Chapter 2
The Extension Principle of Zadeh and Fuzzy Numbers

Everything has numbers and nothing can be understand without numbers.

(Philolaus, Pythagorean-C.470 - C.385 BCE)

Abstract This chapter presents the *Extension Principle of Zadeh*, and as the name suggests, it is a method used to extend to fuzzy set theory the typical operations of classical set theory. It gives the framework to calculate the membership degree of elements of a fuzzy set and functions of fuzzy sets, which are the result of operations. Also, in the context of fuzzy sets, the concepts of fuzzy number and fuzzy number arithmetic are introduced.

2.1 Zadeh's Extension Principle

There is a need to extend concepts from the classical set theory to fuzzy set theory. The extension method proposed by Zadeh, known also as the *Extension Principle*, is one of the basic ideas that induces the extension of nonfuzzy mathematical concepts into fuzzy ones.

The Zadeh's Extension Principle for a function $f : X \longrightarrow Z$ indicates how the image of a fuzzy subset A of X should be computed when the function f is applied. It is expected that this image will be a fuzzy subset of Z.

Definition 2.1 (*Zadeh's Extension Principle*) Let f be a function such that $f : X \longrightarrow Z$ and let A be a fuzzy subset of X. *Zadeh's extension* of f is the function $\widehat{f}$ which applied to A gives us the fuzzy subset $\widehat{f}(A)$ of Z with the membership function given by

$$\varphi_{\widehat{f}(A)}(z) = \begin{cases} \sup\limits_{f^{-1}(z)} \varphi_A(x) & \text{if } f^{-1}(z) \neq \emptyset \\ 0 & \text{if } f^{-1}(z) = \emptyset \end{cases}, \tag{2.1}$$

where $f^{-1}(z) = \{x;\ f(x) = z\}$ is the *preimage* of z.

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We can observe that if f is a bijective function, then

$$\{x : f(x) = z\} = \left\{f^{-1}(z)\right\},$$

where f^{-1} means the inverse function of f. Thus, if A is a fuzzy subset of X, with the membership function φ_A, and if f is bijective, then the membership function of $\widehat{f}(A)$ is given as follows

$$\varphi_{\widehat{f}(A)}(z) = \sup_{\{x:\, f(x)=z\}} \varphi_A(x) \; = \sup_{\left\{x\in f^{-1}(z)\right\}} \varphi_A(x) = \varphi_A(f^{-1}(z)). \tag{2.2}$$

The graph of how to construct the extension $\widehat{f}$ of f is illustrated in Fig. 2.1, where we have used a bijective function f. We observe that if f is injective, then $z = f(x)$ belongs to the fuzzy subset $\widehat{f}(A)$ with the same degree α as x belongs to A. This may not happen if f is not injective.

Let $f : X \to Z$ be an injective function and A a countable (or finite) fuzzy subset of X given by

$$A = \sum_{i=1}^{\infty} \varphi_A(x_i)/x_i.$$

Then the extension principle ensures that $\widehat{f}(A)$ is a fuzzy subset of Z given by

$$\widehat{f}(A) = \widehat{f}\left(\sum_{i=1}^{\infty} \varphi_A(x_i)/x_i\right) = \sum_{i=1}^{\infty} \varphi_A(x_i)/f(x_i).$$

Therefore, the image of A by $\widehat{f}$ can be derived from the knowledge of the images of x_i through f. The membership degree of $z_i = f(x_i)$ in $\widehat{f}(A)$ is the same as x_i in A.

Example 2.1 Let A be a fuzzy set with countable support, $f(x) = x^2$ and $x \geq 0$. Then

$$\widehat{f}(A) = \sum_{i=1}^{\infty} \varphi_A(x_i)/f(x_i) = \sum_{i=1}^{\infty} \varphi_A(x_i)/x_i^2.$$

The extension principle extends the concept to fuzzy sets of a function applied to a classical subset of X. Indeed, let $f : X \longrightarrow Z$ be a function and A a classical subset

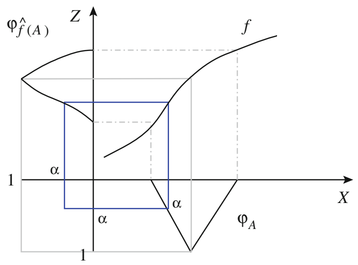


Fig. 2.1 Image of a fuzzy subset from the extension principle for a function f

of X. The membership function of A is its characteristic function. The Zadeh's extension of f applied to A is the subset $\widehat{f}(A)$ of Z, which is the characteristic function

$$\begin{aligned}\varphi_{\widehat{f}(A)}(z) &= \sup_{\{x:f(x)=z\}} \chi_A(x) = \begin{cases} 1 & \text{if } z \in f(A) \\ 0 & \text{if } z \notin f(A) \end{cases} \\ &= \chi_{f(A)}(z)\end{aligned}$$

for all z. Clearly, the membership function of the fuzzy set $\widehat{f}(A)$ is just the characteristic function of the crisp set $f(A)$, that is, the fuzzy set $\widehat{f}(A)$ coincides with the classical set $f(A)$:

$$\widehat{f}(A) = f(A) = \{f(a) : a \in A\}\,.$$

As can be seen in the formula above, when A is a classical set, the image $\widehat{f}(A)$ is clear, i.e., the formula (2.1) is unnecessary since each $f(a)$ belongs to $f(A)$ with membership degree equal to 1. The Zadeh Extension Principle in the context of classical sets is precisely what is called the *united extension* used in set-valued function theory and in interval analysis [1]. The difference between Zadeh's Extension Principle and United Extension is that Zadeh's Extension Principle maps membership functions into membership functions which is equivalent to mapping fuzzy sets to fuzzy sets but does so via a membership functions. Set-valued functions including real-valued intervals, map directly from set to set (without the intermediate membership function).

We can also notice that if A is a classical set, then $[A]^\alpha = A$ for all $\alpha \in]0, 1]$. Consequently,

$$[\widehat{f}(A)]^\alpha = [f(A)]^\alpha = f(A) = f([A]^\alpha).$$

Recall that for $\alpha = 0$ we mean that $[A]^0$ is the closure of A, that is, the smallest closed set containing the support of A, if X is a topological space. This result, stated here as Theorem 2.1, can also be applied to a fuzzy subset of X [2–4].

Theorem 2.1 *Let $f : X \longrightarrow Z$ be a continuous function and let A be a fuzzy subset of X. Then, for all $\alpha \in [0, 1]$, the following equality holds*

$$[\widehat{f}(A)]^\alpha = f([A]^\alpha). \tag{2.3}$$

This result indicates that the α-levels of the fuzzy set obtained by the Zadeh's Extension Principle coincides with the images of the α-levels by the crisp function (see Fig. 2.2). The proof of this theorem uses Weierstrass Theorem and it can be seen in [2–4].

Example 2.2 Let A be a fuzzy set of real numbers whose membership function is given by

$$\varphi_A(x) = \begin{cases} 4(x - x^2) & \text{if } x \in [0, 1] \\ 0 & \text{if } x \notin [0, 1] \end{cases}.$$

The α-levels of A are the intervals

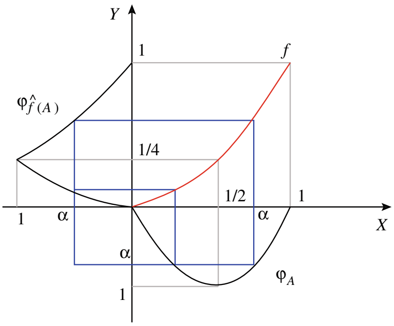
$$[A]^\alpha = \left[\frac{1}{2}(1-\sqrt{1-\alpha}), \frac{1}{2}(1+\sqrt{1-\alpha})\right].$$

Let us now consider the real function $f(x) = x^2$ for $x \geq 0$. Since f is an increasing function, we have

$$\begin{aligned} f([A]^\alpha) &= \left[f(\frac{1}{2}(1-\sqrt{1-\alpha})),\ f(\frac{1}{2}(1+\sqrt{1-\alpha}))\right] \\ &= \left[\frac{1}{4}(1-\sqrt{1-\alpha})^2, \frac{1}{4}(1+\sqrt{1-\alpha})^2\right] \\ &= [\hat{f}(A)]^\alpha. \end{aligned}$$

Figure 2.2 illustrates the fuzzy subset $\widehat{f}(A)$.

Fig. 2.2 Subset $\widehat{f}(A)$ from the Example 2.2



Exercise 2.1 Consider f and A as in Example 2.2. Compute $[\widehat{f}(A)]^\alpha$ for $\alpha = 0$, $\alpha = 3/4$ and $\alpha = 1$.

The relation between classical set functions and fuzzy functions is the following. Let A be a classical set. Its membership function in this context is

$$\chi_A(x) = \mu_A(x) = \begin{cases} 1 & x \in A \\ 0 & \text{otherwise} \end{cases}.$$

Classical set-valued functions compute the range of a classical set A by

$$f(A) = \{y \mid y = f(x),\ x \in A\}.$$

Simply, $f(A)$ is the range set of f over the domain set A. Using Zadeh's Extension Principle, we have for the classical set A,

$$\varphi_{\hat{f}(A)}(z) = \begin{cases} \sup\limits_{x \in f^{-1}(z)} \chi_A(x) & \text{if } f^{-1}(z) \neq \emptyset \\ 0 & \text{if } f^{-1}(z) = \emptyset \end{cases} \tag{2.4}$$

$$= \begin{cases} 1 & z \in f(A) \\ 0 & z \notin f(A) \end{cases}. \tag{2.5}$$

In particular, if $A = [a, b]$, $a \leq b$, $a, b \in \mathbb{R}$,

$$\varphi_{\hat{f}([a,b])}(z) = \begin{cases} \sup\limits_{x \in f^{-1}(z)} \chi_{[a,b]}(x) & \text{if } f^{-1}(z) \neq \emptyset \\ 0 & \text{if } f^{-1}(z) = \emptyset \end{cases} \tag{2.6}$$

$$= \begin{cases} 1 & z \in f([a, b]) \\ 0 & z \notin f([a, b]) \end{cases}. \tag{2.7}$$

This means that Zadeh's Extension Principle is the same as the United Extension on Intervals.

Let us introduce the Extension Principle for functions of two variables looking towards operations between fuzzy numbers that we will present in the next section.

Definition 2.2 Let $f : X \times Y \longrightarrow Z$ be a function and let A and B be fuzzy subsets of X and Y, respectively. The extension $\widehat{f}$ of f, applied to A and B is the fuzzy subset $\widehat{f}(A, B)$ of Z with membership function given by:

$$\varphi_{\widehat{f}(A,B)}(z) = \begin{cases} \sup\limits_{f^{-1}(z)} \min\left[\varphi_A(x), \varphi_B(y)\right] & \text{if } f^{-1}(z) \neq \emptyset \\ 0 & \text{if } f^{-1}(z) = \emptyset \end{cases}, \tag{2.8}$$

where $f^{-1}(z) = \{(x, y) : f(x, y) = z\}$.

Example 2.3 Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function that satisfies $f(x, y) = x + y$. Consider the finite fuzzy sets of $\mathbb{R}$

$$A = 0.4/3 + 0.5/4 + 1/5 + 0.5/6 + 0.2/7$$
$$B = 0.2/6 + 0.5/7 + 1/8 + 0.5/9 + 0.2/10.$$

Let us compute the membership degree of $z = 10$ in $\widehat{f}(A, B)$:

$$\begin{aligned} \varphi_{\widehat{f}(A,B)}(10) = & \sup_{\{x+y=10\}} \min[\varphi_A(x), \varphi_B(y)] = \\ & = \max\{\min[\varphi_A(3), \varphi_B(7)], \min[\varphi_A(4), \varphi_B(6)]\} \\ & = \max\{0.4; 0.2\} = 0.4. \end{aligned}$$

Exercise 2.2 Redo the Example 2.3 for the following two functions:

(a) Defining $f(x, y) = x^2 + y$, determine $\widehat{f}(A, B)$ and the membership degrees of $z = 10$ and $z = 25$ in $\widehat{f}(A, B)$.

(b) Now, if $f(x, y) = 2x + y$, determine $\widehat{f}(A, B)$ and the membership degree of $z = 18$ in $\widehat{f}(A, B)$.

2.2 Fuzzy Numbers

Concrete problems often involve many quantities that are idealizations of inaccurate information involving numerical values. This is the reason why we use words like "around" in such cases. For example, when we measure the height of a person, what we obtain is a numerical value with a level of imprecision. These imprecisions may be caused by the measuring instrument, by the individuals who took the measurements, by the person who was measured, and for many other reasons. In the end, an "accurate value" (real number) h is chosen to indicate the height of the person. However, it would be more prudent to say that the height is *around* or *approximately* h. Mathematically, we represent the expression *around* h by a fuzzy subset A, whose domain of the membership function φ_A is the set of all real numbers. Also, it is reasonable to expect that $\varphi_A(h) = 1$. The choice of real numbers as the domain is due to the fact that, theoretically, the possible values to the height of a person are real numbers.

Definition 2.3 (*Fuzzy Number*) A fuzzy subset A is called a *fuzzy number* when the universal set on which φ_A is defined is the set of all real numbers $\mathbb{R}$ and satisfies the following conditions:

(i) all the α-levels of A are not empty for $0 \leq \alpha \leq 1$;
(ii) all the α-levels of A are closed intervals of $\mathbb{R}$;
(iii) $\text{supp}\, A = \{x \in \mathbb{R} : \varphi_A(x) > 0\}$ is bounded.

Let us represent the α-levels of the fuzzy number A by

$$[A]^\alpha = [a_1^\alpha, a_2^\alpha].$$

We observe that every real number r is a fuzzy number whose membership function is the characteristic function:

$$\chi_r(x) = \begin{cases} 1 & \text{if } x = r \\ 0 & \text{if } x \neq r \end{cases}.$$

We will denote χ_r or just by $\widehat{r}$.

The set of all fuzzy numbers will be denoted by $\mathcal{F}(\mathbb{R})$, and accordingly to what was observed above, the set of the real numbers $\mathbb{R}$ is a subset (classical or crisp) of $\mathcal{F}(\mathbb{R})$. The most common fuzzy numbers are the *triangular, trapezoidal* and the *bell shape* numbers.

Example 2.4 The *fuzzy number* $\widehat{2}$ may be depicted as in Fig. 2.3.

Definition 2.4 A fuzzy number A is said to be *triangular* if its membership function is given by

$$\varphi_A(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{u-a} & \text{if } a < x \leq u \\ \frac{x-b}{u-b} & \text{if } u < x \leq b \\ 0 & \text{if } x \geq b \end{cases}, \tag{2.9}$$

where a, u, b are given numbers. The membership function of a triangular fuzzy number has a graphical representation of a triangle with $[a, b]$ being the base of the triangle and the point $(u, 1)$ as the single vertex. Therefore, the real numbers a, u and b define the triangular fuzzy number A which will be denoted by $(a; u; b)$.

The α-levels of triangular fuzzy numbers have the following simplified form

$$[a_1^\alpha,\ a_2^\alpha] = [(u-a)\alpha + a, (u-b)\alpha + b], \tag{2.10}$$

for all $\alpha \in [0, 1]$.

Notice that a triangular fuzzy number is not necessarily symmetric, since $b - u$ may be different from $u - a$, however, $\varphi_A(u) = 1$. We can say that a fuzzy number A is a reasonable mathematical model for the linguistic expression "*nearly u*". For the expression "*around u*" we expect symmetry. Imposing symmetry results in a simplification of the definition of a triangular fuzzy number. Indeed, let u be symmetric in relation to a and b, that is, $u - a = b - u = \delta$. In this case,

$$\varphi_A(x) = \begin{cases} 1 - \frac{|x-u|}{\delta} & \text{if } u - \delta \leq x \leq u + \delta \\ 0 & \text{otherwise} \end{cases}.$$

Example 2.5 The expression *around four o'clock* can be mathematically modeled by the symmetric triangular fuzzy number A, whose membership function is given by

$$\varphi_A(x) = \begin{cases} 1 - \frac{|x-4|}{0.2} & \text{if } 3.8 \leq x \leq 4.2 \\ 0 & \text{otherwise} \end{cases},$$

and is represented in Fig. 2.4. From (2.10) we obtain the α-levels of this fuzzy subset, which are the intervals $[a_1^\alpha, a_2^\alpha]$, where

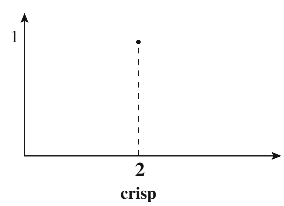


Fig. 2.3 Representation of the fuzzy number $\widehat{2}$

$$a_1^\alpha = 0.2\,\alpha + 3.8 \text{ and } a_2^\alpha = -0.2\,\alpha + 4.2.$$

Definition 2.5 A fuzzy number A is said to be *trapezoidal* if its membership function has the form of a trapezoid and is given by

$$\varphi_A(x) = \begin{cases} \frac{x-a}{b-a} & \text{if } a \le x < b \\ 1 & \text{if } b \le x \le c \\ \frac{d-x}{d-c} & \text{if } c < x \le d \\ 0 & \text{otherwise} \end{cases},$$

where a, b, c, d are given numbers.

The α-levels of a trapezoidal fuzzy set are the intervals

$$[a_1^\alpha, a_2^\alpha] = [(b-a)\alpha + a,\ (c-d)\alpha + d] \tag{2.11}$$

for all $\alpha \in [0, 1]$.

Example 2.6 The fuzzy set of the *teenagers* can be represented by the trapezoidal fuzzy number with the membership function

$$\varphi_A(x) = \begin{cases} \frac{x-11}{3} & \text{if } 11 \le x < 14 \\ 1 & \text{if } 14 \le x \le 17 \\ \frac{20-x}{3} & \text{if } 17 < x \le 20 \\ 0 & \text{otherwise} \end{cases},$$

and it is illustrated in Fig. 2.5. Equation (2.11) provides the α-levels for this example

$$[3\,\alpha + 11,\ \ -3\,\alpha + 20],\ \text{with } \alpha \in [0, 1].$$

Definition 2.6 A fuzzy number has the *bell shape* if the membership function is smooth and symmetric in relation to a given real number. The following membership function has those properties for fixed u, a and δ (see Fig. 2.6).

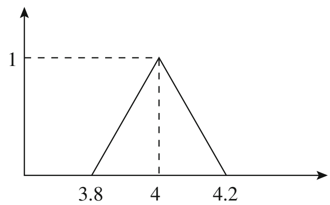


Fig. 2.4 Representation of the fuzzy number "around 4"

$$\varphi_A(x) = \begin{cases} \exp\left(-\left(\frac{x-u}{a}\right)^2\right) & \text{if } u-\delta \le x \le u+\delta \\ 0 & \text{otherwise} \end{cases}.$$

The α-levels of fuzzy numbers in bell shape are the intervals:

$$\left[a_1^\alpha, a_2^\alpha\right] = \begin{cases} \left[u - \sqrt{\ln\left(\frac{1}{\alpha^{a^2}}\right)}, u + \sqrt{\ln\left(\frac{1}{\alpha^{a^2}}\right)}\right] & \text{if } \alpha \ge \overline{\alpha} = e^{-\left(\frac{\delta}{a}\right)^2} \\ [u-\delta, u+\delta] & \text{if } \alpha < \overline{\alpha} = e^{-\left(\frac{\delta}{a}\right)^2} \end{cases}. \qquad (2.12)$$

We next present the arithmetic operations for fuzzy numbers, that is, the operations that allow us "to compute" with fuzzy sets.

2.2.1 Arithmetic Operations with Fuzzy Numbers

The arithmetic operations involving fuzzy numbers are closely linked to the interval arithmetic operations. Let us list some of those operations for closed intervals on the real line $\mathbb{R}$.

Interval Arithmetic Operations

Let λ be a real number and, A and B two closed intervals on the real line given by

$$A = [a_1, a_2] \text{ and } B = [b_1, b_2].$$

Definition 2.7 (*Interval Operations*) The arithmetic operations between intervals can be defined as:

(a) The *sum* between A and B is the interval

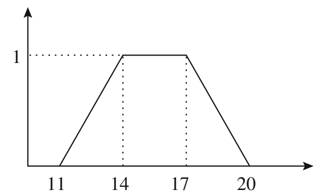


Fig. 2.5 Trapezoidal fuzzy number

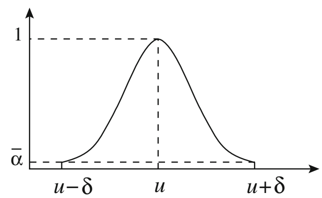


Fig. 2.6 Fuzzy number in the bell shape

$$A + B = [a_1 + b_1, a_2 + b_2].$$

(b) The *difference* between A and B is the interval

$$A - B = [a_1 - b_2, a_2 - b_1].$$

(c) The *multiplication* of A *by a scalar* λ is the interval

$$\lambda A = \begin{cases} [\lambda a_1, \lambda a_2] & \text{if } \lambda \geq 0 \\ [\lambda a_2, \lambda a_1] & \text{if } \lambda < 0 \end{cases}.$$

(d) The *multiplication* of A by B is the interval

$$A \cdot B = [\min P, \max P],$$

where $P = \{a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2\}$.

(e) The *quotient* of A by B, if $0 \notin B$, is the interval

$$A/B = [a_1, a_2] \cdot \left[\frac{1}{b_2}, \frac{1}{b_1}\right].$$

Exercise 2.3 Compute the results of the operations defined above for the intervals

$$A = [-1, 2] \text{ and } B = [5, 6].$$

Notice that the arithmetic operations for intervals extend the respective operations for real numbers. To see this, it is sufficient to observe that each real number can be considered as a closed interval with equal endpoints. Also the membership functions obtained by interval arithmetic operations can be derived directly from the respective operations for real numbers. Such a procedure uses the *extension principle*, which will be a tool to obtain the arithmetic operations of fuzzy numbers.

Let us consider an arbitrary binary operation "$\otimes$" between real numbers. Let χ_A and χ_B be the characteristic functions of the intervals A and B, respectively. The theorem that follows gives us the interval arithmetic operations for the respective operations for real numbers via the *extension principle*.

Theorem 2.2 (Extension Principle for Real Intervals) *Let A and B be two closed intervals of $\mathbb{R}$ and $\otimes$ one of the arithmetic operations between real numbers. Then*

$$\chi_{A \otimes B}(z) = \sup_{\{(x,y): x \otimes y = z\}} \min[\chi_A(x), \chi_B(y)]$$

It is simple to verify that

$$\min(\chi_A(x), \chi_B(y)) = \begin{cases} 1 & \text{if } x \in A \text{ and } y \in B \\ 0 & \text{if } x \notin A \text{ or } y \notin B \end{cases}.$$

Thus, for the sum case ($\otimes = +$), we have

$$\sup_{\{(x,y):x+y=z\}} \min[\chi_A(x), \chi_B(y)] = \begin{cases} 1 & \text{if } x \in A+B \\ 0 & \text{if } x \notin A+B \end{cases}.$$

The other cases can be obtained analogously.

An important consequence of the Theorem 2.2 for operations with fuzzy numbers is the corollary that follows.

Corollary 2.3 *The α-levels of the crisp set $A + B$ with the characteristic function $\chi_{(A+B)}$ are given by*

$$[A+B]^\alpha = A+B$$

for all $\alpha \in [0, 1]$.

Remember that the intervals A and B are fuzzy sets of the real line, so that the result of this corollary is an immediate consequence of the characteristic function definition of a classical set. The arithmetic operations for fuzzy numbers may be defined from the extension principle for fuzzy sets in analogous way. Actually, they are particular cases of the extension principle where the functions that must be extended are traditional operations for real numbers.

Arithmetic Operations with Fuzzy Numbers

The definitions that follow can be interpreted as particular cases of the extension principle, both for a function of one and two variables.

Definition 2.8 Let A and B be two fuzzy numbers and λ a real number.

(a) The *sum* of the fuzzy numbers A and B is the fuzzy number $A + B$, whose membership function is

$$\varphi_{(A+B)}(z) = \sup_{\phi(z)} \min[\varphi_A(x), \varphi_B(y)]$$

where $\phi(z) = \{(x, y) : x + y = z\}$.

(b) The *multiplication* of A by a scalar λ is the fuzzy number λA, whose membership function is

$$\varphi_{\lambda A}(z) = \begin{cases} \sup\limits_{\{x:\lambda x=z\}} [\varphi_A(x)] & \text{if } \lambda \neq 0 \\ \chi_{\{0\}}(z) & \text{if } \lambda = 0 \end{cases} = \begin{cases} \varphi_A(\lambda^{-1}z) & \text{if } \lambda \neq 0 \\ \chi_{\{0\}}(z) & \text{if } \lambda = 0 \end{cases},$$

where $\chi_{\{0\}}$ is the characteristic function of $\{0\}$.

(c) The *difference* $A - B$ is the fuzzy number whose membership function is given by:

$$\varphi_{(A-B)}(z) = \sup_{\phi(z)} \min[\varphi_A(x), \varphi_B(y)]$$

where $\phi(z) = \{(x, y) : x - y = z\}$.

(d) The *multiplication* of A by B is the fuzzy number $A.B$, whose membership function is given by:

$$\varphi_{(A.B)}(z) = \sup_{\phi(z)} \min[\varphi_A(x), \varphi_B(y)]$$

where $\phi(z) = \{(x, y) : xy = z\}$.

(e) The *quotient* is the fuzzy number A/B whose membership function is

$$\varphi_{(A/B)}(z) = \sup_{\phi(z)} \min[\varphi_A(x), \varphi_B(y)]$$

where $\phi(z) = \{(x, y) : x/y = z\}$ and $0 \notin suppB$.

Theorem 2.4 below ensures that the result of the arithmetic operations between fuzzy numbers is a fuzzy number. Moreover, it generalizes Corollary 2.3, relating, via α-levels, arithmetic operations for fuzzy numbers with the respective interval arithmetic operations.

Theorem 2.4 *The α-levels of the fuzzy set $A \otimes B$ are given by*

$$[A \otimes B]^\alpha = [A]^\alpha \otimes [B]^\alpha$$

for all $\alpha \in [0, 1]$, where $\otimes$ is any arithmetic operations $\{+, -, \times, \div\}$.

Although the proof of this theorem will be done here, we can say it is a consequence of the Theorem 2.1 applied to the function $\otimes$, since it is continuous as long as we do not divide by zero. The interested reader can find the proof in the classical books of Klir and Yuan [5], Nguyen [6], Pedrycz and Gomide [7] or more generally in Fuller [8].

The combination of the Theorems 2.1 and 2.4 produces "practical methods" to obtain the results of each operation between fuzzy numbers. We observe again that the α-levels of a fuzzy number is always a closed interval of $\mathbb{R}$ given by:

$$[A]^\alpha = \left[a_1^\alpha, a_2^\alpha\right], \text{ with } a_1^\alpha = \min\{\varphi_A^{-1}(\alpha)\} \text{ and } a_2^\alpha = \max\{\varphi_A^{-1}(\alpha)\},$$

where $\varphi_A^{-1}(\alpha) = \{x \in \mathbb{R} : \varphi_A(x) = \alpha\}$ is the preimage of α.

Hereafter we illustrate such "practical methods" through properties given below.

Proposition 2.5 *Let A and B be fuzzy numbers with α-levels respectively given by $[A]^\alpha = \left[a_1^\alpha, a_2^\alpha\right]$ and $[B]^\alpha = \left[b_1^\alpha, b_2^\alpha\right]$. Then the following properties hold:*

(a) The sum of A and B is the fuzzy number $A + B$ whose α-levels are

$$[A + B]^\alpha = [A]^\alpha + [B]^\alpha = \left[a_1^\alpha + b_1^\alpha, a_2^\alpha + b_2^\alpha\right].$$

(b) The difference of A and B is the fuzzy number $A - B$ whose α-levels are

$$[A - B]^\alpha = [A]^\alpha - [B]^\alpha = \left[a_1^\alpha - b_2^\alpha, a_2^\alpha - b_1^\alpha\right].$$

(c) The multiplication of A by a scalar λ is the fuzzy number λA whose α-levels are

$$[\lambda A]^\alpha = \lambda[A]^\alpha = \begin{cases} [\lambda a_1^\alpha, \lambda a_2^\alpha] & \text{if } \lambda \geq 0 \\ [\lambda a_2^\alpha, \lambda a_1^\alpha] & \text{if } \lambda < 0 \end{cases}.$$

(d) The multiplication of A by B is the fuzzy number $A \cdot B$ whose α-levels are

$$[A \cdot B]^\alpha = [A]^\alpha [B]^\alpha = \left[\min P^\alpha, \max P^\alpha\right],$$

where $P^\alpha = \{a_1^\alpha b_1^\alpha, a_1^\alpha b_2^\alpha, a_2^\alpha b_1^\alpha, a_2^\alpha b_2^\alpha\}$.

(e) The division of A by B, if $0 \notin \operatorname{supp} B$*, is the fuzzy number whose α-levels are*

$$\left[\frac{A}{B}\right]^\alpha = \frac{[A]^\alpha}{[B]^\alpha} = \left[a_1^\alpha, a_2^\alpha\right]\left[\frac{1}{b_2^\alpha}, \frac{1}{b_1^\alpha}\right].$$

Example 2.7 Consider the expressions *nearly* 2 and *nearly* 4 and let A and B be the triangular fuzzy numbers that indicate these expressions. Thus, we define

$$A = (1; 2; 3) \text{ and } B = (3; 4; 5).$$

The results of $A \otimes B$ for each of the arithmetic operations between fuzzy numbers are shown next. First, let us notice that according to formula (2.10)

$$[A]^\alpha = [1 + \alpha, 3 - \alpha] \text{ and } [B]^\alpha = [3 + \alpha, 5 - \alpha].$$

Then by Proposition 2.5 we get

(a) $[A + B]^\alpha = [A]^\alpha + [B]^\alpha = [4 + 2\alpha, 8 - 2\alpha]$. Thus, $A + B = (4; 6; 8)$;
(b) $[A - B]^\alpha = [A]^\alpha - [B]^\alpha = [-4 + 2\alpha, -2\alpha]$. Thus, $A - B = (-4; -2; 0)$;
(c) $[4 \cdot A]^\alpha = 4\,[A]^\alpha = [4 + 4\alpha, 12 - 4\alpha]$. Thus, $4A = (4; 8; 12)$;
(d) $[A \cdot B]^\alpha = [A]^\alpha\,[B]^\alpha = [(1 + \alpha)(3 + \alpha), (3 - \alpha)(5 - \alpha)]$;
(e) $\left[\frac{A}{B}\right]^\alpha = \frac{[A]^\alpha}{[B]^\alpha} = [(1 + \alpha)/(5 - \alpha), (3 - \alpha)/(3 + \alpha)]$.

Notice that the fuzzy numbers obtained in (d) and (e) are not triangular. However, it is easy to verify that with triangular fuzzy numbers, the sum, the difference and the multiplication by a scalar results in a triangular fuzzy number. To see this, it suffices to consider the numbers $A = (a_1; u; a_2)$ and $B = (b_1; v; b_3)$. Then, from Eq. (2.10), we have

$$[A]^\alpha = [(u - a_1)\alpha + a_1, (u - a_2)\alpha + a_2]$$
$$[B]^\alpha = [(v - b_1)\alpha + b_1, (v - b_2)\alpha + b_2].$$

Thus

$$[A+B]^\alpha = [A]^\alpha + [B]^\alpha$$

and then

$$[A+B]^\alpha = [\{(u+v)-(a_1+b_1)\}\alpha + (a_1+b_1), \{(u+v)-(a_2+b_2)\}\alpha + (a_2+b_2)].$$

Using Eq. (2.10) again, we see that these intervals are the α-levels of the following triangular fuzzy number:

$$((a_1+b_1); (u+v); (a_2+b_2)).$$

Finally, it is possible to conclude that $(A-B)+B \neq A$ so that it follows that $A-A \neq 0$. That is, the space of fuzzy numbers is not a vector space since there are no additive (nor multiplication) inverses. This is a property that hampers many areas of the fuzzy mathematics, for example, the area of fuzzy differential equations (see Chap. 8) and is a challenge to fuzzy linear systems to mention but two areas.

Exercise 2.4 Redo the Example 2.7 from the point of view of the extension principle.

The next example presents an explicit way to obtain the function $\widehat{f}$ in the case that f is linear.

Example 2.8 Let $f:\mathbb{R}\to\mathbb{R}$ be the function $f(x)=\lambda x$, with $\lambda \neq 0$ a scalar. If A is a fuzzy number with membership function φ_A, then according to the Definition 2.1, the membership function of $\widehat{f}(A)$ is given by

$$\begin{aligned}\varphi_{\widehat{f}(A)}(z) &= \sup_{\{x:f(x)=z\}} \varphi_A(x) = \sup_{\{x:\lambda x=z\}} \varphi_A(x)\\ &= \sup_{\{z/\lambda\}} \varphi_A(x) = \varphi_A(z/\lambda) = \varphi_A(\lambda^{-1}z),\end{aligned}$$

which, according to Definition 2.9b, is the membership function of λA. Thus, if $f(x)=\lambda x$, then $\widehat{f}:\mathcal{F}(\mathbb{R})\to\mathcal{F}(\mathbb{R})$ is given by $\widehat{f}(A)=\lambda A$.

Exercise 2.5 Verify that if $f(x)=\lambda x+b$, with $\lambda\neq 0$, then for any fuzzy number A the fuzzy set $\widehat{f}(A)$ has the following membership function

$$\varphi_{\widehat{f}(A)}(z) = \varphi_A(\lambda^{-1}(z-b)).$$

After, verify that $\hat{f}(A)$ is a triangular fuzzy number if A is triangular.

Exercise 2.6 From Theorem 2.1 and the proprieties of the arithmetic operations, show that Zadeh's Extension of an affine function $f(x) = ax+b$, is the affine function $\widehat{f}(X) = aX + \widehat{b}$ if $X \in \mathcal{F}(\mathbb{R})$.

Let us use the extension principle to compute the image of a triangular fuzzy number by a known function. Similar to the example of the operations of division and multiplication, the image of a function of triangular numbers may not be triangular even if the function is continuous.

Example 2.9 Consider the function $f(x) = e^x$ and the triangular fuzzy number $A = (0; \ln 2; \ln 3)$.

According to Theorem 1.2, $\widehat{f}(A)$ is determined by its α-levels. From (2.10) it is easy to see that the α-levels of A are the intervals

$$[A]^\alpha = [(\ln 2)\alpha, (\ln 2 - \ln 3)\alpha + \ln 3] = \left[\ln 2^\alpha, \ln\left(3\left(\frac{2}{3}\right)^\alpha\right)\right],$$

with $\alpha \in [0, 1]$.

Now, we obtain the α-levels of $\widehat{f}(A)$ using Theorem 2.1, that is,

$$[\widehat{f}(A)]^\alpha = f([A]^\alpha) = f\left(\left[\ln 2^\alpha, \ln 3\left(\frac{2}{3}\right)^\alpha\right]\right) = \left[2^\alpha, 3\left(\frac{2}{3}\right)^\alpha\right],$$

with $\alpha \in [0, 1]$. Figure 2.7 illustrates $\hat{f}(A)$ of this example. Therefore,

- if $\alpha = 0$ then $[\widehat{f}(A)]^0 = [1, 3]$;
- if $\alpha = \frac{1}{2}$ then $[\widehat{f}(A)]^{\frac{1}{2}} = [\sqrt{2}, \sqrt{6}]$;
- if $\alpha = 1$ then $[\widehat{f}(A)]^1 = \{2\}$.

Now, it is easy to verify that the points $(1, 0)$; $(\sqrt{2}, \frac{1}{2})$ and $(2, 1)$ are not aligned, that is, not a straight line. So $\widehat{f}(A)$ is not a triangular number.

Example 2.10 Consider that a bus trip from the city of Campinas to São Paulo is subject to the following:

- The distance between the two cities is nearly 100 km;
- The speed can not exceed 120 km/h;
- The traffic is usually intense and the speed also decreases at the toll booths;
- The bus usually leaves Campinas late, but the lateness never exceeds more than 30 min.

Question: What is the total time (T) spent on a trip from Campinas to São Paulo by bus?

The solution to this problem from a classical mathematics point of view with an exact real number answer is impossible, since we just have partial information and ill-defined reports. An intuitive approach to solve this problem as may be answered by a person questioned about the solution of this problem may be something like: "*the total time is just a little bit more than one hour*" or "*between one and one hour and a half*". These answers may be based on personal experience of those who have faced the same or similar situations. The reasoning may be as follows: This bus goes fast on the road but the intense traffic and the toll booths force the bus to slow down, besides,

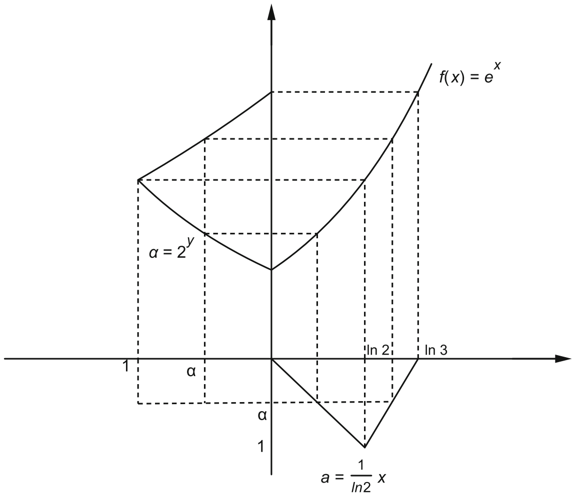


Fig. 2.7 Zadeh's Extension of the triangular fuzzy number $A = (0; \ln 2; \ln 3)$ for $f(x) = e^x$

the bus usually leaves the station late. Therefore, if we want a precise value (real number) for the answer, we need to adopt precise values for the data. For example, a mean speed of 90 km/h and a delay of 15 min (T_1) would result in the answer:

$$T = T_1 + T_2 = 15\,\text{min} + (1\,\text{h and } 6.66\,\text{min}) = 1.36\,\text{h}.$$

But the idea here is to propose a mathematical model for this "intuitive arithmetic" that allows people to compute with imprecise data (as the ones in our problem) to obtain a result with information based on fuzzy numbers, even though they might be linguistic answers and at the same time informative and numerically based. Therefore, we want a model that allows this type of reasoning used by people.

Let approach this example from a fuzzy set theoretic point of view.

- Since the distance (D) of the route is approximate, we can consider it a fuzzy number around 100 km. It can be, for example, a triangular number $D = (90; 100; 110)$ whose membership function is

$$\varphi_D(x) = \begin{cases} 0 & \text{if} \quad x \leq 90 \\ \frac{x}{10} - 9 & \text{if} \;\; 90 < x \leq 100 \\ 11 - \frac{x}{10} & \text{if} \;\; 100 < x \leq 110 \\ 0 & \text{if} \quad x > 110 \end{cases}$$

  and α-levels given by $[D]^\alpha = [10\alpha + 90, -10\alpha + 110]$. Notice that formula (2.6) can be used to get these α-levels.
- The uncertain about the speed of the bus (V) can also be modeled by a triangular fuzzy number. Taking into account that the speed never exceeds 120 km/h and that we have some low speeds in route, we can suppose that $V = (30; 100; 120)$, whose α-levels are $[V]^\alpha = [70\alpha + 30, -20\alpha + 120]$.

- The fact that the bus usually leaves the station late indicates that we should have an extra time (T_1) that does not exceed *half an hour*. This time can be modeled by the triangular fuzzy number $T_1 = (0; 0; 0.5)$, whose α-levels are

$$[T_1]^\alpha = [0, -0.5\alpha + 0.5] = \left[0, \frac{1-\alpha}{2}\right].$$

From physics, the time that is spent on the road (T_2) is obtained by the fuzzy number $T_2 = \frac{D}{V}$. From Proposition 2.5 we have that the α-levels of T_2 are:

$$[T_2]^\alpha = [10\alpha + 90, 110 - 10\alpha]\left[\frac{1}{120 - 20\alpha}, \frac{1}{70\alpha + 30}\right].$$

Therefore, the total time (T) is given by the fuzzy number $T = T_1 + T_2$ whose α-levels are:

$$\begin{aligned}[T]^\alpha &= \left[0, \frac{1-\alpha}{2}\right] + \left[\frac{10\alpha + 90}{120 - 20\alpha}, \frac{110 - 10\alpha}{70\alpha + 30}\right] \\ &= \left[\frac{10\alpha + 90}{120 - 20\alpha}, \frac{1-\alpha}{2} + \frac{110 - 10\alpha}{70\alpha + 30}\right] \\ &= [f(\alpha), g(\alpha)].\end{aligned}$$

This would be the fuzzy solution of the problem which includes the times between $\frac{3}{4}$ h and $\frac{25}{6}$ h. We can also observe that the time with the highest possibility ($\alpha = 1$) is $t = 1$ h. The time $t = 1.36$ h given at the beginning of the discussion (with precise data) would have membership degree of $\alpha^* \simeq 0.8$ in the total time set T, since

$$1.36 = \frac{1-\alpha^*}{2} + \frac{110 - 10\alpha^*}{70\alpha^* + 30}.$$

In the fuzzy context, a real number - obtained a-posteriori by the defuzzification of the fuzzy set T (see Sect. 5.4) or considering the fuzzy expectation of T (Definition 7.10) - could also be used to represent the solution to this problem.

Exercise 2.7 Redo the example, but now, using the arithmetic operations from the extension principle. For that, the reader must consider the membership functions of each fuzzy set. Also, verify that, even though all the sets are triangular, the answer T is not a triangular fuzzy number. Sketch the graphs of $f(\alpha)$ and $g(\alpha)$ to observe that.

Definition 2.9 (*Hukuhara Difference: $A -_H B$*) Let A and B be two fuzzy numbers. If there exists a fuzzy number C such that $A = B + C$, then C is called the Hukuhara Difference of A and B and we denote it by $A -_H B$.

In terms of α-levels this is equivalent to saying that

$$[A -_H B]^\alpha = [a_1^\alpha - b_1^\alpha, a_2^\alpha - b_2^\alpha] \ \forall\, \alpha \in [0, 1].$$

Since

$$[A - B]^\alpha = [a_1^\alpha - b_2^\alpha, a_2^\alpha - b_1^\alpha],$$

it follows that

$$A - B = A -_H B \ \Leftrightarrow \ b_1^\alpha = b_2^\alpha,$$

that is,

$$A - B = A -_H B \ \Leftrightarrow \ B \in \mathbb{R}.$$

Notice that, in general,

$$A - B = A + (-1)B \neq A -_H B.$$

This difference, historically, was first used to study derivatives of fuzzy functions (see Chap. 8).

Let us look at a relationship between the extension principle and the probability theory before we close this chapter. Consider the following Tables 2.1 and 2.2.

The question is: "How can we obtain the uncertain distribution for $Z = X + Y$?". It is clear that the "possible" values for $Z = X + Y$ are the elements of the set $\{5, 6, 7\}$. Table 2.3 shows the values of $\varphi_{X+Y}(z_i)$ and $P_{X+Y}(z_i)$, where P denote the probability and φ the membership function:

According to the formulas

$$\varphi_{X+Y}(z_i) = \sup_{x_j + y_k = z_i} \min(\varphi_A(x_j), \varphi_B(y_k)) \tag{2.13}$$

and

$$P_{X+Y}(z_i) = \sum_{x_j + y_k = z_i} P_{(X,Y)}(X = x_j, Y = y_k), \tag{2.14}$$

where $P_{(X,Y)}(X = x_j, Y = y_k)$ is the joint probability distribution of the random vector (X, Y) (see [9, 10]).

The main observation that we have here is that to obtain the probability of $X + Y$, we need to add the independence hypothesis. However, to compute the membership distribution of $X + Y$ according to the extension principle, the analogous hypothesis is not needed. We stress that if X and Y are independent and we note that the formulas

Table 2.1 Membership and probability distributions of X

$X = x_j$	$\varphi_X(x_j)$	$P_X(x_j)$
2	0.5	0.5
3	0.5	0.5

Table 2.2 Membership and probability distributions of Y

$Y = y_k$	$\varphi_Y(y_k)$	$P_Y(y_k)$
3	0.5	0.5
4	0.5	0.5

Table 2.3 Membership and probability distributions of $X + Y$

$Z = X + Y$	$\varphi_{X+Y}(z_i)$	$P_{X+Y}(z_i)$
5	0.5	0.25
6	0.5	0.50
7	0.5	0.25

(2.13) and (2.14) have some kind of similarity, exchanging **sup** by Σ and **min** by **product**.

Finally, we observe that the last two columns of the Table 2.3 represent, respectively, the "membership" and the "occurrence" of each element of the first column to the sum set. Intuitively, a higher probability for 6 is expected, since its "occurrence" is more than the others. On the other hand, from the fuzzy set theory point of view, which is an extension of the classical set theory, the value 6 belongs to the sum set with the same membership of the others. The number of times that it "occurs" does not matter.

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