



Chapter 2

Center of Gravity, Center of Mass, Centroids

2

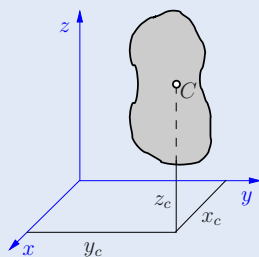
Centroid of a Volume

The coordinates of the Centroid of Volume of a body with volume V are given by

$$x_c = \frac{\int x \, dV}{\int dV},$$

$$y_c = \frac{\int y \, dV}{\int dV},$$

$$z_c = \frac{\int z \, dV}{\int dV}.$$



Centroid of an Area

$$x_c = \frac{\int x \, dA}{\int dA},$$

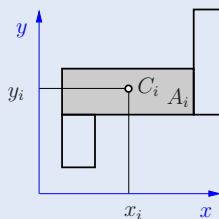
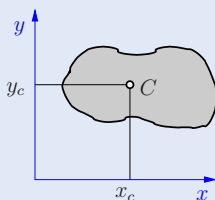
$$y_c = \frac{\int y \, dA}{\int dA}.$$

Here, $\int x \, dA = C_y$ and $\int y \, dA = C_x$ denote the *first moments of the area* with respect to the y - and x -axis, respectively.

For *composite* areas, where the coordinates (x_i, y_i) of the centroids C_i of the individual subareas A_i are known, we have

$$x_s = \frac{\sum x_i A_i}{\sum A_i},$$

$$y_s = \frac{\sum y_i A_i}{\sum A_i}.$$



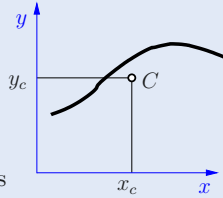
Remarks:

- When analyzing areas (volumes) with holes, it can be expedient to work with “negative” subareas (subvolumes).
- If the area (volume) has an axis of symmetry, the centroid of the area (volume) lies on this axis.

Centroid of a Line

$$x_c = \frac{\int x \, ds}{\int ds},$$

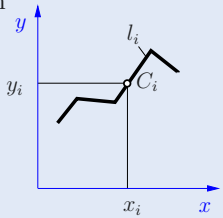
$$y_c = \frac{\int y \, ds}{\int ds}.$$



If a line is composed of several sublines of length l_i with the associated coordinates x_i, y_i of its centroids, the location of the centroid follows from

$$x_c = \frac{\sum x_i l_i}{\sum l_i},$$

$$y_c = \frac{\sum y_i l_i}{\sum l_i}.$$



Center of Mass

The coordinates of the center of mass of a body with density $\rho(x, y, z)$ are given by

$$x_c = \frac{\int x \rho \, dV}{\int \rho \, dV}, \quad y_c = \frac{\int y \rho \, dV}{\int \rho \, dV}, \quad z_c = \frac{\int z \rho \, dV}{\int \rho \, dV}.$$

Consists a body of several subbodies V_i with (constant) densities ρ_i and associated known coordinates x_i, y_i, z_i , of the centroids of the subvolumes then it holds

$$x_c = \frac{\sum x_i \rho_i V_i}{\sum \rho_i V_i}, \quad y_c = \frac{\sum y_i \rho_i V_i}{\sum \rho_i V_i}, \quad z_c = \frac{\sum z_i \rho_i V_i}{\sum \rho_i V_i}.$$

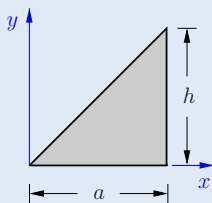
Remark:

For a *homogeneous* body ($\rho = \text{const}$), the center of mass and the centroid of the volume coincide.

Location of Centroids

Areas

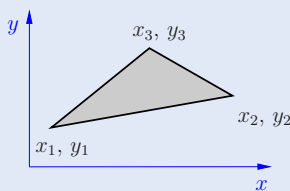
triangle



$$x_c = \frac{2}{3} a$$

$$y_c = \frac{1}{3} h$$

$$A = \frac{1}{2} ah$$

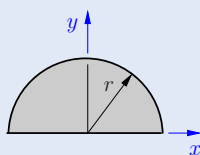


$$x_c = \frac{1}{3} (x_1 + x_2 + x_3)$$

$$y_c = \frac{1}{3} (y_1 + y_2 + y_3)$$

$$A = \frac{1}{2} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}$$

semicircle

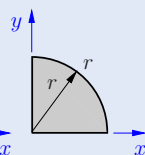


$$x_c = 0$$

$$y_c = \frac{4}{3\pi} r$$

$$A = \frac{\pi}{2} r^2$$

quarter circle

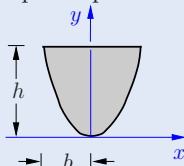


$$= \frac{4}{3\pi} r$$

$$= \frac{4}{3\pi} r$$

$$= \frac{\pi}{4} r^2$$

quadr. parabola

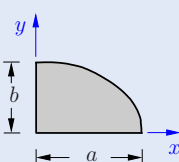


$$= 0$$

$$= \frac{3}{5} h$$

$$= \frac{4}{3} bh$$

quarter ellipse



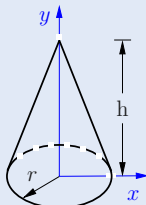
$$= \frac{4}{3\pi} a$$

$$= \frac{4}{3\pi} b$$

$$= \frac{\pi}{4} ab$$

Volumes

cone

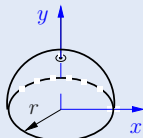


$$x_c = 0$$

$$y_c = \frac{1}{4} h$$

$$V = \frac{1}{3} \pi r^2 h$$

hemisphere



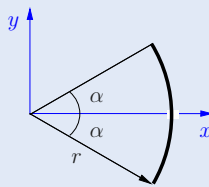
$$x_c = 0$$

$$y_c = \frac{3}{8} r$$

$$V = \frac{2}{3} \pi r^3$$

Line

circular arc



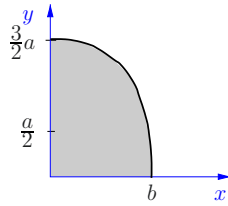
$$x_c = \frac{\sin \alpha}{\alpha} r$$

$$y_c = 0$$

$$l = 2 \alpha r$$

Problem 2.1 The depicted area is bounded by the coordinate axes and the quadratic parabola with its apex at $x = 0$.

Determine the coordinates of the centroid.



P.2.1

Solution The equation of the parabola is given by

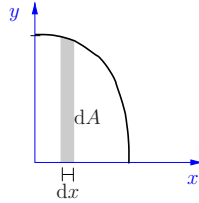
$$y = -\alpha x^2 + \beta.$$

The constants α and β follow with the aid of the points $x_0 = 0$, $y_0 = 3a/2$ and $x_1 = b$, $y_1 = a/2$ as $\beta = 3a/2$ and $\alpha = a/b^2$. Thus, the equation of the can be written as

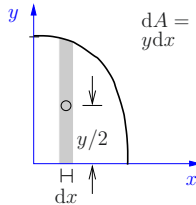
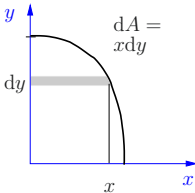
$$y = -a \left(\frac{x}{b} \right)^2 + \frac{3a}{2}.$$

With the infinitesimal area $dA = y dx$, it follows

$$\begin{aligned} \underline{\underline{x_C}} &= \frac{\int x dA}{\int dA} = \frac{\int x y dx}{\int y dx} \\ &= \frac{\int_0^b x \left[-a \left(\frac{x}{b} \right)^2 + \frac{3a}{2} \right] dx}{\int_0^b \left[-a \left(\frac{x}{b} \right)^2 + \frac{3a}{2} \right] dx} = \frac{\frac{1}{2} ab^2}{\frac{7}{6} ab} = \underline{\underline{\frac{3}{7} b}}. \end{aligned}$$



In order to determine the y -coordinate, we choose for simplicity again the infinitesimal area element $dA = y dx$ instead of $dA = x dy$,

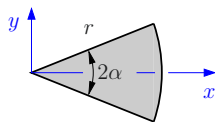


because we have already used it above. Now, we have to take into account that its centroid is located at the height $y/2$. Hence, we obtain

$$\underline{\underline{y_C}} = \frac{\int \frac{y}{2} y dx}{\int y dx} = \frac{6}{14 ab} \int_0^b \left(a^2 \frac{x^4}{b^4} + \frac{3a^2}{b^2} x^2 + \frac{9a^2}{4} \right) dx = \underline{\underline{\frac{87}{140} a}}.$$

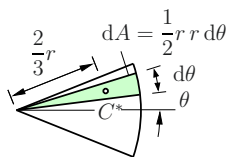
P2.2

Problem 2.2 Locate the centroid of the depicted circular sector with the opening angle 2α .



Solution Due to symmetry reasons, we obtain $y_C = 0$. In order to determine x_C we use the infinitesimal sector of the circle (= triangle) and integrate over the angle θ

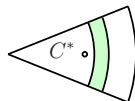
$$\begin{aligned} \underline{\underline{x_C}} &= \frac{\int_{-\alpha}^{\alpha} \left(\frac{2}{3} r \cos \theta \right) \frac{1}{2} r r d\theta}{\int_{-\alpha}^{\alpha} \frac{1}{2} r r d\theta} = \frac{r^3 2 \sin \alpha}{3 r^2 \alpha} \\ &= \underline{\underline{\frac{2}{3} \frac{\sin \alpha}{\alpha} r}}. \end{aligned}$$



In the limit case of a semicircular area ($\alpha = \pi/2$), the centroid is located

$$\underline{\underline{x_C = \frac{4}{3\pi} r}}.$$

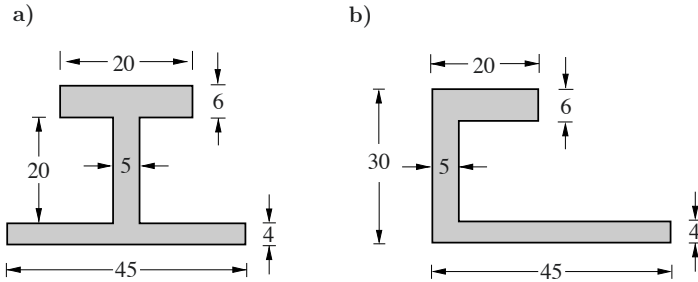
Remark: Alternatively, the determination of the centroid may be done by the decomposition of the area into circular rings and integration over x . In this case the centroid C^* of the circular rings has to be known or determined a priori.



We may determine the centroid of a circular segment with the aid of the above calculations and by subtraction:

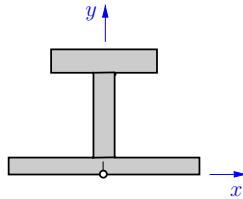
$$\begin{aligned} \underline{\underline{x_C}} &= \frac{x_{C_I} A_I - x_{C_{II}} A_{II}}{A_I - A_{II}} = \frac{\frac{2 \sin \alpha}{3 \alpha} r r^2 \alpha - \frac{1}{2} s r \cos \alpha \frac{2}{3} r \cos \alpha}{r^2 \alpha - \frac{1}{2} s r \cos \alpha} = \underline{\underline{\frac{s^3}{12 A}}}. \end{aligned}$$

Problem 2.3 Locate the centroids of the depicted profiles. The measurements are given in mm.



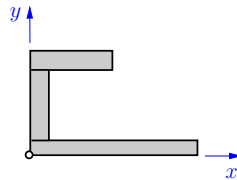
Solution a) The coordinate system is placed, such that the y -axis coincides with the symmetry axis of the system. Therefore, we know $x_C = 0$. In order to determine y_C , the system is decomposed into three rectangles with known centroids and it follows

$$\begin{aligned}\underline{\underline{y_C}} &= \frac{\sum y_i A_i}{\sum A_i} \\ &= \frac{2(4 \cdot 45) + 14(5 \cdot 20) + 27(6 \cdot 20)}{4 \cdot 45 + 5 \cdot 20 + 6 \cdot 20} \\ &= \frac{5000}{400} = \underline{\underline{12.5 \text{ mm}}}.\end{aligned}$$



b) The origin of the coordinate system is placed in the lower left corner. Decomposition of the system into rectangles leads to

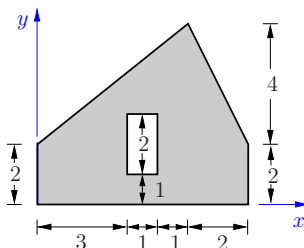
$$\begin{aligned}\underline{\underline{x_C}} &= \frac{22.5(4 \cdot 45) + 2.5(5 \cdot 20) + 10(6 \cdot 20)}{4 \cdot 45 + 5 \cdot 20 + 6 \cdot 20} \\ &= \frac{5500}{400} = \underline{\underline{13.75 \text{ mm}}}, \\ \underline{\underline{y_C}} &= \frac{2(4 \cdot 45) + 14(5 \cdot 20) + 27(6 \cdot 20)}{400} \\ &= \underline{\underline{12.5 \text{ mm}}}.\end{aligned}$$



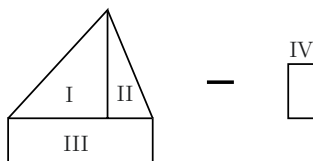
Remark: Note that a displacement of the system in the x -direction does not change the y -coordinate of the centroid.

P2.4

Problem 2.4 Locate the centroid of the depicted area with a rectangular cutout. The measurements are given in cm.



Solution First we decompose the system into two triangles (I, II) and one rectangle (III), from which we subtract the rectangular cutout (IV). The centroids are known for each subsystem.



The calculation is conveniently done by using a table.

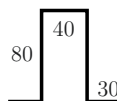
Sub-system i	A_i [cm ²]	x_i [cm]	$x_i A_i$ [cm ³]	y_i [cm]	$y_i A_i$ [cm ³]
I	10	$\frac{10}{3}$	$\frac{100}{3}$	$\frac{10}{3}$	$\frac{100}{3}$
II	4	$\frac{17}{3}$	$\frac{68}{3}$	$\frac{10}{3}$	$\frac{40}{3}$
III	14	$\frac{7}{2}$	49	1	14
IV	-2	$\frac{7}{2}$	-7	2	-4
	$A = \sum A_i = 26$		$\sum x_i A_i = 98$		$\sum y_i A_i = \frac{170}{3}$

Thus, we obtain

$$\underline{\underline{x_C}} = \frac{\sum x_i A_i}{A} = \frac{98}{26} = \underline{\underline{\frac{49}{13} \text{ cm}}}, \quad \underline{\underline{y_C}} = \frac{\sum y_i A_i}{A} = \frac{170/3}{26} = \underline{\underline{\frac{85}{39} \text{ cm}}}.$$

Problem 2.5 A wire with constant thickness is deformed into the depicted figure. The measurements are given in mm.

Locate the centroid.



Solution We choose coordinate axes, such that y is the symmetry axis. Then, due to symmetry reasons, we can identify $x_C = 0$. The y -coordinate of the centroid follows generally by decomposition as

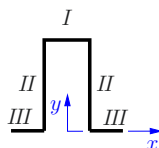
$$y_C = \frac{\sum y_i l_i}{\sum l_i}.$$

Three alternative solutions will be shown. The total length of the wire is

$$l = \sum l_i = 2 \cdot 30 + 2 \cdot 80 + 40 = 260 \text{ mm}.$$

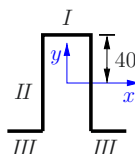
a)

$$\begin{aligned} \underline{\underline{y_C}} &= \frac{1}{260} (\underbrace{80 \cdot 40}_I + \underbrace{2 \cdot 40 \cdot 80}_{II}) \\ &= \frac{9600}{260} = \underline{\underline{36.92 \text{ mm}}}. \end{aligned}$$



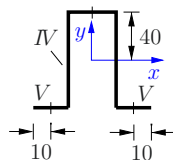
b)

$$\begin{aligned} \underline{\underline{y_C}} &= \frac{1}{260} (\underbrace{40 \cdot 40}_I - \underbrace{2 \cdot 40 \cdot 30}_{III}) \\ &= \underline{\underline{-3.08 \text{ mm}}}. \end{aligned}$$



c) We choose a specific subsystem IV such that its centroid coincides with the origin of the coordinate system:

$$\underline{\underline{y_C}} = \frac{1}{260} [\underbrace{2 \cdot (-40) \cdot 10}_V] = \underline{\underline{-3.08 \text{ mm}}}.$$

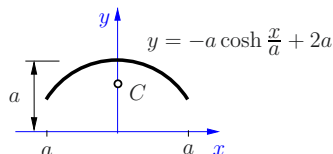


The advantage of alternative c) is, that only the first moment of subsystem V has to be taken into account.

P2.6

Problem 2.6 A thin wire is bent to a hyperbolic function.

Locate the centroid.



Solution The centroid is located on the y -axis due to the symmetry of the system ($x_C = 0$). We obtain the infinitesimal arc length ds with aid of the derivative $y' = -\sinh \frac{x}{a}$ as

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (y')^2} dx = \sqrt{1 + \sinh^2 \frac{x}{a}} dx = \cosh \frac{x}{a} dx.$$

The total arc length follows by integration:

$$s = \int ds = \int_{-a}^{+a} \cosh \frac{x}{a} dx = 2a \sinh 1.$$

The first moment of the line with respect to the x -axis is given by

$$S_x = \int y ds = a^2 \left(4 \sinh 1 - \frac{1}{2} \sinh 2 - 1 \right).$$

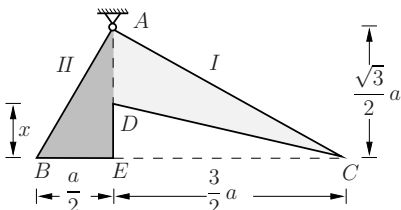
Hence, the centroid is located at

$$\underline{\underline{y_C}} = \frac{\int y ds}{\int ds} = \frac{4a^2 \sinh 1 - \frac{1}{2}a^2 \sinh 2 - a}{2a \sinh 1} = \underline{\underline{0.803 a}}.$$

P2.7

Problem 2.7 From the triangular-shaped metal sheet ABC , the triangle CDE has been cut out. The system is pin supported in A .

Determine x such that \overline{BC} adjusts horizontal.



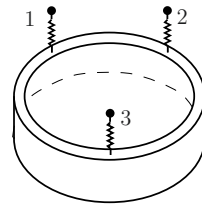
Solution The system is in the required position, if the centroid is located vertically below A . Consequently, the first moments of the triangular-shaped subsystems ADC and ABE have to be equal with respect to the point A :

$$\underbrace{\frac{1}{2} \left(\frac{\sqrt{3}}{2} a - x \right) \frac{3}{2} a}_{\text{area } ADC} \underbrace{\frac{1}{3} \frac{3}{2} a}_{\text{distance}} = \underbrace{\frac{1}{2} \frac{a}{2} \frac{\sqrt{3}}{2} a}_{\text{area } ABE} \underbrace{\frac{1}{3} \frac{a}{2}}_{\text{distance}} \quad \leadsto \quad \underline{\underline{x = \frac{4}{9} \sqrt{3} a}}.$$

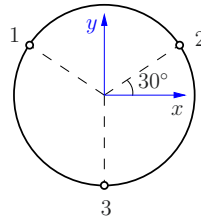
Problem 2.8 A piece of a pipe of weight W is fixed by three spring scales as depicted. The spring scales are equally distributed along the edge of the pipe. They measure the following forces:

$$F_1 = 0.334 W, \quad F_2 = 0.331 W, \\ F_3 = 0.335 W.$$

Now an additional weight shall be attached to the pipe in order to shift the centroid of the total system into the center of the pipe (=static balancing). Determine the location and the magnitude of the additional weight.



Solution We know, due to the different measured forces, that the system is not balanced. Thus, the gravity center C (=location of the resulting weight) is not located in the middle of the ring, but coincides with the location of the resultant of the spring forces. Therefore, in a first step, we determine the location of the center of these forces. This can be done by the equilibrium of moments about the x - and y -axis:



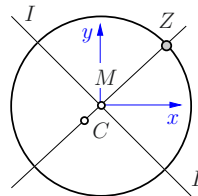
$$y_C W = r \sin 30^\circ (0.334 W + 0.331 W) - r 0.335 W,$$

$$\leadsto y_C = -0.0025 r,$$

$$x_C W = r \cos 30^\circ (0.331 W - 0.334 W),$$

$$\leadsto x_C = -0.0026 r.$$

In order to recalibrate the gravity center into the center M of the ring, the additional required weight Z has to be applied on the intersection point of the ring and the line \overline{CM} . The weight of Z can be determined from the equilibrium of the moments about the perpendicular axis I :



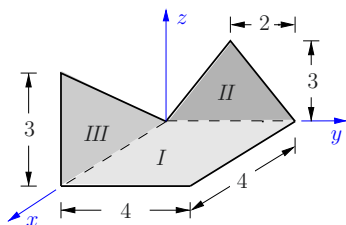
$$r Z = \overline{CM} W \quad \leadsto \quad r Z = \sqrt{x_C^2 + y_C^2} W$$

$$\leadsto \underline{\underline{Z}} = \sqrt{(0.0025)^2 + (0.0026)^2} W = \underline{\underline{0.0036 W}}.$$

P2.9

Problem 2.9 A thin sheet with constant thickness and density, consisting of a square and two triangles, is bent to the depicted figure (measurements in cm).

Locate the center of gravity.



Solution The body is composed by three parts with already known location of centers of mass. The location of the center of mass of the complete system can be determined from

$$x_C = \frac{\sum \rho_i x_i V_i}{\sum \rho_i V_i}, \quad y_C = \frac{\sum \rho_i y_i V_i}{\sum \rho_i V_i}, \quad z_C = \frac{\sum \rho_i z_i V_i}{\sum \rho_i V_i}.$$

Since the thickness and the density of the sheet is constant, these terms cancel out and we obtain

$$x_C = \frac{\sum x_i A_i}{\sum A_i}, \quad y_C = \frac{\sum y_i A_i}{\sum A_i}, \quad z_C = \frac{\sum z_i V_i}{\sum A_i}.$$

The total area is

$$A = \sum A_i = 4 \cdot 4 + \frac{1}{2} \cdot 4 \cdot 3 + \frac{1}{2} \cdot 4 \cdot 3 = 28 \text{ cm}^2.$$

Calculating the first area moments of the total system about each axis, in each case one first moment of a subsystem drops out because of zero distance: $x_{II} = 0$, $y_{III} = 0$, $z_I = 0$. Thus, we obtain

$$\underline{\underline{x_C}} = \frac{x_I A_I + x_{III} A_{III}}{A} = \frac{2 \cdot 16 + \left(\frac{2}{3} \cdot 4\right) 6}{28} = \underline{\underline{1.71 \text{ cm}}},$$

$$\underline{\underline{y_C}} = \frac{y_I A_I + y_{II} A_{II}}{A} = \frac{2 \cdot 16 + 2 \cdot 6}{28} = \underline{\underline{1.57 \text{ cm}}},$$

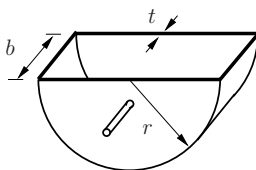
$$\underline{\underline{z_C}} = \frac{z_{II} A_{II} + z_{III} A_{III}}{A} = \frac{\left(\frac{1}{3} \cdot 3\right) 6 + \left(\frac{1}{3} \cdot 3\right) 6}{28} = \underline{\underline{0.43 \text{ cm}}}.$$

Problem 2.10 A semi-circular bucket is produced from a steel sheet with the thickness t and density ρ_S .

a) Determine the required distance of the bearing pivots to the upper edge, such that it is easy to turn the empty bucket around the pivots.

b) Consider a steel bucket which is filled with material of the density ρ_M . How does this change the required distance of the pivots?

Given: $b = r$, $t = r/100$, $\rho_M = \rho_S/3$

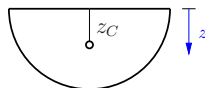


Solution The bucket tilts easiest by positioning the pivots in the axis of the center of mass.

a) In case of an empty bucket (=homogeneous body), the center of mass coincides with the center of volume. Since the sheet thickness is constant, it cancels out. With the centroids of the subareas

$$\text{semi circle} \quad z_1 = \frac{4r}{3\pi}$$

$$\text{semi circular arc} \quad z_2 = \frac{2r}{\pi}$$



we obtain

$$\underline{\underline{z_{CE}}} = \frac{z_1 A_1 + z_2 A_2}{A_1 + A_2} = \frac{\frac{4r}{3\pi} 2 \frac{\pi r^2}{2} + \frac{2r}{\pi} \pi r b}{2 \frac{\pi r^2}{2} + \pi r b} = \underline{\underline{\frac{4r + 6b}{3\pi(r+b)} r}}.$$

b) In case of the filled bucket, we obtain with the mass of the steel bucket $m_S = \pi (r^2 + rb) t \rho_S$ and the mass of the filling $m_M = \frac{1}{2} \pi r^2 b \rho_M$ the location of the mass center as

$$\underline{\underline{z_{CF}}} = \frac{z_{CE} m_S + \frac{4r}{3\pi} m_M}{m_S + m_M} = \frac{4(2r + 3b) t \rho_S + 4rb \rho_M}{3\pi [2(r+b) t \rho_S + r b \rho_M]} r.$$

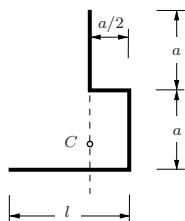
Using the given data $b = r$, $t = r/100$, $\rho_M = \rho_S/3$, it follows

$$z_{CE} = \frac{10}{3\pi \cdot 2} r = 0.53 r, \quad z_{CF} = \frac{4 \cdot 5 \frac{1}{100} + 4 \cdot \frac{1}{3}}{3\pi \left(4 \cdot \frac{1}{100} + \frac{1}{3} \right)} r = 0.44 r.$$

Remark: Since the mass of the filling is much bigger than the mass of the bucket, we find the common center of mass close to the center of mass of the filling: $z_{CM} = 4r/(3\pi) = 0.424 r$.

P2.11

Problem 2.11 The depicted stirrer consists of a homogenous wire that rotates about the sketched vertical axis. Determine the length l , such that the center of mass C is located on the rotation axis.

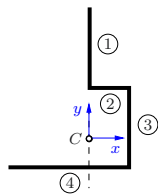


Solution Using the given coordinate system and decomposing the stirrer into four subparts, we obtain the center of mass from

$$x_C = \frac{\sum x_i l_i}{\sum l_i}.$$

For convenience, we use a table.

i	l_i	x_i	$x_i l_i$
1	a	0	0
2	$\frac{a}{2}$	$\frac{a}{4}$	$\frac{a^2}{8}$
3	a	$\frac{a}{2}$	$\frac{a^2}{2}$
4	l	$\frac{a}{2} - \frac{l}{2}$	$\frac{al}{2} - \frac{l^2}{2}$
Σ	$\frac{5a}{2} + l$	—	$\frac{5a^2}{8} + \frac{al}{2} - \frac{l^2}{2}$



The centroid shall lie on the rotation axis. Therefore, from the condition $x_C = 0$, follows the quadratic equation

$$\sum x_i l_i = \frac{5a^2}{8} + \frac{al}{2} - \frac{l^2}{2} = 0 \quad \leadsto \quad l^2 - al - \frac{5a^2}{4} = 0.$$

It has two solutions

$$l_{1,2} = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} + \frac{5a^2}{4}} = \frac{a}{2} \pm \frac{\sqrt{6}}{2}a,$$

from which only the positive one is physically reasonable:

$$\underline{\underline{l = \frac{a}{2}(1 + \sqrt{6}).}}$$

Problem 2.12 Determine the location of the centroid for the depicted surface of a hemisphere with the radius r .

Solution We choose the coordinate system, such that the y -axis coincides with the symmetry axis. Therefore, we know:

$$\underline{\underline{x_C = 0}}, \quad \underline{\underline{z_C = 0}}.$$

The remaining coordinate y_C , follows from

$$y_C = \frac{\int y \, dA}{\int dA}.$$

As infinitesimal area element, we choose the circular ring with the width $r \, d\alpha$ and the circumference $2\pi R$ as our infinitesimal area element:

$$dA = 2\pi R r \, d\alpha.$$

Using $R = r \cos \alpha$ and $y = r \sin \alpha$, it follows

$$dA = 2\pi r^2 \cos \alpha \, d\alpha.$$

Now, we can determine the surface area as

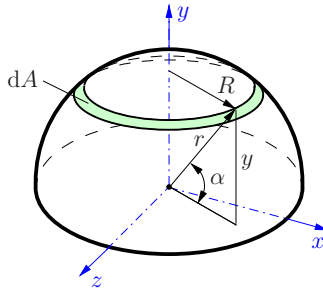
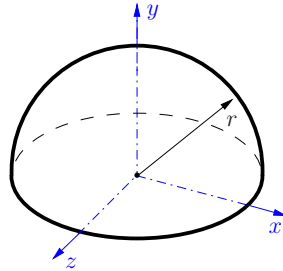
$$A = \int dA = 2\pi r^2 \int_{\alpha=0}^{\pi/2} \cos \alpha \, d\alpha = 2\pi r^2 \sin \alpha \Big|_0^{\pi/2} = 2\pi r^2$$

and the first moment of the area as

$$\int y \, dA = 2\pi r^3 \int_{\alpha=0}^{\pi/2} \sin \alpha \underbrace{\cos \alpha \, d\alpha}_{d \sin \alpha} = 2\pi r^3 \frac{1}{2} \sin^2 \alpha \Big|_0^{\pi/2} = \pi r^3.$$

Thus, the location of the centroid results as

$$\underline{\underline{y_C}} = \frac{1}{A} \int y \, dA = \frac{r}{2}.$$



P2.13

Problem 2.13 Determine the center of the volume for the depicted hemisphere of radius r .

Solution Due to the axisymmetric geometry, we know

$$\underline{\underline{x_c = 0}}, \quad \underline{\underline{z_c = 0}}.$$

The remaining coordinate is determined from

$$y_C = \frac{\int y \, dV}{\int dV}.$$

As infinitesimal volume element we select the circular disk with radius R and thickness dy :

$$dV = R^2 \pi \, dy.$$

By parametrization of the radius R and coordinate y

$$R = r \cos \alpha, \quad y = r \sin \alpha \quad \leadsto \quad dy = r \cos \alpha \, d\alpha,$$

the volume of the hemisphere follows as

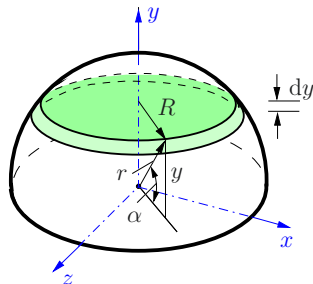
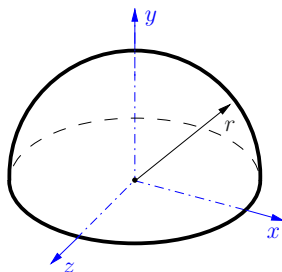
$$\begin{aligned} V &= \int dV = \int_{\alpha=0}^{\pi/2} \pi r^3 \cos^3 \alpha \, d\alpha = \int_{\alpha=0}^{\pi/2} \pi r^3 (1 - \sin^2 \alpha) \underbrace{\cos \alpha \, d\alpha}_{d \sin \alpha} \\ &= \pi r^3 \left(\sin \alpha - \frac{\sin^3 \alpha}{3} \right) \bigg|_0^{\pi/2} = \frac{2}{3} \pi r^3. \end{aligned}$$

With the the first moment of the area as

$$\int y \, dV = \pi r^4 \int_{\alpha=0}^{\pi/2} \cos^3 \alpha \underbrace{\sin \alpha \, d\alpha}_{-d \cos \alpha} = -\frac{\pi r^4}{4} \cos^4 \alpha \bigg|_0^{\pi/2} = \frac{\pi r^4}{4},$$

the center of the volume is determined as

$$\underline{\underline{y_C}} = \frac{1}{V} \int y \, dV = \frac{\pi r^4}{4} \frac{3}{2\pi r^3} = \underline{\underline{\frac{3}{8} r}}.$$



Statics – Formulas and Problems

Engineering Mechanics 1

Gross, D.; Ehlers, W.; Wriggers, P.; Schröder, J.; Müller,
R.

2017, IX, 236 p. 462 illus., 460 illus. in color., Softcover

ISBN: 978-3-662-53853-1