Chapter 1
Mechanics

Abstract From Kepler's laws to hydrodynamics via Lagrange and Hamilton functions this first chapter covers classical mechanics, i.e. without relativistic or quantum effects.

Theoretical physics is the first science to be expressed mathematically: the results of experiments should be predicted or interpreted by mathematical formulae. Mathematical logic, theoretical chemistry and theoretical biology arrived much later. Physics had been understood mathematically in Greece more than 2000 years earlier, for example the law of buoyancy announced by Archimedes—lacking *Twitter*—with *Eureka*! Theoretical Physics first really came into flower, however, with Kepler's laws and their explanation by Newton's laws of gravitation and motion. We also shall start from that point.

1.1 Point Mechanics

1.1.1 Basic Concepts of Mechanics and Kinematics

A point mass is a mass whose spatial dimension is negligibly small in comparison with the distances involved in the problem under consideration. Kepler's laws, for example, describe the earth as a point mass "circling" the sun. We know, of course, that the earth is not really a point, and geographers cannot treat it in their field of work as a point. Theoretical physicists, however, find this notion very convenient for describing approximately the motion of the planets: theoretical physics is the science of successful approximations. Biologists often have difficulties in accepting similarly drastic approximations in their field.

The motion of a point mass is described by a position vector $\boldsymbol{r}$ as a function of time t, where $\boldsymbol{r}$ consists of the three components (x, y, z) of a rectangular coordinate

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system. (A boldface variable represents a vector. The same variable not in boldface represents the absolute magnitude of the vector, thus for example $r = |\boldsymbol{r}|$). Its velocity $\boldsymbol{v}$ is the time derivative

$$\boldsymbol{v}\,(t) = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = (\dot{x},\ \dot{y},\ \dot{z})\,, \tag{1.1}$$

where a dot over a variable indicates the derivative with respect to time t. The acceleration $\boldsymbol{a}$ is

$$\boldsymbol{a}\,(t) = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = \left(\dot{v}_{\mathrm{x}},\ \dot{v}_{\mathrm{y}},\ \dot{v}_{\mathrm{z}}\right), \tag{1.2}$$

the second derivative of the position vector with respect to time.

Galileo Galilei (1564–1642) discovered by experimentally dropping objects, presumably not from the Leaning Tower of Pisa, that all objects fall to the ground equally "fast", with the constant acceleration

$$\boldsymbol{a} = \boldsymbol{g} \quad \text{and } g = 9.81\,m/s^2. \tag{1.3}$$

Nowadays this law can be conveniently "demonstrated" in the university lecture room by allowing a piece of chalk and a scrap of paper to drop simultaneously: both reach the floor at the same time ... don't they?

It will be observed that theoretical physics is often concerned with asymptotic limiting cases: (1.3) is valid only in the limiting case of vanishing friction, never fully achieved experimentally, just as good chemistry can be carried out only with "chemically pure" materials. Nature is so complex that natural scientists prefer to observe unnatural limiting cases, which are easier to understand. A realistic description of Nature must strive to combine the laws so obtained, in such a way that they describe the reality, and not the limiting cases.

The differential equation (1.3), $\mathrm{d}^2\boldsymbol{r}/\mathrm{d}t^2 = (0, 0, -g)$ has for its solution the well known parabolic trajectory

$$\boldsymbol{r}\,(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + (0, 0, -g)t^2/2,$$

where the z-axis is taken as usual to be the upward vertical. Here $\boldsymbol{r}_0$ and $\boldsymbol{v}_0$ are the position and the velocity initially (at $t = 0$); the number 1.3 is an equation number, denoted as 1.3 or eq. (1.3) etc. in other publications. It is more complicated to explain the motion of the planets around the sun; in 1609 and 1619 Johann Kepler accounted for the observations known at that time with the three Kepler laws:

1. Each planet moves on an ellipse with the sun at a focal point.
2. The radius vector $\boldsymbol{r}$ (from the sun to the planet) sweeps out equal areas in equal times.
3. The ratio (orbital period)^2/(major semi-axis)^3 has the same value for all planets in our solar system.

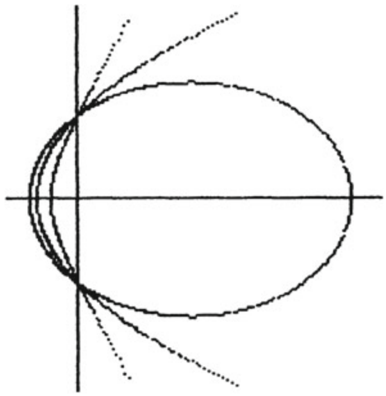


Fig. 1.1 Examples of an ellipse, an hyperbola, and a parabola as limiting case ($\varepsilon = 1/2$, 2 and 1, respectively)

Ellipses are finite conic sections and hence differ from hyperbolae; the limiting case between ellipses and hyperbolae is the parabola. In polar coordinates (distance r, angle ϕ) we have

$$r = p/(1 + \varepsilon \cos \phi),$$

where $\varepsilon < 1$ is the eccentricity of the ellipse and the planetary orbit. (Circle $\varepsilon = 0$; parabola $\varepsilon = 1$; hyperbola $\varepsilon > 1$; see Fig. 1.1). Hyperbolic orbits are exhibited by comets; mathematically, Halley's Comet is not a comet *in this sense*, but a very eccentric planet.

It is remarkable, especially for modern science politicians, that from these laws of Kepler for the motion of remote planets, theoretical physics and Newton's law of motion resulted. Modern mechanics was derived, not from practical, "down to earth" research, but from a desire to understand the motion of the planets in order to produce better horoscopes. Kepler also occupied himself with snowflakes (see Chap. 5). That many of his contemporaries ignored Kepler's work, and that he did not always get his salary, places many of us today on a par with him, at least in this respect.

1.1.2 *Newton's Law of Motion*

Regardless of fundamental debates on how one defines "force" and "mass", we designate a reference system as an inertial system if a force-free body moves in a straight line with a steady velocity. We write the law of motion discovered by Isaac Newton (1642–1727) thus:

$$\boldsymbol{f} = m\boldsymbol{a}$$
$$\text{force} = \text{mass} \times \text{acceleration}. \tag{1.4}$$

For free fall we state Galileo's law (1.3) as

$$\text{weight} = mg. \tag{1.5}$$

Forces are added as vectors ("parallelogram of forces"), for two bodies we have action = −reaction, and masses are added arithmetically. So long as we do not need to take account of Einstein's theory of relativity, masses are independent of velocity.

The *momentum* $\boldsymbol{p}$ is defined by $\boldsymbol{p} = m\boldsymbol{v}$, so that (1.4) may also be written as

$$\boldsymbol{f} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}, \tag{1.6}$$

which remains valid even with relativity. The law action = −reaction then states for two mutually interacting point masses that

> The sum of the momenta of the two masses remains constant. (1.7)

It is crucial to these formulae that the force is proportional to the acceleration and not to the velocity. For thousands of years it was believed that there was a connection with the velocity, as is suggested by one's daily experience dominated by friction. For seventeenth century philosophers it was very difficult to accept that force-free bodies would continue to move with constant velocity; children of the space age have long been familiar with this idea.

It is not stipulated which of the many possible inertial systems is used: one can specify the origin of coordinates in one's office or in the government's Department of Education. Transformations from one inertial system to another ("Galileo transformations") are written mathematically as:

$$\boldsymbol{r}' = \mathcal{R}\boldsymbol{r} + \boldsymbol{v}_0 t + \boldsymbol{r}_0; \quad t' = t + t_0 \tag{1.8}$$

with arbitrary parameters $\boldsymbol{v}_0$, $\boldsymbol{r}_0$, t_0 (Fig. 1.2). Here $\mathcal{R}$ is a rotational matrix with three "degrees of freedom" (three angles of rotation); there are three degrees of freedom also in each of v_0 and r_0, and the tenth degree of freedom is t_0. Corresponding to these ten continuous variables in the general Galileo transformation we shall later find ten laws of conservation.

There are interesting effects if the system of reference is not an inertial system. For example we can consider a flat disk rotating (relative to the fixed stars) with an angular velocity $\omega = \omega(t)$ (Fig. 1.3). The radial forces then occurring are well known from rides on a carousel. Let the unit vector in the r direction be $\boldsymbol{e}_\mathrm{r} = \boldsymbol{r}/|r|$, and the unit vector perpendicular to it in the direction of rotation be $\boldsymbol{e}_\phi$, where ϕ is the angle with the x-axis: $x = r\cos\phi$, $y = r\sin\phi$. The time derivative of $\boldsymbol{e}_\mathrm{r}$ is ωe_ϕ, that of $\boldsymbol{e}_\phi$ is $-\omega\boldsymbol{e}_\mathrm{r}$, with the *angular velocity* $\omega = \mathrm{d}\phi/\mathrm{d}t$. The velocity is

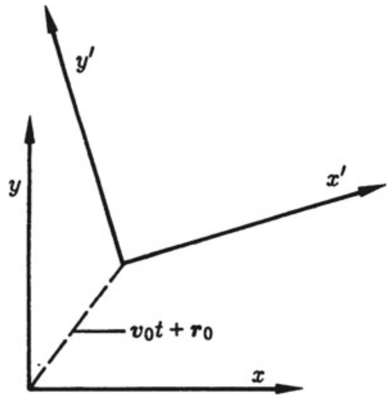


Fig. 1.2 Example of a transformation (1.8) in two-dimensional space

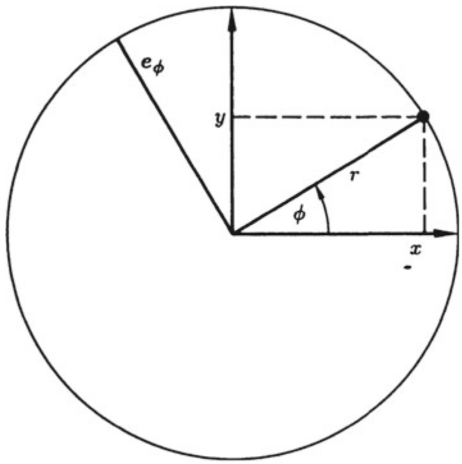


Fig. 1.3 Polar coordinates (r, ϕ) on a flat disk rotating with angular velocity ω, viewed from above

$$\boldsymbol{v} = \mathrm{d}\left(r\boldsymbol{e}_\mathrm{r}\right)/\mathrm{d}t = \boldsymbol{e}_\mathrm{r}\mathrm{d}r/\mathrm{d}t + r\omega\boldsymbol{e}_\phi$$

according to the rule for the differentiation of a product Similarly for the acceleration a and the force $\boldsymbol{f}$ we have

$$\frac{\boldsymbol{f}}{m} = \boldsymbol{a} = \dot{\boldsymbol{v}} = \left(\frac{\mathrm{d}^2r}{\mathrm{d}t^2} - \omega^2 r\right)\boldsymbol{e}_\mathrm{r} + (2\dot{r}\omega + r\dot{\omega})\boldsymbol{e}_\phi. \tag{1.9}$$

Of the four terms on the right hand side the third is especially interesting. The first is "normal", the second is "centrifugal", the last occurs only if the angular velocity varies. In the case when, as at the north pole on the rotating earth, the angular velocity is constant, the last term disappears. The penultimate term in (1.9) refers to the Coriolis force and implies that in the northern hemisphere of the earth swiftly moving objects are deflected to the right, as observed with various phenomena on the rotating earth: Foucault's pendulum (1851), the precipitous right bank of the Volga, the direction of spin on the weather map for European depressions, Caribbean hurricanes and Pacific typhoons. For example, in an area of low pressure in the North Atlantic the air flows inwards; if the origin of our polar coordinates is taken at the centre of the depression (and for the sake of simplicity this is taken at the north pole), $\mathrm{d}r/\mathrm{d}t$ is then negative, ω is constant, and the "deflection" of the wind observed from the rotating earth is always towards the right; at the south pole it is reversed. (If the observer is not at the north pole, ω has to be multiplied by $\sin\psi$, where ψ is the latitude: at the equator there is no Coriolis force.)

1.1.3 Simple Applications of Newton's Law

(a) Energy Law
Since $\boldsymbol{f} = m\boldsymbol{a}$ we have:

$$\boldsymbol{f}\,\mathrm{d}\boldsymbol{r}/\mathrm{d}t = m\,(\mathrm{d}\boldsymbol{r}/\mathrm{d}t)\left(\mathrm{d}^2\boldsymbol{r}/\mathrm{d}t^2\right) = \mathrm{d}\left(mv^2/2\right)/\mathrm{d}t = \mathrm{d}T/\mathrm{d}t,$$

where $T = mv^2/2$ is the *kinetic energy*. Accordingly the difference between the kinetic energy at position 1 (or time 1) and that at position 2 is given by

$$T\,(t_2) - T\,(t_1) = \int_1^2 \boldsymbol{f}\,\boldsymbol{v}\,\mathrm{d}t = \int_1^2 \boldsymbol{f}\,\mathrm{d}\boldsymbol{r},$$

which corresponds to the mechanical work done on the point mass ("work = force times displacement"). (The product of two vectors such as $\boldsymbol{f}$ and $\boldsymbol{v}$ is here the scalar product, viz. $f_x v_x + f_y v_y + f_z v_z$. The multiplication point is omitted. The cross product of two vectors such as $\boldsymbol{f} \times \boldsymbol{v}$ comes later.) The power $\mathrm{d}T/\mathrm{d}t$ ("power = work/time") is therefore equal to the product of force $\boldsymbol{f}$ and velocity $\boldsymbol{v}$, as one appreciates above all on the motorway, but also in the study.

A three-dimensional force field $\boldsymbol{f}(\boldsymbol{r})$ is called *conservative* if the above integral over $\boldsymbol{f}\,\mathrm{d}\boldsymbol{r}$ between two fixed endpoints 1 and 2 is independent of the path followed from 1 to 2. The gravity force $\boldsymbol{f} = m\boldsymbol{g}$, for example, is conservative:

$$\int \boldsymbol{f}\,\mathrm{d}\boldsymbol{r} = -mgh,$$

where the height h is independent of the path followed. Defining the *potential energy*

$$U(\boldsymbol{r}) = -\int \boldsymbol{f}\,\mathrm{d}\boldsymbol{r}$$

we then have:

The force $\boldsymbol{f}$ is conservative if and only if a potential U exists such that

$$f = -\text{grad}\, U = -\nabla U. \tag{1.10}$$

Here we usually have conservative forces to deal with and often neglect frictional forces, which are not conservative. If a point mass now moves from 1 to 2 in a conservative field of force, we have:

$$T_2 - T_1 = \int_1^2 \boldsymbol{f}\, \mathrm{d}\boldsymbol{r} = -\left(U_2 - U_1\right),$$

so that $T_1 + U_1 = T_2 + U_2$, i.e. $T + U = \text{const}$:

The energy $T + U$ is constant in a conservative field of force. (1.11)

Whoever can find an exception to this law of energy so central to our daily life can produce perpetual motion. We shall later introduce other forms of energy besides T and U, so that frictional losses ("heat") etc. can also be introduced into the energy law, allowing non-conservative forces also to be considered. Equation (1.11) shows mathematically that one can already predict important properties of the motion without having to calculate explicitly the entire course of the motion ("motion integrals").

(b) One-Dimensional Motion and the Pendulum

In one dimension all forces (depending on x only and thus ignoring friction) are automatically conservative, since there is only a unique path from one point to another point in a straight line. Accordingly $E = U(x) + mv^2/2$ is always constant, with $\mathrm{d}U/\mathrm{d}x = -f$ and arbitrary force $f(x)$. (Mathematicians should know that physicists pretend that all reasonable functions are always differentiable and integrable, and only now consider that known mathematical monsters such as "fractals" (see Chap. 5) also have physical meaning.) One can also see this directly:

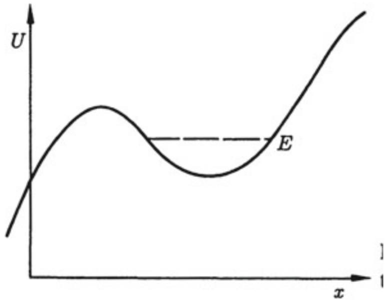
$$\mathrm{d}E/\mathrm{d}t = (\mathrm{d}U/\mathrm{d}x)(\mathrm{d}x/\mathrm{d}t) + mv\,\mathrm{d}v/\mathrm{d}t = -f\,v + mva = 0.$$

Moreover we have $\mathrm{d}t/\mathrm{d}x = 1/v = [(E-U)2/m]^{-1/2}$, and hence

$$t = t(x) = \int \frac{\mathrm{d}x}{\sqrt{(E - U(x))2/m}}. \tag{1.12}$$

Accordingly, to within an integration constant, the time is determined as a function of position x by a relatively simple integral. Many pocket calculators can already carry out integrations automatically at the push of a button. For harmonic oscillators,

Fig. 1.4 Periodic motion between the points a and b, when the energy E lies in the trough of the potential $U(x)$



such as the small amplitude pendulum, or the weight oscillating up and down on a spring, $U(x)$ is proportional to x^2, and this leads to sine and cosine oscillations for $x(t)$, provided that the reader knows the integral of $(1 - x^2)^{-1/2}$. In general, if the energy E results in a motion in a potential trough of the curve $U(x)$, there is a periodic motion (Fig. 1.4), which however need not always be $\sin(\omega t)$. In the anharmonic pendulum, for example, the restoring force is proportional to $\sin(x)$ (here x is the angle), and the integral (1.12) leads to elliptic functions, which we do not propose to pursue any further.

Notwithstanding the exact solution by (1.12), it is also useful to consider a computer program, with which one can solve $f = ma$ directly. Quite basically (I leave better methods to the numerical mathematicians) one divides up the time into individual time steps Δt. If we know the position x at that time we can calculate the force f and hence the acceleration $a = f/m$. The velocity v varies in the interval Δt by $a\Delta t$, the position x by $v\Delta t$. We thus construct the command sequence of the program PENDULUM, which is constantly to be repeated

> calculate $f(x)$
> replace v by $v + (f/m)\Delta t$
> replace x by $x + v\Delta t$
> return to calculation of f.

At the start we need an initial velocity v_0 and an initial position x_0. By suitable choice of the unit of time the mass can be set equal to unity. Programmable pocket calculators can be eminently suitable for executing this program. It is presented here in the computer language BASIC for $f = -\sin x$. It is clear that programming can be very easy; one should not be frightened by textbooks, where a page of programming may be devoted merely to the input of the initial data.

```
PROGRAM PENDULUM
10 x =0.0
20 v =1.0
30 dt=0.1
40 f=-sin(x)
50 v =v+f*dt
60 x =x+v*dt
70 print x,v
80 goto 40
90 end
```

In BASIC and FORTRAN

a = b + c (a := b + c); (in PASCAL)

signifies that the sum of b and c is to be stored at the place in store reserved for the variable a. The command

$$n = n + 1$$

is therefore not a sensational new mathematical discovery, but indicates that the variable n is to be increased by one from its previous value. By "goto" one commands the computer control to jump to the program line corresponding to the number indicated. In the above program the computer must be stopped by a command. In line 40 the appropriate force law is declared. It is of course still shorter if one simply replaces lines 40 and 50 by

$$40\, v = v - \sin(x) * \mathrm{d}t.$$

About computer programming: The programs in this book are supposed to be *understood*, not to be merely used as black boxes. The language BASIC is used for them since that name suggests that the language is simple. FORTRAN is quite similar while C^{++} is different. Translating a BASIC program into your preferred programming language will help understanding it. For much longer programs than those in this book, structured programming with subroutines is recommended. Graphic commands have to be adjusted to the computer you use except if you print out all the numbers and then plot them by hand.

(c) Angular Momentum and Torque

The cross product $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ of position and momentum is the *angular momentum*, and $\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{f}$ is the *torque*. Pedantic scientists might maintain that the cross product is not really a vector but an antisymmetric 3×3 matrix. We three-dimensional physicists can quite happily live with the pretence of handling $\boldsymbol{L}$ and $\boldsymbol{M}$ as vectors. As the analogue of $\boldsymbol{f} = \mathrm{d}\boldsymbol{p}/\mathrm{d}t$ we have

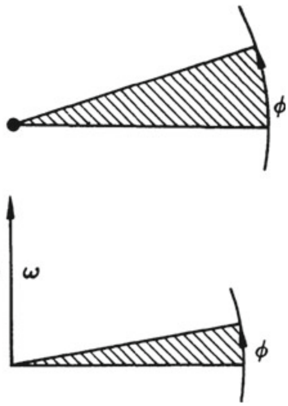


Fig. 1.5 The triangular area swept out by the radius vector $\boldsymbol{r}$ per unit time is a half of the cross-product $\boldsymbol{r} \times \boldsymbol{v}$. The upper picture is as seen, looking along the axis. The lower picture shows in three dimensions the angle ϕ and the vectors $\boldsymbol{L}$ and $\boldsymbol{\omega}$

$$\boldsymbol{M} = \frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t}, \tag{1.13}$$

which can also be written as

$$\boldsymbol{M} = \boldsymbol{r} \times \dot{\boldsymbol{p}} = \mathrm{d}(\boldsymbol{r} \times \boldsymbol{p})/\mathrm{d}t - \dot{\boldsymbol{r}} \times \boldsymbol{p} = \dot{\boldsymbol{L}},$$

and since the vector $\mathrm{d}\boldsymbol{r}/\mathrm{d}t$ is parallel to the vector $\boldsymbol{p}$, the cross product of the two vectors vanishes. Geometrically $\boldsymbol{L}/m = \boldsymbol{r} \times \boldsymbol{v}$ is twice the rate at which area is swept out by the radius vector $\boldsymbol{r}$ (Fig. 1.5); the second law of Kepler therefore states that the sun exerts no torque on the earth and therefore the angular momentum and the rate at which area is swept out remain constant.

(d) Central Forces

Central Forces are those forces $\boldsymbol{F}$ which act in the direction of the radius vector $\boldsymbol{r}$, thus $\boldsymbol{F}(\boldsymbol{r}) = f(\boldsymbol{r})\boldsymbol{e}_\mathrm{r}$ with an arbitrary scalar function f of the vector $\boldsymbol{r}$. Then the torque $\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} = (\boldsymbol{r} \times \boldsymbol{r})f(\boldsymbol{r})/|\boldsymbol{r}| = 0$:

> Central forces exert no torque and leave the angular momentum unchanged. (1.14)

For all central forces the motion of the point mass lies in a plane normal to the constant angular momentum $\boldsymbol{L}$:

$$\boldsymbol{r}\boldsymbol{L} = \boldsymbol{r}(\boldsymbol{r} \times \boldsymbol{p}) = \boldsymbol{p}(\boldsymbol{r} \times \boldsymbol{r}) = 0$$

using the triple product rule

$$\boldsymbol{a}(\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{c}(\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b}(\boldsymbol{c} \times \boldsymbol{a}).$$

The calculation of the angular momentum in polar coordinates shows that for this motion ωr^2 remains constant: the nearer the point mass is to the centre of force, the faster it orbits round it *Question*: Does this mean that winter is always longer than summer?

(e) Isotropic Central Forces

Most central forces with which theoretical physicists have to deal are isotropic central forces. These are central forces in which the function $f(r)$ depends only on the magnitude $|\boldsymbol{r}| = r$ and not on the direction: $\boldsymbol{F} = f(r)\boldsymbol{e}_{\mathrm{r}}$. With

$$U(r) = -\int f(r)\mathrm{d}r$$

we then have $\boldsymbol{F} = -\mathrm{grad}U$ and $f = -\mathrm{d}U/\mathrm{d}r$: the potential energy U also depends only on the distance r. Important examples are:

$U \sim 1/r$, so $f \sim 1/r^2$:	gravitation, Coulomb's law;
$U \sim \exp(-r/\xi)/r$:	Yukawa potential; screened Coulomb potential;
$U = \infty$ for $r < a$, $U = 0$ for $r > a$:	hard spheres (billiard balls);
$U = \infty,\ -U_0$ and 0 for $r < a,\ a < r < b$ and $r > b$:	spheres with potential well;
$U \sim (a/r)^{12} - (a/r)^6$:	Lennard-Jones or "6–12" potential;
$U \sim r^2$:	harmonic oscillator.

(Here $\sim$ is the symbol for proportionality, also denoted by the symbol $\propto$.)

For the computer simulation of real gases such as argon the Lennard-Jones potential is the most important: one places 10^6 such point masses in a computer and moves each according to force = mass $\times$ acceleration, where the force is the sum of the Lennard-Jones forces from the neighbouring particles. This method is called "molecular dynamics" and uses a lot of computer time.[1]

Since there is always a potential energy U, isotropic central forces are always conservative. If one constructs any apparatus in which only gravity and electrical forces occur, then the energy $E = U + T$ is necessarily constant. In a manner similar to the one-dimensional case the equation of motion can here be solved exactly, by resolving the velocity v into a component $\mathrm{d}r/\mathrm{d}t$ in the r-direction and a component $r\mathrm{d}\phi/\mathrm{d}t = r\omega$ perpendicular thereto and applying $L = m\omega r^2$:

$$\begin{aligned} E &= U + T = U + \frac{1}{2}(mv^2) \\ &= U + \frac{1}{2}m\left[(\mathrm{d}r/\mathrm{d}t)^2 + r^2\omega^2\right] = U + \frac{1}{2}m\left[(\mathrm{d}r/\mathrm{d}t)^2 + L^2/m^2r^2\right]. \end{aligned}$$

[1] W.G. Hoover, Computational Statistical Mechanics (Elsevier, Amsterdam 1991).

(In order to economise on parentheses, physicists often write a/bc for the fraction $a/(bc)$). Accordingly, with $U_{\text{eff}} = U + L^2/2mr^2$, we have:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{2(E - U_{\text{eff}}/m)}, \quad t = \int \frac{\mathrm{d}r}{\sqrt{2(E - U_{\text{eff}}/m)}}. \tag{1.15}$$

By defining the effective potential U_{eff} we can thus reduce the problem to the same form as in one dimension (1.12). However, we now want to calculate also the angle $\phi(t)$, using

$$L = mr^2\omega = mr^2 \frac{\mathrm{d}\phi}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} : \frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{L}{mr^2\sqrt{2(E - U_{\text{eff}})/m}}. \tag{1.16}$$

Integration of this yields $\phi(r)$ and everything is solved.

(f) Motion in a Gravitational Field

Two masses M and m separated by a distance r attract each other according to Newton's law of gravity

$$U = -GMm/r \text{ and } f = -GMm/r^2, \tag{1.17}$$

where G the gravitational constant is equal to 6.67×10^{-8} in cgs units. (The old familiar centimetre-gram-second units such as ergs and dynes are still in widespread use in theoretical physics; 1 dyne $= 10^{-5}$newton $= 1\,\mathrm{g\,cm/s^2}$; 1 erg $= 10^{-7}$joule or watt $-$ second $= 1\,\mathrm{g\,cm^2/s^2}$.) Unlike mutually repulsive electrical charges, mutually repulsive masses have so far not been discovered. For planets M is the mass of the sun and m is the mass of the planet.

Since

$$\int (1 - x^2)^{-1/2}\mathrm{d}x = -\arccos x$$

integration of (1.16) leads to the result

$$r = p/(1 + \varepsilon \cos \phi)$$

corresponding to Kepler's ellipse law of Sect. 1.1.1 with the parameter $p = L^2/GMm^2$ and the eccentricity $\varepsilon = (1 + 2Ep/GMm)^{1/2}$. For large energies $\varepsilon > 1$ and we obtain a hyperbola (comet) instead of an ellipse ($\varepsilon < 1$). Kepler's second law states, as mentioned above, the conservation of angular momentum, a necessary consequence of isotropic central forces such as gravitation. The third law, moreover, states that

$$\frac{(\text{period})^2}{(\text{major semi} - \text{axis})^3} = \frac{4\pi^2}{GM}. \tag{1.18}$$

(The derivation can be made specially simple by using circles instead of ellipses and then setting the radial force $m\omega^2 r$ equal to the gravitational force GMm/r^2 : period $= 2\pi/\omega$.)

The computer simulation also makes it possible to allow hypothetical deviations from the law of gravitation, e.g., $U \sim 1/r^2$ instead of $1/r$. The computer simulation shows that there are then no closed orbits at all. The BASIC program PLANET illustrates only the correct law, and with the inputs 0.5, 0, 0. 01 leads to a nice ellipse, especially if one augments the program with the graphic routine appropriate for the computer in use. In contrast to our first program, we are here dealing with two dimensions, using x and y for the position and v_x and v_y for the velocity; the force also must be resolved into x- and y-components: $f_x = xf/r$, $f_y = yf/r$. ("Input" indicates that one should key in the numbers for the start of the calculation, and "sqr" is the square root.) For an artificial law of gravitation with $U \sim 1/r^2$ one has only to replace the root "sqr(r2)" in line 50 by its argument "r2"; the graphics will then show that nothing works so well any more.

PROGRAM PLANET

```
10 input "vx,vy,dt ="; vx,vy,dt
20 x =0.0
30 y =1.0
40 r2=x*x+y*y
50 r3=dt/(r2*sqr(r2))
60 vx=vx-x*r3
70 vy=vy-y*r3
80 x =x+dt*vx
90 y =y+dt*vy
100 print x,y
110 goto 40
120 end
```

1.1.4 Harmonic Oscillator in One Dimension

The harmonic oscillator appears as a continuous thread through theoretical physics and is defined in mechanics by

$$T = mv^2/2, \; U = Kx^2/2, \; E = T + U = p^2/2m + Kx^2/2. \qquad (1.19)$$

For example, a weight hanging on a spring moves in this way, provided that the displacement x is not too great, so that the restoring force is proportional to the displacement.

(a) Without Friction
The calculation of the integral (1.12) with $\omega^2 = K/m$ gives the solution

$$x = x_0 \cos(\omega t + \text{const.}),$$

which one can however obtain directly: it follows from (1.4) that

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + Kx = 0, \tag{1.20}$$

and the sine or the cosine is the solution of this differential equation. The potential energy oscillates in proportion to the square of the cosine, the kinetic energy in proportion to the square of the sine; since $\cos^2\psi + \sin^2\psi = 1$ the total energy $E = U + T$ is constant, as it must be.

In electrodynamics we shall come across light waves, where the electric and magnetic field energies oscillate. In quantum mechanics we shall solve (1.19) by the Schrodinger equation and show that the position x and the momentum p cannot both be exactly equal to zero ("Heisenberg's Uncertainty Principle"). In statistical physics we shall calculate the contribution of vibrations to the specific heat, for application perhaps in solid state physics ("Debye Theory"). Harmonic oscillations are also well known in technology, for example as oscillating electrical currents in a coil ($\approx$ kinetic energy) and a condenser ($\approx$ potential energy), with friction corresponding to the electrical resistance ("Ohm's Law").

(b) With Friction
In theoretical physics (not necessarily in reality) frictional forces are usually proportional to the velocity. We therefore assume a frictional force $-R\mathrm{d}x/\mathrm{d}t$,

$$m\mathrm{d}^2x/\mathrm{d}t^2 + R\mathrm{d}x/\mathrm{d}t + Kx = 0.$$

This differential equation (of the second order) is linear, i.e. it involves no powers of x, and has constant coefficients, i.e. m, R and K are independent of t. Such differential equations can generally be solved by complex exponential functions $\exp(\mathrm{i}\phi) = \cos\phi + \mathrm{i}\sin\phi$, of which one eventually takes the real part. In this sense we try the solution

$$x = a\,\mathrm{e}^{\mathrm{i}\omega t} \to \mathrm{d}x/\mathrm{d}t = \mathrm{i}\omega x \to \mathrm{d}^2x/\mathrm{d}t^2 = -\omega^2 x$$

and try to find the complex numbers a and ω. For the case without friction (1.20) is quite simple:

$$-m\omega^2 x + Kx = 0, \text{ or } \omega^2 = K/m.$$

With friction we now obtain

$$-m\omega^2 x + \mathrm{i}\omega Rx + Kx = 0.$$

This quadratic equation has the solution

$$\omega = \mathrm{i}R/2m \pm \sqrt{K/m - R^2/4m^2}.$$

If we resolve ω into its real part Ω and its imaginary part $1/r$, $\omega = \Omega + \mathrm{i}/r$, we obtain

$$x = a\mathrm{e}^{\mathrm{i}\Omega t}\mathrm{e}^{-t/r}.$$

Quite generally, with a complex frequency $\omega = \Omega + i/\tau$ the real part Ω corresponds to a cosine oscillation and the imaginary part to a damping with a decay time r. In the above expression, if we set $a = 1$ for simplicity, the real part is

$$x = \cos(\Omega t)\mathrm{e}^{-t/r}. \tag{1.21}$$

Similarly other linear differential equations of n-th order with constant coefficients can be reduced to a normal equation with powers up to ω^n. The imaginary and complex numbers, with $\mathrm{i}^2 = -1$, existing originally only in the imagination of mathematicians, have thus become a useful tool in practical physics.

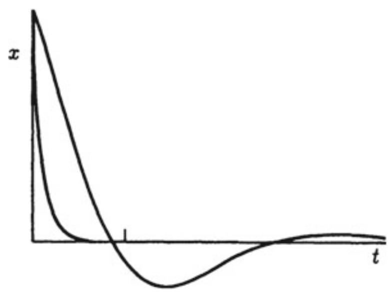
So what does the above result mean? If $4K/m > R^2/m^2$, then the square root is real and equal to Ω, and $1/\tau = R/2m$. Then (1.21) describes a damped oscillation. If on the other hand $4K/m < R^2/m^2$, then the square root is purely imaginary, ω no longer has a real part Ω, and we have an overdamped, purely exponentially decaying motion. The "aperiodic limiting case" $4Km = R^2$ involves a further mathematical difficulty ("degeneracy") which we willingly leave to the shock absorption engineers. Figure 1.6 shows two examples.

(c) Resonance

We shall discuss resonance effects when a damped harmonic oscillator moves under the influence of a periodic external force. "As everyone knows", resonance presupposes that the oscillator and the force have about the same oscillation frequency.

We again use complex numbers in the calculation; the external force, which obeys a sine or cosine law, is accordingly expressed as a complex oscillation $f\exp(\mathrm{i}\omega t)$, and not as proportional to $\cos(\omega t)$. Then the inhomogeneous differential equation

Fig. 1.6 Displacement x as a function of time t for $K = 1, m = 1$ and $R = 1$ (oscillation) and $R = 4$ (damping). In the first case the decay period τ is marked



becomes:

$$m\mathrm{d}^2x/\mathrm{d}t^2 + R\mathrm{d}x/\mathrm{d}t + Kx = f\mathrm{e}^{\mathrm{i}\omega\mathrm{t}}.$$

The trial solution $x = a\exp(i\omega t)$ leads again to the algebraic equation

$$-m\omega^2 a + R\mathrm{i}\omega a + Ka = f;$$

the factor $\exp(\mathrm{i}\omega t)$ has dropped out, retrospectively justifying the trial solution. It is clearer if we put $\omega_0^2 = K/m$ and $1/\tau = R/m$, since ω_0 is the eigenfrequency of the oscillator without the external force, and τ is its decay time. The above equation can be solved quite simply:

$$a = (f/m)/\left(\omega_0^2 - \omega^2 + \mathrm{i}\omega/\tau\right).$$

This amplitude a is a complex number, $a = |a|\exp(-\mathrm{i}\psi)$, where the "phase" ψ represents the angle by which the oscillation x lags behind the force f. The modulus, given by $|a|^2 = (\mathrm{Re}\,a)^2 + (\mathrm{Im}\,a)^2$, is of greater interest:

$$|a| = \frac{(f/m)}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2/\tau^2}}. \tag{1.22}$$

This function $|a|$ of ω is an even function, i.e. its value is independent of the sign of ω, so we can now assume $\omega \geq 0$. If the friction is small, so that τ is large, then this function looks something like Fig. 1.7: a relatively narrow peak has its maximum in the neighbourhood of $\omega = \omega_0$, the width of this maximum being of the order $1/\tau$. (Experts will know that it is not the amplitude, but the energy loss through friction, that is maximal when $\omega = \omega_0$; for weak damping the difference is unimportant.) Similar phenomena often occur in physics: the eigenfrequency is given approximately by the position of the resonance maximum, the reciprocal of the decay time by the width.

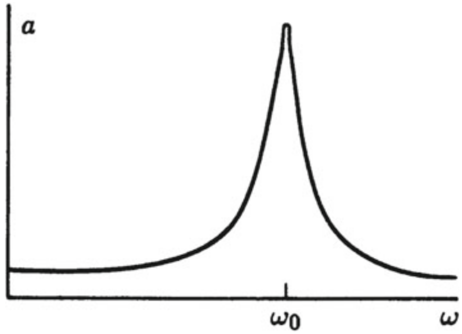


Fig. 1.7 Representation of the resonance function (1.22) for small damping. The maximum lies close to the eigenfrequency ω_0, the width is given by $1/\tau$

When $\omega = \omega_0$ then $|a| = (f/m)/(\omega_0/\tau) = f\tau/(Km)^{1/2}$. The smaller the damping, the longer is the decay time, and so the higher is the maximum near $\omega = \omega_0$. In the limiting case of infinitely small damping, $\tau = \infty$, there is an infinitely high and infinitely sharp maximum, and a resonance experiment is then impossible in practice: one would have to hit the correct frequency ω_0 exactly. It is therefore realistic to have only a very weak damping, and the effect of radio tuning is well known. One has to set the frequency approximately right in order to obtain a perceptible amplification. The smaller is the damping, the more exactly one has to hit the required frequency. One who finds radio programs too boring might well study the film of a particularly elegant road bridge (Tacoma Narrows, U.S.A.), which collapsed decades ago, as the wind produced oscillation frequencies which coincided with the eigenfrequencies of the torsional oscillations of the bridge. (A historian compared this instability with the international situation leading to the First World War; another compared Newton's law of gravity with the difficulty of keeping even far-away parts of an empire under control of the central government.)

1.2 Mechanics of Point Mass Systems

Up to this point we have considered a single point mass in a steady force field; in this section we pass on to several moving point masses, exerting forces upon each other. We shall find a complete solution for two such point masses; for more than two we restrict ourselves to general conservation laws.

1.2.1 The Ten Laws of Conservation

(a) Assumptions

Let there be N point masses with masses m_i, $i = 1, 2, ..., N$, which exert the forces $F_\mathrm{ik} = -F_\mathrm{ki}$ mutually between pairs. All these forces are isotropic central forces, i.e. mass k exerts on mass i the force

$$\boldsymbol{F}_\mathrm{ki} = f_\mathrm{ki}\,(r_\mathrm{ki})\,\boldsymbol{r}_\mathrm{ki}/\,|\boldsymbol{r}_\mathrm{ki}| = f_\mathrm{ki}\,(r_\mathrm{ki})\,\boldsymbol{e}_\mathrm{ki}$$

with $\boldsymbol{r}_\mathrm{ki} = \boldsymbol{r}_\mathrm{k} - \boldsymbol{r}_\mathrm{i}$. For convenience we define $f_\mathrm{ii} = 0$ and then have to solve the following equations of motion:

$$m_\mathrm{i}\mathrm{d}^2\boldsymbol{r}_\mathrm{i}/\mathrm{d}t^2 = \sum_\mathrm{k} f_\mathrm{ki}\boldsymbol{e}_\mathrm{ki}.$$

(b) Energy Law

Let the kinetic energy $T = \Sigma_\mathrm{i}T_\mathrm{i} = \Sigma_\mathrm{i}m_\mathrm{i}v_\mathrm{i}^2/2$ be the sum of all the particle energies T_i, the potential energy U the double sum $\Sigma_\mathrm{i}\,\Sigma_\mathrm{k}\,U_\mathrm{ik}/2$ of all the two-particle potentials

U_{ik} and let there be no explicit dependence on time. Then the energy conservation law is:

$$\text{The energy } E = U + T \text{ is constant in time.} \tag{1.23}$$

Proof

$$\begin{aligned} \mathrm{d}T/\mathrm{d}t &= \sum_{i} m_i \boldsymbol{v}_i \dot{\boldsymbol{v}}_i = \sum_{ik} f_{ki} \boldsymbol{e}_{ki} \boldsymbol{v}_i = \sum_{ik} \left(f_{ki} \boldsymbol{e}_{ki} \boldsymbol{v}_i + f_{ik} \boldsymbol{e}_{ik} \boldsymbol{v}_k \right)/2 \\ &= \sum_{ik} \boldsymbol{e}_{ki} \left(\boldsymbol{v}_i - \boldsymbol{v}_k \right) f_{ki}/2 = -\sum_{ik} \boldsymbol{e}_{ik} \dot{\boldsymbol{r}}_{ki} f_{ki}/2 \\ &= -\sum_{ik} \left(\partial \mathrm{U}_{ki} / \partial r_{ki} \right) \dot{r}_{ki}/2 = -\mathrm{d}U/\mathrm{d}t, \end{aligned}$$

where $f_{ki} = f_{ik}$ and $\boldsymbol{e}_{ki} = -\boldsymbol{e}_{ik}$ has been used. Energy conservation with its associated problems is therefore based here on the chain-rule of differentiation and the exchange of indices in the double sums. (The partial derivative $\partial f/\partial x$ of a function $f(x, y, z, ...)$ is the derivative at constant $y, z, ...$)

(c) Momentum Law

The total momentum P, hence the sum $\Sigma_i p_i$ of the individual momenta, is likewise constant:

$$\mathrm{d}\boldsymbol{P}/\mathrm{d}t = \sum_{i} \sum_{k} \boldsymbol{e}_{ki} f_{ki} = \sum_{ik} (\boldsymbol{e}_{ki} + \boldsymbol{e}_{ik}) f_{ki}/2 = 0,$$

$$\text{The momentum } \boldsymbol{P} \text{ is constant in time.} \tag{1.24}$$

(d) Law of Centre of Mass

$\boldsymbol{R} = \Sigma_i m_i \boldsymbol{r}_i / \Sigma_i m_i$ is the *centre of mass*, and $M = \Sigma_i m_i$ is the total mass. Since both $\boldsymbol{P}$ and M are constant, the velocity $\boldsymbol{V}$ of the centre of mass is also constant, because $\boldsymbol{P} = \Sigma_i m_i \boldsymbol{v}_i = M \mathrm{d}\boldsymbol{R}/\mathrm{d}t = M\boldsymbol{V}$. Hence

$$\boldsymbol{R} = \boldsymbol{R}_0 + \boldsymbol{V} t. \tag{1.25}$$

It is often appropriate to choose the “centre of mass system” as the system of reference, in which the centre of mass lies always at the origin: $\boldsymbol{V} = \boldsymbol{R}_0 = 0$ (Fig. 1.8).

Fig. 1.8 Divorce in outer space: the two point masses fly asunder, but their centre of mass remains fixed

(e) Angular Momentum Law
For the constancy of the total angular momentum $\boldsymbol{L} = \Sigma_{\rm i}\boldsymbol{L}_{\rm i}$ we use

$$\boldsymbol{r}_{\rm i} \times \boldsymbol{F}_{\rm ki} + \boldsymbol{r}_{\rm k} \times \boldsymbol{F}_{\rm ik} = \boldsymbol{r}_{\rm ik} \times \boldsymbol{F}_{\rm ik} = 0.$$

Hence one can show that

The angular momentum $\boldsymbol{L}$ is constant in time. (1.26)

Altogether we have here found ten conservation laws, since the constants $\boldsymbol{P}$, $\boldsymbol{V}$ and $\boldsymbol{L}$ each have three components; E has only one component. Later we shall explain how these ten conservation laws are associated with ten "invariants"; for example, the total angular momentum is constant since no external torque is present and since therefore the total potential is invariant (unchanged) in a rotation of the whole system through a specified angle.

1.2.2 The Two-Body Problem

Systems with two point masses have simple and exact solutions. Let there be two point masses with isotropic central forces. We have to reconcile 12 unknowns ($\boldsymbol{r}$ and $\boldsymbol{v}$ for each of the two particles), the ten conservation quantities given above and Newton's laws of motion for the two particles. The problem should therefore be solvable. We use the centre of mass reference system already recommended above.

In this system we have $\boldsymbol{r}_1 = -(m_2/m_1)\boldsymbol{r}_2$, so that $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ and $\boldsymbol{e}_{\rm r} = \boldsymbol{e}_{21}$ lie in the direction to $\boldsymbol{r}_1$ from $\boldsymbol{r}_2$. We therefore have:

$$\mathrm{d}^2\boldsymbol{r}/\mathrm{d}t^2 = \boldsymbol{e}_{21} f_{21}/m_1 - \boldsymbol{e}_{12} f_{21}/m_2 = \boldsymbol{e}_{\rm r} f_{21}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \boldsymbol{e}_{\rm r} f(r)/\mu.$$

Accordingly Newton's law of motion is valid for the difference vector $\boldsymbol{r}$, with an effective or reduced mass μ:

$$\mu\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = \boldsymbol{e}_{\rm r} f(r) \quad \text{with} \quad \mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{1.27}$$

The problem of the two point masses has therefore been successfully reduced to the already solved problem of a single point mass.

In the motion of the earth around the sun the latter does not, of course, stand still but also moves, despite Galileo, around the combined centre of mass of the sun-earth system, which however lies inside the surface of the sun. The earth, like the sun, rotates on an ellipse, whose focal point lies at the centre of mass. Kepler's second law also applies to this centre of mass, both for the earth and for the sun. In Kepler's third law, where different planets are compared, there is now introduced a correction factor $m/\mu = (M + m)/M$, which is close to unity if the planetary mass m is very much smaller than the solar mass M. This correction factor was predicted theoretically and confirmed by more exact observations: a fine, if also rare, example of successful collaboration between theory and experiment.

In reality this is of course still inaccurate, since many planets are simultaneously orbiting round the sun and all of them are exerting forces upon each other. This many-body problem can be simulated numerically on the computer for many millions of years; but eventually the errors can become very large because the initial positions and velocities are not known exactly (and also because of the limited accuracy of the computer and the algorithm). Physicists call it "chaos" (see Chap. 5) when small errors can increase exponentially and make the eventual behaviour of the system unpredictable.[2] If the exhausted reader therefore lets this book fall to the ground, that tiny tremor will later cause so great an effect in the planetary system (supposing that this system is chaotic) that the decay of the planets' accustomed orbits will thereby be affected (positively or negatively). This will, however, not take place before your next exams!

1.2.3 Constraining Forces and d'Alembert's Principle

In reality the earth is not an inertial system, even if we "ignore" the sun, because of the gravitational force with which the earth attracts all masses. Billiard balls on a smooth table nevertheless represent approximately free masses, since they are constrained to move on the horizontal table. The force exerted by the smooth table on the balls exactly balances the force of gravity. This is a special case of the general conditions of constraint now to be considered, in which the point masses are kept within certain restrictive conditions (in this case on the table).

(a) Restrictive Conditions

We shall deal only with holonomic-scleronomic restrictive conditions, which are given by a condition $f(x, y, z) = 0$. Thus the motion on a smooth table of height $z = h$ means that the condition $0 = f(x, y, z) = z - h$ is fulfilled, whereas $0 = f =$

[2]H.G. Schuster, Deterministic Chaos (Physik Verlag, Weinheim, second edition 1989); M. Schroeder, Fractals, Chaos, Power Laws (Freeman, New York 1991); J. Guckenheimer and P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields (Springer, Berlin, Heidelberg 1983).

$z \cdot \tan(\alpha) - x$ represents a sloping plane with inclination angle α. In general $f = 0$ indicates a surface, whereas the simultaneous fulfillment of two conditions $f_1 = 0$ and $f_2 = 0$ characterises a line (intersection of two surfaces).

The opposite of scleronomic (fixed) conditions are rheonomic (fluid) conditions of the type $f(x, y, z, t) = 0$. Non-holonomic conditions on the other hand can only be represented differentially: $0 = a \cdot dx + b \cdot dy + c \cdot dz + e \cdot dt$, and not by some function $f = 0$. Railways run on fixed tracks, whose course can be described by an appropriate function $f(x, y, z, t) = 0$: holonomic. Cars are, in contrast, non-holonomic: the motion $d\boldsymbol{r}$ follows the direction of the wheels, which one can steer. So in parking, for example, one can alter the y-coordinate at will, for a specified x-coordinate, by shuffling backwards and forwards in the x-direction (and more or less skillful steering). This shunting is not describable holonomically by $f(x, y) = 0$. The car is rheonomic, because one turns the steering-wheel, the railway is scleronomic.

(b) Constraining Forces

Those forces which hold a point mass on a prescribed path by the (holonomic-scleronomic) restrictive conditions are called *constraining forces* $\boldsymbol{Z}$. The billiard balls are held on the horizontal table by the constraining forces which the table exerts on them and which sustain the weight. The other forces, which are not constraining forces, are called *imposed forces* $\boldsymbol{F}$. We accordingly have: $m d^2\boldsymbol{r}/dt^2 = \boldsymbol{F} + \boldsymbol{Z}$, the constraining forces act perpendicularly to the surface (or curve) on which the point mass has to move, and only the imposed forces can cause accelerations along the path of the point masses.

Mathematically the gradient grad $f = \nabla f = (\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$ is perpendicular to the surface defined by $f(x, y, z) = 0$. The constraining force is therefore parallel to grad f

$$\begin{array}{ll} \boldsymbol{Z} = \lambda \nabla f & \text{(one condition)} \\ \boldsymbol{Z} = \lambda_1 \nabla f_1 + \lambda_2 \nabla f_2 & \text{(two conditions),} \end{array}$$

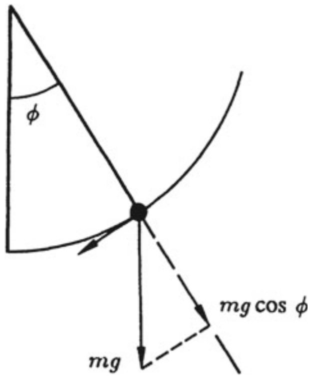
with $\lambda = \lambda(\boldsymbol{r}, t)$. We accordingly have the Lagrange equations of the first kind for one and two conditions, respectively:

$$m\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{F} + \lambda \nabla f, \quad m\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{F} + \lambda_1 \nabla f_1 + \lambda_2 \nabla f_2. \tag{1.28}$$

after Joseph Louis Comte de Lagrange (born in 1736 as Guiseppe Luigi Lagrangia in Turin, working also in Berlin.)

In practice one can solve this equation by resolving the imposed force $\boldsymbol{F}$ into one component $\boldsymbol{F}_t$ tangential and another component $\boldsymbol{F}_\mathrm{n}$ normal (perpendicular) to the surface or curve of the restrictive condition: $\boldsymbol{F} = \boldsymbol{F}_\mathrm{t} + \boldsymbol{F}_\mathrm{n}$, $\boldsymbol{Z} = \boldsymbol{Z}_\mathrm{n} + 0 = -\boldsymbol{F}_\mathrm{n}$. Something on an inclined plane is treated quite simply in this way, as you learn at school; we use it instead to treat the pendulum (Fig. 1.9):

Fig. 1.9 Constraining force and imposed gravity force in a pendulum, with resolution into normal and tangential components



A mass m hangs on a string of length l; the string is tied to the origin of coordinates. Mathematically this is signified by the restriction $0 = f(\boldsymbol{r}) = |\boldsymbol{r}| - l$; hence grad $f = \boldsymbol{e}_\mathrm{r}$ and $\boldsymbol{Z} = \lambda\boldsymbol{e}_\mathrm{r}$: the string force acts along the string. The resolution of the imposed gravity force $\boldsymbol{F} = m\boldsymbol{g}$ into tangential component $F_\mathrm{t} = -mg\sin\phi$ and normal component $F_\mathrm{n} = -mg\cos\phi$ (ϕ = angular displacement) gives $ml\mathrm{d}^2\phi/\mathrm{d}t^2 = ma_\mathrm{t} = F_\mathrm{t} = -mg\sin\phi$. The mass cancels out (since gravity mass = inertial mass), and there remains only the pendulum equation already treated in Sect. 1.1.3b. Monsieur Lagrange has therefore told us nothing new, but we have demonstrated with this familiar example that the formalism gives the correct result.

(c) Virtual Displacement and d'Alembert's Principle

We define a *virtual displacement* as an infinitely small displacement of the point mass such that the restrictive conditions are not violated. ("Infinitely small" in the sense of the differential equation: in $f'(x) = \mathrm{d}y/\mathrm{d}x$, $\mathrm{d}y$ is the variation in the function $y = f(x)$ caused by an infinitely small variation $\mathrm{d}x$.) With an inclined plane this virtual displacement is therefore a displacement along the plane, without leaving it.

A virtual displacement $\delta\boldsymbol{r}$ accordingly occurs along the surface or the curve representing the restrictive conditions and is therefore perpendicular to the constraining force $\boldsymbol{Z}$. Constraining forces therefore do no work: $\boldsymbol{Z}\delta\boldsymbol{r} = 0$, as is known from curriculum reform. Since $-\boldsymbol{Z} = \boldsymbol{F} - m\boldsymbol{a}$ we have:

$$\left(\boldsymbol{F} - m\mathrm{d}^2\boldsymbol{r}/\mathrm{d}t^2\right)\boldsymbol{r} = 0; \tag{1.29}$$

$$\text{in equilibrium:} \qquad \boldsymbol{F}\delta\boldsymbol{r} = 0; \tag{1.30}$$

$$\text{if } \boldsymbol{F} \text{ is conservative:} \quad \delta U = \nabla U\delta\boldsymbol{r} = 0. \tag{1.31}$$

One generalises this principle to a system of N point masses m_i ($i = 1, 2, ..., N$) with ρ restrictive conditions $f_\mu = 0$, ($\mu = 1, 2, ..., \rho$), so we have

$$\text{Lagrange 1st kind:} \quad m_\mathrm{i}\mathrm{d}^2\boldsymbol{r}_\mathrm{i}/\mathrm{d}t^2 = \boldsymbol{F}_\mathrm{i} + \sum_\mu \lambda_\mu\nabla_\mathrm{i}f_\mu(\boldsymbol{r}_1, ..., \boldsymbol{r}_\mathrm{N}); \tag{1.32}$$

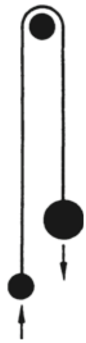


Fig. 1.10 Atwood's Machine or: How the theoretical physicist presents an experimental apparatus

$$\text{d'Alembert:} \quad \sum_i \left(\boldsymbol{F}_i - m_i \frac{d^2 \boldsymbol{r}_i}{dt^2} \right) \delta \boldsymbol{r}_i = 0; \tag{1.33}$$

$$\text{in equilibrium:} \quad \sum_i \boldsymbol{F}_i \delta \boldsymbol{r}_i = 0; \tag{1.34}$$

$$\text{if } F_i \text{ conservative:} \quad \delta U = 0, \tag{1.35}$$

where U is the total potential energy.

The last equation $\delta V = 0$ summarises in only four symbols all the equilibrium questions of point mechanics. A machine may be arbitrarily complicated, with struts between the different masses, and rails on which the masses must move: nevertheless with this machine in equilibrium it is still true that a quite small displacement of any part cannot change the total potential U: the principle of virtual work. So this part of theoretical physics is seen to be not only elegant, but also practical. The law of levers is a particularly simple application: if the left-hand arm of a balance has length a and the righthand one length b, then the changes in height with a small rotation are as $a : b$. The potential energies $m_a g z$ and $m_b g z$ do not change in sum if $m_a g a = m_b g b$ or $m_a a = m_b b$. As an example for d'Alembert we can take Atwood's machine in Fig. 1.10: two point masses hang from a string which passes over a frictionless pulley. With what acceleration does the heavier mass sink?

Since the length of the string is constant, we have $\delta z_1 = -\delta z_2$ for the virtual displacements in the z-direction (upwards). The imposed gravity forces in the z-direction are $F_1 = -m_1 g$ and $F_2 = -m_2 g$. Hence we have

$$\begin{aligned} 0 &= \sum_i \left(F_i - m_i d^2 z_i / dt^2 \right) \delta z_i \\ &= \left(-m_1 g - m_1 d^2 z_1 / dt^2 \right) \delta z_1 + \left(-m_2 g - m_2 d^2 z_2 / dt^2 \right) \delta z_2 \\ &= \delta z_1 \left(-m_1 g + m_1 a + m_2 g + m_2 a \right) \end{aligned}$$

for arbitrary δz_1. So the contents of the brackets must be zero:

$$a = -g(m_2 - m_1)/(m_2 + m_1),$$

which as a clearly sensible result confirms the d'Alembert formalism.

In the next section we present this formalism in more detail; even this last section could be counted as analytical mechanics.

1.3 Analytical Mechanics

In this section we present the discussion, already begun, in more general formal methods. Later in quantum mechanics we shall become acquainted with their practical uses, e.g., the Hamilton function of position and momentum.

1.3.1 The Lagrange Function

(a) Generalised Coordinates and Velocities

Now we renumber the coordinates of all the N particles thus: instead of x_1, y_1, z_1, x_2, y_2, z_2, ..., x_N, y_N, z_N we write x_1, x_2, x_3, x_4, x_5, x_6, ..., x_{3N-1}, and x_{3N}. Now d'Alembert's principle from (1.33) has the form

$$\sum_\mathrm{i} \left(F_\mathrm{i} - m_\mathrm{i}\mathrm{d}^2 x_\mathrm{i}/\mathrm{d}t^2\right)\delta x_\mathrm{i} = 0.$$

These coordinates x_i, however, are not very convenient if constraints limit the motions. Then we should rather use generalised coordinates $q_1, q_2, ..., g_\mathrm{f}$, if there are $3N - f$ restrictive conditions and hence f "degrees of freedom". These generalised coordinates should automatically fulfill the restrictive conditions, so that on inserting any numerical values for the q_μ there is no violation of the restrictive conditions, while on the other hand the declaration of all the q_μ completely specifies the system. If, for example, a motion follows a plane circular orbit with radius R, then instead of the traditional coordinates x_1 and x_2 with the condition $x_1^2 + x_2^2 = R^2$ it is much simpler to write the angle ϕ as the single generalised coordinate q. These generalised coordinates therefore do not necessarily have the dimension of a length; we usually restrict ourselves in practice to lengths and angles for the q_μ.

(b) Lagrange Equation of the Second Kind

The d'Alembert's principle mentioned above can now be rewritten in the new variables q_μ. For economy of effort we give the result immediately:

$$\frac{\mathrm{d}\left[\partial L/\partial \dot{q}_\mu\right]}{\mathrm{d}t} = \frac{\partial L}{\partial q_\mu}, \tag{1.36}$$

where the *Lagrange Function* $L = T - U$ is the difference between the kinetic and the potential energies, written as a function of the q_μ and their time derivatives. It is easy to decide where the dot for timewise differentiation occurs in (1.36) from dimensional considerations; and if one does not believe the whole equation it is easy to demonstrate it using the example $L = \Sigma_i m_i v_i^2/2 - U(x_1, x_2, ..., x_{3N})$, in the absence of restrictive conditions (hence $q_\mu = x_i$, and $v_i = dq_\mu/dt$). Then we obtain from (1.36): $m_i dv_i/dt = -\partial U/\partial x_i$, as required by Newton. If there are restrictive conditions, then they are elegantly eliminated from the Lagrange equation of the second kind (1788) by concealing them in the definition of the generalised coordinates $q\mu$.

Accordingly one proceeds in general as follows:

- choice of coordinates q_μ corresponding to the restrictive conditions;
- calculation of dx_i/dt as a function of q_μ and dq_μ/dt;
- substitution of the results in the kinetic energy T;
- calculation of the potential energy U as a function of the q_μ;
- derivation of $L = T - U$ with respect to q_μ and dq_μ/dt, substitution in (1.36).

We have therefore found a general method of calculating systems with arbitrarily complex restrictive conditions. In practice it often looks simpler than these general rules: for the pendulum of length l the coordinate q is the angle ϕ, the kinetic energy is $mv^2/2 = ml^2(d\phi/dt)^2/2$ and the potential energy is $-mgl\cos\phi$, if $\phi = 0$ is the rest position.

We accordingly have

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + mgl\cos\phi,$$

so that (1.36) gives the usual pendulum equation $ml^2 d^2\phi/dt^2 = -mgl\sin\phi$ from Sect. 1.1.3b. Lagrange turns out to be correct.

(c) the Hamilton Principle of Least Action

We have here an extremal principle similar to many others in physics: the actual motion of a system is such that the action W is extremal, i.e. it is either a maximum or a minimum, when one considers all the possible motions from a specified starting point "1" to a specified endpoint "2". Here action is defined by the integral

$$W = \int_{t_1}^{t_2} L\left(q_\mu, \dot{q}_\mu\right) dt$$

along the motion path $q_\mu = q_\mu(t), \dot{q}_\mu = \dot{q}_\mu(t)$. With some calculation, and application of (1.36) and of partial integration one can show that with fixed endpoints "1" and "2":

$$\delta W = 0. \tag{1.37}$$

This Hamilton principle (1834) accordingly states that the action does not change if one alters the actual motion of the system very slightly. Vanishing of small variations is a well known characteristic of a maximum or a minimum. To experts in variational analysis (1.36) is readily recognised as the indication of an extremal principle.

Similarly, light "moves" in such a way that another integral, namely the traveling time, is minimal: Fermat's principle. From this follows, for example, the principle of geometric optics.

1.3.2 The Hamilton Function

It seems strange that the Lagrange function L is the difference and not the sum of the kinetic and the potential energies. This is different in the Hamilton function $H = T + U$; so this does not differ from the total energy, only we write it as a function of the (generalised) coordinates and momenta: $L = L(x, v)$, but $H = H(x, p)$ for a particle with position x, velocity v and momentum $p = mv$ in one dimension. The partial derivative $\mathrm{d}/\mathrm{d}x$ accordingly leaves unchanged the velocity v in L, but the momentum p in H.

In case constraints are again present we define a generalised momentum

$$p_\mu = \partial L/\partial \dot{q}_\mu,$$

which in the absence of constraints coincides with the ordinary momentum $m\mathrm{d}q_\mu/\mathrm{d}t$. The Lagrange equation of the second kind now has the form $\mathrm{d}p_\mu/\mathrm{d}t = \partial L/\partial q_\mu$. Accordingly if a coordinate q_μ does not appear in the Lagrange function L of the system under consideration, so that L is invariant to changes in the variable q_μ, then the corresponding momentum p_μ is constant. For every invariance with respect to a continuous variable q_μ there accordingly is a conservation law. This was demonstrated more rigorously by Emmy Noether in 1918. Thus the constancy of the angular momentum follows from invariance with respect to a rotation of the total system, and the invariance of the total momentum from invariance with respect to a translation, as discussed in Sect. 1.2.1.

The total time derivative of the Lagrange function L is given by

$$\begin{aligned}\frac{\mathrm{d}L}{\mathrm{d}t} &= \sum_\mu \left(\frac{\partial L}{\partial q_\mu}\frac{\mathrm{d}q_\mu}{\mathrm{d}t} + \frac{\partial L}{\partial \dot{q}_\mu}\frac{\mathrm{d}\dot{q}_\mu}{\mathrm{d}t}\right)\\ &= \sum_\mu \left[\frac{\mathrm{d}(\partial L/\partial \dot{q}_\mu)}{\mathrm{d}t}\frac{\mathrm{d}q_\mu}{\mathrm{d}t} + \frac{\partial L}{\mathrm{d}\dot{q}_\mu}\frac{\partial \dot{q}_\mu}{\mathrm{d}t}\right]\\ &= \mathrm{d}\left(\sum_\mu \dot{q}_\mu \frac{\partial L}{\partial \dot{q}_\mu}\right)/\mathrm{d}t.\end{aligned}$$

Since the energy $E = -L + \Sigma_\mu \dot{q}_\mu \partial L/\partial \dot{q}_\mu$ we therefore have $\mathrm{d}E/\mathrm{d}t = 0$: the energy is constant. The fact that this E is actually the total energy $T + U$, shows that U is independent of the velocities, whereas T depends quadratically on the (generalised) velocities $\mathrm{d}q_\mu/\mathrm{d}t$, and hence

$$\sum_\mu \dot{q}_\mu p_\mu = \sum_\mu \dot{q}_\mu \frac{\partial T}{\partial \dot{q}_\mu} = 2T.$$

We can therefore summarise as follows:

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \quad \dot{p}_\mu = \frac{\partial L}{\partial q_\mu}, \quad T + U = E = H = \sum_\mu p_\mu \dot{q}_\mu - L, \tag{1.38}$$

and this energy E is constant:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0. \tag{1.39}$$

The energy is conserved here, because external forces and time dependent potentials were neglected.

Comparing now the differential $\mathrm{d}H = \Sigma_\mu(\partial H/\partial q_\mu)\mathrm{d}q_\mu + (\partial H/\mathrm{d}p_\mu)\mathrm{d}p_\mu$ with the analogous differential $\mathrm{d}L$, and taking account of (1.38), we find the *canonical equations*

$$\dot{p}_\mu = -\frac{\partial H}{\partial q_\mu}, \quad \dot{q}_\mu = \frac{\partial H}{\partial p_\mu}, \quad H = H(q_\mu, p_\mu). \tag{1.40}$$

It is evident from the one-dimensional example of the free particle, $H = p^2/2m$, that these equations lead to the correct results $\mathrm{d}p/\mathrm{d}t = 0$, $\mathrm{d}q/\mathrm{d}t = p/m$. One also finds from this example where the minus sign is needed in (1.40).

As already mentioned, the Hamilton function plays an important role in quantum mechanics. The so-called commutator of quantum mechanics resembles the Poisson bracket of classical physics, defined by

$$\{F, G\} = \sum_\mu \left(\frac{\partial F}{\partial q_\mu}\frac{\partial G}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu}\frac{\partial G}{\partial q_\mu} \right), \tag{1.41}$$

where F and G are any functions dependent on the positions q and the momenta p. Using the chain rule of differentiation it then follows that

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, H\} \tag{1.42}$$

just as the timewise variation of the quantum mechanical average value of a quantity F is given by the commutator $FH - HF$ (where F and H are "operators", i.e. a sort of matrices).

As example we take once again the one-dimensional harmonic oscillator: $T = mv^2/2, U = Kx^2/2$, with no restrictions, so that $q = x$, $p = mv$. Then the Hamilton function is

$$H\,(q,p) = p^2/2m + Kq^2/2.$$

From the canonical equations (1.42) it follows that $\mathrm{d}p/\mathrm{d}t = -Kq$ and $\mathrm{d}q/\mathrm{d}t = p/m$, which is correct. From (1.42), with $F = p$ in the Poisson bracket, it follows that

$$\mathrm{d}p/\mathrm{d}t = \{p, H\} = \frac{\partial p}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial p}{\partial p}\frac{\partial H}{\partial q} = -\partial H/\partial q = -Kq.$$

which is also a correct result. We have thus successfully transcribed the simple law that force = mass times acceleration into a more complicated form, but one which is also more elegant, and suitable for the reader interested in practical applications in quantum mechanics. The next section, however, presents a different application.

1.3.3 Harmonic Approximation for Small Oscillations

A very commonly used approximation in theoretical physics is the harmonic approximation, where one develops a complicated function as a Taylor series and then truncates the series after the quadratic term. Applied to the potential energy U of a particle this gives

$$U(x) = U_0 + x\mathrm{d}U/\mathrm{d}x + (x^2/2)\mathrm{d}^2U/\mathrm{d}x^2 + \ldots,$$

where U_0 and $\mathrm{d}U/\mathrm{d}x$ drop out if we take the origin of coordinates at the minimum of the energy $U(x)$. The Hamilton function is then $H = p^2/2m + Kx^2/2$ with $K = \mathrm{d}^2U/\mathrm{d}x^2 + \ldots$ (derivatives at the point $x = 0$), i.e. the well known function of the harmonic oscillator. In a solid body there are many atoms, which exert complicated forces upon each other. If we develop the total potential energy U about the equilibrium position of the atoms and truncate this Taylor series after the quadratic term, then this harmonic approximation leads to a large number of coupled harmonic oscillators. These are the lattice vibrations or phonons of the solid body. Before we mathematically decouple these 10^{24} oscillators, we must first learn with just two oscillators.

(a) Two Coupled Oscillators

Let two point masses of mass m be connected to one another by a spring, and connected to two rigid walls, each by a further spring (Fig. 1.11). The three springs all have the force constant K. Let the system be one-dimensional, the coordinates x_1 and x_2 giving the displacements of the two point masses from their rest positions. Then the Hamilton function, with $v_\mathrm{i} = \mathrm{d}x_\mathrm{i}/\mathrm{d}t$, is:

$$H = (m/2)\left[v_1^2 + v_2^2\right] + (K/2)\left[x_1^2 + x_2^2 + (x_1 - x_2)^2\right].$$

Fig. 1.11 Two coupled one-dimensional oscillators between two fixed walls. All three spring constants are equal

The kinetic energy is here a straight sum of two quadratic terms, but the potential energy on account of the coupling is proportional to $(x_1 - x_2)^2$. What is to be done about it?

Although there are no restrictive conditions here, we make use of the possibility discussed above of mathematically simplifying ("diagonalising") the problem by appropriate choice of coordinates q_μ. Thus, with $q_1 = x_1 + x_2$ and $q_2 = x_1 - x_2$, so that $x_1 = (q_1 + q_2)/2$ and $x_2 = (q_1 - q_2)/2$, we obtain

$$H = \frac{m}{4}\left[\dot{q}_1^2 + \dot{q}_2^2\right] + \frac{K}{4}\left[q_1^2 + 3q_2^2\right] = H_1^{\text{osc}} + H_2^{\text{osc}},$$

where H_1^{osc} depends only on q_1 and $\dot{q}_1$ and has the structure of the Hamilton function of a normal harmonic oscillator, similarly H_2. With the generalised momenta

$$p_\text{i} = \partial L/\partial \dot{q}_\text{i} = \partial H/\partial \dot{q}_\text{i} = m\dot{q}_\text{i}/2$$

and the canonical equations (1.40)

$$(m/2)\text{d}^2 q_\text{i}/\text{d}t^2 = \dot{p}_\text{i} = -\partial H/\partial q_\text{i}$$

we find the two equations of motion

$$m\text{d}^2 q_\text{i}/\text{d}t^2 = -K_\text{i} q_\text{i}$$

with $K_1 = K$ and $K_2 = 3K$. They are solved by $q_1 \sim \exp(\text{i}\omega t)$ and $q_2 \sim \exp(\text{i}\Omega t)$ with $\omega^2 = K/m$ and $\Omega^2 = 3K/m$. If $q_2 = 0$, so that $x_1 = x_2$, then the system oscillates with the angular velocity ω; if on the other hand $q_1 = 0$, so that $x_1 = -x_2$ then it oscillates with the larger $\Omega = \omega\sqrt{3}$. The masses therefore oscillate together with a lower frequency than if they swing against each other. In solid state physics one speaks of acoustic phonons when the vibrations are sympathetic, and of optical phonons when they are opposed. The general oscillation is a superposition (addition, or linear combination) of these two normal oscillations. The essential aspects of the harmonic vibrations in a solid body are therefore represented by this simple example; the next section does the same, only in a more complicated case.

(b) Normal Vibrations in the Solid State

We now calculate in a similar way the vibration frequencies of the atoms in a solid body which has atoms of mass m and only one type. Let the equilibrium position of the

i-th atom be r_i^0, and let $q_i = r_i - r_i^0$ be the displacement from equilibrium. We expand the potential U quadratically ("harmonic approximation") about the equilibrium position $q_i = 0$ and again number the coordinates i from 1 to $3N$:

$$U(q) = U(0) + \sum_{ik} \left(\partial^2 U/\partial q_i \partial q_k\right) q_i q_k/2,$$

since the first derivatives vanish at equilibrium (minimum of the potential energy U). With the "Hesse matrix"

$$K_{ik} = \partial^2 U/\partial q_i \partial q_k = K_{ki}$$

the Hamilton function has the form

$$H = U(0) + \sum_i \left(p_i^2/2m\right) + \sum_{ik} K_{ik} q_i q_k/2.$$

The canonical equation (1.40) then gives

$$-m\mathrm{d}^2 q_j/\mathrm{d}t^2 = -\dot{p}_j = \partial H/\partial q_j = \partial U/\partial q_j = \sum_k K_{jk} q_k,$$

which can of course also be derived directly from

$$\text{mass} \cdot \text{acceleration} = \text{force} = -\text{grad}\, U.$$

(In the differentiation of the double sum there are two contributions, one from $i = j$ and the other from $k = j$; since $K_{ik} = K_{ki}$ the two terms are equal, so the factor 1/2 disappears.) For this system of linear differential equations (constant coefficients) we try the usual exponential solution: $q_j \sim \exp(\mathrm{i}\omega t)$. This leads to

$$m\omega^2 q_j = \sum_k K_{jk} q_k. \tag{1.43}$$

Mathematicians recognise that on the right-hand side the $3N$-dimensional vector with the components q_k, $k = 1, 2, \ldots, 3N$ is multiplied by the Hesse matrix $\mathcal{K}$ of the K_{jk} and that the result (on the left-hand side) should be equal to this vector, to within a constant factor $m\omega^2$. Problems of this type

$$\text{factor} \cdot \text{vector} = \text{matrix} \cdot \text{vector}$$

are called eigenvalue equations (here the eigenvalue of the matrix is the factor, and the vector is the eigenvector). In general the equation system matrix $\cdot$ vector $= 0$ has a solution (with a vector which is not null) only if the determinant of the matrix is zero. If $\mathcal{E}$ is the unit matrix of Kronecker deltas, so that $\mathcal{E}_{jk} = \delta_{jk} = 1$ for $j = k$ and $= 0$ otherwise, then eigenvalue equations have the form

$$(\text{matrix} \; - \; \text{factor} \; \cdot \mathcal{E}) \cdot \; \text{vector} \; = 0,$$

which leads to

$$\text{determinant of (matrix} \; - \; \text{factor} \cdot \mathcal{E}) = 0,$$

as the condition for a solution. The determinant det of a two-dimensional matrix is

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc;$$

the reader will find larger matrices treated in books on linear algebra.

In the case of rigid body vibrations we therefore have to set to zero the determinant of a $3N$-dimensional matrix:

$$\det(K - m\omega^2 \mathcal{E}) = 0. \tag{1.44}$$

From linear algebra it is well known that the eigenvalues of a symmetric matrix ($K_{\text{jk}} = K_{\text{kj}}$) are real and not complex. If the potential energy in equilibrium is a minimum, which it must be for a stable equilibrium, then no eigenvalues $m\omega^2$ are negative, so that ω is also not imaginary. We therefore have true vibrations, and not disturbances decaying exponentially with time.

The so-called secular equation (1.44) is a polynomial of degree $3N$, which is really troublesome to calculate with $N = 10^{24}$ atoms. It is easier if one assumes that in equilibrium all atoms lie in positions on a periodic lattice. Then one makes the trial solution of a plane wave:

$$q_{\text{j}} \sim \exp(\mathrm{i}\omega t - \mathrm{i}\,\boldsymbol{Q}\boldsymbol{r}_{\text{j}}^0), \tag{1.45}$$

where $\boldsymbol{q}_{\text{j}}$ is now a three-dimensional vector, $j = 1, 2, ..., N$, and $\boldsymbol{Q}$ indicates a wave vector. With this simplification the eigenvalue problem is reduced to that of a three-dimensional "polarisation vector" $\boldsymbol{q}$ with the associated eigenvalue $m\omega^2$, both of which depend on the wave vector $\boldsymbol{Q}$. (See textbooks on solid state physics.) To determine the eigenvalues of a 3×3 matrix leads to an equation of the third degree; in two dimensions one has a quadratic equation to solve. Typical solutions for the frequency ω as a function of the wave vector $\boldsymbol{Q}$ in three dimensions have the form of Fig. 1.12, where A stands for "acoustic" (sympathetic vibrations), O for "optical" (opposed vibrations), and L for longitudinal (displacement $\boldsymbol{q}$ in the direction of the wave vector $\boldsymbol{Q}$) and T for transverse. With only one sort of atom there are only three acoustic branches (left), with two different sorts of atoms there are also three optical branches (right). In quantum mechanics these vibrations are called phonons.

(c) Linear Chains

We now wish to calculate explicitly the frequency spectrum $\omega(Q)$ in one dimension, i.e. in an infinitely long chain of identical point masses m. Between the points j and

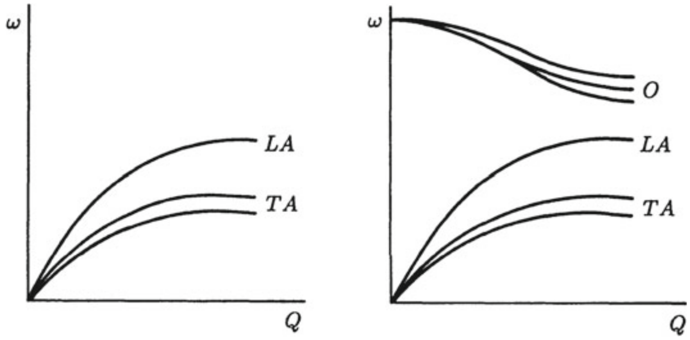


Fig. 1.12 Typical phonon spectra in three-dimensional crystals

$j+1$ there is a spring with the force constant K; if neighbouring point masses are separated by the distance a the spring force is zero and the atoms are in equilibrium: $x_j^0 = aj$ for $-\infty < j < +\infty$.

The Hamilton function or total energy is then

$$H = \sum_j (p_j^2/2m) + \frac{K}{2}\sum_j (q_{j+1} - q_j)^2 = \sum_j (p_j^2/2m) + \sum_{jk} K_{jk} q_j q_k r/2$$

with $q_j = x_j - x_j^0$ and the matrix elements $K_{jk} = 0, -K, 2K, -K$ and 0 for $k < j-1, k = j-1, k = j, k = j+1$, and $k > j+1$, respectively. The trial solution of a plane wave (1.45) with wave vector $\boldsymbol{Q}$, $q_j \sim \exp(\mathrm{i}\omega t - \mathrm{i}Qaj)$, using (1.43), gives

$$\begin{aligned} m\omega^2 \exp(\mathrm{i}\omega t - \mathrm{i}Qaj) &= \sum_k K_{jk} \exp(\mathrm{i}\omega t - \mathrm{i}Qak) \text{ or} \\ m\omega^2 &= \sum_k K_{jk} \exp(\mathrm{i}Qa(j-k)) \\ &= -K\exp(\mathrm{i}Qa) + 2K - K\exp(-\mathrm{i}Qa) \\ &= -K(\exp(\mathrm{i}Qa/2) - \exp(-\mathrm{i}Qa/2))^2 = 4K\sin^2(Qa/2) \end{aligned}$$

so that

$$\omega = \pm 2(K/m)^{1/2}\sin(Qa/2). \tag{1.46}$$

To be meaningful the wave vector Q is limited to the region $0 \le |Q| \le \pi/a$, because in a periodic chain the wave vectors Q and $Q + 2\pi/a$, for example, are completely equivalent (between the atoms there is nothing that could move). In this so-called Brillouin zone between $Qa = 0$ and $Qa = \pi$ the sine in (1.46) rises from 0 to 1, just as it does schematically for the longitudinal acoustic phonon branch in Fig. 1.12.

1.4 Mechanics of Rigid Bodies

The theme of this section is the motion of solid bodies as entities. With an iron plate we no longer consider this plate as a point mass, as in Sects. 1.1 and 1.2, nor as a system of 10^{25} or more inter-vibrating atoms, as on the previous pages, but we ask, for example, what forces act on the plate if it is attached to a motor and then rotated. Why do gyroscopes behave in the way they do? In general, then, we consider rigid bodies, in which the distances and the angles between different atoms are *fixed* (more precisely: in which the changes in distances and angles are negligible).

1.4.1 Kinematics and Inertia Tensor

(a) Rotations

If a rigid body rotates about an axis with the angular velocity $\omega = \partial\phi/\partial t$, then the vector $\boldsymbol{\omega}$ lies in the direction of the axis (Fig. 1.5). The body rotates in the clockwise direction when regarded in the direction of $+\boldsymbol{\omega}$: the rule of the thumb of the right hand. The fact that here right has precedence over left is due, not to politics, but to the cheating of physicists: they regard certain asymmetric 3×3 matrices as axial vectors, although they are not true vectors. These pseudo-vectors correspond to cross-products, magnetic fields, and vectors, such as $\boldsymbol{\omega}$ itself, defined by the direction of rotation. With the definition of tensors later on, and in Sect. 2.3.2 (Relativistic Electrodynamics), we shall see more of these imposters.

The velocity $\boldsymbol{v}$ of a point on the rotating rigid body at a position $\boldsymbol{r}$ relative to the origin is the cross-product

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}, \tag{1.47}$$

assuming (as we shall always assume in future) that the origin of coordinates lies on the axis of rotation. Not only $\boldsymbol{v}$ but also $\boldsymbol{r}$ are genuine polar vectors, $\boldsymbol{\omega}$ and the cross-product of two polar vectors are axial vectors. Axial vectors, unlike polar vectors, change their sign if the x-, y- and z-axes all change their signs ("inversion" of the coordinate system). The two sign changes in $\boldsymbol{\omega}$ and the cross-product therefore cancel out in (1.47). In general (1.47) can best be made clear by considering points on a plane which is at right-angles to the axis of rotation; points on the rotation axis have no velocity v.

If one holds the axle of the front wheel of a bicycle, sets it spinning rapidly, and then tries to turn the axle into a different direction, one will notice the tendency of the axle to turn at right-angles to the force being exerted on it. This evasion at right-angles is easy to explain in principle: the timewise variation of the angular momentum $\boldsymbol{L}$ is according to (1.13) equal to the torque $\boldsymbol{M}$. This again is $\boldsymbol{r} \times \boldsymbol{f}$; if then the force $\boldsymbol{f}$ is applied perpendicular to the axle at the position $\boldsymbol{r}$, then the torque $\boldsymbol{M}$ and the change in the angular momentum are perpendicular both to the axle and to the force (Fig. 1.13). We should find this easy to understand. In the following section

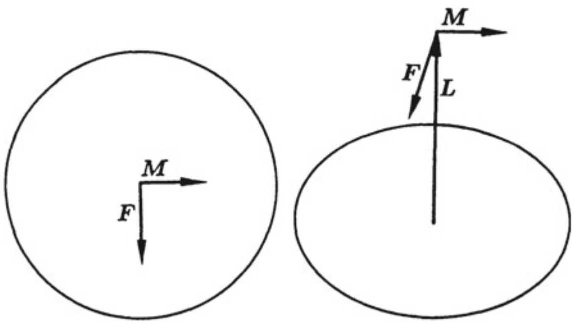


Fig. 1.13 Simple explanation of the perpendicular evasion of the applied force $\boldsymbol{F}$. The angular momentum $\boldsymbol{L}$ points upwards. On the *left* the gyro is viewed from above, on the *right* from the side. $\boldsymbol{L}$ changes in the direction of the torque $\boldsymbol{M}$

we shall replace this qualitative explanation by a more precise, but unfortunately more complicated, argument.

The gyrocompass is a practical application. Since the earth is not an inertial system, but spins daily on its axis, this terrestrial rotation exerts a torque on every rotating rigid body having its axis of rotation fixed to the earth's surface. If instead the axis of rotation is suspended in such a way that it can rotate horizontally to the earth's surface, but not vertically, then the torque from the terrestrial rotation leads in general to the above mentioned deflection perpendicular to the axis of the gyroscope. This continuing deflection of the axis of the gyroscope ("precession") causes frictional losses; gradually the gyroscope axis sets itself in the north-south direction, where the precession no longer occurs. The flight of the boomerang is also based on the gyroscopic effect; its demonstration by a theoretical physicist in a fully occupied lecture hall, however, has certain disadvantages.

(b) Angular Momentum and Inertia Tensor

For the cross-product with a cross-product we have the transformation $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(ac) - \boldsymbol{c}(ab)$ into scalar products. We apply this rule to the angular momentum L_i, of the ith atom or mass element:

$$\begin{aligned} \boldsymbol{L}_\mathrm{i} m_\mathrm{i} &= \boldsymbol{r}_\mathrm{i} \times \boldsymbol{v}_\mathrm{i} = \boldsymbol{r}_\mathrm{i} \times (\boldsymbol{\omega} \times \boldsymbol{r}_\mathrm{i}) \\ &= \boldsymbol{\omega}(\boldsymbol{r}_\mathrm{i}\boldsymbol{r}_\mathrm{i}) - \boldsymbol{r}_\mathrm{i}(\boldsymbol{r}_\mathrm{i}\boldsymbol{\omega}) = \boldsymbol{\omega} r_\mathrm{i}^2 - \sum_{\nu=1}^{3} \o\,\omega_\nu r_{\mathrm{i}\nu} \boldsymbol{r}_\mathrm{i}, \end{aligned}$$

or in components ($\mu, \nu = 1, 2, 3$):

$$L_{\mathrm{i}\mu}/m_\mathrm{i} = \omega_\mu r_\mathrm{i}^2 - \sum_\mathrm{v} \omega_\nu r_{\mathrm{i}\mu} r_{\mathrm{i}\nu} = \sum_\nu \omega_\nu \left(r_\mathrm{i}^2 \delta_{\mu\nu} - r_{\mathrm{i}\mu} r_{\mathrm{i}\nu} \right)$$

with again the Kronecker delta $\delta_{\mu\nu} = 1$ for $\mu = \nu$ and $= 0$ otherwise. For the components of the total angular momentum $\boldsymbol{L} = \Sigma_\mathrm{i} \boldsymbol{L}_\mathrm{i}$ we have

$$L_\mu = \sum_\nu \omega_\nu \Theta_{\mu\nu} \text{ or } \boldsymbol{L} = \Theta \boldsymbol{\omega}, \;\; \Theta_{\mu\mathrm{v}} = \sum_\mathrm{i} m_\mathrm{i} \left(r_\mathrm{i}^2 \delta_{\mu\mathrm{v}} - r_{\mathrm{i}\mu} r_{\mathrm{i}\nu} \right). \tag{1.48}$$

The matrix Θ of the $\Theta_{\mu\nu}$ so defined is called the *inertia tensor*. Overlooking this matrix property the relation $\boldsymbol{L} = \Theta\boldsymbol{\omega}$ for the rotation of a rigid body is quite analogous to the momentum definition $\boldsymbol{p} = m\boldsymbol{v}$ for its translational motion. Tensors are "true" matrices with physical meaning. More precisely: a vector for a *computer program* is any combination of (in three dimensions) three numbers, e.g., the number triplet consisting of, in the first place the Dow Jones Index from Wall Street, in the second place the body weight of the reader, and in the third place the size of the university budget. For *physics* this is gibberish, whereas, for example, the position vector is a true vector. For the physicist, moreover, true vectors are those number triplets which transform like a position vector under a rotation of the coordinate system. Similarly, not every square array of numbers which a computer could store as a matrix would be regarded by a physicist as a tensor. Tensors are only those matrices whose components are transformed under a rotation of the coordinate system in such a way that the tensor before and after links the same vectors. Accordingly, for true vectors and tensors the relation: $\text{vector}_1 = \text{tensor} \cdot \text{vector}_2$ is independent of the direction of the coordinate axes. Only then do equations such as (1.48) make sense.

Since the inertia tensor Θ is symmetric, $\Theta_{\mu\nu} = \Theta_{\nu\mu}$, it has only real eigenvalues. Moreover, for all symmetric tensors one can choose the eigenvectors so that they are mutually perpendicular. If we therefore set our coordinate axes in the directions of these three eigenvectors, then any vector lying in the x-axis will, after multiplication by the tensor Θ, again lie in the x-axis, but with its length multiplied by the first eigenvalue, called Θ_1. Similarly, any vector in the y-direction, after application of the matrix Θ, will be stretched or shortened by the factor Θ_2, without change of direction. The third eigenvalue Θ_3 applies to vectors in the z-direction. General vectors are made up of their components in the x-, y- and z-directions, and after multiplication by Θ are again the sum of their three components multiplied by Θ_μ. Accordingly in the new coordinate system with its axes in the direction of the eigenvectors we have

$$\boldsymbol{L} = \begin{pmatrix} \Theta_1 & 0 & 0 \\ 0 & \Theta_2 & 0 \\ 0 & 0 & \Theta_3 \end{pmatrix} \cdot \boldsymbol{\omega} = \Theta\boldsymbol{\omega} \quad \text{or} \quad L_\mu = \Theta_\mu \omega_\mu \tag{1.49}$$

for $\mu = 1, 2, 3$. The tensor Θ therefore has a diagonal form in the new coordinate system; outside the diagonal the matrix consists of zeros.

Mathematicians call this choice of coordinate system, possible with any symmetric matrix, its *principal axes* form; one has referred the tensor to its principal axes, or "diagonalised" it. Physicists call the eigenvalues Θ_1 of the inertia tensor the *principal moments of inertia*.

If one uses these principal axes one has

$$\Theta_\mu = \Theta_{\mu\mu} = \sum_{\mathrm{i}} m_{\mathrm{i}} \rho_{\mathrm{i}}^2 \quad \text{with} \quad \rho_{\mathrm{i}}^2 = r_{\mathrm{i}}^2 - r_{\mathrm{i}\mu}^2, \tag{1.50}$$

where ρ is the distance of the position r from the μ-axis; when $\mu = 1$, i.e. the x-axis, we accordingly have $\rho^2 = x^2 + y^2 + z^2 - x^2 = y^2 + z^2$, as it should be. If one rotates the rigid body about a fixed axle, not just about an imaginary axis, then (1.50) is likewise valid with the axle in place of the μ-axis: $L = \Sigma_i m_i \rho_i^2 \omega$. In this case one calls $\Sigma_i m_i \rho_i^2$ the moment of inertia ϑ; ϑ is then a number and no longer a tensor. Do you still remember the Steiner rule? If not, have you at least come across Frisbee disks?

(c) Kinetic Energy

If the centre of mass of the rigid body of mass M does not lie at the origin then its kinetic energy is $T' = T + P^2/2M$, where P is the total momentum and T the kinetic energy in the coordinate system whose origin coincides with the centre of mass of the rigid body. It is therefore practical, here and elsewhere, to use the latter system at once and calculate T.

We have

$$2T = \sum_i m_i v_i^2 = \sum_i m_i v_i(\boldsymbol{\omega} \times \boldsymbol{r}_i) = \boldsymbol{\omega}\boldsymbol{L},$$

where at the end we have again applied the "triple product" formula (volume of a parallelepiped) $\boldsymbol{a}(\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c}(\boldsymbol{a} \times \boldsymbol{b})$. Hence:

$$2T = \boldsymbol{\omega}\boldsymbol{L} = \boldsymbol{\omega}\Theta\boldsymbol{\omega} = \sum_{\mu\nu} \omega_\mu \Theta_{\mu\nu} \omega_\nu = \omega_1^2\Theta_1 + \omega_2^2\Theta_2 + \omega_3^2\Theta_3, \tag{1.51}$$

where the last relation is valid only in the principal axes system of the body. If the body rotates with moment of inertia ϑ about a fixed axis, (1.51) is simplified to $2T = \vartheta\omega^2$. Since in the absence of external forces the kinetic energy is constant, $\Sigma_{\mu\nu}\omega_\mu\Theta_{\mu\nu}\omega_\nu$ is therefore constant. This condition describes an *ellipsoid of inertia* in $\boldsymbol{\omega}$-space. If the three principal moments of inertia are equal this "ellipsoid" clearly degenerates into a sphere. One usually calls any rigid body with three equal principal moments of inertia a "spherical gyroscope", although besides the sphere a homogeneous cube also qualifies for this. "Symmetric" *gyroscopes* are those with two of the three principal moments of inertia Θ_μ equal.

The angular momentum $\boldsymbol{L}$ is according to (1.49) the gradient (in ω-space) of the kinetic energy T according to (1.51) and is therefore normal to this ellipsoid of inertia; in general grad f is normal to the surface defined by $f(\boldsymbol{r}) = \text{const}$. In general, therefore, as shown by Fig. 1.14, the vectors $\boldsymbol{\omega}$ and $\boldsymbol{L}$ are not parallel. Only when the ellipsoid degenerates into a sphere, i.e. all three moments of inertia are equal, are $\boldsymbol{\omega}$ and $\boldsymbol{L}$ always parallel. Mathematicians will see this directly from the relation (1.49).

1.4.2 Equations of Motion

(a) Fundamentals

The rigid body is in equilibrium only if no external torque nor any external force acts on it. This is not quite trivial for individual point masses, as is a well known

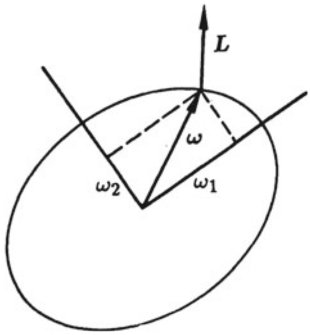


Fig. 1.14 Two-dimensional illustration of (1.49) and (1.51). If the inertia ellipse (in principal axis form $\Theta_1\omega_1^2 + \Theta_2\omega_2^2 = 2T$) does not degenerate into a circle, the vectors $\boldsymbol{\omega}$ and $\boldsymbol{L}$ are in general not parallel

fact from daily experience, since enormous constraining forces, and perhaps torques also, act between the atoms of the rigid body. However, these all balance out, as one sees from the principle of virtual work (1.32). If the whole body is displaced through the distance $\delta\boldsymbol{R}$ and rotated through the angle $\delta\boldsymbol{\phi}$, then $\delta r_\mathrm{i} = \delta\boldsymbol{R} + \delta\boldsymbol{\phi} \times \boldsymbol{r}_\mathrm{i}$, so since these virtual displacements do no work we have:

$$0 = \sum_\mathrm{i} \boldsymbol{F}_\mathrm{i}\delta\boldsymbol{r}_\mathrm{i} = \delta\boldsymbol{R}\sum_\mathrm{i} \boldsymbol{F}_\mathrm{i} + \delta\boldsymbol{\phi}\sum_\mathrm{i} \boldsymbol{r}_\mathrm{i} \times \boldsymbol{F}_\mathrm{i}$$

for all small $\delta\boldsymbol{R}$ and $\delta\boldsymbol{\phi}$. Then the sums must also vanish: $\Sigma_\mathrm{i}\boldsymbol{F}_\mathrm{i} = 0 = \Sigma_\mathrm{i}\boldsymbol{r}_\mathrm{i} \times \boldsymbol{F}_\mathrm{i}$. Accordingly the total force and also the total torque vanish.

If an external force $\boldsymbol{F}$ and an external torque $\boldsymbol{M}$ act on the rigid body, these determine the changes in the total momentum $\boldsymbol{P}$ and the total angular momentum $\boldsymbol{L}$, precisely because the inner forces and torques all cancel out:

$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t}, \quad \boldsymbol{M} = \frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \frac{\mathrm{d}(\Theta\boldsymbol{\omega})}{\mathrm{d}t} \tag{1.52}$$

in an inertial system. These are six equations for six unknowns, so we find ourselves in a promising situation.

(b) Euler's Equations

If a body rotates, then all its principal axes rotate with it, and also the entire inertia tensor. We consider this body from an inertial system under the influence of an external torque $\boldsymbol{M}$ and denote by $\boldsymbol{e}_\mu$ the unit vectors in the directions of the principal axes (eigenvectors of the inertia tensor). Then these $\boldsymbol{e}_\mu$ change in time at the rate $\mathrm{d}\boldsymbol{e}_\mu/\mathrm{d}t = \boldsymbol{\omega} \times \boldsymbol{e}_\mu$, $\mu = 1, 2, 3$. The angular momentum is then seen from the inertial system, taking account of the diagonal form (1.49) of the tensor Θ, to be:

$$\boldsymbol{L} = \Theta\boldsymbol{\omega} = \sum_\mu \Theta_\mu\omega_\mu\boldsymbol{e}_\mu.$$

Here we use $\boldsymbol{\omega} = \Sigma_\mu \omega_\mu \boldsymbol{e}_\mu$, which sounds trivial but establishes that now the three ω_μ are the components relative to the $\boldsymbol{e}_\mu$ system of reference fixed in the body, and not relative to the inertial system.

For the time derivative of $\boldsymbol{L}$ we therefore have

$$\boldsymbol{M} = \dot{\boldsymbol{L}} = \sum_\mu \left(\Theta_\mu \dot{\omega}_\mu \boldsymbol{e}_\mu + \Theta_\mu \omega_\mu \dot{\boldsymbol{e}}_\mu\right).$$

Substituting

$$\mathrm{d}\boldsymbol{e}_1/\mathrm{d}t = (\omega_1 \boldsymbol{e}_1 + \omega_2 \boldsymbol{e}_2 + \omega_3 \boldsymbol{e}_3) \times \boldsymbol{e}_1 = \omega_3 \boldsymbol{e}_2 - \omega_2 \boldsymbol{e}_3,$$

and similar relations for the two other components, finally reduces the above expression for $\boldsymbol{M}$ to the Euler equations:

$$\begin{aligned} M_1 &= \Theta_1 \mathrm{d}\omega_1/\mathrm{d}t + (\Theta_3 - \Theta_2)\,\omega_2\omega_3 \\ M_2 &= \Theta_2 \mathrm{d}\omega_2/\mathrm{d}t + (\Theta_1 - \Theta_3)\,\omega_3\omega_1 \\ M_3 &= \Theta_3 \mathrm{d}\omega_3/\mathrm{d}t + (\Theta_2 - \Theta_1)\,\omega_1\omega_2. \end{aligned} \tag{1.53}$$

If one knows one of these equations, the others follow from it naturally by cyclic exchange of the indices: 1 by 2, 2 by 3, 3 by 1. In a spherical gyroscope all three Θ_μ are equal and hence we simply have

$$\boldsymbol{M} = \Theta_\mu \mathrm{d}\boldsymbol{\omega}/\mathrm{d}t.$$

A remarkable thing about these equations is first of all that they are not linear, but quadratic in ω. Since we often can only solve linear differential equations exactly, we program their simulation by the method already described in Sect. 1.1, for the case $\boldsymbol{M} = 0$ (see Program EULER).

PROGRAM EULER

```
10 input "omega="; w1, w2, w3
20 dt = 0.01
30 t1 =10.0
40 t2 = 1.0
50 t3 = 0.1
60 d1 =dt*(t2-t3)/t1
70 d2 =dt*(t3-t1)/t2
80 d3 =dt*(t1-t2)/t3
90 w1 =w1+d1*w2*w3
100 w2=w2+d2*w3*w1
110 w3=w3+d3*w1*w2
120 print w1, w2
130 goto 90
140 end
```

Whether linear or nonlinear, it is all the same to the BASIC program; it is more important that we must denote ω by w. The three principal moments of inertia are 10, 1 and 1/10, the time-step dt is 1/100. If we allow the body to rotate about a principal axis, e.g. by the input 0, 1, 0, then nothing changes at all. If we introduce small disturbances, however, inputing one of the three ω_μ as 1, the other two as 0.01, then the picture changes. A rotation about the principal axis with the greatest moment of inertia (here axis 1) is stable, i.e. the small disturbances in the two other components oscillate about zero and remain small, while ω_1 remains in the neighbourhood of 1. This is obtained by inputing 1.0, 0.01, 0.01. With the input 0.01, 1, 0.01, on the other hand, i.e. with a rotation about the axis with the middle moment of inertia, the body totters about all over the place: the initially small ω_3 becomes up to ten times larger, but above all ω_2 changes its sign. The initially dominant rotation component ω_2 is thus influenced quite critically by the small disturbances. Of course, none of these disturbances grows exponentially without limit (contrary to linear differential equations), since the kinetic energy must be conserved. The rotation about the third principal axis with the smallest moment of inertia is again stable.

One can try experimentally to demonstrate the stability (instability) for rotation about the axis with the greatest (middle) moment of inertia by skilful throws of filled matchboxes. But here one comes close to the border between accurate observation and hopeful faith. Instead, one can treat the Euler equations in harmonic approximation and theoretically distinguish clearly between instability (exponential increase of disturbances) and stability; Euler (1707–1783) knew no BASIC.

(c) Nutation

The stable rocking of a gyroscope without external torque is called *nutation*; we call *precession* the gradual rotation of the rotation axis under the influence of a weak torque. Other definitions also occur in the literature. We now calculate the nutation frequency, which we could observe empirically in the quantity ω_2 with the above computer program. We consider the symmetric gyroscope, $\Theta_1 = \Theta_2$, and assume that the gyroscope spins fast about the third axis with moment Θ_3, but that ω_1 and ω_2 are not exactly zero. In this stable case how do the two components ω_1 and ω_2 oscillate, i.e. how does the instantaneous axis of rotation rock, when seen from the rigid body? The ω_μ in (1.53) and here are still always the components in the principal axes system fixed in the rigid body.

The Euler equations with the abbreviation $\tau = (\Theta_1 - \Theta_3)/\Theta_1$ now become

$$\mathrm{d}\omega_1/\mathrm{d}t = \tau\omega_2\omega_3, \quad \mathrm{d}\omega_2/\mathrm{d}t = -\tau\omega_3\omega_1, \quad \mathrm{d}\omega_3/\mathrm{d}t = 0.$$

The principal component ω_3 therefore remains constant and we have

$$\mathrm{d}^2\omega_1/\mathrm{d}t^2 = \tau\omega_3\mathrm{d}\omega_2/\mathrm{d}t = -r_2\omega_3^2\omega_1.$$

This is once again the equation of the harmonic oscillator and is similarly valid for ω_2. The well known solution is

$$\omega_\mu \sim \mathrm{e}^{\mathrm{i}\Omega t} \; (\mu = 1, 2), \quad \Omega/\omega_3 = \tau = (\Theta_1 - \Theta_3)/\Theta_1. \tag{1.54}$$

Accordingly, if the axis of rotation in the direction of ω does not exactly coincide with the body axis $\boldsymbol{e}_3$ of the symmetrical gyroscope, so that ω_1 and ω_2 are not zero, then the axis of rotation wobbles with the nutation frequency Ω about the body axis $\boldsymbol{e}_3$. This nutation frequency is proportional to the actual rotation frequency ω_3; the factor of proportionality is a ratio of the moments of inertia.

Since the three ω-components are measured from the rotating reference system fixed in the body, one can easily become dizzy. It is safer if we take the planet earth as the example of a rigid body. It is known that the earth is not a sphere, but is slightly flattened; the principal moment of inertia Θ_3 relating to the north-south axis is therefore somewhat greater than the other two in the equatorial plane. The reference system fixed in the body is now our longitude and latitude, familiar from maps. If we define the south pole by the direction of the principal moment of inertia $\boldsymbol{e}_3$, the instantaneous axis of rotation will not coincide exactly with this pole, but will nutate about it. Since $\tau = 1/300$ the nutation frequency Ω must correspond to a period of about 300 days. Actually the pole is observed to wobble with a period of 427 days, as the earth is not a rigid body. Volcanic eruptions show that it is fluid inside.

If one observes the nutating symmetric gyroscope from the inertial system instead of from the system fixed in the body, it is no longer the axis e_3 of the body that is always in the same direction, but the angular momentum (provided that there is no external torque). Around this fixed direction of the angular momentum the axis e_3 of the body describes the "nutation cone". The instantaneous axis of rotation ω rotates on the "rest cone" or "herpolhode cone", while ω itself spins about the body axis on the "rolling cone" or "polhode cone".

(d) Precession

What happens when an external torque acts on a symmetric gyroscope? For example, this is the situation for a toy spinning-top whose tip rests on the ground and which is not perfectly upright. For the reader whose youth was so occupied with video-games that he had no time for such toys, Fig. 1.15 shows a sketch of this experimental apparatus.

If m is the mass of the top, R the distance of its centre of mass from its tip (point of support) and $\boldsymbol{g}$ the downward acceleration of terrestrial gravity, then the weight $m\boldsymbol{g}$ exerts the torque $m\boldsymbol{R} \times \boldsymbol{g}$ on the top. The vector $\boldsymbol{R}$ lies in the direction of the body axis (at least when the top is perfectly round), and this is in the direction of the angular momentum $\boldsymbol{L}$ if we neglect the nutation. (We thus assume that the top is spinning exactly about its axis of symmetry.) We therefore have

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \boldsymbol{M} = \boldsymbol{\omega}_\mathrm{L} \times \boldsymbol{L}, \tag{1.55}$$

where the vector $\boldsymbol{\omega}_\mathrm{L}$, acts upwards and has the magnitude mRg/L. The solution of this equation is simple: the horizontal component of the angular momentum (and therefore also that of the body axis) rotates with the angular velocity ω_L about the

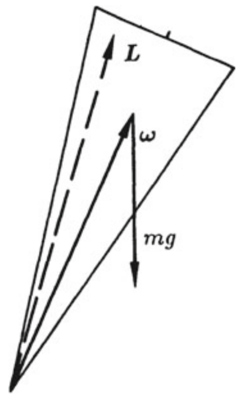


Fig. 1.15 Example of a symmetric spinning-top in the field of gravity. The angular momentum $\boldsymbol{L}$ is almost parallel to $\boldsymbol{\omega}$, the torque $\boldsymbol{M}$ is perpendicular to the plane of the paper

vertical. This slow rotation proportional to the external torque is called precession (other names also occur in the literature for our nutation and precession). The magnitude of $\boldsymbol{L}$ and its vertical component L_3 accordingly remains constant. With real tops, of course, there are also frictional forces. This explains why the top does not fall over, but moves perpendicular to the direction in which it would be expected to fall.

Another example of the application of (1.55) is the "Larmor precession" of magnetic moments ("spins"). In the classical atom an electron orbits around the nucleus, and because of its electrical charge produces a circular electrical current and hence a magnetic dipole moment $\boldsymbol{\mu}$. An atom therefore usually has a moment of inertia and an angular momentum $\boldsymbol{L}$, and also a magnetic dipole moment $\boldsymbol{\mu}$ proportional to $\boldsymbol{L}$. (We shall learn later in electrodynamics that charges at rest cause electric fields, while moving charges cause magnetic fields and oscillating charges waves.) In a magnetic field $\boldsymbol{B}$ a magnetic dipole moment experiences a torque $\boldsymbol{B} \times \boldsymbol{\mu}$. Then (1.55) is again valid, with the Larmor frequency $\omega_{\mathrm{L}} = |\boldsymbol{B} \times \boldsymbol{\mu}|/L$. In the absence of quantum mechanics, elementary magnets would therefore continually precess if they were not exactly parallel to the magnetic field. The "gyromagnetic" ratio μ/L is proportional to the ratio of the elementary electric charge e to the product of mass m and the velocity of light $c : \omega_L = eB/mc$ in appropriate units.

Such effects are applied in spin resonance (NMR, MRI: since 1946) to the study of solid bodies and biological macromolecules, but more recently also in medicine to diagnosis without an operation and without X-radiography (NMR tomography, NMR = Nuclear Magnetic Resonance).

Precession is also important for *horoscopes*. Because of the flattening of the earth the gravitation of the sun exerts a torque on the earth, and the angular momentum of the earth processes with a period of 26,000 years. Accordingly the agreement between the stellar constellations and the calendar months becomes worse and worse with the passage of time; every 26,000/12 years the signs of the zodiac move along by one sign. Since the signs of the zodiac had already been fixed a long time ago, they are no longer correctly placed today. Modern foretellers of the future therefore always have

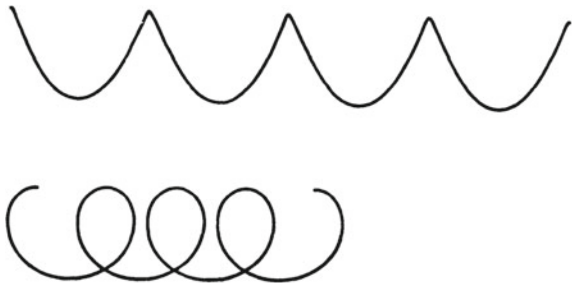


Fig. 1.16 Motion of the peak of a *spinning-top* with weak (*bottom*) and with strong (*top*) precession, as well as steady nutation. This garland may be regarded as a laurel wreath (Nobel Prize substitute) fashioned for the reader

to read between the signs, casting the horoscope according to the average value of the two predictions of two neighbouring signs of the zodiac. In this way we arrived at the prediction that this textbook would be a great success.

Whether it concerns mechanical gyroscope or magnetic spin, the torque precesses with the angular velocity ω_L on a cone about the vertical, if the gravitational force or magnetic field acts downwards. This simple result holds only, of course, when both friction and nutation are neglected. If there is a weak nutation, since the symmetric gyroscope does not rotate exactly around the body axis $\boldsymbol{e}_3$, the vector $\boldsymbol{e}_3$ no longer moves on the cone, so its end no longer moves on a circle. Instead, the end of $\boldsymbol{e}_3$ moves in a series of loops (strong amplitude of nutation) or waves (weak nutation) formed by the superposition of two circular motions (Fig. 1.16).

1.5 Continuum Mechanics

1.5.1 Basic Concepts

(a) Continua

Elastic solids, flowing liquids and drifting gases are the continua of this Section on *elasticity* and *hydrodynamics*. If in this sense a solid is not rigid, then one has actually to treat all the molecules separately. In a glass of beer there are about 10^{25} particles, and there are more congenial methods to go about this than to solve Newton's equations of motion for all of them simultaneously. Instead, we once again use an approximation: we average over many atoms. If we wish to describe the flow of air round a motor-car or the deformation of an iron plate supporting a heavy load, then in these engineering applications we are scarcely interested in the thermal motion of the air molecules or the high frequency phonons in the iron. We wish to find a mean velocity of the air molecules and a mean displacement of the atoms of iron from their equilibrium positions. We need therefore to average over a "mesoscopic" region, containing many molecules, but small compared with the deformation of the solid or with the distances over which the velocity of the fluid flow changes significantly.

Actually we do not really carry out this averaging; only in recent years has hydrodynamics been studied on the computer by the simulation of every individual atom. We accordingly restrict ourselves here to postulating that there is a mean deformation and a mean velocity. On this assumption we construct the whole of continuum mechanics, without actually calculating these mean values from the individual molecules. We shall later use similar tricks with Maxwell's equations in matter and in thermodynamics. If we do not know a quantity which would in principle be calculated from the individual molecules, then we give this quantity a name ("density", "viscosity", "susceptibility", "specific heat") and assume that it can be measured concurrently by experimental physics. We then work with this measured value, in order to predict other measured values and phenomena. This may be regarded as cheating, but this method has been vindicated over hundreds of years. A theory is generally called "phenomenological" if certain material properties are not calculated, but only measured experimentally.

Almost all the formulae in this section hold in common for gases, liquids and solids. In any case, one cannot always distinguish clearly between these phases, since iron plates can be deformed even more easily than glass, and at the critical point (see van der Waals' equation) the difference between vapour and liquid disappears. Nevertheless, when the discussion is about strain, the reader can think of a single-crystal solid; in velocity fields it is best to think of the flow of "incompressible" water, and shock waves can be envisaged in "compressible" air.

(b) Strain Tensor ε

In an elastic solid let $\boldsymbol{u}$ be the mean displacement of the molecules from the equilibrium configuration; $\boldsymbol{u}$ depends on the position $\boldsymbol{r}$ in the solid under consideration. (For liquids and gases $\boldsymbol{u}$ is the displacement from the position at time $t = 0$.) For sufficiently small distances $\boldsymbol{r}$ between two points in the solid we have the Taylor expansion:

$$u(\boldsymbol{r}) = \boldsymbol{u}(0) + \sum_{\mathrm{k}} x_{\mathrm{k}} \partial \boldsymbol{u}/\partial x_{\mathrm{k}}, \quad k = 1, 2, 3.$$

We define

$$\operatorname{div} \boldsymbol{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \tag{1.56}$$

$$\operatorname{curl} \boldsymbol{u} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \tag{1.57}$$

as the *divergence* and the *curl* of the quantity $\boldsymbol{u}(r)$. Many authors write div $\boldsymbol{u}$ as the scalar product of the nabla operator $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$ with the vector $\boldsymbol{u}$; in this sense curl $\boldsymbol{u}$ is the cross-product $\nabla \times \boldsymbol{u}$. Many rules concerning scalar and cross-products are also valid here. The point of prime importance is that the curl is a vector, and the divergence is not.

After some manipulation the above Taylor expansion becomes

$$\boldsymbol{u}(\boldsymbol{r}) = \boldsymbol{u}(0) + \mathrm{curl}(\boldsymbol{u}) \times \boldsymbol{r}/2 + \varepsilon \boldsymbol{r} \tag{1.58}$$

with the *strain* tensor ε, a 3×3 matrix, defined by

$$\varepsilon_{\mathrm{ik}} = (\partial u_{\mathrm{i}}/\partial x_{\mathrm{k}} + \partial u_{\mathrm{k}}/\partial x_{\mathrm{i}})/2 = \varepsilon_{\mathrm{ki}}. \tag{1.59}$$

This shows clearly that the displacement $\boldsymbol{u}$ can be represented in small regions ($\boldsymbol{r}$ not too large) as a superposition of a translation $\boldsymbol{u}(0)$, a rotation through the angle $\mathrm{curl}(\boldsymbol{u})/2$, and a distortion or strain of the elastic solid. For the rigid solids of the previous section the distortion is absent, and $\mathrm{curl}(\boldsymbol{u})$ is uniform over space.

Since the strain tensor ε is always symmetric, there is a rectangular coordinate system in which the matrix of the $\varepsilon_{\mathrm{ik}}$ is diagonal: $\varepsilon_{\mathrm{ik}} = 0$ except when $i = k$. In this coordinate system the volume change ΔV of a distorted prism of length x, breadth y and height z is especially convenient to calculate, since now $\Delta x = \varepsilon_{11} x$, etc.:

$$\Delta V/V = [(x + \Delta x)(y + \Delta y)(z + \Delta z) - xyz]/xyz \approx \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \mathrm{Tr}(\varepsilon)$$

with the trace $\mathrm{Tr}(\varepsilon) = \Sigma_{\mathrm{i}}\varepsilon_{\mathrm{ii}}$. Mathematicians have proved that the trace of a matrix does not change with a rotation of the coordinate system. The trace of the unit tensor $\mathcal{E}$, defined as the matrix of the Kronecker delta δ_{ik}, is trivially equal to 3. With the definition

$$\varepsilon = \varepsilon' + \mathrm{Tr}(\varepsilon)\mathcal{E}/3$$

the strain tensor is partitioned into a shear ε' without volume change (since $\mathrm{Tr}(\varepsilon') = 0$) and a volume change without change of shape (since it is proportional to the unit matrix). This analysis of the general displacement $\boldsymbol{u}$ into a translation, a rotation, a change of shape and a change of volume is very plausible even without mathematics.

(c) Velocity Field

In gases and liquids the displacement field $\boldsymbol{u}(\boldsymbol{r})$ can be described as the displacement of the molecules from their positions at time $t = 0$; there is no equilibrium position. It is more appropriate, however, to talk of a mean velocity $\boldsymbol{v}(\boldsymbol{r})$ of the molecules: $\boldsymbol{v} = \mathrm{d}\boldsymbol{u}/\mathrm{d}t$. The velocity field $\boldsymbol{v}$ depends on the time t, as well as on the position $\boldsymbol{r}$.

A clear distinction must be made between the total time derivative $\mathrm{d}/\mathrm{d}t$ and the partial time derivative $\partial/\partial t$. This distinction can be clarified physically by considering the temperature T in a stream of water. If one measures it at a fixed position, e.g., at a bridge, then the position $\boldsymbol{r}$ is held constant and the measured rate of temperature change is consequently $\partial T/\partial t$. If, on the other hand, one drops the thermometer into the stream, so that it drifts along with the current, then one measures the heating or cooling of the portion of water in which the thermometer remains all the time it is drifting. This rate of change of temperature, with varying position, is therefore $\mathrm{d}T/\mathrm{d}t$.

Mathematically the two derivatives are connected via the temperature gradient grad T:

$$\begin{aligned} \mathrm{d}T/\mathrm{d}t &= \partial T/\partial t + (\partial T/\partial x)(\partial x/\partial t) + (\partial T/\partial y)(\partial y/\partial t) + (\partial T/\partial z)(\partial z/\partial t) \\ &= \partial T/\partial t + \sum_{\mathrm{i}} v_{\mathrm{i}} \partial T/\partial x_{\mathrm{i}} = \partial T/\partial t + (\boldsymbol{v}\,\mathrm{grad})T, \end{aligned}$$

where $(\boldsymbol{v}$ grad) is the scalar product of the velocity with the nabla operator ∇. Another notation for this operator $(\boldsymbol{v}$ grad) is $(\boldsymbol{v} \cdot \nabla)$; anybody who finds this operator notation difficult can always replace the expression $(\boldsymbol{v}\,\mathrm{grad})T$ by $\Sigma_{\mathrm{i}} v_{\mathrm{i}} \partial T/\partial x_{\mathrm{i}}$ with $i = 1, 2, 3$ for the three directions.

What was said for temperature is equally true for any other quantity A:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\partial A}{\partial t} + \sum_{\mathrm{i}} v_{\mathrm{i}} \frac{\partial A}{\partial x_{\mathrm{i}}}. \tag{1.60}$$

One speaks also of the Euler notation, working with $\partial/\partial t$, and of the Lagrange notation, working with the total derivative $\mathrm{d}/\mathrm{d}t$. Simple dots as symbols for derivatives with respect to time are dangerous in hydrodynamics.

If we now apply Newton's law of motion

$$\text{force} = \text{mass} \cdot \text{acceleration},$$

then the acceleration is the total time derivative of the velocity, since the particles of water are accelerating ("*substantial* derivative" $\mathrm{d}\boldsymbol{v}/\mathrm{d}t$):

$$\text{force} = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = m\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\,\mathrm{grad})\boldsymbol{v}\right].$$

Here $(\boldsymbol{v}\,\mathrm{grad})\boldsymbol{A}$ with a vector $\boldsymbol{A}$ means that $(\boldsymbol{v}$ grad) is applied to each of the three components and that the result is a vector:

$$\left[(\boldsymbol{v}\,\mathrm{grad})A\right]_{\mathrm{k}} = \sum_{\mathrm{i}} v_{\mathrm{i}} \partial A_{\mathrm{k}}/\partial x_{\mathrm{i}}.$$

It is important to notice that the velocity $\boldsymbol{v}$ now occurs in Newton's law of motion not just linearly, but quadratically. Many problems in hydrodynamics accordingly are no longer soluble exactly for high velocities, but use up much calculation time on supercomputers. Clearly we measure $\mathrm{d}\boldsymbol{v}/\mathrm{d}t$ if we throw a scrap of paper into the stream and follow its acceleration; $\partial \boldsymbol{v}/\partial t$ is being assessed if we hold a finger in the stream and feel the changing force on it. In both cases a bath-tub as a measuring environment is more practical than a bridge over the Mississippi.

Just as in the whole of continuum mechanics, we do not wish to consider the atoms individually, but to average them. We define therefore the density ρ as the ratio of mass to volume. More precisely ρ is the limiting value of the ratio of mass to volume, when the mass is determined in a notionally defined partial volume of the liquid, and this volume is very much greater than the volume of a single atom, but very much smaller than the total volume or the volume within which the density changes significantly. We take ρ simply to be the mass per cm^3, since the Mississippi is broader than a centimetre.

In a completely analogous manner we define the *force density* $\boldsymbol{f}$ as the force per cm^3 acting on a fluid ($\boldsymbol{f}$ = force/volume). Newton's law now has the form

$$\boldsymbol{f} = \rho \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \text{ grad})\boldsymbol{v} \right]. \tag{1.61}$$

An example of the force density $\boldsymbol{f}$ is the force of gravity, $\boldsymbol{f} = \rho \boldsymbol{g}$. Later we shall also meet areal forces such as the pressure.

A "universally known" law is that of Gauss:

$$\oint \boldsymbol{j} \boldsymbol{d}^2 \boldsymbol{S} = \int \text{div}(\boldsymbol{j}) \text{d}^3 r \tag{1.62}$$

for a vector field $\boldsymbol{j} = \boldsymbol{j}(r)$. The left-hand side is a two-dimensional integral over the surface of the volume, over which the right-hand side is integrated three-dimensionally. The areal element $\boldsymbol{d}^2\boldsymbol{S}$ is normal to this surface and points outwards. *Notation.* Two- or three-dimensional integrals, taken over a plane or a space, we denote by just an integral sign, and write the integration variable, for example, as $\text{d}^3 r$. An area integral, which extends, for example, over the closed surface of a three-dimensional volume, is denoted by an integral sign with a circle, as in (1.62); the area element is then a vector $\boldsymbol{d}^2\boldsymbol{S}$, in contrast to $\text{d}^3 r$. In Stokes's law (1.81) will occur a closed one-dimensional line integral, which is also marked with a circle; these line integrals have a vector $\boldsymbol{dl}$ as integration variable pointing in the direction of the line. The notation $\text{d}V$ for $\text{d}^3 r$ will be avoided here; in the section on heat the quantity V will be the magnitude of the volume in the mechanical work $-PdV$.

We now apply this calculation rule (1.62) to the current density $\boldsymbol{j} = \rho \boldsymbol{v}$ of the fluid stream; $\boldsymbol{j}$ thus represents how many grams of water flow per second through a cross-sectional area of one square centimetre, and points in the direction of the velocity $\boldsymbol{v}$. Then the surface integral (1.62) is the difference between the outward and the inward flowing masses per second in the integration volume, and hence in the limit of a very small volume

$$-\partial(\text{mass})/\partial t = \text{div}(\boldsymbol{j}) \cdot \text{volume}.$$

Accordingly after division by the volume we obtain the *equation of continuity*

$$\frac{\partial \rho}{\partial t} + \text{div}(\boldsymbol{j}) = 0. \tag{1.63}$$

This fundamental relation between density variation and divergence of the relevant current density is valid similarly in many fields of physics, e.g., with electrical charge density and electrical current density. It is also familiar in connection with bank accounts: the divergence of outgoings and ingoings determines the growth of the reader's overdraft, and the growth in wealth of the textbook author.

A medium is called *incompressible* if its density ρ is constant:

$$\mathrm{div}(\boldsymbol{j}) = 0, \quad \mathrm{div}(\boldsymbol{v}) = 0. \tag{1.64}$$

Water is usually approximated as being *incompressible*, whereas air is rather compressible. Elastic solids also may be incompressible; then div $\boldsymbol{u} = 0$.

1.5.2 Stress, Strain and Hooke's Law

The force of gravity is, as mentioned, a volume force, which is measured by

$$\text{force density} = \text{force/volume}.$$

The pressure on the other hand has the dimension of force/area, and is therefore an areal force. In general we define an *areal force* as the limiting value of force/area for small area. Like force it is a vector, but the area itself can have various orientations. The areal force is therefore defined as a stress tensor σ:

> σ_{ik} is the force (per unit area) in the i-direction
> on an area at right-angles to the k-direction; $i, k = 1, 2, 3$. (1.65)

This tensor also is, like nearly all physical matrices, symmetric. Its diagonal elements σ_{ii} describe the pressure, which can indeed depend on the direction i in compressed solids; the non-diagonal elements such as σ_{12} describe the shear stresses. In liquids at rest the pressure P is equal in all directions, and there are no shear stresses: $\sigma_{\mathrm{ik}} = -P\delta_{\mathrm{ik}}$.

In the case when in a certain volume there is not only a volume force $\boldsymbol{f}$ but also an areal force σ acting on its surface, then the total force is

$$\boldsymbol{F} = \oint \sigma \boldsymbol{d}^2\boldsymbol{S} + \int \boldsymbol{f}\mathrm{d}^3r = \int(\mathrm{div}\,\sigma + \boldsymbol{f})\mathrm{d}^3r,$$

where we understand the divergence of a tensor to be the vector whose components are the divergences of the rows (or columns) of the tensor:

$$(\mathrm{div}\ \sigma)_\mathrm{i} = \sum_\mathrm{k} \partial\sigma_\mathrm{ik}/\partial x_\mathrm{k} = \sum_\mathrm{k} \partial\sigma_\mathrm{ki}/\partial x_\mathrm{k}.$$

In this sense we can apply Gauss's law (1.62) in the above formula. In the limiting case of small volume we therefore have

$$\frac{\text{areal force}}{\text{volume}} = \mathrm{div}\ \sigma, \tag{1.66}$$

e.g., for the force which pressure differences exert on a cm^3.

Because of (1.66) the equation of motion now becomes

$$\rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \mathrm{div}\ \sigma + \boldsymbol{f} \tag{1.67}$$

with the total derivative according to (1.60), for solids as for liquids and gases. In a liquid at rest under the influence of gravity $\boldsymbol{f} = \rho\boldsymbol{g}$ we therefore have $\boldsymbol{f} = -\mathrm{div}\,\sigma = \mathrm{div}(P\delta_\mathrm{ik}) = \mathrm{grad}\ P$, and accordingly at height h: $P = \mathrm{const.} - \rho g h$. For every ten metres of water depth the pressure increases by one "atmosphere" $\approx$ 1000 millibars $= 10^5$ pascals. Anybody who dives in the sea for treasure or coral must therefore surface very slowly, as the sudden lowering of pressure would allow dangerous bubbles to grow in the blood vessels. The relation $\mathrm{div}\,\sigma = -\mathrm{grad}\ P$ is also generally valid in "ideal" fluids without frictional effects (Euler 1755):

$$\rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\mathrm{grad}(P) + \boldsymbol{f}\,. \tag{1.68}$$

Equation (1.68) gives three equations for four unknowns, $\boldsymbol{v}$ and ρ. If the flow is compressible we need also to know how the density depends on the pressure. As a rule we use a linear relation: $\rho(P) = \rho(P = 0)(1 + \kappa P)$, the compressibility κ being defined thereby.

In an elastic solid the *stress tensor* σ is no longer given by a unique pressure P, and instead of a unique compressibility we now need many elastic constants C. We again assume a linear relationship, only now between the stress tensor σ and the strain tensor ε,

$$\sigma = C\varepsilon, \tag{1.69}$$

analogous to Hooke's law: restoring force = $C\cdot$ displacement. Robert Hooke (1635–1703) would be somewhat surprised to be regarded as the father of (1.69), since σ_ik and ε_mn are indeed tensors (matrices). Consequently C is a tensor of the fourth order (the only one in this book), i.e. a quantity with four indices:

$$\sigma_\mathrm{ik} = \sum_\mathrm{mn} C^\mathrm{ik}_\mathrm{mn}\varepsilon_\mathrm{mn} \quad (i, k, m, n = 1, 2, 3).$$

These 81 elements of the fourth order tensor C reduce to two Lamé constants μ and λ in isotropic solids:

$$\sigma = 2\mu\varepsilon + \lambda\mathcal{E}\mathrm{Tr}(\varepsilon) \tag{1.70}$$

with the unit matrix $\mathcal{E}$, and hence $\sigma_{\mathrm{ik}} = 2\mu\varepsilon_{\mathrm{ik}} + \lambda\delta_{\mathrm{ik}}\Sigma_{\mathrm{j}}\varepsilon_{\mathrm{jj}}$. The compressibility is then (see Exercise) $\kappa = 3/(3\lambda + 2\mu)$, the ratio of pressure to relative change of length is the Young's modulus $E = \mu(2\mu + 3\lambda)/(\mu + \lambda)$. The ratio: relative change of length perpendicular to the direction of force divided by the relative change of length parallel to the direction of force is the Poisson's ratio $\lambda/(2\mu + 2\lambda)$. Accordingly, without proof, the elastic energy is given by $\Sigma_{\mathrm{ik}}\mu(\varepsilon_{\mathrm{ik}})^2 + (\lambda/2)(\mathrm{Tr}\varepsilon)^2$.

1.5.3 Waves in Isotropic Continua

Sound waves (long-wave acoustic phonons) propagate in air, water and solids with different velocities. How does it function? The mathematical treatment is the same in all cases, so long as frictional effects (acoustic damping) are ignored and we are dealing only with isotropic media, in which sound propagates with the same velocity in all directions. Then we have (1.70), but with $\mu = 0$, $\lambda = 1/\kappa$ for gases and liquids.

Acoustic vibrations have such small amplitudes (in contrast to shock waves) that they are treated in the harmonic approximation; quadratic terms such as $(\boldsymbol{v}\,\mathrm{grad})\boldsymbol{v}$ accordingly drop out: $\mathrm{d}\boldsymbol{v}/\mathrm{d}t = \partial\boldsymbol{v}/\partial t$. Therefore, taking account of (1.70), after some manipulation (1.67) takes the form

$$\rho\frac{\partial\boldsymbol{v}}{\partial t} = \mathrm{div}\,\sigma + \boldsymbol{f} = \mu\nabla^2\boldsymbol{u} + (\mu + \lambda)\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} + \boldsymbol{f}. \tag{1.71}$$

Here even for gases and liquids the displacement $\boldsymbol{u}$ makes sense, in that $\boldsymbol{v} = \partial\boldsymbol{u}/\partial t$, since all vibrations do indeed have a rest position. The Laplace operator ∇^2 is the scalar product of the nabla operator ∇ with itself:

$$\nabla^2 A = \nabla(\nabla A) = \mathrm{div}\,\mathrm{grad}\,A = \Sigma_{\mathrm{i}}\partial^2 A/\partial x_{\mathrm{i}}^2$$

for a scalar A. For a vector $\boldsymbol{u}$, $\nabla^2\boldsymbol{u}$ is a vector with the three components $\nabla^2 u_1$, $\nabla^2 u_2$, $\nabla^2 u_3$. One should notice also the difference between div grad and grad div: operators are seldom commutative.

For the calculation of the sound velocity we neglect the gravity force $\boldsymbol{f}$ and assume sound propagation in the x-direction:

$$\rho\frac{\partial^2\boldsymbol{u}}{\partial t^2} = \mu\frac{\partial^2\boldsymbol{u}}{\partial x^2} + (\mu + \lambda)\mathrm{grad}\left(\frac{\partial u_{\mathrm{x}}}{\partial x}\right) \tag{1.72}$$

or

$$\rho\partial^2 u_{\mathrm{x}}/\partial t^2 = (2\mu + \lambda)\partial^2 u_{\mathrm{x}}/\partial x^2, \quad \rho\partial^2 u_{\mathrm{y}}/\partial t^2 = \mu\partial^2 u_{\mathrm{y}}/\partial x^2,$$

in components, with the z-component analogous to the y-component. These equations have the form of the general *wave equation*

$$\frac{\partial^2 \Psi}{\partial t^2} = c^2 \frac{\partial^2 \Psi}{\partial x^2} \quad (\text{or} = c^2 \nabla^2 \Psi) \tag{1.73}$$

for the vibration Ψ, which for a plane wave has the solution

$$\Psi \sim \mathrm{e}^{\mathrm{i}(Qx - \omega t)} \quad \text{with} \quad \omega = cQ. \tag{1.74}$$

(For arbitrary direction of propagation Qx is to be replaced in the trial solution by $\boldsymbol{Qr}$.) The sound velocity is given by $c = \omega/Q$, the velocity with which a definite phase is propagated, such as, for example, a zero of the real part $\cos(Qx - \omega t)$. (This phase velocity is to be distinguished from the group velocity $\mathrm{d}\omega/\mathrm{d}Q$, which may be smaller for high frequency phonons, but here coincides with ω/Q.) In three dimensions $\boldsymbol{Q}$ is the wave vector with magnitude $Q = 2\pi$/(wavelength); it is often denoted by $\boldsymbol{q}$, $\boldsymbol{k}$ or $\boldsymbol{K}$.

If we compare (1.73) with (1.72) in their three components we immediately see that

$$c^2 = (2\mu + \lambda)/\rho \tag{1.75}$$

for the case when the displacement u is parallel to the x-direction (longitudinal vibrations), and

$$c^2 = \mu/\rho \tag{1.76}$$

for transverse vibrations perpendicular to the x-direction. In general the sound is a superposition of longitudinal and transverse types of vibration. The longitudinal sound velocity is greater than the transverse velocity in solids, since in the longitudinal vibrations the density must also be compressed. In liquids and gases with $\mu = 0$ and $\lambda = 1/\kappa$ only longitudinal sound waves can exist (at low frequencies, as here assumed) with

$$c^2 = 1/(\kappa\rho). \tag{1.77}$$

Since the densities ρ can be different even in gases with the same compressibility κ, the sound velocity c always depends on the material. Usually one naturally thinks of sound in air under normal conditions.

1.5.4 Hydrodynamics

In this section we think less about solids but rather of isotropic liquids and gases. Nearly always we shall assume the flow to be incompressible, as suggested by water (hydro- comes from the Greek word for water).

(a) Bernoulli's Equation and Laplace's Equation

We call the flow static if $\boldsymbol{v} = 0$, and steady if $\partial \boldsymbol{v}/\partial t = 0$. (Is zero growth in the economy static or steady?) If the volume force $\boldsymbol{f}$ is conservative there is a potential ϕ with $\boldsymbol{f} = -\mathrm{grad}\,\phi$. Then in a steady incompressible flow with conservative volume force we have according to Euler's equation (1.68): $\rho(\boldsymbol{v}\,\mathrm{grad})\boldsymbol{v} = -\mathrm{grad}(\phi + P)$; here the pressure clearly becomes a sort of energy density (erg per cm^3).

Streamlines are the (averaged) velocity direction curves of the water molecules, and thus given mathematically by $\mathrm{d}x/v_\mathrm{x} = \mathrm{d}y/v_\mathrm{y} = \mathrm{d}z/v_\mathrm{z}$. If l is the length coordinate along a streamline, and $\partial/\partial l$ the derivative with respect to this coordinate in the direction of the streamline (hence in the direction of the velocity $\boldsymbol{v}$), then we have $|(\boldsymbol{v}\,\mathrm{grad})\boldsymbol{v}| = v\partial v/\partial l$, and hence for steady flows

$$-\partial(\phi + P)/\partial l = |-\mathrm{grad}(\phi + P)| = \rho v \partial v/\partial l = \rho\partial(v^2/2)/\partial l$$

analogous to the derivative of the energy law in one dimension (see Sect. 1.1.3a). Along a steady streamline we therefore have

$$\phi + P + \rho v^2/2 = \mathrm{const.} \tag{1.78}$$

(Bernoulli 1738). This is a conservation law for energy if one interprets the pressure, which derives from the forces between the molecules, as energy per cm^3; then ϕ is, for example, the gravitational energy and $\rho v^2/2$ the kinetic energy of a unit volume. This mechanical energy is therefore constant along a streamline, since friction is neglected. By measurement of the pressure difference one can then calculate the velocity.

A flow $\boldsymbol{v}$ is called a *potential flow* if there is a function Φ whose negative gradient is everywhere equal to the velocity $\boldsymbol{v}$. Since quite generally the curl of a gradient is zero, for potential flows curl $\boldsymbol{v} = 0$, i.e. the flow is "vortex free". If a potential flow is also incompressible, then we have $0 = \mathrm{div}\,\boldsymbol{v} = -\mathrm{div}\,\mathrm{grad}\Phi = -\nabla^2\Phi$ and

$$\nabla^2\Phi = 0 \quad \text{(Laplace Equation)}. \tag{1.79}$$

It can also be shown that (1.78) is then valid not only along a streamline, but also when comparing different streamlines:

$$\phi + P + \rho v^2/2 = \mathrm{const.} \tag{1.80}$$

in the whole region of an incompressible steady flow without friction, with conservative forces.

(b) Vortex Flows

The well-known Stokes's law states that

$$\Gamma = \oint \boldsymbol{v}\boldsymbol{d}l = \iint \mathrm{curl}(\boldsymbol{v})\boldsymbol{d}^2\boldsymbol{S}, \tag{1.81}$$

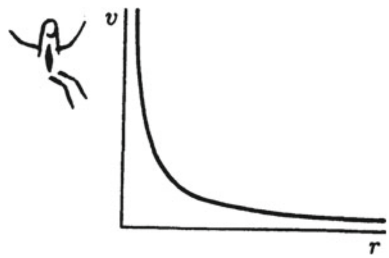


Fig. 1.17 Author's dream of the Lorelei and her whirlpool (schematic). Below and to the right is seen a vortex: velocity v as a function of distance r from the vortex, with core radius $a \to 0$

with the line integral $\boldsymbol{dl}$ along the rim of the area over which the integral $\boldsymbol{d}^2\boldsymbol{S}$ is integrated. In hydrodynamics Γ is called the circulation or *vortex strength*; it vanishes in a potential flow. Since Thomson (1860) it is known that

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = 0, \tag{1.82}$$

for incompressible fluids without friction (even unsteady), i.e. the circulation moves with the water particles.

With vortex lines, such as are realised approximately in a tornado, the streamlines are circles about a vortex axis, similar to the force lines of the magnetic field round a wire carrying a current. The velocity v of the flow is inversely proportional to the distance from the vortex axis, as can be observed at the drain hole of a bath-tub. In polar coordinates (r', ϕ) about the vortex axis an ideal vortex line therefore has the velocity

$$\boldsymbol{v} = \boldsymbol{e}_\phi \Gamma / 2\pi r', \quad r' > a$$
$$\boldsymbol{v} = \boldsymbol{e}_\phi \omega r', \quad r' < a$$

with the core radius a and the angular velocity $\omega = \Gamma/2\pi a^2$ within the core. Under these conditions curl $\boldsymbol{v} = 0$ outside the core and $= 2\omega$ in the core: the vortex strength is concentrated almost like a point mass at the core, assumed small.

For a hurricane, the core is called the eye, and there it is relatively calm; the film *Key Largo* is a good hurricane teacher. In the out-flowing bath-tub the core is replaced by air. In modern physical research vortices are of interest no longer because of the Lorelei, which enticed the Rhine boatmen into the whirlpool of old (Fig. 1.17), but because of the infinitely long lifetime of vortices in superfluid helium[3] at low temperatures because of quantum effects (Onsager, Feynman, around 1950). Also the lift of an aeroplane wing arises from the circulation about the wing; the wing is therefore the core of a sort of vortex line.

If two or more vortex lines are parallel side by side in the fluid, the core of each vortex line must move in the velocity field arising from the other vortex lines. For

[3]E.L. Andronikashvili and Yu.G. Mamaladze, p. 79 in: Progress in Low Temperature Physics, vol. V, edited by C.J. Gorter (North Holland, Amsterdam 1967).

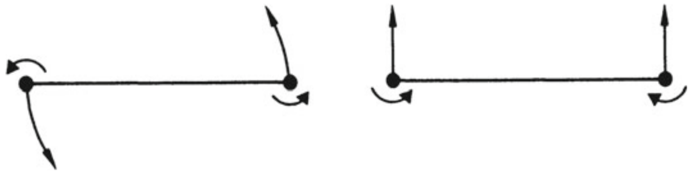
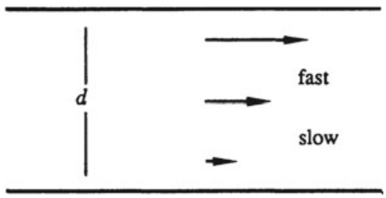


Fig. 1.18 Motion of a vortex pair with equal (*left*) and opposite (*right*) circulations

Fig. 1.19 Flow between two plates (side view) for determination of the viscosity η. The *upper plate* moves with velocity v_0 to the *right*, in contrast to the *lower plate*, which is fixed



the circulation is concentrated on a thin core and must move with the fluid, as stated above. So two parallel vortex filaments with $\Gamma_1 = -\Gamma_2$ follow a straight line course side by side, whereas if $\Gamma_1 = +\Gamma_2$ they dance round each other (Fig. 1.18). If one bends a vortex line into a closed ring, then for similar reasons this vortex ring moves with unchanging shape in a straight line: each part of the ring must move in the velocity field of all the other pans. This vorticity is also the reason why one can blow out candles, but not suck them out (danger of burning in proving this experimentally!). Also experienced smokers can blow smoke-rings (if the non-smokers let them).

(c) Fluids with Friction

In the "ideal" fluids investigated up to this point there is no friction, and so the stress tensor σ consists only of the pressure P: $\sigma_{ik} = -P\delta_{ik}$. If, however, we stir honey with a spoon we create shear stresses such as σ_{12}, which are proportional to the velocity differences.

Just as two elastic constants μ and λ sufficed in the elasticity theory for isotropic solids in (1.70), we need only two *viscosities*, η and ζ (with $\mathcal{E}$ = unit tensor) for the stresses caused by friction:

$$\sigma' = 2\eta\varepsilon' + (\zeta - 2\eta/3)\mathcal{E}\,\mathrm{Tr}(\varepsilon'). \tag{1.83}$$

Here σ' is the stress tensor without the pressure term, and ε' has the matrix elements $(\partial v_i/\partial x_k + \partial v_k/\mathrm{d}x_i)/2$, since the corresponding expression with $\boldsymbol{u}$ in (1.59) makes little sense for fluids. The trace of the tensor ε' is then simply div $\boldsymbol{v}$, so that in incompressible flows the complicated second term in (1.83) drops out. Thus hydrodynamics usually requires only one friction coefficient whereas elasticity theory needs two.

Let us consider as an example the flow between two parallel plates perpendicular to the z-axis (Fig. 1.19). The upper plate at $z = d$ moves with velocity v_0 to the right, the lower plate at $z = 0$ is fixed. After some time a steady fluid flow is established between the plates: $\boldsymbol{v}$ points only in the x-direction to the right, with $v_x(z) = v_0 z/\mathrm{d}$, independently of x and y. Accordingly div $\boldsymbol{v} = 0$: the flow is incompressible then, even if the fluid itself is compressible. The tensor $\sigma' = 2\eta\varepsilon'$ according to (1.83) contains many zeros, since only $\varepsilon'_{13} = \varepsilon'_{31} = (\partial v_x/\partial z + 0)/2 = v_0/2\mathrm{d}$ is different from zero:

$$\sigma'_{13} = \eta v_0/d.$$

This is therefore the force in the x-direction, which is exerted on each square centimetre of the plates perpendicular to the z-direction, in order to overcome the frictional resistance of the fluid. In principle the viscosity η can be measured in this way, although falling spheres (see below) are a more practical method for determining the viscosity. The other viscosity ζ only comes into it if the density changes, as for example in the damping of shock waves.

With this stress tensor σ' and the pressure P, (1.67) has the form

$$\rho \mathrm{d}\boldsymbol{v}/\mathrm{d}t = \mathrm{div}\,\sigma' - \mathrm{grad}\,P + \boldsymbol{f},$$

which can be rewritten (see (1.83)) to be analogous to (1.71)

$$\rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \eta\nabla^2\boldsymbol{v} + (\zeta + \eta/3)\mathrm{grad}\ \mathrm{div}\boldsymbol{v} - \mathrm{grad}\ P + \boldsymbol{f}. \tag{1.84}$$

In the special case of incompressible flow div $\boldsymbol{v} = 0$ and $\boldsymbol{f} = 0$ this yields the celebrated *Navier-Stokes* equation (1822):

$$\rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \eta\nabla^2\boldsymbol{v} - \mathrm{grad}\ P, \tag{1.85}$$

which has already used up much work and storage in many computers. Since ρ is now constant we can, if the pressure is also constant, define the kinematic viscosity $\nu = \eta/\rho$, and write

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \nu\nabla^2\boldsymbol{v}. \tag{1.86}$$

This equation has the form of a *diffusion* or heat conduction equation, ignoring the difference (negligible for small velocities) between $\mathrm{d}\boldsymbol{v}/\mathrm{d}t$ and $\partial\boldsymbol{v}/\partial t$. A high velocity concentrated in one place is therefore propagated outwards by friction just like the temperature of a solid briefly heated in one place, until eventually the whole fluid has the same velocity. The solution is $\exp(-t/\tau)\sin(\boldsymbol{Q}r)$ with $1/\tau = \nu Q^2$, if $\sin(\boldsymbol{Q}r)$ is the starting condition, whether it is the propagation of small velocities in a viscous fluid, heat in a solid, or molecules in a porous material. In air, water and glycerine ν is of order 10^{-1}, 10^{-2} and $10\,\mathrm{cm}^2/\mathrm{s}$, respectively.

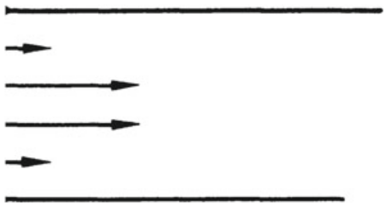


Fig. 1.20 Poiseuille flow through a long tube, with parabolic velocity profile $v_x(r), 0 < r < R$

(d) Poiseuille Law (1839)

A somewhat more complicated flow than that described above between moving and fixed plates is that through a long tube (Fig. 1.20). In the middle the water flows fastest, at the walls it "sticks". For the steady solution we require the Navier-Stokes equation: $0 = -\text{grad}\, P + \eta\nabla^2 v$, or since all the flow is only towards the right in the x-direction: $\partial P/\partial x = \eta\nabla^2 v_x$. P is independent of y and z, whereas v_x is a function of the distance r from the centre of the tube; $v_x(r = R) = 0$ at the wall of the tube with radius R.

For a quantity A independent of angle we have in general

$$\nabla^2 A = \mathrm{d}^2 A/\mathrm{d}r^2 + \frac{d-1}{r}\mathrm{d}A/\mathrm{d}r$$

in d dimensions. Here $d = 2$ (polar coordinates for the cross-section of the tube); moreover $P' = \partial P/\partial x = -\Delta P/L$ in a tube with length L and pressure difference ΔP. Accordingly, we have to solve

$$P' = \eta\left(\mathrm{d}^2 v_x/\mathrm{d}r^2 + \frac{1}{r}\mathrm{d}v_x/\mathrm{d}r\right) = \frac{\eta}{r}\mathrm{d}(r\mathrm{d}v_x/\mathrm{d}r)/\mathrm{d}r,$$

(these transformations of ∇^2 are also useful elsewhere). We find that

$$\begin{aligned} r\mathrm{d}v_x/\mathrm{d}r &= P'r^2/2\eta + \text{const.}, \\ v_x &= P'r^2/4\eta + \text{const}\ \ln(r) + \text{const.}'. \end{aligned}$$

Since the velocity at $r = 0$ must be finite, the const is zero, and since at $r = R$ the velocity must be zero, $\text{const.}' = -P'R^2/4\eta$, so that

$$v_x = \frac{\Delta P}{4L\eta}(R^2 - r^2), \tag{1.87}$$

and the velocity profile is a parabola. The total flow through the tube (grams per second) is

$$J = \rho\int\!\!\int v_x(r)\mathrm{d}z\mathrm{d}y = (\rho\Delta P\pi/8L\eta)R^4$$

so that

$$J \sim R^4. \tag{1.88}$$

The flow of water through a tube is therefore not proportional to the cross-section, but to the square of the cross-section, since the maximal velocity in the centre of the tube, (1.87), is itself proportional to the cross-section. This law also can be applied to the measurement of the viscosity. It no longer holds when the steady flow becomes unstable at high velocities because of turbulence.

Modern research in hydrodynamics has to do with, for example, the flow of oil and water through porous media. When an oil well "runs dry" there is still a great deal of oil in the porous sand. When one tries to squeeze it out by pumping water into the sand, complex instabilities arise, with beautiful, but unhelpful, fractal structures (see Chap. 5). Hydrodynamics is no dead formalism!

Fractal[4] is the name given to objects with masses proportional to $(\text{radius})^D$ and a fractal dimension D differing from the space dimension d; other fractals are snowflakes, the path of a diffusing particle, polymer chains in solutions, geographical features, and also the "clusters" which the magnetism program of Sect. 2.2.2 produces on the computer near the Curie point. Since about 1980, fractals (see Chap. 5) have been a rapidly expanding research field in physics.

(e) Similarity Laws

Quite generally, one should always try first to solve complicated differential equations in dimensionless form. Thus, if one divides all velocities by a velocity typical of the flow v_0, all lengths by a typical length l, etc., setting $r/l = r'$, $v/v_0 = v'$, $t/(l/v_0) = t'$, $P/(\rho v_0^2) = P'$ then (1.85) takes the dimensionless form

$$\mathrm{d}\boldsymbol{v}'/\mathrm{d}t' = \nabla'^2\boldsymbol{v}'/\mathrm{Re} - \mathrm{grad}'\, P',$$

where Re is the so-called *Reynolds number*, defined as

$$\mathrm{Re} = v_0 l\rho/\eta = v_0 l/\nu \tag{1.89}$$

We can study the Navier-Stokes equation without knowing v_0 and l; one only needs to know the value of Re. If one has found a solution (exact, on the computer, or by experiment) of the Navier-Stokes equation for a certain geometry, the flow for a similar geometry (uniform magnification or shrinking factor) is similar, if only the Reynolds number is the same. A tanker captain can therefore get initial experience in the control of a ship in a small tank, if the flow conditions in the tank reproduce the full-scale flow with the same Reynolds number (if we neglect gravity).

It turns out, for example, that the steady solutions obtained so far are stable only up to Reynolds numbers of about 10^3. Above that value turbulence sets in, with the spontaneous formation of vortices. This also is a current field of research.

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[4]See, e.g., B. Mandelbrot: The Fractal Geometry of Nature (Freeman, New York, San Francisco 1982); also Physica D 38 (1989).

If, for example, one heats a flow between two plates from below, "Rayleigh-Benard" instabilities occur with large temperature differences Δ, and these are also observed in the atmosphere (spacewise periodic clouds). With particularly large Δ the heat flow increases with an experimentally determined $\Delta^{1.28}$ (Libchaber and co-workers 1988) in contrast to normal heat conduction; theoretically an exponent 9/7 is predicted.

If a sphere of radius R sinks under its own weight through a viscous fluid with velocity v_0, then the ratio: force$/(\rho v_0^2 R^2)$ is dimensionless and therefore according to the Navier-Stokes law is a function only of the Reynolds number $\mathrm{Re} = v_0 R/\nu$. For small Re this frictional force F is proportional as usual to the velocity: $F = \mathrm{const.}(\rho v_0^2 R^2)/\mathrm{Re} = \mathrm{const.} v_0 R\eta$. Exact calculation gives const. $= 6\pi$ and hence the Stokes law

$$F = 6\pi\eta v_0 R. \tag{1.90}$$

Our dimensional analysis has thus spared us much calculation, but of course does not provide the numerical factor 6π. The Stokes law provides a convenient method for measuring η.

Another dimensionless ratio is the Knudsen number $\mathrm{Kn} = \lambda/l$, where λ is the mean free path length of gas molecules. Our hydrodynamics is valid only for small Knudsen numbers. Other examples are the Peclet number, the Nusselt number and the Rayleigh number.

In conclusion it should be noticed that the forces acting on solids, liquids and gases, such as we have been treating here, are quite generally linked by linear combinations of the tensors ε and σ, their traces and their time derivatives. Our results up to now are therefore special cases: our simple equation $\rho(P) = \rho(P = 0)(1 + \kappa P)$ uses only $\mathrm{Tr}(\sigma)$ and $\mathrm{Tr}(\varepsilon)$: the much more complicated equation (1.69) links σ and ε and (1.83) also does this (only ε is then defined by the time derivative of the position).

Questions

Section 1.1

1. State Kepler's third law.
2. When do force-free bodies move in a straight line?
3. What force does a stone exert on a string when it is whirled round at constant speed?
4. With what speed must we throw a stone upwards, in order that it should escape the earth's gravity field? (Energy conservation: potential energy is $-GMm/r$, where r is the distance from the centre of the earth.)
5. Estimate the numerical value of the mean density of the earth ρ, from G, g and the earth's radius R.

Section 1.2

6. What is the "reduced mass" in the two-body problem?
7. State d'Alembert's Principle on constraining forces.
8. State the principle of virtual displacement with constraining forces.

Section 1.3

9. Why does Hamilton's principle apply only with fixed endpoints?
10. What are the variables of the Lagrange function L, and those of the Hamilton function H?
11. What are optic and acoustic phonons?

Section 1.4

12. What are the relationships between torque $\boldsymbol{M}$, angular momentum $\boldsymbol{L}$, inertia tensor Θ and angular velocity $\boldsymbol{\omega}$? Is $\boldsymbol{\omega}$ a vector?
13. What are the "principal axes" of an inertia tensor, and what are the (principal) moments of inertia?
14. What is the nutation frequency of a cube rotating about an axis of symmetry?
15. Do the Euler equations determine the amplitude of the nutation of a symmetrical gyroscope?
16. Why does the axis of the gyroscope move perpendicularly to the applied force?
17. What is "Larmor precession" and what is it used for?

Section 1.5

18. What is the difference between $\partial/\partial t$ and $\mathrm{d}/\mathrm{d}t$ in continuum physics?
19. What is an equation of continuity?
20. What are the relationships between pressure, stress tensor and strain tensor?
21. What is the difference between: hurricane, typhoon and tornado?
22. What is the meaning of: incompressible, vortex-free, ideal, steady, static?
23. For what values of the "Knudsen number" λ/R is Stokes's formula for the motion in air of spheres (radius R) valid?
24. With what diffusion constant D does a cluster of spheres disperse in a viscous fluid, when according to Einstein diffusivity/mobility $= k_{\mathrm{B}}T$?

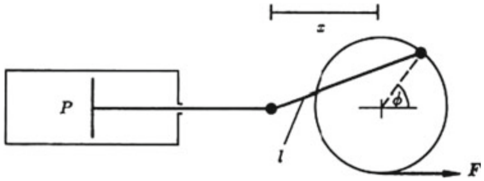
Problems

Section 1.1

1. Is a uniform motion in a straight line transformed into uniform motion in a straight line by a Galileo transformation?
2. Describe in one or two pages the Coriolis force, e.g., when shooting polar bears at the north pole.
3. A point mass moves on a circular orbit round an isotropic force centre with potential $\sim r^{-x}$. For what values of x is this orbit stable, i.e., at a minimum of the effective potential energy?
4. With what velocity does a point mass fall from the height h to earth, first if $h \ll$ earth radius, then generally?

Section 1.3

5. Using the principle of virtual displacements, calculate the pressure on the piston, if the force $\boldsymbol{F}$ acts on the wheel.



6. Lagrange equation of the first kind in cylindrical coordinates: a point mass moves in a gravity field on a rotationally symmetric tube $\rho = W(z)$, with $\rho^2 = x^2 + y^2$, where the height h and the angular velocity ω are constant ("centrifuge"). What shape $W(z)$ must the tube have, if ω is to be independent of z? *Hint*: resolve the acceleration into components e_{r}, e_ϕ and e_{z} in cylindrical coordinates.
7. Study the Lagrange equation for a thin hoop rolling down a hillside. *Hint*: $T = T_{\mathrm{trans}} + T_{\mathrm{rot}}$; all point masses are at the same distance from the centre.
8. Prove in the general case that $\{q_\mu, p_\nu\} = \delta_{\mu\nu}$, $\{p_\mu, p_{\mathrm{v}}\} = \{q_\mu, q_{\mathrm{v}}\} = 0$, $\{F_1 F_2, G\} = F_1\{F_2, G\} + F_2\{F_1, G\}$ and hence study $\mathrm{d}\boldsymbol{p}/\mathrm{d}t = \{\boldsymbol{p}, H\}$ for the three-dimensional oscillator, $U = Kr^2/2$.
9. Using the harmonic approximation calculate the vibration frequencies of a particle in a two-dimensional potential $U(x, y)$.

Section 1.4

10. Discuss in the harmonic approximation the stability of free rotation of a rigid body about its principal axes, with $\Theta_1 > \Theta_2 > \Theta_3$.
11. Which properties of matrices does one need in theoretical mechanics? Is $(\Theta_1, \Theta_2, \Theta_3)$ a vector? Is $\Theta_1 + \Theta_2 + \Theta_3$ a "scalar", i.e., invariant under rotation of the coordinate axes?
12. Calculate the inertia tensor of a cylinder with mass M, radius R and height H in a convenient system of reference.
Hint: $\int_0^1 (1 - x^2)\mathrm{d}x = \pi/4$ and $\int_0^1 (1 - x^2)^{3/2}\mathrm{d}x = 3\pi/16$.

Section 1.5

13. What is the form of the strain tensor if a specimen expands in the x-direction by one part in a thousand, shrinks in the y-direction by one part in a thousand, and stays unchanged in the z-direction? What is the volume change? What is changed if we also have $\varepsilon_{13} = 10^{-3}$? What follows generally from $\varepsilon_{\mathrm{ki}} = \varepsilon_{\mathrm{ik}}$?
14. An iron wire is extended by a tension ΔP (= force/area). Prove that $\Delta P = (\Delta l/l)E$ for the change in length, with $E = (2\mu + 3\lambda)\mu/(\mu + \lambda)$, and $\Delta V/V = \Delta P\kappa/3$ for the volume change, with $\kappa = 3/(2\mu + 3\lambda)$.

15. Use the Gaussian theorem $\nabla^2 1/r = -4\pi\delta(r)$ to show that a vortex line is (almost) a potential flow. At what speed do two vortices with the same circulation move around each other?

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