the edge of the boundary layer, the free-stream velocity does not depend on the coordinate in the main flow direction, and the resultant constant pressure is imprinted on the boundary layer in the direction transverse to the main flow. In the boundary layer, therefore, the pressure does not depend on the coordinate transverse to the main flow direction as well. In these cases, the pressure gradient disappears from the momentum equation in the boundary-layer form identically. Therefore, after introduction of the stream function, there is no need to take the curl of the momentum equation. In these cases, we obtain from the momentum equation in the main flow direction x the PDE for the stream function in Cartesian coordinates

$$\frac{\partial \psi_r}{\partial y}\frac{\partial^2 \psi_r}{\partial x \partial y} - \frac{\partial \psi_r}{\partial x}\frac{\partial^2 \psi_r}{\partial y^2} = \nu \frac{\partial^3 \psi_r}{\partial y^3} \ . \tag{2.38}$$

The stream function is composed of a mapping function of the spatial coordinate x only and a function of a *self-similar* coordinate. The *self-similar* coordinate is a y coordinate transverse to the main flow direction, normalised by the boundary-layer thickness, which itself depends on the x position in the field. The corresponding profiles of the normalised velocity component in the main flow direction x turn out *self-similar*, i.e. they are determined by the self-similar function only. The reference velocity for the normalisation may be the incoming free-stream velocity in that flow direction. In jets, it is the maximum value of the x velocity, located in the symmetry plane or on the symmetry axis of the jet. This *self-similar* behaviour of the flow may be made use of for deriving analytical solutions of the differential equation (2.38). For this purpose, we define the stream function with a self-similar function f_r of a self-similar coordinate η_r. We define the stream function ψ_r as the product of a power x^α of the coordinate in the main flow direction, which plays the role of the

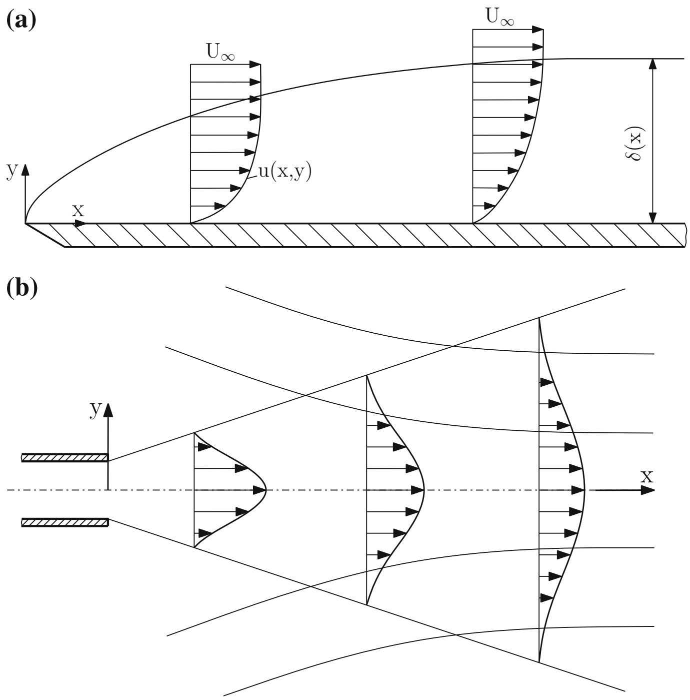

Fig. 2.1 Two examples of plane flow field geometries with boundary-layer type: **a** flow along a plane submerged body, and **b** submerged plane free jet

mapping function, and the function $f_r(\eta_r)$. The self-similar coordinate is defined as proportional to the ratio of the coordinate y transverse to the main flow direction and the width of the flow domain influenced by diffusive viscous momentum transport. This width is assumed to be proportional to a power x^β of the coordinate in the main flow direction, with a positive exponent β. So we assume that this flow domain widens in the direction of the flow. Figure 2.1 shows two examples of this flow field geometry: (a) the flat-plate boundary-layer flow and (b) a plane submerged free jet. The stream function and the self-similar coordinate have the forms

$$\psi_r = C_r x^\alpha f_r(\eta_r), \quad \eta_r = D_r \frac{y}{x^\beta} . \tag{2.39}$$

In these definitions, the coefficients C_r and D_r are introduced for dimensional reasons: C_r to allow the velocity u in the x-direction to be obtained as the derivative $\partial\psi_r/\partial y$, and D_r to render the self-similar coordinate η_r non-dimensional. Substituting these definitions into Eq. (2.38), we obtain

$$(\alpha - \beta)\, f_r'^2 - \alpha f_r f_r'' = \nu \frac{D_r}{C_r} x^{1-\alpha-\beta} f_r''' . \tag{2.40}$$

For the concept of self-similarity to work we require that this equation is an ODE for the function $f_r(\eta_r)$. This is the case if and only if the dependency on the coordinate x disppears, i.e. if the exponent of x equals zero. We therefore require that $\alpha + \beta = 1$ and obtain the self-similar differential equation

$$(1 - 2\beta)\, f_r'^2 - (1 - \beta)\, f_r f_r'' = \nu \frac{D_r}{C_r} f_r''' \,. \tag{2.41}$$

A second condition for the exponents α and β, and expressions relating C_r and D_r to the flow situation, are found by adapting Eq. (2.41) to the respective flow. In order to obtain the differential equation for the self-similar function f_r in a form free of coefficients, we may require that $\nu D_r / C_r = 1$. Furthermore, the dimension of D_r must be $m^{\beta-1}$ in order that η_r is non-dimensional. Since this adaptation determining the final form of Eq. (2.41) and the solutions of the equation depend on the actual flow situation, we will go into these details in Chap. 5, where boundary-layer flows are discussed.

2.2 The Equation for the Stream Function in Cylindrical Coordinates

Polar cylindrical flow

The *cylindrical coordinates* (r, θ, z) are related to the Cartesian coordinates as per $x = r\cos\theta,\ y = r\sin\theta,\ z = z$. For cylindrical flows in r, θ planes (normal to the z axis), where the velocity vector is $(u_r, u_\theta, 0)$ (with zero z velocity component), the stream function ψ_{cz} is introduced by the definitions of the r and θ velocity components

$$u_r = -\frac{1}{r}\frac{\partial \psi_{cz}}{\partial \theta}, \quad u_\theta = \frac{\partial \psi_{cz}}{\partial r} \,. \tag{2.42}$$

Introducing this two-dimensional velocity vector into the momentum equation (1.3) with the material law (1.8), and taking the curl of the resulting equation to make the gradient fields of pressure and body force potential disappear, we obtain the fourth-order PDE [2]

$$\frac{\partial}{\partial t}\left(\nabla^2 \psi_{cz}\right) + \frac{1}{r}\frac{\partial(\psi_{cz}, \nabla^2 \psi_{cz})}{\partial(r, \theta)} = \nu \nabla^4 \psi_{cz} \tag{2.43}$$

for the stream function ψ_{cz}. The nabla operator squared reads

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \,, \tag{2.44}$$

which is the *Laplace* operator in cylindrical coordinates for a scalar quantity independent of the coordinate z.

2.2.1 Polar, Linear, Unsteady Flow

Analytical solutions of Eq. (2.43) are found for linear flow fields, where the *Jacobi*an is either negligible or vanishes exactly. The linearisation for cylindrical flows in the r, θ plane leads to the equation

$$\left(-\frac{1}{\nu}\frac{\partial}{\partial t} + \nabla^2\right)\nabla^2\psi_{cz} = 0\,. \tag{2.45}$$

In solving this equation, we use the approach of *Tomotika* developed for the axisymmetric cylindrical case for the polar cylindrical problem as well [3]. The stream function may be composed of two parts as per $\psi_{cz} = \psi_{cz,1} + \psi_{cz,2}$, where $\psi_{cz,1}$ and $\psi_{cz,2}$ are solutions of the two PDEs

$$\nabla^2\psi_{cz,1} = 0 \tag{2.46}$$

and

$$-\frac{1}{\nu}\frac{\partial\psi_{cz,2}}{\partial t} + \nabla^2\psi_{cz,2} = 0\,, \tag{2.47}$$

respectively. The two Eqs. (2.46) and (2.47) are solved by separation of variables, which reveals the functions $\psi_{cz,1}$ and $\psi_{cz,2}$ as products of eigenfunctions of the two operators in the two cylindrical coordinates and in time. For $\psi_{cz,1}$ we obtain the solution

$$\psi_{cz,1,m} = \left(C_1 r^m + C_2 r^{-m}\right)e^{im\theta-\alpha t}\,, \tag{2.48}$$

where m is a mode number and plays the role of a wave number in the direction of the polar angle θ. The dependency of the flow on the polar angle θ and time t is represented by the exponential function with an imaginary argument in θ and a complex argument in t. The latter allows for periodic disturbances which are damped or grow in time. For the function $\psi_{cz,2}$ we obtain the solution

$$\psi_{cz,2,m} = \left[C_1' J_m(qr) + C_2' Y_m(qr)\right]e^{im\theta-\alpha t}\,, \tag{2.49}$$

where we have defined $q = (\alpha/\nu)^{1/2}$. The functions J_m and Y_m are Bessel functions of the first and second kinds and order m. The stream function for mode m can therefore be written as

$$\psi_{cz,m} - \psi_{cz,m,0} = \left[C_{1,m}r^m + C_{2,m}r^{-m} + C_{1,m}' J_m(qr) + C_{2,m}' Y_m(qr)\right]e^{im\theta-\alpha t}\,, \tag{2.50}$$

where $\psi_{cz,m,0}$ is a constant. The general form of the stream function is a sum over all the modes m. When applying the general description for the *Stokes*ian stream function to flows in special geometries, the values of the coefficients $C_{i,m}$ are determined by initial and boundary conditions of the problem. Terms of the solution must be discarded in regions of the flow field where they diverge by setting the related coefficients $C_{i,m}$ to zero. This is the case, e.g. for a flow field including the axis $r = 0$, where the functions r^{-m} and $Y(qr)$ diverge. The actual need for discarding terms, however, is seen only in the resulting equations for the corresponding velocity components. We will go into the details of this in Chap. 6.

2.2.2 Polar, Linear, Steady Flow

For linear steady flows in cylindrical coordinates, which do not depend on the coordinate z, the equation determining the stream function reads

$$\nabla^4 \psi_{cz} = 0 \,, \tag{2.51}$$

where ∇^4 emerges by applying ∇^2 in Eq. (2.44) to itself. The operator reads

$$\begin{aligned}\nabla^4 = &\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]\right\} + \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\right] + \\ &+ \frac{1}{r^3}\frac{\partial}{\partial r}\left(r\frac{\partial^3}{\partial r \partial \theta^2}\right) + \frac{1}{r^4}\frac{\partial^4}{\partial \theta^4} \,.\end{aligned} \tag{2.52}$$

Equation (2.51) is the biharmonic equation. Separation of the stream function ψ_{cz} into a function $f(r)$ of the radial coordinate r and a function of the polar angle θ, which is assumed to be periodic in the angular direction with a wave number m, leads to the following ordinary differential equation for the radial dependency of the stream function

$$r\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(rf'\right) - m^2\frac{f}{r^2}\right]\right\} - m^2\left[\frac{1}{r}\frac{d}{dr}\left(rf'\right) - m^2\frac{f}{r^2}\right] = 0 \,, \tag{2.53}$$

where $f' = df/dr$. This equation has the solution

$$f(r) = C_1 r^m + C_2 r^{-m} + C_3 r^{m+2} + C_4 r^{-m+2} \,, \tag{2.54}$$

which, together with the periodic solution in the angular coordinate θ, yields the stream function

$$\psi_{czs,m} - \psi_{czs,m,0} = \left(C_{1,m} r^m + C_{2,m} r^{-m} + C'_{1,m} r^{m+2} + C'_{2,m} r^{-m+2}\right) e^{im\theta} \,. \tag{2.55}$$

In case of axial symmetry of the flow field, where the dependency of the field on the polar angle θ vanishes and there is no component of a pressure gradient in that direction, the value of the mode number m is zero and the highest order derivative is the third. This leads to the differential equation for the stream function

$$r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi_{czs,0}}{dr}\right)\right] = 0 \tag{2.56}$$

with the solution

$$\psi_{czs,0} = \frac{C_1}{2}r^2 + C_2 \ln r + C_3\ . \tag{2.57}$$

The corresponding flow exhibits a velocity component in the angular direction θ only, which depends only on the radial coordinate. The velocity component reads

$$u_\theta(r) = C_1 r + \frac{C_2}{r}\ . \tag{2.58}$$

The integration constants C_1 and C_2 are determined by boundary conditions, as will be detailed in Chap. 3 below.

2.2.3 Polar, Nonlinear, Steady Flow

Nonlinear steady flow in a cylindrical geometry, which does not depend on the axial coordinate z, could be, e.g. the flow around a steady or steadily spinning cylinder. Flows in that geometry which would allow the boundary-layer concept to be applied, would require a high Reynolds number. Such flow, however, tends to separate from the cylinder contour on its back side, with the location of separation depending on the Reynolds number of the flow. The separation immediately makes the boundary-layer concept break down, since the boundary-layer thickness cannot be taken as small as compared to a length along the contour any more. A boundary-layer flow in such a geometry would therefore be restricted to a region around the upstream stagnation zone where, however, the boundary layer may still not be regarded as slender in the sense of the boundary-layer approximation. Such flow is, therefore, not considered in the present context.

Axisymmetric cylindrical flow

For cylindrical flows in (meridional) r, z planes which do not depend on the coordinate θ, i.e. which are axially symmetric around the z axis of the flow field, where the velocity vector is $(u_r, 0, u_z)$ (with zero swirl component), the stream function $\psi_{c\theta}$ is introduced by the definitions of the r and z velocity components

$$u_r = -\frac{1}{r}\frac{\partial\psi_{c\theta}}{\partial z}, \quad u_z = \frac{1}{r}\frac{\partial\psi_{c\theta}}{\partial r}\ . \tag{2.59}$$

Introducing this formulation of the velocity vector with $u_\theta = 0$ into the momentum equation (1.3), with the material law (1.8), and taking the curl of the resulting equation to make the gradient fields of the pressure and the body force potential disappear, we obtain the fourth-order partial differential equation [2]

$$\frac{\partial}{\partial t}\left(E^2_{c\theta}\psi_{c\theta}\right) - \frac{1}{r}\frac{\partial(\psi_{c\theta}, E^2_{c\theta}\psi_{c\theta})}{\partial(r,z)} - \frac{2}{r^2}\frac{\partial\psi_{c\theta}}{\partial z}E^2_{c\theta}\psi_{c\theta} = \nu E^4_{c\theta}\psi_{c\theta} \tag{2.60}$$

for the stream function $\psi_{c\theta}$. In this equation, the differential operator $E^2_{c\theta}$ reads

$$E^2_{c\theta} = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2} \tag{2.61}$$

and the operator to the fourth power is

$$E^4_{c\theta} = r\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)\right]\right\} + 2r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial z^2}\right)\right] + \frac{\partial^4}{\partial z^4}\,. \tag{2.62}$$

2.2.4 Axisymmetric, Linear, Unsteady Flow

Analytical solutions of Eq. (2.60) are found for linear flow fields, where the *Jacobi*an is either negligible or vanishes exactly and the product of $E^2_{c\theta}\psi_{c\theta}$ with the z derivative of $\psi_{c\theta}$ is small of the same order. The linearisation for cylindrical flows in the r, z plane leads to the equation

$$\left(-\frac{1}{\nu}\frac{\partial}{\partial t} + E^2_{c\theta}\right)E^2_{c\theta}\psi_{c\theta} = 0\,. \tag{2.63}$$

In his 1935 paper on the instability of a liquid jet immersed in an immiscible viscous fluid, *Tomotika* used the following idea for the stream function [3]: since the operators $-1/\nu\,\partial/\partial t + E^2_{c\theta}$ and $E^2_{c\theta}$ in Eq. (2.63) are commutative with each other, the stream function may be composed of two parts as per $\psi_{c\theta} = \psi_{c\theta,1} + \psi_{c\theta,2}$, where $\psi_{c\theta,1}$ and $\psi_{c\theta,2}$ are solutions of the two partial differential equations

$$E^2_{c\theta}\psi_{c\theta,1} = 0 \tag{2.64}$$

and

$$-\frac{1}{\nu}\frac{\partial\psi_{c\theta,2}}{\partial t} + E^2_{c\theta}\psi_{c\theta,2} = 0\,, \tag{2.65}$$

respectively. Equations (2.64) and (2.65) are solved by separation of variables, which yields the functions $\psi_{c\theta,1}$ and $\psi_{c\theta,2}$ as products of eigenfunctions of the two operators in the radial and axial coordinates and in time. The operator in the radial coordinate yields as the related ordinary differential equation of the separated function, say, $f(r)$ a Bessel-type equation which, however, is of the form $r^2 f'' - r f' - k^2 r^2 f = 0$, i.e. the first minus sign in the equation differs from the plus in the Bessel differential equation. The solution of this equation has the form $C_1\, r I_1(kr) + C_2\, r K_1(kr)$, where I_1 and K_1 are modified Bessel functions of the first and second kinds, respectively [1]. For $\psi_{c\theta,1}$ we therefore obtain

$$\psi_{c\theta,1} = [C_1 r I_1(kr) + C_2 r K_1(kr)]\, e^{ikz-\alpha t} + \mathrm{const}_1 \,. \tag{2.66}$$

The quantity $k = 2\pi/\lambda$ is the wave number of a spatially periodic variation of the stream function with the wavelength λ in the direction of the axial coordinate z. The dependency of the flow on the axial coordinate z and time t is represented by the exponential function with an imaginary argument in z and a complex argument in t. The latter therefore allows for damped or growing periodic disturbances, as encountered in the temporal instability of cylindrical flows. For the function $\psi_{c\theta,2}$ we obtain

$$\psi_{c\theta,2} = [C_3 r I_1(lr) + C_4 r K_1(lr)]\, e^{ikz-\alpha t} + \mathrm{const}_2 \,, \tag{2.67}$$

where $l^2 = k^2 - \alpha/\nu$. The stream function can therefore be written in a general form as

$$\psi_{c\theta} - \psi_{c\theta,0} = [C_1 r I_1(kr) + C_2 r K_1(kr) + C_3 r I_1(lr) + C_4 r K_1(lr)] \cdot e^{ikz-\alpha t} \,, \tag{2.68}$$

where $\psi_{c\theta,0}$ is a constant. When applying this *Stokes*ian stream function in axially symmetric cylindrical flows to special geometries, the values of the coefficients C_i are determined by initial and boundary conditions of the problem. Terms of the solution must be discarded in regions of the flow field where they diverge by setting the related coefficients C_i to zero. This is the case, e.g. for the flow field of a jet at $r = 0$, i.e. on its symmetry axis, where the modified Bessel functions of the second kind diverge. Analogously, in the flow field of the medium in which the jet is immersed, which may extend to infinity, the modified Bessel functions of the first kind must be discarded, since they diverge for infinite values of their arguments. The actual need for setting the related coefficients C_i to zero, however, is seen only in the resulting equations for the corresponding velocity components. We will go into the details of this in Chap. 6.

A special case of the axisymmetric linear unsteady flow may be independent on the coordinate z in the main flow direction, which corresponds to a flow hydraulically developed in the direction of its motion. The partial differential equation for the stream function then simplifies to the form

$$\left[-\frac{1}{\nu}\frac{\partial}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]\left(\frac{1}{r}\frac{\partial \psi_{c\theta}}{\partial r}\right) = C \,, \tag{2.69}$$

where C may be a function of time or a constant, which may be zero as well. We may be interested in a solution of this equation for the case of, e.g. a hydraulically developed pulsating ($C = C(t)$) or starting ($C = \text{constant} \neq 0$) / fading ($C = 0$) pipe flow. The quantity C represents a pressure gradient driving the flow.

We first solve Eq. (2.69), for the case that $C = C_1 + C_t e^{-\alpha t}$. We seek for a stream function of the form

$$\psi_{c\theta} = \psi_{c\theta s}(r) + f(r)e^{-\alpha t} \ , \tag{2.70}$$

which is composed of a steady and an unsteady part. The steady part $\psi_{c\theta s}(r)$ of the stream function satisfies the ODE

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left(\frac{\psi'_{c\theta s}}{r}\right)\right] = C_1 \ , \tag{2.71}$$

where the prime denotes the derivative with respect to the coordinate r. The solution reads

$$\psi_{c\theta s} = C_1 r^4 + C_2 r^2\left(\ln r - \frac{1}{2}\right) + C_3 r^2 + C_4 \ . \tag{2.72}$$

Substituting the formulation (2.70) of the stream function into Eq. (2.69), and using (2.71) for the steady part, we obtain the following ODE for the unknown function $f(r)$:

$$\frac{\alpha}{\nu}\left(\frac{f'}{r}\right) + \frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left(\frac{f'}{r}\right)\right] = C_t \ . \tag{2.73}$$

Replacing the function f'/r by a new function $h(r)$, we obtain the equation

$$\frac{\alpha}{\nu}h + \frac{1}{r}\frac{d}{dr}\left(rh'\right) = C_t \ , \tag{2.74}$$

which we rewrite to obtain

$$r^2 h'' + rh' + \frac{\alpha}{\nu}r^2 h = C_t r^2 \ . \tag{2.75}$$

Defining a new variable $\xi = qr$ with $q = (\alpha/\nu)^{1/2}$ and substituting $h(r) =: k(\xi)$, we obtain

$$\xi^2 k'' + \xi k' + \xi^2 k = C_t r^2 \ . \tag{2.76}$$

This is an inhomogeneous form of a *Bessel*-type ordinary differential equation. The solution of the homogeneous equation is composed of the two zero-order Bessel functions of the first and second kinds, $J_0(\xi)$ and $Y_0(\xi)$. For the inhomogeneous equation (2.76) we find a particular solution $k_p(r)$ with the ansatz that the solution must be a constant, $k_p(r) = C$, which satisfies the inhomogeneous equation, so that the general solution of the differential equation (2.76), expressed in the function $h(r)$ reads

$$h(r) = C_1 J_0(qr) + C_2 Y_0(qr) + C_3 \ . \tag{2.77}$$

From this function we derive the function $f(r)$ in the ansatz (2.70) as

$$f(r) = C_1 \int r J_0(qr) dr + C_2 \int r Y_0(qr) dr + \frac{C_3}{2} r^2 + C_4 \ . \tag{2.78}$$

The final form of the stream function for the pipe flow with an imposed time-dependent pressure gradient driving the flow follows as

$$\begin{aligned} \psi_{c\theta} - \psi_{c\theta,0} &= C_1' r^4 + C_2' r^2 + C_3' r^2 \ln r + \\ &+ \left[C_4 \int r J_0(qr) dr + C_5 \int r Y_0(qr) dr + C_6 r^2 + C_7 \right] e^{-\alpha t} \ , \end{aligned} \tag{2.79}$$

where $\psi_{c\theta,0}$ is a constant.

The case of an impulsively started pipe flow corresponds to the imposition of a constant pressure gradient driving the flow at some time instant $t = 0$. The quantity C in Eq. (2.69) is then a non-zero constant. The stream function has again the structure (2.70), since the flow converges to the steady *Hagen–Poiseuille* flow. The steady part of the stream function is given by the same function (2.72) as above. The function $f(r)$ in the unsteady part is determined by the differential equation

$$\frac{\alpha}{\nu} \left(\frac{f'}{r} \right) + \frac{1}{r} \frac{d}{dr} \left[r \frac{d}{dr} \left(\frac{f'}{r} \right) \right] = 0 \ . \tag{2.80}$$

Its solution derived along the same lines as above is

$$f(r) = C_1 \int r J_0(qr) dr + C_2 \int r Y_0(qr) dr + C_3 \ . \tag{2.81}$$

The final form of the stream function for the starting pipe flow follows as

$$\begin{aligned} \psi_{c\theta} - \psi_{c\theta,0} &= C_1' r^4 + C_2' r^2 + C_3' r^2 \ln r + \\ &+ \left[C_4 \int r J_0(qr) dr + C_5 \int r Y_0(qr) dr + C_6 \right] e^{-\alpha t} \ , \end{aligned} \tag{2.82}$$

where $\psi_{c\theta,0}$ is a constant.

In the corresponding case of a fading pipe flow, the pressure gradient formerly driving the flow is removed at some time instant $t = 0$. The quantity C in Eq. (2.69) is then zero. A steady part of the stream function does not exist in this case, so that the stream function consists of the unsteady part of (2.70) only. The function $f(r)$ is determined by the same differential equation as for the impulsively started flow, with the same solution as there. The final form of the stream function for the fading pipe flow therefore reads

$$\psi_{c\theta} - \psi_{c\theta,0} = \left[C_4 \int r J_0(qr) dr + C_5 \int r Y_0(qr) dr + C_6 \right] e^{-\alpha t} \; , \tag{2.83}$$

where $\psi_{c\theta,0}$ is a constant.

These flows will be discussed in detail in Chap. 3.

2.2.5 Axisymmetric, Linear, Steady Flow

For linear steady cylindrical flow problems which are swirl-free and independent from the coordinate θ, i.e. which are axisymmetric, the equation determining the stream function reads

$$E^4_{c\theta} \psi_{c\theta} = 0 \; , \tag{2.84}$$

where $E^4_{c\theta}$ is given in Eq. (2.62). Separation of the stream function $\psi_{c\theta}$ into functions of the radial coordinate r and the axial coordinate z, with the assumption that the solution is periodic in the axial direction with a wavenumber $k = 2\pi/\lambda$, where λ is the wavelength, leads to the following ODE for the function $f(r)$ representing the radial dependency of the stream function

$$r \frac{d}{dr} \left\{ \frac{1}{r} \frac{d}{dr} \left[r \frac{d}{dr} \left(\frac{f'}{r} \right) \right] \right\} - 2k^2 r \frac{d}{dr} \left(\frac{f'}{r} \right) + k^4 f = 0 \; . \tag{2.85}$$

In the equation, the prime denotes the derivative with respect to the radial coordinate. This equation has the solution

$$f(r) = k^2 r^2 \left[C_1 I_0(kr) + \frac{C_2}{kr} I_1(kr) + C_3 K_0(kr) + \frac{C_4}{kr} K_1(kr) \right] \; , \tag{2.86}$$

which, together with the periodic function in the axial coordinate z, yields the stream function

$$\psi_{c\theta} - \psi_{c\theta,0} = k^2 r^2 \left[C_1 I_0(kr) + \frac{C_2}{kr} I_1(kr) + C_3 K_0(kr) + \frac{C_4}{kr} K_1(kr) \right] e^{ikz} \; . \tag{2.87}$$

A special case is a linear steady flow field with the fluid motion in the z direction, which is hydraulically developed and therefore does not depend on the z coordinate. The stream function, therefore, cannot depend on that coordinate. The reduced form of Eq. (2.84) is the statement that

$$\frac{d}{dr} \left\{ \frac{1}{r} \frac{d}{dr} \left[r \frac{d}{dr} \left(\frac{1}{r} \frac{d\psi_{c\theta}}{dr} \right) \right] \right\} = 0 \; . \tag{2.88}$$

The solution is

$$\psi_{c\theta} = C_1 \frac{r^4}{16} + C_2 \frac{r^2}{2} \left(\ln r - \frac{1}{2} \right) + C_3 \frac{r^2}{2} + C_4 \, . \tag{2.89}$$

Three of the four constants are determined by boundary conditions, while C_4 is a free reference value.

Since the values of all the integration constants in the above solutions depend on the particular flow problem, we will go into the details of their determination in Chap. 3.

2.2.6 Axisymmetric, Nonlinear, Steady Flow with Constant Pressure

Another class of flows accessible with analytical methods are boundary-layer flows in simple geometries with constant pressure throughout, such as flows along bodies of revolution, round submerged jets, etc. The pressure is constant in these flow fields, since, in the region far from the edge of the boundary layer, the free-stream velocity does not depend on the coordinate in the main flow direction, and the resultant constant pressure is imprinted on the boundary layer in the direction transverse to the main flow. In the boundary layer, therefore, the pressure does not depend on the coordinate transverse to the main flow direction as well. In these cases, the pressure gradient disappears from the momentum equation in the boundary-layer form identically. Therefore, after introduction of the stream function, there is no need to take the curl of the momentum equation. In these cases, we obtain from the momentum equation in the main flow direction z the PDE for the stream function in cylindrical coordinates

$$\frac{\partial \psi_{c\theta}}{\partial z} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi_{c\theta}}{\partial r} \right) - \frac{1}{r} \frac{\partial \psi_{c\theta}}{\partial r} \frac{\partial^2 \psi_{c\theta}}{\partial r \partial z} = -\nu \frac{\partial}{\partial r} \left[r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi_{c\theta}}{\partial r} \right) \right] . \tag{2.90}$$

The stream function is composed of a mapping function of the spatial coordinate z only and a universal function of a *self-similar* coordinate. The *self-similar* coordinate is an r coordinate transverse to the main flow direction, normalised by the boundary-layer thickness, which itself depends on the z position in the field. The corresponding profiles of the normalised velocity component in the main flow direction z turn out *self-similar*, i.e. they are determined by the self-similar function only. The reference velocity for the normalisation may be the incoming free-stream velocity in that flow direction. In jets, it is the maximum value of the z velocity, located on the symmetry axis of the jet. This *self-similar* behaviour of the flow may be made use of for deriving analytical solutions of the differential equation (2.90). For this purpose, we define the stream function with a self-similar function f_c of a self-similar coordinate η_c. We define the stream function ψ_c as the product of a power z^α of the coordinate

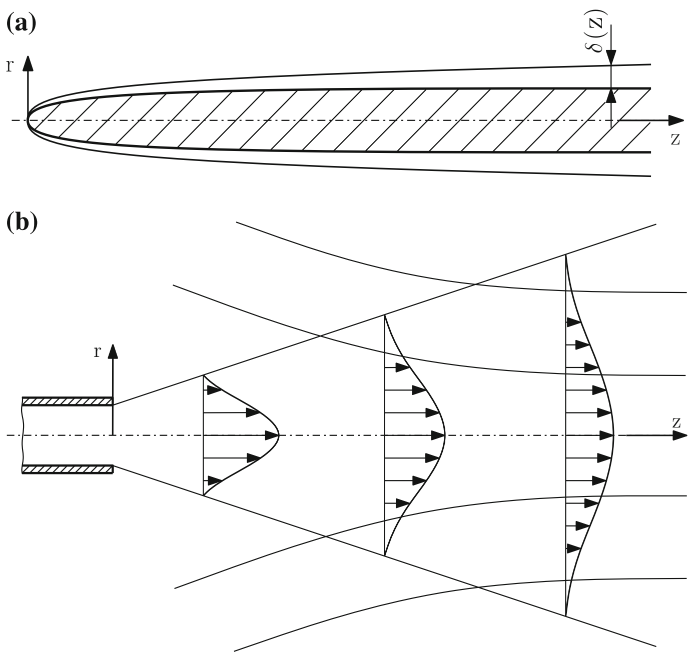

Fig. 2.2 Two examples of cylindrical flow field geometries with boundary-layer type: **a** flow along a submerged body of revolution, and **b** submerged round free jet

in the main flow direction, which plays the role of the mapping function, and the function $f_c(\eta_c)$. The self-similar coordinate is defined as proportional to the ratio of the coordinate r transverse to the main flow direction and the width of the flow domain influenced by diffusive viscous momentum transport. This width is assumed to be proportional to a power z^β of the coordinate in the main flow direction, with a positive exponent β. So we assume that the flow domain widens in the direction of the flow. Figure 2.2 shows two examples of this flow field geometry: (a) the boundary-layer flow along a slender body of revolution and (b) a round submerged free jet. The stream function and the self-similar coordinate have the forms

$$\psi_{c\theta} = C_c z^\alpha f_c(\eta_c), \quad \eta_c = D_c \frac{r}{z^\beta} \; . \tag{2.91}$$

In these definitions, the coefficients C_c and D_c are introduced for dimensional reasons: C_c to allow the velocity u_z in the z direction to be obtained as the derivative $(1/r)\partial\psi_{c\theta}/\partial r$, and D_r to render the self-similar coordinate η_c non-dimensional. Substituting these definitions into Eq. (2.90), we obtain

$$(\alpha - 2\beta)\, f_c'^2 - \alpha f_c f_c'' + \alpha \frac{f_c f_c'}{\eta_c} = \frac{\nu}{C_c z^{\alpha-1}} \eta_c \left[\eta_c \left(\frac{f_c'}{\eta_c} \right)' \right]' \; . \tag{2.92}$$

For the concept of self-similarity to work, we require that this equation is an ODE for the function $f_c(\eta_c)$. This is the case if and only if the dependency on the coordinate z disappears, i.e. if its exponent equals zero. We therefore require that $\alpha = 1$ and obtain the self-similar differential equation

$$(1-2\beta)\, f_c'^2 - f_c f_c'' + \frac{f_c f_c'}{\eta_c} = \frac{\nu}{C_c}\eta_c \left[\eta_c \left(\frac{f_c'}{\eta_c}\right)'\right]' . \tag{2.93}$$

A second condition for the exponents α and β, and expressions relating C_c and D_c to the flow situation, are found by adapting Eq. (2.93) to the respective flow. In order to obtain the differential equation for the self-similar function f_c in a form free of coefficients, we may require that $C_c = \nu$, which is consistent with the requirement that the dimension of C_c is m^2/s. Furthermore, the dimension of D_c must be $m^{\beta-1}$ in order that η_c is non-dimensional. Since this adaptation determining the final form of Eq. (2.93) and the solutions of the equation depend on the actual flow situation, we will go into these details in Chap. 5, where the related boundary-layer flows are discussed.

2.3 The Equation for the Stream Function in Spherical Coordinates

The *spherical coordinates* (r, θ, ϕ) are related to the Cartesian coordinates as per $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$. For spherical flow fields where the velocity vector is $(u_r, u_\theta, 0)$ (with zero swirl component), which are axially symmetric around the z axis, the stream function ψ_s is introduced by the definitions of the r and θ velocity components

$$u_r = -\frac{1}{r^2\sin\theta}\frac{\partial\psi_s}{\partial\theta}, \quad u_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi_s}{\partial r} \tag{2.94}$$

Introducing this two-dimensional velocity vector into the momentum equation (1.3), with the material law (1.8), and taking the curl of the resulting equation to make the gradient fields of pressure and body force potential disappear, we obtain the fourth-order partial differential equation [2]

$$\frac{\partial}{\partial t}\left(E_s^2\psi_s\right) + \frac{1}{r^2\sin\theta}\frac{\partial(\psi_s, E_s^2\psi_s)}{\partial(r,\theta)} - \frac{2E_s^2\psi_s}{r^2\sin^2\theta}\left(\frac{\partial\psi_s}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial\psi_s}{\partial\theta}\sin\theta\right) = \nu E_s^4\psi_s \tag{2.95}$$

for the stream function ψ_s. The differential operator E_s^2 reads

$$E_s^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) . \tag{2.96}$$

2.3.1 Linear, Unsteady Flow

Analytical solutions of Eq. (2.95) are found for linear flow fields, where the *Jacobi*an is either negligible or vanishes exactly. The linearisation for spherical flows in the r, θ plane leads to the equation

$$\left(-\frac{1}{\nu}\frac{\partial}{\partial t} + E_s^2\right) E_s^2 \psi_s = 0 \; . \tag{2.97}$$

In solving this equation, we use the approach of *Tomotika* developed for the cylindrical case for the spherical problem as well [3]. The details are given in Sect. 2.2.4. The stream function may be composed of two parts as per $\psi_s = \psi_{s,1} + \psi_{s,2}$, where $\psi_{s,1}$ and $\psi_{s,2}$ are solutions of the two partial differential equations

$$E_s^2 \psi_{s,1} = 0 \tag{2.98}$$

and

$$-\frac{1}{\nu}\frac{\partial \psi_{s,2}}{\partial t} + E_s^2 \psi_{s,2} = 0 \; , \tag{2.99}$$

respectively. The two Eqs. (2.98) and (2.99) are solved by separation of variables, which reveals the functions $\psi_{s,1}$ and $\psi_{s,2}$ as products of eigenfunctions of the two operators in the radial and angular coordinates and in time. The dependencies on the polar angle θ exhibit a modal structure. For $\psi_{s,1}$ of mode m we obtain

$$\psi_{s,1,m} = \left[C_{1,m} r^{-m} + C_{2,m} r^{m+1}\right]\left[A_m P_m'(\cos\theta) + B_m Q_m'(\cos\theta)\right]\sin^2\theta e^{-\alpha t} + C_1 \; , \tag{2.100}$$

where $P_m'(\cos\theta)$ and $Q_m'(\cos\theta)$ are the first-order derivatives of the *Legendre* polynomial (*Legendre* function of the first kind) $P_m(\cos\theta)$ and the *Legendre* function of the second kind $Q_m(\cos\theta)$ with respect to their arguments. The mode number m is a natural number or zero. For the function $\psi_{s,2}$ of mode m we obtain

$$\begin{aligned}\psi_{s,2,m} = &\left[C_{1,m}' qr j_m(qr) + C_{2,m}' qr y_m(qr)\right] \cdot \\ &\cdot \left[A_m P_m'(\cos\theta) + B_m Q_m'(\cos\theta)\right]\sin^2\theta e^{-\alpha t} + C_2 \; ,\end{aligned} \tag{2.101}$$

where j_m and y_m are spherical Bessel functions of the first and second kinds, respectively. In their arguments we have defined

$$q = (\alpha/\nu)^{1/2} \; . \tag{2.102}$$

The stream function can therefore be written in a general form as

$$\begin{aligned}\psi_{s,m} - \psi_{s,m,0} = &\left[C_{1,m} r^{-m} + C_{2,m} r^{m+1} + C_{1,m}' qr j_m(qr) + C_{2,m}' qr y_m(qr)\right] \cdot \\ &\cdot \left[A_m P_m'(\cos\theta) + B_m Q_m'(\cos\theta)\right]\sin^2\theta e^{-\alpha t} \; ,\end{aligned} \tag{2.103}$$

where $\psi_{s,m,0}$ is a constant. The general form of the stream function is a sum over all the modes m. When applying the general description for the *Stokes*ian stream function to flows in special geometries, the values of the coefficients $C_{i,m}$ are determined by initial and boundary conditions of the problem. Terms of the solution must be discarded in regions of the flow field where they diverge by setting the related coefficients $C_{i,m}$ to zero. This is the case, e.g. for a flow field including the origin $r = 0$ of the coordinate system, where the functions r^{-m} and $y_m(qr)$ diverge. Furthermore, the powers of the radial coordinate represented in the function (2.103) put limits to allowable values of the mode number m which depend on the geometry. The actual need for discarding terms, however, is seen only in the resulting equations for the corresponding velocity components. We will go into the details of this in Chaps. 3 and 6.

2.3.2 Linear, Steady Flow

For linear steady flows in spherical coordinates independent on the coordinate ϕ, and without a (swirl) velocity in this coordinate direction, the equation determining the stream function reads

$$E_s^4 \psi_s = 0 \,. \tag{2.104}$$

Given the form (2.96) of the operator E_s^2, Eq. (2.104) reads

$$\begin{aligned} E_s^4 \psi_s = {} & \frac{\partial^2}{\partial r^2}\left[\frac{\partial^2 \psi_s}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi_s}{\partial\theta}\right)\right] + \\ & + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial^2\psi_s}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi_s}{\partial\theta}\right)\right)\right] = 0 \,. \end{aligned} \tag{2.105}$$

Seeking for a solution of this partial differential equation for ψ_s in the form of eigenfunctions of the differential operators involved, we use the separation ansatz defining $\psi_s := f(r)\, g(\theta)$. This enables the equation to be rewritten in the form

$$\begin{aligned} g f^{IV} + {} & \left[\frac{d^2}{dr^2}\left(\frac{f}{r^2}\right) + \frac{f''}{r^2}\right]\sin\theta \frac{d}{d\theta}\left(\frac{g'(\theta)}{\sin\theta}\right) + \\ & + \frac{f}{r^4}\sin\theta\frac{d}{d\theta}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\left(\frac{g'(\theta)}{\sin\theta}\right)\right)\right] = 0 \,, \end{aligned} \tag{2.106}$$

where the primes denote derivatives with respect to the independent variables of the functions. Denoting furthermore $H(\theta) := \sin\theta d/d\theta \left(g'(\theta)/\sin\theta\right)$, (2.106) becomes

$$r^4\frac{f^{IV}}{f} + \frac{r^4}{f}\left[\frac{d^2}{dr^2}\left(\frac{f}{r^2}\right) + \frac{f''}{r^2}\right]\frac{H(\theta)}{g(\theta)} + \frac{\sin\theta}{g(\theta)}\frac{d}{d\theta}\left(\frac{H'(\theta)}{\sin\theta}\right) = 0 \,. \tag{2.107}$$

A separable form of this equation is achieved if $H(\theta)/g(\theta)$ is a constant, which we denote $-m(m+1)$, where m may be a natural mode number or zero. This turns the relation between $g(\theta)$ and $H(\theta)$ into a *Legendre*-type ODE with the solution

$$g(\theta) = -\left[A_m P_m'(\cos\theta) + B_m Q_m'(\cos\theta)\right]\sin^2\theta\ . \tag{2.108}$$

This function turns the term $(\sin\theta/g)d/d\theta\left(H'(\theta)/\sin\theta\right)$ into a constant with the value $m^2(m+1)^2$. Consequently, Eq. (2.107) becomes an *Euler*-type ODE for the function $f(r)$, which reads

$$r^4 f^{IV} - 2m(m+1)r^2 f'' + 4m(m+1)rf' + m(m+1)(m-2)(m+3)f = 0\ . \tag{2.109}$$

Introducing the ansatz $f(r) = Cr^k$ into Eq. (2.109) and determining the roots of the resulting characteristic equation

$$k\,(k-3)\left[(k-1)\,(k-2) - 2m\,(m+1)\right] = -m\,(m+1)\,(m-2)\,(m+3) \tag{2.110}$$

for the exponent k with a given mode number m, we find a linear combination of powers of the radial coordinate r which constitutes the general solution of the equation. The resulting stream function ψ_s for a given mode number m may therefore be written in the form

$$\psi_{s,m} - \psi_{s,m0} = \sum_{i=1}^{4} C_{i,m} r^{k_i} \cdot \left[A_m P_m'(\cos\theta) + B_m Q_m'(\cos\theta)\right]\sin^2\theta\ . \tag{2.111}$$

The coefficients $C_{i,m}$ and the values of the exponents k_i are determined by the respective details of the flow. We will discuss this in further detail in Chap. 6.

2.3.3 Nonlinear, Steady Flow with Constant Pressure

The class of boundary-layer flow with constant pressure throughout the flow field does not exist for the spherical geometry, since the curvature of the body in all directions always leads to a pressure gradient in the field. A boundary-layer flow in such a geometry would therefore be restricted to a region around the upstream stagnation zone where, however, the boundary layer may still not be regarded as slender in the sense of the boundary-layer approximation. Such flow is, therefore, not considered in the present context.

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