Chapter 2

Kinetics of a Point Mass

2

NEWTON's 2nd Law (law of motion): The motion of a point mass under the action of forces is described by

$$\boxed{\frac{\mathrm{d}(m\,\boldsymbol{v})}{\mathrm{d}t} = \dot{\boldsymbol{p}} = \boldsymbol{F}}$$

with $\boldsymbol{F} = \sum \boldsymbol{F}_i$ and the *momentum*

$$\boxed{\boldsymbol{p} = m\,\boldsymbol{v}\ .}$$

Since the mass is constant, Newton's law can also be expressed as

$$\boxed{m\,\boldsymbol{a} = \boldsymbol{F}} \qquad \textit{mass} \times \textit{acceleration} = \textit{force}\,.$$

As an example, this leads for cartesian coordinates to

$$ma_x = \sum F_x\ , \qquad ma_y = \sum F_y\ , \qquad ma_z = \sum F_z\ .$$

Remarks:

- Newton's law is valid in this form only in an inertial reference frame (= reference system that is absolutely at rest or in uniform, rectilinear motion, see also chapter 8),
- Bodies with finite dimensions can be regarded as point masses if their dimensions have no influence on the motion.

Impulse Law: Time integration of the law of motion leads to

$$\boxed{m\boldsymbol{v} - m\boldsymbol{v}_0 = \int_{t_0}^{t} \boldsymbol{F}\,\mathrm{d}\bar{t}} \qquad \text{bzw.} \qquad \boxed{\boldsymbol{p} - \boldsymbol{p}_0 = \widehat{\boldsymbol{F}}}$$

where $\widehat{\boldsymbol{F}} = \int_{t_0}^{t} \boldsymbol{F}\mathrm{d}\bar{t}$ is the *linear impulse*. When no forces are acting ($\boldsymbol{F} = 0$), the linear momentum is conserved:

$$\boldsymbol{p} = m\,\boldsymbol{v} = \text{const}\ .$$

Angular Momentum Theorem: The vector product of Newton's law with the position vector $\boldsymbol{r}$ yields

$$\boxed{\frac{\mathrm{d}\boldsymbol{L}^{(0)}}{\mathrm{d}t} = \boldsymbol{M}^{(0)}\,,}$$

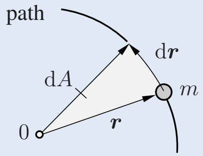
where

$\boldsymbol{L}^{(0)} = \boldsymbol{r} \times \boldsymbol{p}$ = angular momentum with respect to the fixed point 0,

$\boldsymbol{M}^{(0)} = \boldsymbol{r} \times \boldsymbol{F}$ = moment with respect to the fixed point 0.

If the moment vanishes ($\boldsymbol{M}^{(0)} = 0$), the angular momentum is conserved:

$$\boldsymbol{L}^{(0)} = \boldsymbol{r} \times m\,\boldsymbol{v} = \text{const} \ .$$



In this case, with

$$\mathrm{d}\boldsymbol{A} = \frac{1}{2}\,\boldsymbol{r} \times \mathrm{d}\boldsymbol{r} \quad \leadsto \quad \frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = \frac{1}{2}\,\boldsymbol{r} \times \boldsymbol{v}$$

the *law of areas (Kepler's 2nd law)* is obtained (see page 4):

$$\dot{\boldsymbol{A}} = \text{const} \ .$$

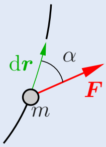
Work–Energy Theorem: Path integration of the law of motion yields

$$\boxed{\frac{mv_1^2}{2} - \frac{mv_0^2}{2} = \int_{\boldsymbol{r}_0}^{\boldsymbol{r}_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}} \qquad \text{or} \qquad \boxed{T_1 - T_0 = U \ ,}$$

Kinetic Energy : $\quad T = \frac{1}{2}\,m\,v^2,$

Work of Force $\boldsymbol{F}$: $\quad U = \int \mathrm{d}U = \int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}\,,$

$$\mathrm{d}U = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = |\boldsymbol{F}|\,|\mathrm{d}\boldsymbol{r}| \cos\alpha.$$



Remarks:
- Forces orthogonal to the path ($\alpha = \pi/2$), do not execute work.
- For a rotation holds $\mathrm{d}U = \boldsymbol{M} \cdot \mathrm{d}\boldsymbol{\varphi}$.

Conservation-of-Energy Law: If the forces according to

$$\boldsymbol{F} = -\,\text{grad}\ V = -\Big(\frac{\partial V}{\partial x}\,\boldsymbol{e}_x + \frac{\partial V}{\partial y}\,\boldsymbol{e}_y + \frac{\partial V}{\partial z}\,\boldsymbol{e}_z\Big)$$

can be derived from a potential V ($\widehat{=}$ *conservative forces*), the work is path independent, i.e. given by the potential difference:

$$U = \int_{\boldsymbol{r}_0}^{\boldsymbol{r}_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = V_0 - V_1 \ .$$

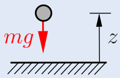
From the Work-Energy Theorem then follows

$$\boxed{T_1 + V_1 = T_0 + V_0 = \text{const} \ .}$$

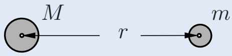
In words: *When the applied forces possess a potential, then the sum of potential energy V and kinetic energy T remains constant during the motion.*

Several Potentials

Gravitational Potential (near earth's surface) $\quad V = mg\,z$

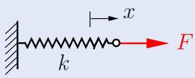


Gravitational Potential (general) $\quad V = -G\,\dfrac{Mm}{r}$



Gravitational constant $\quad G = 6,673 \cdot 10^{-11}\ \mathrm{m}^3/\mathrm{kg\,s}^2$

Potential of a spring $\quad V = \dfrac{1}{2}\,kx^2$



Power

$$P = \frac{\mathrm{d}U}{\mathrm{d}t} = \boldsymbol{F} \cdot \boldsymbol{v} \qquad = \text{Power of a force,}$$

$$P = \boldsymbol{M} \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \boldsymbol{M} \cdot \boldsymbol{\omega} = \text{Power of a moment.}$$

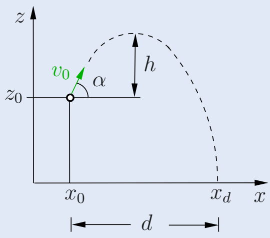
Projectile Motion

Parabolic trajectory of motion:

$$z = z_0 - \frac{g}{2}\left(\frac{x - x_0}{v_0 \cos\alpha}\right)^2 + (x - x_0)\tan\alpha\ ,$$

Maximum height:

$$h = \frac{1}{2g}(v_0 \sin\alpha)^2\ ,$$



Flight time:

$$t_d = \frac{v_0 \sin\alpha}{g}\left[1 + \sqrt{1 + \frac{2gz_0}{v_0^2 \sin^2\alpha}}\,\right]\ ,$$

Flight distance:

$$d = v_0^2\,\frac{\sin\alpha\cos\alpha}{g}\left[1 + \sqrt{1 + \frac{2gz_0}{v_0^2 \sin^2\alpha}}\,\right]\ .$$

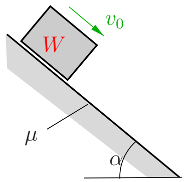
Special case $z_0 = 0$:

$$t_d = \frac{2}{g}\ v_0 \sin\alpha\ , \qquad d = \frac{1}{g}\ v_0^2 \sin 2\alpha\ .$$

Problem 2.1 A box of weight W is pushed downwards a rough inclined plane (kinetic friction coefficient μ) with an initial velocity v_0.

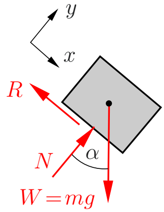
a) Determine the velocity in dependence on the distance.

b) At what distance x_E the box comes to rest? Under what circumstances is this possible?



P2.1

Solution **a)** The law of motion yields in x- and in y-direction



$$\searrow: \quad m\ddot{x} = G\sin\alpha - R\,,$$

$$\nearrow: \quad 0 = N - G\cos\alpha\,.$$

In conjunction with the friction law $R = \mu N$, the acceleration follows as

$$\ddot{x} = g(\sin\alpha - \mu\cos\alpha) = a_0\,.$$

Twice integration, taking into account the initial conditions $x(0) = 0$, $v(0) = v_0$, leads to:

$$v(t) = \dot{x} = v_0 + a_0 t\,, \qquad x(t) = v_0 t + \frac{1}{2}\,a_0 t^2\,.$$

Therewith, by eleminating the time, we obtain

$$t = \frac{v - v_0}{a_0} \quad \leadsto \quad x = v_0\frac{v - v_0}{a_0} + \frac{a_0}{2}\,\frac{v^2 - 2vv_0 + v_0^2}{a_0^2} = \frac{v^2 - v_0^2}{2a_0}$$

$$\leadsto \quad \underline{\underline{v(x) = \sqrt{v_0^2 + 2a_0 x}}}\,.$$

b) From the condition $v(x_E) = 0$ (rest), the covered distance x_E is determined:

$$0 = v_0^2 + 2a_0 x_E \qquad \leadsto \qquad \underline{\underline{x_E = -\frac{v_0^2}{2a_0}}}\,.$$

From the condition $x_E > 0$ follows $a_0 < 0$, i.e. $\mu > \tan\alpha$.

The same results can be found easier by applying the work-energy theorem $T_1 - T_0 = U$. It leads with

$$U = (mg\sin\alpha)x - Rx, \quad T_0 = \frac{1}{2}mv_0^2, \quad T_1 = \frac{1}{2}mv^2, \quad R = \mu mg\cos\alpha$$

and by solving for v directly to

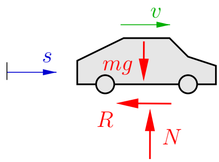
$$\underline{\underline{v(x) = \sqrt{v_0^2 + 2g(\sin\alpha - \mu\cos\alpha)x}}}\,.$$

P2.2 **Problem 2.2** Two cars, one with and one without ABS, are stopping from the speed $v_0 = 100$ km/h by full breaking. The first without ABS by blocking the wheels, i.e. sliding ('kinetic friction'), the second with ABS with still rolling wheels (ideal 'limiting static friction' assumed).

Determine for both cars the time t_B and the distance s_B for stopping if the coefficients of static and kinetic friction between pavement and tire are $\mu_0 = 0.7$ and $\mu = 0.45$, respectively.

Solution From the equation of motion of the first car (sliding)

$$\rightarrow: \quad m\dot{v} = m\ddot{s} = -R\,, \qquad \uparrow: \quad 0 = N - mg$$



and the friction law

$$R = \mu N\,,$$

it follows

$$\dot{v} = -\mu g\,.$$

Integration yields with $v(t{=}0) = v_0$ and $s(t{=}0) = 0$

$$v(t) = v_0 - \mu g t\,, \qquad s(t) = v_0 t - \frac{1}{2}\,\mu g t^2\,.$$

The stopping time and distance are calculated from the condition $v = 0$:

$$t_B = \frac{v_0}{\mu g}\,, \qquad s_B = s(t_B) = \frac{v_0^2}{\mu g} - \frac{v_0^2}{2\mu g} = \frac{v_0^2}{2\mu g}\,.$$

With the given coefficient of kinetic friction, we obtain

$$\underline{\underline{t_B}} = \frac{100}{3.6 \cdot 0.45 \cdot 9.81} = \underline{\underline{6.3\,\text{s}}}\,, \qquad \underline{\underline{s_B}} = \frac{100^2}{3.6^2 \cdot 2 \cdot 0.45 \cdot 9.81} = \underline{\underline{87\,\text{m}}}\,.$$

For the second car the wheels are still rolling under the limit condition of static friction (ABS), i.e. the friction force is now given by

$$H = H_0 = \mu_0 N\,.$$

This means that in the calculation above only R must replaced by H_0 and μ by μ_0, respectively. Thus, for the car with ABS, we obtain

$$\underline{\underline{t_B}} = 6.3 \cdot \frac{0.45}{0.7} = \underline{\underline{4.05\,\text{s}}}\,, \qquad \underline{\underline{s_B}} = 87 \cdot \frac{0.45}{0.7} = \underline{\underline{56\,\text{m}}}\,.$$

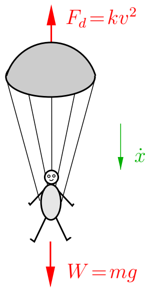
Remark: Note that stopping time and distance are inverse proportional to the friction coefficient. Note also that, because of the neglected reaction time, the numbers for t_B and s_B in reality might be higher!

Problem 2.3 A parachutist (weight W including parachute) has the initial velocity v_0 imediately after the parachute opens. **P2.3**

a) Determine the velocity v in dependence on t if the air drag is assumed to obey the law $F_d = kv^2$.

b) What limit speed v_l reaches the parachutist?

Solution The law of motion yields



$$\downarrow: \quad ma = m\ddot{x} = mg - kv^2$$

or

$$\ddot{x} = \frac{\mathrm{d}v}{\mathrm{d}t} = g - k_1 v^2 \qquad \text{with} \qquad k_1 = \frac{k}{m}\ .$$

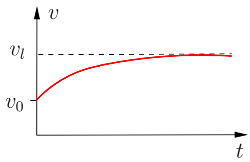
a) Separation of variables and integration leads to

$$\int_{v_0}^{v} \frac{\mathrm{d}\bar{v}}{g - k_1 \bar{v}^2} = \int_0^t \mathrm{d}\bar{t}\ ,$$

where the time t is counted from the opening of the parachute. With the basic integral

$$\int \frac{\mathrm{d}z}{A - Bz^2} = \frac{1}{\sqrt{AB}}\,\mathrm{artanh}(\sqrt{B/A}\; z)$$

we obtain



$$\left[\frac{1}{\sqrt{gk_1}}\,\mathrm{artanh}\sqrt{k_1/g}\;\bar{v}\right]_{v_0}^{v} = t$$

or by solving for $v(t)$

$$\underline{\underline{v(t) = \sqrt{\frac{g}{k_1}}\tanh\left(\sqrt{gk_1}\;t + \mathrm{artanh}\sqrt{\frac{k_1}{g}}\;v_0\right)}}\ .$$

b) For $t \to \infty$, it follows ($\tanh z \to 1$ for $z \to \infty$)

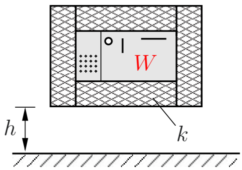
$$\underline{\underline{v_l}} = \sqrt{\frac{g}{k_1}} = \underline{\underline{\sqrt{\frac{W}{k}}}}\ .$$

The same result can be found from the consideration that in the limit case, the acceleration is zero:

$$a = g - k_1 v_l^2 = 0 \quad \rightsquigarrow \quad \underline{\underline{v_l}} = \sqrt{\frac{g}{k_1}} = \underline{\underline{\sqrt{\frac{W}{k}}}}\ .$$

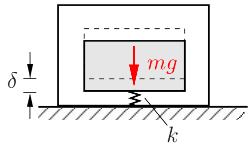
P2.4

Problem 2.4 A computer (weight W=100 N) in a packing case is protected against impact by foam plastics (spring stiffness $k = 100$ N/cm).



From what height h the case may impinge a hard surface, if the acceleration of the computer shall not be bigger than four times the gravity?

Solution During free fall, the case experiences the acceleration g. After impinging the surface, the foam plastics ($\widehat{=}$ linear spring) will be compressed and the computer will be accelerated upwards. Then the motion is described by



$$\uparrow: \quad m\,a = -mg + k\,\delta\ .$$

From the condition $a_{\max} = 4g$ follows the maximum spring compression

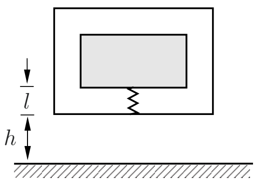
$$\delta_{\max} = \frac{5mg}{k} = 5\ \text{cm}\ .$$

Knowing this limit compression, the allowable height of fall can be determined from the conservation of energy law

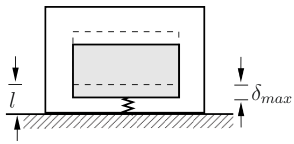
$$T_1 + V_1 = T_2 + V_2$$

as follows:

state 1



state 2



$$T_1 = 0\,, \quad V_1 = mg(l+h)\,, \quad T_2 = 0\,, \quad V_2 = mg(l-\delta_{\max}) + \frac{1}{2}k\delta_{\max}^2\ .$$

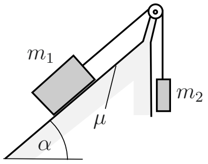
Introducing these quantities yields

$$\underline{\underline{h}} = \frac{1}{2}\,\frac{k}{mg}\,\delta_{\max}^2 - \delta_{\max} = \frac{15}{2}\,\frac{mg}{k} = \underline{\underline{7.5\ \text{cm}}}\ .$$

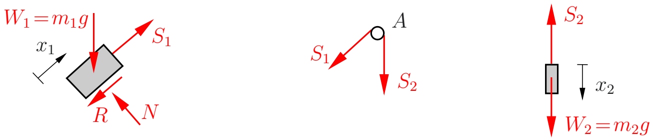
Problem 2.5 On a rough inclined plane (inclination angle α, kinetic friction constant μ), a block (mass m_1) is moving which is connected by a rope with a body of mass m_2. Pulley and rope are regarded as massless. **P2.5**

a) Determine the accelerations when m_1 slides upwards and downwards, respectively.

b) What force acts in the rope?



Solution **a)** We cut the rope and formulate for the 3 parts the basic equations, where we first assume upward sliding:



$$\nearrow: \ m_1 a_1 = S_1 - R - W_1 \sin\alpha, \quad \overset{\curvearrowleft}{A}: \ S_1 = S_2, \quad \downarrow: \ m_2 a_2 = W_2 - S_2,$$

$$\nwarrow: \ N = W_1 \cos\alpha, \qquad R = \mu N\,.$$

With the kinematic condition (unextensible rope) $v_1 = v_2$ and consequently $a_1 = a_2 = a$, we obtain

$$\underline{\underline{a^{(u)} = a = g\,\frac{m_2 - m_1(\sin\alpha + \mu\cos\alpha)}{m_1 + m_2}}}\,.$$

For upward sliding, the acceleration must be positive, $a > 0$, and therefore $m_2 > m_1(\sin\alpha + \mu\cos\alpha)$!

For downward sliding, only the direction of R must be changed. Then it follows

$$\underline{\underline{a^{(d)} = a = -g\,\frac{m_1(\sin\alpha - \mu\cos\alpha) - m_2}{m_1 + m_2}}}\,.$$

This case occurs for $a < 0$, i.e. for $m_1(\sin\alpha - \mu\cos\alpha) > m_2$.

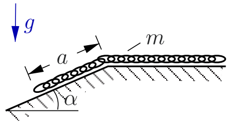
b) Independent on the sliding direction, the force in the rope is

$$S = S_2 = S_1 = W_2 - m_2 a_2 = m_2(g - a)\,.$$

Introducing the respective accelerations yields

$$\underline{\underline{S^{(u)} = \frac{m_1 m_2 g(1 + \sin\alpha + \mu\cos\alpha)}{m_1 + m_2}}}, \qquad \underline{\underline{S^{(d)} = \frac{m_1 m_2 g(1 + \sin\alpha - \mu\cos\alpha)}{m_1 + m_2}}}.$$

P2.6 **Problem 2.6** A chain (mass m, length l) lying on an inclined frictionless support starts sliding from the sketched position.

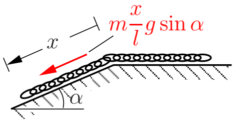


a) Determine the position and velocity as functions of time.

b) Determine the velocity as function of the position by applying the conservation-of-energy law.

Solution a) Each chain link experiences the same velocity and acceleration. We therefore consider the chain as a single body of mass m which is driven by a force which depends on the length x of the overhanging part. Thus, the equation of motion reads

$$m\,\ddot{x} = m\,\frac{x}{l}\,g\sin\alpha \quad \rightsquigarrow \quad \ddot{x} - \kappa^2 x = 0\,.$$



where $\kappa^2 = g\sin\alpha/l$. This differential equation has the solution

$$x(t) = A\cosh\kappa t + B\sinh\kappa t$$
$$\dot{x} = A\kappa\sinh\kappa t + B\kappa\cosh\kappa t\,.$$

The integration constants follow from the initial conditions

$$\dot{x}(0) = 0 \quad \rightsquigarrow \quad B = 0\,, \qquad x(0) = a \quad \rightsquigarrow \quad A = a$$

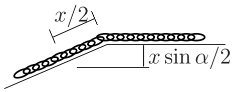
what finally leads to

$$\underline{\underline{x(t) = a\cosh\kappa t}}\,, \qquad \underline{\underline{\dot{x}(t) = a\kappa\sinh\kappa t}}\,, \qquad \kappa^2 = g\sin\alpha/l.$$

Note that this solution is only valid for $a \le x \le l$. When the complete chain is on the inclined plane, the driving force remains constant!

b) If we use as reference position for zero potential energy the upper horizontal plane, the energy terms in the initial and in the displaced position are given by

$$V_0 = -\frac{a}{2}\sin\alpha\,m\frac{a}{l}g\,, \qquad T_0 = 0\,,$$
$$V_1 = -\frac{x}{2}\sin\alpha\,m\frac{x}{l}g\,, \qquad T_1 = \frac{1}{2}\dot{x}^2 m\,.$$

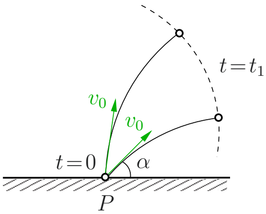


Introducing into $V_0 + T_0 = V_1 + T_1$ and solving for $\dot{x}$ leads to

$$\underline{\underline{\dot{x}(x) = \sqrt{(x^2 - a^2)g\sin\alpha}}}\,.$$

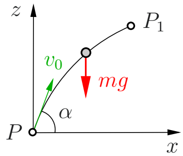
Again, this solution is only valid for $a \le x \le l$.

Problem 2.7 Determine the geometric locus of all points P_1 that is given by the position of all point masses at time $t = t_1$, that are thrown at time $t = 0$ with the *same* initial velocity v_0 from a point P under *different* angles α with respect to the horizontal. Assume that all trajectories are located in the same vertical plane and that there is no air drag. **P2.7**



Solution For convenience, the origin of the coordinates is chosen at P. Then it follows from

$$\uparrow: \quad m\ddot{z} = -mg \,, \qquad \rightarrow: \quad m\ddot{x} = 0$$



with the initial conditions

$$x(0) = z(0) = 0 \,,$$

$$\dot{x}(0) = v_0 \cos\alpha \,,$$

$$\dot{z}(0) = v_0 \sin\alpha$$

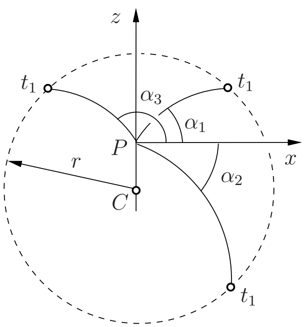
by integration

$$x = v_0 t \cos\alpha \,, \qquad z = -\frac{1}{2} g t^2 + v_0 t \sin\alpha \,.$$

Since the solution is sought at time t_1 for arbitrary angles, the angle α must be eliminated. Squaring and adding yields

$$\left.\begin{aligned} x_1^2 &= (v_0 t_1)^2 \cos^2\alpha \,, \\ \left(z_1 + \frac{g}{2}\, t_1^2\right)^2 &= (v_0 t_1)^2 \sin^2\alpha \end{aligned}\right\}$$

$$\leadsto \quad \underline{\underline{x_1^2 + \left(z_1 + \frac{g}{2} t_1^2\right)^2 = (v_0 t_1)^2}} \,.$$

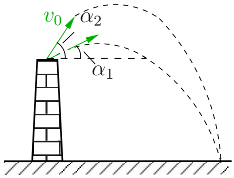


Accordingly, all points P_1 are located on a circle with the radius $r = v_0 t_1$ and the center C at

$$z = -\frac{g}{2}\, t_1^2 \,.$$

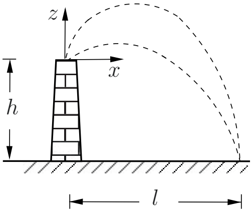
P2.8 **Problem 2.8** From the top of a tower, two point masses are thrown with the same initial velocity v_0 under two different angles α_1 and α_2. It is recognized that both masses impinge the surface at the same location.

Determine the height of the tower.



Solution The parabolic trajectory of motion is given by (see page 20)

$$z - z_0 = -\frac{g}{2}\left(\frac{x - x_0}{v_0 \cos\alpha}\right)^2 + (x - x_0)\tan\alpha \,.$$



We chose the origin of the coordinates at the top of the tower. Then we have $x_0 = z_0 = 0$, and the point where the masses impinge the surface has the unknown coordinates $x = l$, $z = -h$. For both throws holds:

$$-h = -\frac{g}{2}\,\frac{l^2}{v_0^2\cos^2\alpha_1} + l\tan\alpha_1 \,,$$

$$-h = -\frac{g}{2}\,\frac{l^2}{v_0^2\cos^2\alpha_2} + l\tan\alpha_2 \,.$$

Equating both expressions leads for the horizontal distance l to

$$l = \frac{2v_0^2}{g}\,\frac{1}{\tan\alpha_1 + \tan\alpha_2} \,.$$

Herewith, from the 1st equation, the height is determined as

$$\underline{\underline{h}} = +\frac{g}{2v_0^2}\left(\frac{2v_0^2}{g}\right)^2 \frac{1}{\cos^2\alpha_1}\left(\frac{1}{\tan\alpha_1 + \tan\alpha_2}\right)^2 - \frac{2v_0^2}{g}\,\frac{\tan\alpha_1}{\tan\alpha_1 + \tan\alpha_2}$$

$$= \underline{\underline{\frac{2v_0^2}{g}\,\frac{1}{(\tan\alpha_1 + \tan\alpha_2)\tan(\alpha_1 + \alpha_2)}}} \,.$$

Remark: For the solution, the following formulas are used:

$$\frac{1}{\cos^2\alpha_2} - \frac{1}{\cos^2\alpha_1} = \tan^2\alpha_2 - \tan^2\alpha_1$$
$$= (\tan\alpha_2 - \tan\alpha_1)(\tan\alpha_2 + \tan\alpha_1) \,,$$

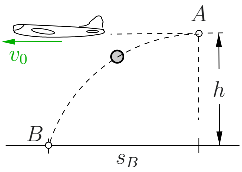
$$\frac{\tan\alpha_1 + \tan\alpha_2}{1 - \tan\alpha_1\tan\alpha_2} = \tan(\alpha_1 + \alpha_2) \,.$$

Problem 2.9 To rescue shipwrecked persons, a rescue package of mass m is dropped from an airplane, flying with speed $v_0 = 200\,\mathrm{km/h}$ in a height of $h = 150\,\mathrm{m}$. **P2.9**

a) Determine the distance s_B from launching the package until it impacts on sea surface.

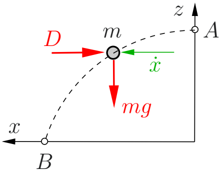
b) Calculate the impact velocity v_B

Assuming a high horizontal velocity component v_h, the air drag shall be taken into account by a horizontal drag force $D = \kappa m\, v_h^2$ with $\kappa = 0.003\,\mathrm{m}^{-1}$.



Solution **a)** We introduce an appropriate coordinate system and sketch the free body diagram with the acting drag force D and the weight mg. With $v_h = \dot{x}$, the equations of motion in x- and in z direction read

$$\leftarrow: \quad m\ddot{x} = -m\kappa\,\dot{x}^2\,, \qquad \uparrow: \quad m\ddot{z} = -mg\,.$$



Integration yields with the initial conditions $x(0) = 0$, $z(0) = h$, $\dot{x}(0) = v_0$, $\dot{z}(0) = 0$

$$\int_{v_0}^{\dot{x}} \frac{\mathrm{d}\dot{\bar{x}}}{\dot{\bar{x}}^2} = -\kappa \int_0^t \mathrm{d}\bar{t} \quad \rightsquigarrow \quad \dot{x} = \frac{1}{\dfrac{1}{v_0} + \kappa t}\,,$$

$$\int_0^x \mathrm{d}\bar{x} = \int_0^t \frac{1}{\dfrac{1}{v_0} + \kappa\bar{t}}\,\mathrm{d}\bar{t} \quad \rightsquigarrow \quad x = \frac{1}{c_0}\ln(1 + \kappa v_0 t)$$

$$\dot{z} = -gt \quad , \quad z = -\frac{g}{2}t^2 + h\,.$$

The impact time t_B follows from $z_B = 0$ as

$$t_B = \sqrt{2h/g}\,,$$

and thus, we obtain for the distance

$$\underline{\underline{s_B = x(t_B) = \frac{1}{\kappa}\ln\left(1 + \kappa v_0\sqrt{2h/g}\right) = 218\,\mathrm{m}}}\,.$$

b) From the velocity components at impact,

$$\dot{x}(t_B) = \frac{v_0}{1 + \kappa v_0 t_B} = 104\,\mathrm{km/h}\,,$$

$$\dot{z}(t_B) = -g t_B = -54.2\,\mathrm{m/s} = -195\,\mathrm{km/h}\,,$$

results the velocity as

$$\underline{\underline{v_B = \sqrt{\dot{x}^2(t_B) + \dot{z}^2(t_B)} = 221\,\mathrm{km/h}}}\,.$$

P2.10 **Problem 2.10** A rocket without an own propulsion is catapulted vertically upwards from earth's surface with an initial velocity v_0.

a) Determine the maximum flight hight H by considering the change of gravitation and neglecting drag forces.

b) What magnitude of v_0 is required when the rocket shall escape from the gravitation field of earth? (Earth's radius $R = 6370$ km)

Solution **a)** Since only conservative forces are acting, the conservation-of-energy law

$$T_1 + V_1 = T_0 + V_0$$

is appropriate as solution method. The gravitational potential $V = -GMm/r$ according to (force on earth's surface = weight mg)

$$mg = -\frac{\mathrm{d}V}{\mathrm{d}r}\bigg|_{r=R} = G\frac{Mm}{R^2} \quad \rightsquigarrow \quad GM = gR^2$$

can be written as

$$V = -mg\frac{R^2}{r} \,.$$

Thus, the different energies on earth's surface ($r = R$) and final flight height ($r = R + H$) are

$$T_0 = \frac{1}{2}mv_0^2 \,, \qquad V_0 = -mgR \,,$$

$$T_1 = 0 \,, \qquad V_1 = -mg\frac{R^2}{R+H} \,.$$

Introduction into the energy conservation law yields

$$-mg\frac{R^2}{R+H} = \frac{1}{2}mv_0^2 - mgR \quad \rightsquigarrow \quad \underline{\underline{H = R\,\frac{v_0^2}{2gR - v_0^2}}} \,.$$

b) The 'escape velocity' v_0^* is found from

$$H \to \infty \quad \rightsquigarrow \quad \underline{\underline{v_0^* = \sqrt{2gR}}} = 11180\ \frac{\mathrm{m}}{\mathrm{s}} \approx \underline{\underline{40000\ \frac{\mathrm{km}}{\mathrm{h}}}} \,.$$

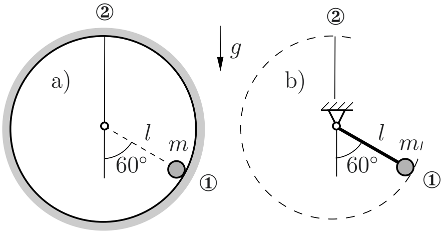
Remarks:

- Note that a rocket in real cases is *not* launched from earth's surface without an own propulsion!
- The required kinetic energy to reach the 'escape velocity' is $T_0 = mgR$.

Problem 2.11 Which minimum initial velocity v_0 in position ① is necessary, such that the body with mass m reaches position ② if **P2.11**

a) it slides along a frictionless circular path (radius l),

b) it is fixed at a rigid massless rod (length l) ?

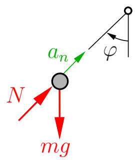


Solution In both cases the initial velocity v_0 is connected with v_2 at position ② by the energy-conservation law $T_2 + V_2 = T_1 + V_1$. Choosing zero potential energy at position ①, we obtain with $V_1 = 0$

$$\frac{m}{2}\,v_2^2 + mg(l + l\cos 60^\circ) = \frac{m}{2}\,v_0^2 \quad \rightsquigarrow \quad v_0^2 = v_2^2 + 3gl\ .$$

a) The necessary velocity v_2 is obtained from the condition that the normal force N between the path and the body is non-negative (otherwise the body looses contact with the path). From the law of motion

$$\nearrow:\quad ma_n = N - mg\cos\varphi$$



with $a_n = v^2/l$ we obtain at position ② ($\varphi = \pi$) for the limit case $N = 0$:

$$\frac{mv_2^2}{l} = mg \quad \rightsquigarrow \quad v_2^2 = gl\ .$$

Hence, it follows

$$v_0^2 = gl + 3gl \quad \rightsquigarrow \quad \underline{\underline{v_0 = 2\sqrt{gl}}}\ .$$

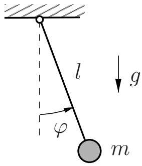
b) The initial velocity for the mass fixed at the rod will take a minimum if it comes to rest in position ②. For $v_2 = 0$, the energy-conservation law directly yields

$$\underline{\underline{v_0 = \sqrt{3}\,\sqrt{gl}}}\ .$$

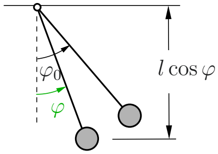
Remark: In case b) the force S in the rod may get negative. For example, in ② ($v_2 = 0$) the force is $S = -mg$.

P2.12 **Problem 2.12** Determine the velocity $v(\varphi)$ of mass m of a simple pendulum in dependence on the maximum amplitude φ_0.

Discuss the result for characteristic angles φ.



Solution Since the velocity shall be determined in dependence on the position, the energy-conservation law is the first choice as solution method. As reference position for the potential energy, we choose the horizontal position $\varphi = \pi/2$ and find from



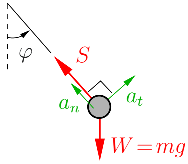
$$T(\varphi) + V(\varphi) = T_0 + V_0$$

with $v(\varphi_0) = 0$, i.e. $T_0 = 0$:

$$\frac{m}{2}\,v^2 - mgl\cos\varphi = 0 - mgl\cos\varphi_0 \quad \rightsquigarrow \quad \underline{\underline{v = \pm\sqrt{2gl(\cos\varphi - \cos\varphi_0)}}}\,.$$

The same result can be found by integrating the law of motion in tangential direction. From

$$\nearrow: \quad ma_t = ml\ddot{\varphi} = -mg\sin\varphi$$



with $\ddot{\varphi}\,\mathrm{d}\varphi = \dot{\varphi}\,\mathrm{d}\dot{\varphi}$ and the initial condition $\dot{\varphi}(\varphi_0) = 0$, we obtain

$$\int_{\dot{\varphi}}^{0} \dot{\bar{\varphi}}\,\mathrm{d}\dot{\bar{\varphi}} = -\frac{g}{l}\int_{\varphi}^{\varphi_0} \sin\bar{\varphi}\,\mathrm{d}\bar{\varphi} \quad \rightsquigarrow \quad -\frac{\dot{\varphi}^2}{2} = \frac{g}{l}(\cos\varphi_0 - \cos\varphi)$$

or with $v = l\dot{\varphi}$ again

$$\underline{\underline{v^2 = 2gl(\cos\varphi - \cos\varphi_0)}}\,.$$

The maximum speed occurs for $\cos\varphi = 1$, i.e. at $\varphi = 0$:

$$v_{\max} = \sqrt{2gl(1 - \cos\varphi_0)} = \sqrt{2gl2\sin^2\frac{\varphi_0}{2}} = 2\sqrt{gl}\,\sin\frac{\varphi_0}{2}\,.$$

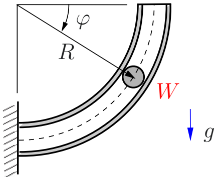
For a *small* maximum amplitude φ_0 also the angle φ remains small, and we obtain by truncated series approximation

$$\cos\varphi \approx 1 - \varphi^2/2\,, \qquad \sin(\varphi_0/2) \approx \varphi_0/2$$
$$\rightsquigarrow \quad v^2 = gl(\varphi_0^2 - \varphi^2)\,, \qquad v_{\max} = \sqrt{gl}\,\varphi_0\,.$$

Problem 2.13 In a clamped *frictionless* pipe elbow (radius R) glides a sphere (weight $W = mg$) with zero initial velocity downwards from the top. **P2.13**

Determine the support reactions at the clamping in dependence on the position φ of the sphere.

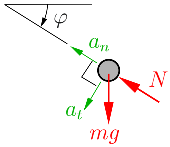
At which φ the reactions take extreme values?



Solution NEWTON's law yields in components:

$$\swarrow: \quad ma_t = mg\cos\varphi\,,$$

$$\nwarrow: \quad ma_n = N - mg\sin\varphi\,.$$



With $a_t = R\,\ddot{\varphi}$, $a_n = R\,\dot{\varphi}^2$ and $\ddot{\varphi}\mathrm{d}\varphi = \dot{\varphi}\mathrm{d}\dot{\varphi}$, it follows from the 1st equation by integration

$$\int\limits_0^{\dot{\varphi}} \dot{\bar{\varphi}}\,\mathrm{d}\dot{\bar{\varphi}} = \int\limits_0^{\varphi} \frac{g}{R}\cos\bar{\varphi}\,\mathrm{d}\bar{\varphi} \quad \leadsto \quad \frac{\dot{\varphi}^2}{2} = \frac{g}{R}\sin\varphi\,.$$

Therewith, we obtain the normal force from the 2nd equation as

$$N(\varphi) = mg\sin\varphi + mR\dot{\varphi}^2 = 3W\sin\varphi\,.$$

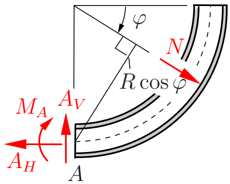
The equilibrium conditions for the elbow lead to the support reactions:

$$\uparrow: \quad \underline{\underline{A_V}} = N\sin\varphi = \underline{\underline{3W\sin^2\varphi}}\,,$$

$$\leftarrow: \quad \underline{\underline{A_H}} = N\cos\varphi = 3W\sin\varphi\cos\varphi$$

$$= \underline{\underline{-\frac{3}{2}W\sin 2\varphi}}\,,$$

$$\overset{\curvearrowright}{A}: \quad \underline{\underline{M_A}} = -NR\cos\varphi = \underline{\underline{-\frac{3}{2}WR\sin 2\varphi}}\,.$$

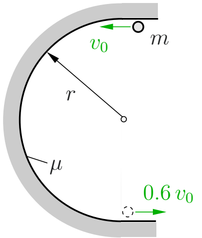


The clamping moment M_A and the horizontal force A_H take their maximum, when the sphere is at $\varphi = \pi/4$. The vertical force A_V is maximal for $\varphi = \pi/2$:

$$M_{A\,\max} = -\frac{3}{2}WR\,, \qquad A_{H\,\max} = \frac{3}{2}W\,, \qquad A_{V\,\max} = 3W\,.$$

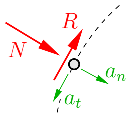
P2.14 **Problem 2.14** A hockey puck (mass m) in an ideally smooth ice field (no friction) is shot with speed v_0 into the half-circular part of the boards and slides along the boards. At the end of the curved part, the speed is measured to be $0.6\,v_0$.

Determine the friction coefficient μ of the boards.



Solution With aid of the free body diagram, we obtain the equations of motion

$$\swarrow: \quad ma_t = -R\,, \qquad \searrow: \quad ma_n = N\,.$$



Introducing the kinematic relations $a_t = \dot{v}$, $a_n = v^2/r$ and the friction law $R = \mu N$ yields

$$m\dot{v} = -\mu m\,\frac{v^2}{r}\,.$$

Separation of variables and integration leads to the velocity $v(t)$:

$$\int_{v_0}^{v} \frac{\mathrm{d}\bar{v}}{\bar{v}^2} = -\int_0^t \frac{\mu}{r}\,\mathrm{d}\bar{t} \quad \rightsquigarrow \quad v(t) = \frac{v_0}{1 + \frac{\mu v_0}{r}\,t}\,.$$

From repeated integration, the path $s(t)$ is determined:

$$\int_0^s \mathrm{d}\bar{s} = v_0 \int_0^t \frac{\mathrm{d}\bar{t}}{1 + \mu\frac{v_0}{r}\,\bar{t}} \quad \rightsquigarrow \quad s(t) = \frac{r}{\mu}\ln\left(1 + \frac{\mu v_0}{r}\,t\right)\,.$$

The time t_1, until the speed is reduced to $0.6\,v_0$, is calculated as

$$0.6\,v_0 = \frac{v_0}{1 + \frac{\mu v_0}{r}\,t_1} \quad \rightsquigarrow \quad t_1 = \frac{2}{3}\,\frac{r}{\mu v_0}\,.$$

Thus, we obtain for the corresponding path length at the end of the curved boards

$$s_1 = s(t_1) = \frac{r}{\mu}\ln\left(1 + \frac{\mu v_0}{r}\,t_1\right) = \frac{r}{\mu}\,\ln\frac{5}{3}\,.$$

Equalizing it with the length of the half-circle yields

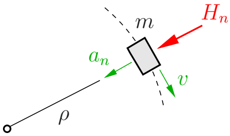
$$s_1 = r\pi \quad \rightsquigarrow \quad \underline{\underline{\mu = \frac{\ln(5/3)}{\pi} = 0.16}}\,.$$

Problem 2.15 A car (weight $W = mg$) passes a bend (static friction coefficient μ_0), whose curvature $1/\rho$ increases proportional to the covered distance s, i.e. $s = A^2/\rho$ (clothoid). At time $t_0 = 0$, the car is at $s_0 = 0$ and has an initial speed v_0. **P2.15**

At which speed v, where and when the car 'skids off the bend' if
a) it moves with constant speed,
b) it brakes with constant deceleration a_0?

Given.: $A = 35\,\text{m}$, $\mu_0 = 0.6$, $a_0 = g/4$, $v_0 = 72\,\text{km/h}$.

Solution **a)** For $a_t = 0$, Newton's law yields with $a_n = v_0^2/\rho$ the friction force



$$\swarrow: \quad H = H_n = m a_n = m\frac{v_0^2}{\rho} .$$

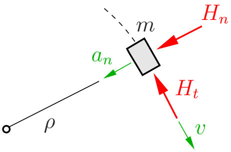
The car leaves its path when the limit friction force is attained:

$$H = \mu_0 m g \quad \rightsquigarrow \quad m\frac{v_0^2}{\rho_1} = \mu_0 m g \quad \rightsquigarrow \quad \rho_1 = \frac{v_0^2}{\mu_0 g} .$$

This leads with $s = A^2/\rho = v_0 t$ to $(v_1 = v_0)$

$$\underline{\underline{s_1 = \frac{\mu_0 g A^2}{\rho_1} = 18\,\text{m}}} , \qquad \underline{\underline{t_1 = \frac{s_1}{v_0} = 0.9\,\text{s}}} .$$

b) When the motion is decelerated, an additional force acts in tangential direction. With $a_t = -a_0$ follows



$$\searrow: \quad m a_t = -m a_0 = -H_t ,$$

$$\swarrow: \quad m a_n = m\frac{v^2}{\rho} = H_n .$$

Static limit friction is attained for

$$H = \sqrt{H_n^2 + H_t^2} = \mu_0 m g \quad \rightsquigarrow \quad \sqrt{\left(\frac{m v^2}{\rho_2}\right)^2 + (m a_0)^2} = \mu_0 m g$$

$$\rightsquigarrow \quad \frac{v^2}{\rho_2} = \sqrt{\mu_0^2 g^2 - a_0^2} .$$

Thus, with $v = \sqrt{v_0^2 - 2 a_0 s} = v_0 - a_0 t$ (constant deceleration) and $s = A^2/\rho$, we obtain

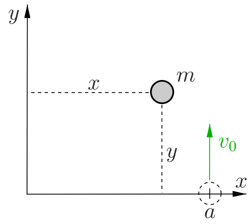
$$\underline{\underline{s_2}} = \frac{v_0^2}{4a_0} \overset{(+)}{-} \sqrt{\left(\frac{v_0^2}{4a_0}\right)^2 - \frac{A^2}{2a_0}\sqrt{\mu_0^2 g^2 - a_0^2}} = \underline{\underline{22.7\,\text{m}}} ,$$

$$\underline{\underline{v_2}} = \sqrt{v_0^2 - 2 a_0 s_2} = \underline{\underline{61.2\,\text{km/h}}} , \qquad \underline{\underline{t_2}} = \frac{v_0 - v_2}{a_0} = \underline{\underline{1.22\,\text{s}}} .$$

P2.16 **Problem 2.16** The potential of a free movable point mass m in a horizontal plane is given by $V(x,y) = k(x^2+y^2)/2$.

a) Determine the acting forces and formulate the equations of motion.

b) Determine the path of the point mass in parameter and implicit representation for the initial conditions $x(0) = a$, $\dot{x}(0) = 0$, $y(0) = 0$, $\dot{y}(0) = v_0$.



Solution **a)** The forces follow from the derivatives of the potential as

$$F_x = -\frac{\partial V}{\partial x} = -kx\,, \qquad F_y = -\frac{\partial V}{\partial y} = -ky\,,$$

which leads to the equations of motion

$$\rightarrow:\quad m\ddot{x} = F_x = -kx\ , \qquad \uparrow:\quad m\ddot{y} = F_y = -ky$$

or

$$\underline{\underline{\ddot{x} + \omega^2 x = 0}}\,, \qquad \underline{\underline{\ddot{y} + \omega^2 y = 0}}\,, \qquad \text{where} \quad \omega^2 = k/m\ .$$

b) Both differential equations describe free undamped vibrations, whose solutions are given by (cf. chapter 7)

$$x = A\cos\omega t + B\sin\omega t\ , \qquad y = C\cos\omega t + D\sin\omega t\ ,$$

where A, B, C, D are constants. They are determined by using the initial conditions:

$$x(0) = a \quad \rightsquigarrow \quad A = a\ , \quad y(0) = 0 \quad \rightsquigarrow \quad C = 0\ ,$$
$$\dot{x}(0) = 0 \quad \rightsquigarrow \quad B = 0\ , \quad \dot{y}(0) = v_0 \quad \rightsquigarrow \quad D = \frac{v_0}{\omega}\ .$$

Thus, in parameter representation, the path is described by

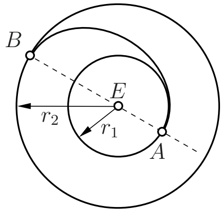
$$\underline{\underline{x(t) = a\cos\omega t}}\ , \qquad \underline{\underline{y(t) = \frac{v_0}{\omega}\sin\omega t}}\ .$$

The implicit representation is found by eliminating t through squaring and adding, resulting in

$$\underline{\underline{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{v_0/\omega}\right)^2 = 1}}\ .$$

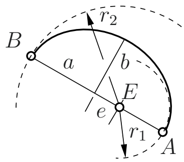
Accordingly, the point mass moves counterclockwise along an ellipse with half-axes a and v_0/ω.

Problem 2.17 A space vessel shall be lifted from a circular path with radius r_1 around the earth E (earth's radius R) to a more distant circular orbit of radius r_2. The transition is carried out by changing in A and in B suddenly the magnitude of the velocity of the vessel. **P2.17**



Determine the necessary velocity change in A.

Solution According to Kepler's 1st law, the vessel moves between A and B along an ellipse, whose one focus coincides with the earth. Since in A and B only the magnitude of velocity and not its direction shall be changed, the transition ellipse in A and in B must be tangential to the circles. From this condition the ellipse parameters follow as



$$a = \frac{r_1 + r_2}{2}\,, \quad e = a - r_1 = \frac{r_2 - r_1}{2}\,, \quad b^2 = a^2 - e^2 = r_1 r_2\,,$$

and the curvature radius at A (vertex of the ellipse) yields

$$\rho = \frac{b^2}{a} = \frac{2 r_1 r_2}{r_1 + r_2}\,.$$

In A, the gravitational force has the magnitue $F = mg(R/r_1)^2$. At this location, the law of motion normal to the circular path (before velocity change) leads to

$$m\,\frac{v_1^2}{r_1} = mg\left(\frac{R}{r_1}\right)^2 \quad \rightsquigarrow \quad v_1 = R\sqrt{\frac{g}{r_1}}$$

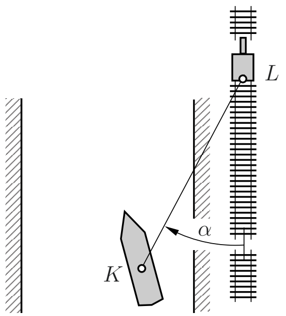
and normal to the elliptic orbit (after velocity change) to

$$m\,\frac{v_A^2}{\rho} = mg\left(\frac{R}{r_1}\right)^2 \quad \rightsquigarrow \quad v_A = R\sqrt{\frac{g}{r_1}}\,\sqrt{\frac{2r_2}{r_1 + r_2}}\,.$$

Thus, the necessary velocity change is given by

$$\underline{\underline{\Delta v_A}} = v_A - v_1 = \underline{\underline{R\sqrt{\frac{g}{r_1}}\left\{\sqrt{\frac{2r_2}{r_1 + r_2}} - 1\right\}}}\,.$$

P2.18 **Problem 2.18** A barge K is towed in a channel by a haul engine L. In the towing rope acts a force $S = 9\,\text{kN}$, which is inclined by an angle $\alpha = 28^\circ$ with respect to the rail track.



Determine
a) the work for a covered distance $s = 3\,\text{km}$,
b) the power for a towing speed $v = 9\,\text{km/h}$.

Solution **a)** For the work

$$U = \int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int |\boldsymbol{F}| \cos\alpha \, |\mathrm{d}\boldsymbol{r}| \,,$$

we obtain with $|\boldsymbol{F}| = \text{const} = S$, $\cos\alpha = \text{const} = \cos 28^\circ$ and $|\mathrm{d}\boldsymbol{r}| = \mathrm{d}s$:

$$\underline{\underline{U}} = S\cos\alpha\; s = 9 \cdot 0.883 \cdot 3000 = 23800\,\text{kNm} = 23800\,\text{kJ} = \underline{\underline{23.8\,\text{MJ}}}\,.$$

b) The power is given by

$$\underline{\underline{P}} = \boldsymbol{F} \cdot \boldsymbol{v} = S\cos\alpha\, v = 9 \cdot 0.883 \cdot \frac{9}{3.6} = 19.9\,\text{kJ/s} = \underline{\underline{19.9\,\text{kW}}}\,.$$

P2.19 **Problem 2.19** Determine the necessary work for lifting a body of weight $W = 1\,\text{N}$ from erth's surface (earth's radius R) into the distance r_0 of the moon ($r_0 = 60\,R$).

Solution According to the gravitational law, the gravitational force varies inverse to the squared distance from the earth's surface. Thus, the 'weight' in distance r is

$$F = W\left(\frac{R}{r}\right)^2 .$$

Therewith, the work follows as

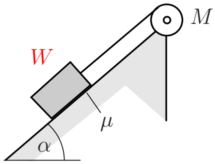
$$U = W\int_R^{60R} \left(\frac{R}{r}\right)^2 \mathrm{d}r = \frac{59}{60}\,WR\,.$$

With $R = 6370$ km and $W = 1$ N, we obtain

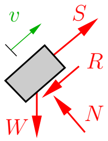
$$\underline{\underline{U}} = \frac{59}{60} \cdot 6370\,\text{Nkm} = 6264\,\text{kJ} = \underline{\underline{6.3\,\text{MJ}}}\,.$$

Problem 2.20 A motor winch M tows a body of weight $W = mg$ with constant speed v_0 upwards a rough inclined plane (coefficient of kinetic friction μ). **P2.20**

Determine the necessary electric power P_A of the winch if its efficiency η is known.



Solution For uniform motion ($\dot{v} = 0$), the force in the rope S follows from the equilibrium conditions



$$\nearrow: \quad S = W \sin\alpha + R\,, \qquad \nwarrow: \quad N = W\cos\alpha$$

and the friction law $R = \mu N$ as

$$S = W(\sin\alpha + \mu\cos\alpha)\ .$$

Thus, the power generated by the winch is

$$P = S\,v_0 = W(\sin\alpha + \mu\cos\alpha)v_0\ .$$

The power absorbed by the winch having an effienciency η is given by

$$\underline{\underline{P_A}} = \frac{P}{\eta} = \underline{\underline{\frac{W}{\eta}(\sin\alpha + \mu\cos\alpha)v_0}}\ .$$

Problem 2.21 A big container vessel with a drive power of 80000 kW covers in 7 days 4000 nautical miles. **P2.21**

Determine the average drag force F_d

Solution Using the conversion 1 nautical mile = 1.852 km and 1 kW = 1 kNm/s, we obtain from

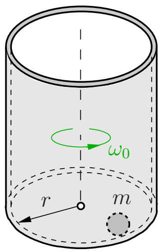
$$P = F_d\,v \qquad \text{with} \qquad v = \frac{4000 \cdot 1852}{7 \cdot 24 \cdot 3600} = 12.25\ \frac{\text{m}}{\text{s}}$$

the drag force

$$\underline{\underline{F_d}} = \frac{P}{v} = \frac{80000\ \text{kNm/s}}{12.25\ \text{m/s}} = 6531\ \text{kN} = \underline{\underline{6.53\ \text{MN}}}\ .$$

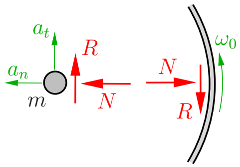
P2.22

Problem 2.22 In a centrifuge of radius r, rotating with constant angular velocity ω_0, a body (point mass m) is accelerated by dynamic friction (friction coefficient μ) from its initial angular velocity $\omega(0) = \omega_0/2$ to the final angular velocity ω_0.



Determine the required acceleration time t_r, the drive torque $M(t)$, the power $P(t)$ and the work U done by the centrifuge.

Solution During acceleration, the point mass rotates with angular velocity $\omega(t)$. With the accelerations $a_t = r\dot{\omega}$, $a_n = r\omega^2$, the equations of motion are given by



$$\uparrow: \quad mr\dot{\omega} = R\,, \qquad \leftarrow: \quad mr\omega^2 = N\,.$$

Introducing the friction law $R = \mu N$, eliminating N and using $\omega(0) = \omega_0/2$ leads to

$$\dot{\omega} = \mu\omega^2 \quad \rightsquigarrow \quad \int \frac{\mathrm{d}\omega}{\omega^2} = \mu \int \mathrm{d}t \quad \rightsquigarrow \quad \frac{2}{\omega_0} - \frac{1}{\omega} = \mu t\,.$$

The acceleration time t_r is obtained from the condition $\omega(t_r) = \omega_0$:

$$\frac{2}{\omega_0} - \frac{1}{\omega_0} = \mu t_r \quad \rightsquigarrow \quad \underline{\underline{t_r = \frac{1}{\mu\omega_0}}}\,.$$

Since the centrifuge is not accelerated, the driving torque is given by the moment of the friction force R:

$$\underline{\underline{M(t)}} = rR = mr^2\dot{\omega} = \mu m r^2 \omega^2 = \underline{\underline{\frac{\mu m r^2 \omega_0^2}{(2-\mu\omega_0 t)^2}}}\,.$$

Because M and ω_0 are coaxial, the required power is given by

$$\underline{\underline{P(t)}} = \boldsymbol{M} \cdot \boldsymbol{\omega}_0 = M\omega_0 = \underline{\underline{\frac{\mu m r^2 \omega_0^3}{(2-\mu\omega_0 t)^2}}}\,.$$

The total work U done by the centrifuge (strictly speaking, the friction force) is calculated easiest from the difference of kinetic energies:

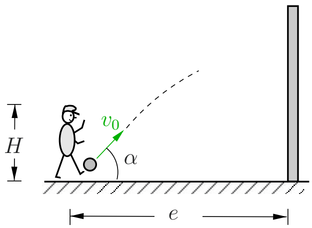
$$\underline{\underline{U}} = \frac{1}{2}mv^2(t_r) - \frac{1}{2}mv^2(0) = \frac{1}{2}m(r\omega_0)^2 - \frac{1}{2}m(r\omega_0/2)^2 = \underline{\underline{\frac{3}{8}mr^2\omega_0^2}}\,.$$

Problem 2.23 A soccer player kicks the ball from a distance e with a kick-off angle $\alpha = 45^\circ$ against a vertical wall. The impact at the wall is assumed to be ideal-elastic. **P2.23**

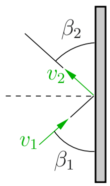
What initial velocity v_0 of the ball is necessary if

a) it shall bounce back exactly to the foot of the player,

b) it shall bounce back to the head (height H) of the player?

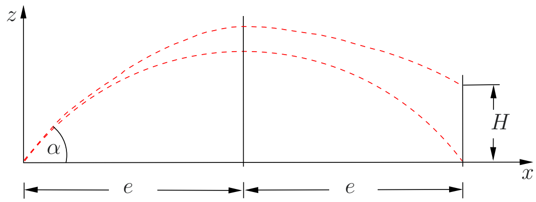


Solution Since no energy gets lost when the impact against the wall is ideal-elastic, the magnitudes of impact velocity v_1 and rebound velocity v_2 must be equal. Then, from the impulse law follows (reflection law)



$$\uparrow: \quad m v_2 \cos\beta_2 - m v_1 \cos\beta_1 = 0 \quad \rightsquigarrow \quad \underline{\underline{\beta_1 = \beta_2}}\,.$$

Hence, we can replace the problem of reflection at the wall by a mirroring problem, where we imagine the trajectory being continued through the wall.



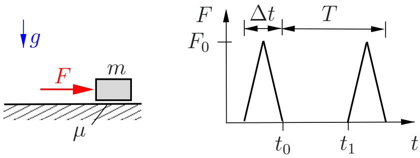
a) The 'flight distance' $d = 2e$ follows with $\alpha = 45^\circ$ and $z_0 = 0$ as (see page 32)

$$d = 2e = v_0^2 \frac{\sin 2\alpha}{g} \quad \rightsquigarrow \quad \underline{\underline{v_0}} = \sqrt{\frac{2ge}{\sin 2\alpha}} = \underline{\underline{\sqrt{2ge}}}\,.$$

b) We introduce the coordinates of the kick-off point $x_0 = z_0 = 0$ and end point $x = 2e$, $z = H$ into the parabolic trajectory of motion (page 32) and obtain

$$H = -\frac{g}{2}\left(\frac{2e}{v_0 \cos 45^\circ}\right)^2 + 2e \tan 45^\circ \quad \rightsquigarrow \quad \underline{\underline{v_0 = 2e\sqrt{\frac{g}{2e - H}}}}\,.$$

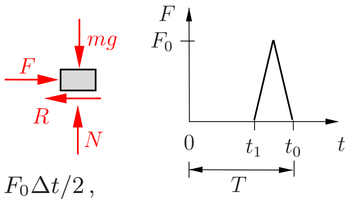
P2.24 **Problem 2.24** A body (mass m) is driven along a rough horizontal path (kinetic friction coefficient μ) by a periodically acting force, such that $v(t_0) = v_0$ and $v_1 = v(t_1) = v_0/2$. During the driving phase Δt, the force profile $F(t)$ is triangular.



a) Determine the period T and the required peak force F_0 for a given Δt.

b) Calculate the work U done by F during a period T.

Solution **a)** We consider one period and start counting time at the end of the driving phase. First we apply the impulse law over the full period T. With $R = \mu N = \mu mg = \text{const}$ and the given $F(t)$ profile, the linear impulses of the friction force and the driving force are given by



$$\widehat{R} = \mu mgT\,, \qquad \widehat{F} = \int_{\Delta t} F(t)\mathrm{d}t = F_0\Delta t/2\,,$$

Thus, with $v(0) = v(t_0) = v_0$, the impulse law leads to

$$\rightarrow: \quad m\,v_0 - m\,v_0 = 0 = -\mu mgT + \frac{1}{2}F_0\Delta t\,.$$

In the same way, we obtain with $v(t_0) = v_0$ and $v(t_1) = v_0/2$ from the impulse law applied over the driving phase Δt

$$\rightarrow: \quad m\,v_0 - m\,v_0/2 = -\mu mg\Delta t + \frac{1}{2}F_0\Delta t\,.$$

From these two equations for the two unknowns T and F_0, it follow

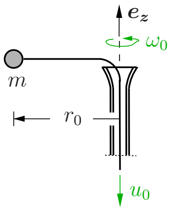
$$\underline{\underline{T = \Delta t + \frac{v_0}{2\mu g}}}\,, \qquad \underline{\underline{F_0 = 2\mu mg\left[1 + \frac{v_0}{2\mu mg\Delta t}\right]}}\,.$$

b) The work U done by $F(t)$ during time T is determined by using the work-energy theorem, i.e. from the difference of kinetic energies at t_0 and t_1:

$$\underline{\underline{U}} = T(t_0) - T(t_1) = \frac{1}{2}\,mv_0^2 - \frac{1}{2}\,m(v_0/2)^2 = \underline{\underline{\frac{3}{8}\,mv_0^2}}\,.$$

Remark: The work of F and of R during T are equal but have opposite signs. This easily allows calculating the covered distance: $l = 3v_0^2/(8\mu g)$.

Problem 2.25 A point mass, fixed at a massless thread, rotates along a horizontal circular path. At time $t = 0$, the radius is r_0 and the angular velocity is ω_0. P2.25



a) Determine $r(t)$ and $\omega(t)$, when the thread is pulled with constant speed u_0 downwards through the sketched vertical pipe.

b) At what time t_1 the angular velocity has doubled and how big is the associated radius r_1?

c) Determine for this case the change of kinetic energy ΔT of the point mass.

Solution **a)** Because there acts no external moment on the point mass with respect to the center of its path, the angular momentum remains conserved:

$$\boldsymbol{L} = \boldsymbol{r} \times m\,\boldsymbol{v} = \text{const} \ .$$

With $\boldsymbol{r} \times \boldsymbol{v} = r v_\varphi\, \boldsymbol{e}_z$ and $v_\varphi = r\omega$, it follows

$$L = m\,r^2\omega = m\,r_0^2\omega_0 \quad \rightsquigarrow \quad \omega = \omega_0\,\frac{r_0^2}{r^2} \ .$$

The dependence of $r(t)$ on time is given by the constant thread speed $\dot{r} = -u_0$:

$$\underline{\underline{r(t) = r_0 - u_0 t}} \ .$$

Inserting into ω leads to

$$\underline{\underline{\omega(t) = \frac{\omega_0 r_0^2}{(r_0 - u_0 t)^2}}} \ .$$

b) From the condition $\omega(t_1) = 2\omega_0$, it follows

$$\underline{\underline{t_1 = \frac{r_0}{u_0}\left(1 - \frac{1}{2}\sqrt{2}\right)}} \qquad \text{and} \qquad \underline{\underline{r_1 = r_0 - u_0 t_1 = \frac{\sqrt{2}}{2}\,r_0}} \ .$$

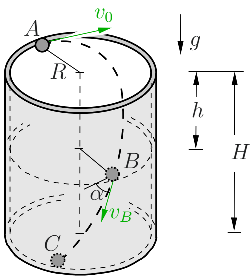
c) The energy change is calculated as

$$\begin{aligned}\Delta T &= \frac{m}{2}(v_{\varphi 1}^2 + u_0^2) - \frac{m}{2}(v_{\varphi 0}^2 + u_0^2)\\ &= \frac{m}{2}\left(\frac{\sqrt{2}}{2}\,r_0 2\omega_0\right)^2 - \frac{m}{2}(r_0\omega_0)^2 = \underline{\underline{\frac{1}{2}\,m r_0^2\omega_0^2}} \ .\end{aligned}$$

The kinetic energy has doubled.

P2.26

Problem 2.26 In an upright standing frictionless hollow cylinder (radius R), a little sphere (point mass m) is inserted at point A with a horizontal initial speed v_0.



a) What angle α to the horizontal plane has the velocity v_B at point B lying in height distance h below A?

b) What speed v_0 is necessary such that the sphere impinges on ground at C with an angle of 45° and what magnitude has v_C in this case?

Solution **a)** The speed of the sphere in point B follows from the energy conservation law $T_A + V_A = T_B + V_B$:

$$\frac{1}{2}\,mv_0^2 + mgh = \frac{1}{2}\,mv_B^2 \quad \rightsquigarrow \quad v_B = \sqrt{v_0^2 + 2gh}\,.$$

Since there acts no external moment with respect to the cylinder axis, the angular momentum (moment of momentum) with respect to this axis is conserved:

$$L = \text{const} \quad \rightsquigarrow \quad L_A = L_B\,.$$

With $L_A = R(mv_0)$ and $L_B = R(mv_B\cos\alpha)$, the angle α follows as

$$\underline{\underline{\cos\alpha}} = \frac{v_0}{v_B} = \underline{\underline{\frac{v_0}{\sqrt{v_0^2 + 2gh}}}}\,.$$

b) With $\alpha = 45°$ and $h = H$, we obtain at C

$$\cos 45° = \frac{1}{2}\sqrt{2} = \frac{v_0}{\sqrt{v_0^2 + 2gH}}$$

or after squaring and solving for v_0

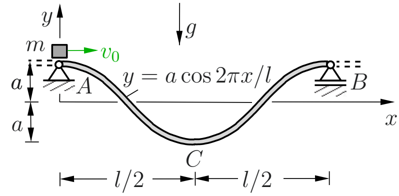
$$\underline{\underline{v_0 = \sqrt{2gH}}}\,.$$

Thus, after falling the height distance H, the velocity is

$$\underline{\underline{v_C}} = \sqrt{v_0^2 + 2gH} = \sqrt{2gH + 2gH} = \underline{\underline{2\sqrt{gH}}}\,.$$

Remark: Because on the sphere acts only the weight (vertical) and the normal force from the cylinder wall (normal to the wall) the horizontal velocity component remains unchanged v_0.

Problem 2.27 A cosine-shaped arch of a roller coaster is in A and in B pin-supported. The arch is passed without friction by a car (mass m) that has in point A the initial velocity $v_0 = \sqrt{ga/10}$. **P2.27**



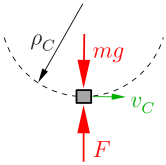
Determine the support reactions and the bending moment at C when the car just passes point C. The weight of the arch shall be disregarded.

Solution The velocity of the car at point C follows from the energy conservation law $T_A + V_A = T_C + V_C$:

$$\frac{1}{2}\,mv_0^2 + mga = \frac{1}{2}\,mv_C^2 - mga \qquad \rightsquigarrow \qquad v_C^2 = v_0^2 + 4ga = \frac{41}{10}\,ga\,.$$

The derivatives $y' = -(2\pi a/l)\sin 2\pi x/l$ and $y'' = -(4\pi^2 a/l^2)\cos 2\pi x/l$ yield $y'(l/2) = 0$ and $y''(l/2) = 4\pi^2 a/l^2$. Hence, the curvature radius ρ of the path at C is given by

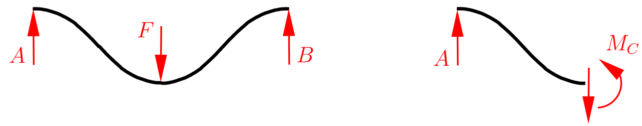
$$\frac{1}{\rho_C} = \frac{y''(l/2)}{[1 + y'^2(l/2)]^{3/2}} = \frac{4\pi^2 a}{l^2}\,.$$



With the normal acceleration $a_n = v^2/\rho$, the law of motion allows to determine the force F at C, which acts from the arch onto the car:

$$\uparrow: \quad m\,\frac{v_C^2}{\rho_C} = F - mg \quad \rightsquigarrow \quad F = mg + m\,\frac{v_C^2}{\rho_C} = mg\Big(1 + \frac{164\,\pi^2 a^2}{10\,l^2}\Big).$$

Knowing F, the support reactions and the bending moment in C can be calculated:



$$\underline{\underline{A = B = \frac{mg}{2}\Big(1 + \frac{164\,\pi^2 a^2}{10\,l^2}\Big)}}, \qquad \underline{\underline{M_C}} = A\,\frac{l}{2} = \underline{\underline{\frac{mgl}{4}\Big(1 + \frac{164\,\pi^2 a^2}{10\,l^2}\Big)}}.$$

Remark: When the results are evaluated for the data $m = 500\,\mathrm{kg}$, $2a = 10\,\mathrm{m}$, $l = 50\,\mathrm{m}$, we obtain $v_0 = 7{,}97\,\mathrm{km/h}$, $v_C = 51{,}05\,\mathrm{km/h}$, $\rho_C = 12{,}66\,\mathrm{m}$, $F = 12{,}84\,\mathrm{kN}$, $A = B = 6{,}42\,\mathrm{kN}$, $M_C = 160{,}55\,\mathrm{kNm}$.

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