

Chapter 2

Planning Repeated Degradation Testing for Degrading Products

2.1 Introduction

Degradation information regarding the system's health state, especially from highly reliable items, has been a useful alternative for the system's remaining useful life (RUL) estimation, as well as a valuable basis for condition-based maintenance (CBM). Once the degradation information of a system is available by the degradation test, one well-recognized method is to establish a stochastic degradation model to predict the distributions of the future degradation and the associated lifetime, based on the relationship between the degradation and failure time. However, the accuracy of the aforementioned degradation or lifetime distributions is heavily influenced by the accuracy of the parameter estimation, which is affected by the number of items and the sampling frequency of each item. Therefore, to achieve a satisfactory prognosis accuracy, engineers need to decide how many items should be measured and how often should the measurements be made, before the degradation studies are performed [1]. In addition, the degradation test is usually costly, particularly for highly valued systems or vital components. In this case, how to achieve a tradeoff between the limited fund and the required estimation accuracy for important statistics of interest is also an interesting problem deserving in-depth studies.

Researchers and engineers have paid much attention to the degradation test design, particularly in the field of accelerated degradation test planning. Meeker et al. in [2] discussed the modeling and analysis issues of accelerated degradation test. Tseng and Yu in [3, 4] proposed several appropriate termination rules for degradation experiments. The works in [5, 6] considered optimal step-stress accelerated degradation test design for Wiener process and Gamma process, respectively. In the works of [5, 6], the temporal variability in stochastic degradation characteristics is involved, while both the unit-to-unit diversity and the measurements variability are ignored. Shi et al. in [7] studied the test planning methods for accelerated destructive degradation, where only the measurement errors were taken into account. Recently, Bayesian methods for designing accelerated destructive degradation test have also been developed by [8]. In addition, Weaver et al. in [1] documented several useful methods for

planning repeated degradation tests. In these two works, both diversities among units and measurement errors are considered, but the temporal variability is ignored in the process of stochastic degradation modeling.

In general, the degradation process of an item is stochastic. As a result, the lifetime and the degradation in the future are also random variables, resulting in the difficulty to predict the degradation in the future and estimate the lifetime with certainty. As summarized by [9], there are three sources of variability contributing to the uncertainties of degradation modeling and prognosis: (1) temporal variability; (2) unit-to-unit variability (usually modeled as random effect); and (3) measurement variability. The temporal variability is referred to as the inherent stochastic characteristics of the associated degradation process over time [10]. The unit-to-unit variability determines the heterogeneity among the degradation paths of different units [11]. The measurement variability describes the randomness in the measured data of the degradation, which might be contaminated by the uncertainty during the measurement process [12]. Therefore, a reasonable and appropriate degradation model for prognosis has to take into account these sources of variability. It has been found in [9] that degradation modeling with three-source variability shows great potentiality in improving the accuracy of the lifetime estimation.

By the above survey over recent advances in degradation modeling and degradation test design planning, it can be observed that, though a significant volume of research regarding planning degradation test has appeared so far, there is no literature addressing the problem of planning repeated degradation test for products whose degradation measurements exhibiting three-source variability. Such a method for planning repeated degradation test is useful and desired, particularly for the case that the concerned system is highly valued but with limited fund conducting extensive degradation tests.

In response to the above desire, this chapter considers the problem of planning repeated degradation test for degrading products with three-source variability. The degradation process is modeled as a Wiener process with a random linear drift coefficient and a constant volatility coefficient, while the measurement errors are described as additive zero-mean random variables. Based on the presented model, the lifetime distribution is formulated under the concept of the first passage time (FPT). After the model parameters are estimated by the expectation maximization (EM) algorithm, the large-sample approximate standard errors (ASE) of the maximum likelihood estimation (MLE) for the mean failure time and the quantile of degradation distribution are derived, respectively. Then, we take into account the relationship between the performance of the measurement errors and its cost, after which we propose a constrained optimum designing model by minimizing the test cost under the condition of a maximum acceptable ASE. An example is provided to illustrate the procedure and advantages of the proposed planning method.

In summary, the contributions of the chapter mainly include two aspects. On the one hand, based on a relatively general deteriorating model, we proposed a method for repeated degradation test planning for systems with three-source variability, which has not been considered before but such gap is filled by this work. On the other hand, we introduced the measurement error into the constrained optimization model for the

first time. As such, the relationship between the accuracy and cost of measurement has been modeled and brought into the optimization of the test plans. Through such modeling, the proposed method can provide some practical guidance on repeated degradation tests panning for both researchers and engineers.

The remainder of this chapter is structured as follows. Section 2.2 describes the degradation model with three-source variability. The parameter estimation and information matrix derivation framework are presented in Sect. 2.3. Section 2.4 proposes the estimation methods for both the degradation distribution and the lifetime distribution. The method to choose an optimal degradation test plan is discussed in Sect. 2.5. Section 2.6 provided a numerical example for illustration.

2.2 Degradation Modeling with Three-Source Variability

In this chapter, the degradation process is modeled as a Wiener process with a linear drift, which has been widely used in the field of reliability. As one kind of stochastic-process-based degradation models, Wiener process has many favorable properties including the ability to handle nonmonotonous deteriorating process, an analytical result for the FPT distribution, and strong Markovian property, etc. In addition, it is very easy to extend Wiener process to nonlinear diffusion processes, and to incorporate the measurement errors, covariates as well as random effects. A comprehensive review of Wiener process as a degradation model can be founded in [13].

In this study, suppose that there are totally n units for the test and the i -th unit is measured m_i times. Let D_{ij} be the degradation state of the i -th unit at time t_{ij} , where $i = 1, \dots, n$ and $j = 1, \dots, m_i$. Then, based on the Wiener process, D_{ij} can be modeled as

$$D_{ij} = \lambda_{0i} + \lambda_{1i}t_{ij} + \sigma_B B(t_{ij}), \quad (2.1)$$

where λ_{0i} and λ_{1i} are the initial state (intercept) and the deteriorating rate (slope) of the i th unit, respectively; σ_B is the diffusion parameter which is a constant used to capture the level of the temporal variability, and $B(t)$ is a standard Brownian motion. Note that a linear drift $\lambda_{0i} + \lambda_{1i}t_{ij}$ is used here for the reason that many nonlinear degradation processes can be converted into an approximated linear form as Eq. (2.1) by some transformation techniques (see [14] for more details).

To describe the unit-to-unit variability, random effects are incorporated. Particularly, it is assumed that the intercept λ_{0i} and the slop λ_{1i} are treated as the random realizations following the bivariate-normal distribution $[\lambda_{0i}, \lambda_{1i}]^T \sim BVN(\boldsymbol{\lambda}, \mathbf{V})$, where $\boldsymbol{\lambda} = [\lambda_0, \lambda_1]^T$ represents the common properties of a specific kind of units in interpret and slope, $[\cdot]^T$ denotes the transpose of a vector or a matrix, and $\mathbf{V}$ is the covariance matrix characterizing the unit-to-unit variability with

$$\mathbf{V} = \begin{bmatrix} \sigma_{\lambda_0}^2 & \rho\sigma_{\lambda_0}\sigma_{\lambda_1} \\ \rho\sigma_{\lambda_0}\sigma_{\lambda_1} & \sigma_{\lambda_1}^2 \end{bmatrix}.$$

It is worth noting that, for convenience, the normally distributed random variables are used to depict the random effects. Actually, any other forms of distributions can be treated as random effects [11], but some difficulty in derivation may be involved.

In practice, the degradation state is usually partly observable. In other words, the actual observation of D_{ij} , denoted by y_{ij} is frequently influenced by the measurement errors. Thus, the degradation measurement model can be formulated as

$$y_{ij} = D_{ij} + \varepsilon_{ij}, \quad (2.2)$$

where ε_{ij} is also normally distributed with the mean 0 and variance σ^2 , representing the measurement errors.

By now, the temporal variability, unit-to-unit variability, and measurement variability are all incorporated into the degradation model. For the notation convenience, we gather the observations of unit i into $\mathbf{Y}_i = [y_{i1}, y_{i2}, \dots, y_{im_i}]^T$ and give an equivalent formulation of Eq. (2.2) as

$$\mathbf{Y}_i = \mathbf{Z}_i \boldsymbol{\lambda}_i^* + \boldsymbol{\xi}_i + \boldsymbol{\varepsilon}_i, \quad (2.3)$$

where $\boldsymbol{\lambda}_i^* = [\lambda_{0i}^*, \lambda_{1i}^*]^T$ is a random vector drawn from the bivariate distribution $BVN(\boldsymbol{\lambda}, \mathbf{V})$, $\boldsymbol{\xi}_i$ is another multivariate-normal distributed random vector following $MVN(\mathbf{0}, \sigma_B^2 \mathbf{I}_i)$ with

$$\mathbf{T}_i = \begin{bmatrix} t_{i1} & t_{i1} & \cdots & t_{i1} \\ t_{i1} & t_{i2} & \cdots & t_{i2} \\ \vdots & \vdots & \ddots & \vdots \\ t_{i1} & t_{i2} & \cdots & t_{im_i} \end{bmatrix}.$$

$\mathbf{Z}_i$ are the design matrix of the form

$$\mathbf{Z}_i = \begin{bmatrix} 1 & t_{i1} \\ \vdots & \vdots \\ 1 & t_{im_i} \end{bmatrix},$$

and $\boldsymbol{\varepsilon}_i = [\varepsilon_{i1}, \dots, \varepsilon_{im_i}]^T$.

It is reasonable to assume that $\boldsymbol{\varepsilon}_i$, $\boldsymbol{\lambda}_i^*$ and $\boldsymbol{\xi}_i$ are mutually independent, and the components of $\boldsymbol{\varepsilon}_i$ are also independent and normally distributed with $\boldsymbol{\varepsilon}_i \sim MVN(\mathbf{0}, \sigma^2 \mathbf{I}_i)$, where $\mathbf{I}_i$ is a $m_i \times m_i$ identity matrix. As such, the observation $\mathbf{Y}_i$ follows a multivariate normal distribution $MVN(\mathbf{Z}_i \boldsymbol{\lambda}, \boldsymbol{\Sigma}_i)$ with

$$\boldsymbol{\Sigma}_i = \mathbf{Z}_i \mathbf{V} \mathbf{Z}_i^T + \sigma_B^2 \mathbf{T}_i + \sigma^2 \mathbf{I}_i.$$

Based on the above general model description, we elaborate the issues of the parameter estimation and the associated derivation of the information matrix in the following section.

2.3 Parameter Estimation and Information Matrix

Based on the structure of the presented degradation model involving three-source variability, the EM algorithm is chosen here to estimate the model parameters. Usually, the parameters with random effects are treated as unobservable variables. After an initial guess of the parameters, the expectation of the complete likelihood function for both observable and unobservable variables is computed conditional on the available observations in the E-Step, followed by the M-Step which updates the guess of the parameters by maximizing the conditional expectation. The specific algorithm for estimating the model parameters is omitted here due to the limited space. For the details, see [15–17].

Since the focus of this chapter is on test planning, we now check the variance-covariance matrix of the estimated parameters. With n independent observations $[\mathbf{y}_1, \dots, \mathbf{y}_n]$ from $\mathbf{Y}_i$, the log-likelihood for unit i

$$\mathcal{L}_i = -\frac{1}{2} \log [\det \boldsymbol{\Sigma}_i] - \frac{1}{2} (\mathbf{y}_i - \mathbf{Z}_i \boldsymbol{\lambda})^\tau \boldsymbol{\Sigma}_i^{-1} (\mathbf{y}_i - \mathbf{Z}_i \boldsymbol{\lambda}), \quad (2.4)$$

and the total log-likelihood for n units is

$$\begin{aligned} \mathcal{L} &= \sum_{i=1}^n \mathcal{L}_i \\ &= -\frac{1}{2} \sum_{i=1}^n \log [\det \boldsymbol{\Sigma}_i] - \frac{1}{2} \sum_{i=1}^n (\mathbf{y}_i - \mathbf{Z}_i \boldsymbol{\lambda})^\tau \boldsymbol{\Sigma}_i^{-1} (\mathbf{y}_i - \mathbf{Z}_i \boldsymbol{\lambda}). \end{aligned} \quad (2.5)$$

Denote $\boldsymbol{\theta}^\tau = [\boldsymbol{\lambda}^\tau, \boldsymbol{\vartheta}^\tau]$ as the parameter vector to be estimated, where $\boldsymbol{\vartheta} = [\sigma_{\lambda_0}, \sigma_{\lambda_1}, \rho, \sigma_B, \sigma]^\tau$. From the large-sample theory, the large-sample approximate covariance matrix of the MLE can be formulated as

$$\text{Avar}(\hat{\boldsymbol{\theta}}) = [\mathcal{J}(\boldsymbol{\theta})]^{-1}, \quad (2.6)$$

where $\hat{\boldsymbol{\theta}}$ is the ML estimator of $\boldsymbol{\theta}$, and $\mathcal{J}(\boldsymbol{\theta}) = \sum_{i=1}^n \mathcal{J}_i$ is the Fisher information matrix with the definition

$$\mathcal{J}_i(\boldsymbol{\theta}) = E_Y \left[-\frac{\partial^2 \mathcal{L}_i}{\partial \boldsymbol{\theta}^2} \right]. \quad (2.7)$$

Setting $\boldsymbol{\mu}_i = \mathbf{Z}_i \boldsymbol{\lambda}$ for simplicity, the specific form of the information matrix can then be expressed as

$$\mathcal{J}_i(\boldsymbol{\lambda}, \boldsymbol{\vartheta}) = \text{diag} [\mathcal{J}_i(\boldsymbol{\lambda}), \mathcal{J}_i(\boldsymbol{\vartheta})], \quad (2.8)$$

where $\text{diag}(\cdot)$ is a diagonal matrix with blocks, and

$$\mathcal{J}_i(\boldsymbol{\lambda})_{j,k} = \frac{\partial \boldsymbol{\mu}_i^\tau}{\partial \boldsymbol{\lambda}_j} \boldsymbol{\Sigma}_i \frac{\partial \boldsymbol{\mu}_i}{\partial \boldsymbol{\lambda}_k}, 1 \leq j, k \leq 2,$$

$$\mathcal{J}_i(\boldsymbol{\vartheta})_{j,k} = \frac{1}{2} \text{tr} \left(\boldsymbol{\Sigma}_i^{-1} \frac{\partial \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\vartheta}_j} \boldsymbol{\Sigma}_i^{-1} \frac{\partial \boldsymbol{\Sigma}_i}{\partial \boldsymbol{\vartheta}_k} \right), 1 \leq j, k \leq 5,$$

respectively.

An estimator of $\text{Avar}(\hat{\boldsymbol{\theta}})$ denoted by $\widehat{\text{var}}(\hat{\boldsymbol{\theta}})$ can be achieved by substituting ML estimator $\hat{\boldsymbol{\theta}}$ into Eq. (2.6). It is worth pointing out here that $\hat{\boldsymbol{\theta}}$ is the MLE of the model parameters and such estimating process is achieved by the EM algorithm. $\widehat{\text{var}}(\hat{\boldsymbol{\theta}})$ is the foundation to evaluate the standard error of some other important quantiles, which will be shown in the following sections.

2.4 Estimating the Degradation Distribution and Lifetime Distribution

The degradation distribution and the lifetime distribution, without which many statistical inference will be impossible, are two most important concerns in prognosis. For the mean of lifetime and the quantile of the degradation distribution, a smaller standard error corresponds to a better degradation test plan. Therefore, we evaluate the degradation distribution and lifetime distribution, as well as some important quantiles which will be used as criteria of test planning.

2.4.1 The Quantiles of Degradation Distribution and Its Variance

We begin with the quantile of the degradation distribution. From Eq. (2.1), it follows that the degradation at time t is drawn from a normal distribution with mean $E[D(t)] = \lambda_0 + \lambda_1 t$ and variance $\text{var}[D(t)] = \sigma_{\lambda_0}^2 + \sigma_{\lambda_1}^2 t^2 + 2t\rho\sigma_{\lambda_0}\sigma_{\lambda_1} + t\sigma_B^2$. As a result, the p quantile of the degradation distribution at time t can be formulated by

$$d_p(t) = \lambda_0 + \lambda_1 t + \Phi^{-1}(p) \sqrt{\sigma_{\lambda_0}^2 + \sigma_{\lambda_1}^2 t^2 + 2t\rho\sigma_{\lambda_0}\sigma_{\lambda_1} + t\sigma_B^2}, \quad (2.9)$$

with the inverse standard normal cumulative distribution function $\Phi^{-1}(p)$.

Therefore, the ML estimator of $d_p(t)$, denoted by $\hat{d}_p(t)$, can be accessed by computing Eq. (2.9) at the MLE $\hat{\boldsymbol{\theta}}$.

From Eq. (2.9), we note that the quantile $d_p(t)$ is a function of parameters $\boldsymbol{\theta}$, which means the formula for the approximated standard error (ASE) of $\hat{d}_p(t)$ can be derived through the delta method. Thus, the large-sample approximate variance of $\hat{d}_p(t)$ can be formulated as

$$Avar(\widehat{d}_p(t)) = \left[\frac{\partial d_p(t)}{\partial \theta} \right]^\tau Avar(\widehat{\theta}) \left[\frac{\partial d_p(t)}{\partial \theta} \right], \quad (2.10)$$

where $\frac{\partial d_p(t)}{\partial \theta}$ is the partial deviation of $d_p(t)$ with respect to θ .

Equation (2.10) indicates that the standard error of $\widehat{d}_p(t)$ is $ASE_{\widehat{d}_p} = \sqrt{Avar(\widehat{d}_p(t))}$, which can be estimated by evaluating Eq. (2.10) at $\widehat{\theta}$ as $\widehat{SE}_{\widehat{d}_p} = \sqrt{\widehat{var}(\widehat{d}_p)}$. Note that $\widehat{\theta}$ is the estimation for the standard error of $\widehat{d}_p(t)$ and $\widehat{var}(\widehat{d}_p)$ is the estimation of $Avar(\widehat{d}_p(t))$. The specific forms of $\partial d_p(t)/\partial \theta_k$ are listed as

$$\begin{aligned} \frac{\partial d_p(t)}{\partial \lambda_0} &= 1, \quad \frac{\partial d_p(t)}{\partial \lambda_1} = t, \\ \frac{\partial d_p(t)}{\partial \sigma_{\lambda_0}} &= \frac{\Phi^{-1}(p)(\sigma_{\lambda_0} + t\rho\sigma_{\lambda_1})}{\xi}, \\ \frac{\partial d_p(t)}{\partial \sigma_{\lambda_1}} &= \frac{\Phi^{-1}(p)(\sigma_{\lambda_1} + t\rho\sigma_{\lambda_0})}{\xi}, \\ \frac{\partial d_p(t)}{\partial \rho} &= \frac{\Phi^{-1}(p)t\sigma_{\lambda_0}\sigma_{\lambda_1}}{\xi}, \\ \frac{\partial d_p(t)}{\partial \sigma_B} &= \frac{\Phi^{-1}(p)\sigma_B t}{\xi}, \quad \frac{\partial d_p(t)}{\partial \sigma} = 0, \end{aligned}$$

with $\xi = \sqrt{\sigma_{\lambda_0}^2 + \sigma_{\lambda_1}^2 t^2 + 2t\rho\sigma_{\lambda_0}\sigma_{\lambda_1} + t\sigma_B^2}$.

Based on the availability for $\widehat{d}_p(t)$ and $\widehat{SE}_{\widehat{d}_p}$, a large-sample approximation 100(1- α)% confidence interval for $d_p(t)$ can be also achieved by

$$[\underline{d}_p(t), \overline{d}_p(t)] = \widehat{d}_p(t) \pm z_{(1-\alpha/2)} \widehat{SE}_{\widehat{d}_p}, \quad (2.11)$$

where $z_{(1-\alpha/2)}$ is the $(1 - \alpha/2)$ standard normal quantile.

2.4.2 The Lifetime Distribution

For the degradation process with soft failure, the life of a unit ends when its performance degradation process hits a preset threshold ω , known as the failure threshold. Therefore, the lifetime is defined as the FPT of the degradation process crossing ω :

$$T = \inf \{t : D(t) \geq \omega | D(0) < \omega\}.$$

The work in [9] developed a useful method to derive the lifetime distribution for units with the degradation process described by Eq. (2.1). The probability density function (PDF) of lifetime $f_L(l)$ and the cumulative density function (CDF) $F_L(l)$

can be evaluated using the law of total probability, given the corresponding results of the degradation model ignoring the random effect in parameters.

Thus, we begin the discussion with a simplified model without considering random parameters. Then, the PDF and CDF of lifetime corresponding to the simplified model are respectively formulated as

$$f_{L|S}(l) = \frac{\omega - \lambda_0}{\sqrt{2\pi l^3 \sigma_B^2}} \exp\left(-\frac{(\omega - \lambda_0 - \lambda_1 l)^2}{2\sigma_B^2 l}\right), \quad (2.12)$$

and

$$\begin{aligned} F_{L|S}(l) &= 1 - \Phi\left(\frac{\omega - \lambda_0 - \lambda_1 l}{\sigma_B \sqrt{l}}\right) \\ &+ \exp\left(\frac{2\lambda_1(\omega - \lambda_0)}{\sigma_B^2}\right) \Phi\left(\frac{-\omega + \lambda_0 - \lambda_1 l}{\sigma_B \sqrt{l}}\right). \end{aligned} \quad (2.13)$$

Equation (2.12) shows that the lifetime is an inverse Gaussian distribution with mean $\frac{\omega - \lambda_0}{\lambda_1}$ and covariance $\frac{(\omega - \lambda_0)\sigma_B^2}{\lambda_1^3}$. Due to the law of total probability, and the results in [17], when the random effect is considered, the PDF of lifetime $f_L(l)$ can be formulated as

$$\begin{aligned} f_L(l) &= \frac{W(\sigma_B^2 l + \mathbf{b}^T \mathbf{V} \mathbf{b} - \mathbf{a}^T \mathbf{V} \mathbf{b}) - \sigma_B^2 l \mathbf{a}^T \boldsymbol{\lambda} - \mathbf{b}^T \mathbf{V} \mathbf{b} \mathbf{a}^T \boldsymbol{\lambda} + \mathbf{a}^T \mathbf{V} \mathbf{b} \mathbf{b}^T \boldsymbol{\lambda}}{\sqrt{2\pi (\sigma_B^2 l + \mathbf{b}^T \mathbf{V} \mathbf{b})^3}} \\ &\times \exp\left[-\frac{(W - \mathbf{b}^T \boldsymbol{\lambda})^2}{2(\sigma_B^2 l + \mathbf{b}^T \mathbf{V} \mathbf{b})}\right], \end{aligned} \quad (2.14)$$

where $\mathbf{a}^T = [1, 0]$ and $\mathbf{b}^T = [1, l]$ for brevity.

Accordingly, the CDF of the lifetime $F_L(l)$ can be obtained as

$$\begin{aligned} F_L(l) &= 1 - \Phi\left(\frac{c_1 + \mathbf{d}_1^T \boldsymbol{\lambda}}{\sqrt{1 + \mathbf{d}_1^T \mathbf{V} \mathbf{d}_1}}\right) \\ &+ \frac{1}{|\mathbf{A} \mathbf{V}|^{\frac{1}{2}}} \exp\left[\frac{2\boldsymbol{\lambda}^T \mathbf{B} \boldsymbol{\lambda} + 2\boldsymbol{\lambda}^T \mathbf{a}_2 + (2\mathbf{B} \boldsymbol{\lambda} + \mathbf{a}_2)^T \mathbf{a}_2^{-1} (2\mathbf{B} \boldsymbol{\lambda} + \mathbf{a}_2)}{2}\right] \\ &\times \Phi\left(\frac{c_2 + \mathbf{d}_2^T \boldsymbol{\lambda} + \mathbf{d}_2^T \mathbf{A}^{-1} (2\mathbf{B} \boldsymbol{\lambda} + \mathbf{a}_2)}{\sqrt{1 + \mathbf{d}_2^T \mathbf{A} \mathbf{d}_2}}\right), \end{aligned} \quad (2.15)$$

with parameters $c_1 = \frac{W}{\sigma_B \sqrt{l}}$, $\mathbf{d}_1 = -\frac{1}{\sigma_B^2 \sqrt{l}}[1 \ l]$, $c_2 = -\frac{W}{\sigma_B \sqrt{l}}$, $\mathbf{a}_2 = [0 \ W]$, $\mathbf{d}_2 = -\frac{1}{\sigma_B^2 \sqrt{l}}[-1 \ l]$, $\mathbf{A} = \mathbf{V}^{-1} - 2\mathbf{B}$, and

$$\mathbf{B} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

Even in the situation where the unit-to-unit variability is ignored, no analytical-form quantile of the lifetime distribution can be obtained, which leads to either the use of iterative algorithm or an approximation of the distribution itself [13]. By contrast, the mean of the lifetime, which is frequently used in statistical inference and decision-making, should be estimated with a requirement of the degradation test performance. In this chapter, the standard error of the MLE for the mean lifetime is considered in the planning of the degradation test.

Thanks to the law of total probability, we can obtain the mean lifetime as

$$E_L = \rho\sigma_{\lambda_0} + \frac{\sqrt{2}(\omega + \rho\lambda_1\sigma_{\lambda_0} - \lambda_0\sigma_{\lambda_1})}{\sigma_{\lambda_1}} G\left(\frac{\lambda_1}{\sqrt{2}\sigma_{\lambda_1}}\right), \quad (2.16)$$

where $G(x) = \exp(-x^2) \int_0^x \exp(u^2) du$ is the Dawson integral for all real x .

Furthermore, the delta method is employed to estimate the standard error for the MLE of the mean lifetime $\hat{E}_L$. Then, the large-sample approximate variance of $\hat{E}_L$ is formulated as

$$\text{Avar}(\hat{E}_L) = \left[\frac{\partial E_L}{\partial \boldsymbol{\theta}} \right]^T \text{Avar}(\hat{\boldsymbol{\theta}}) \left[\frac{\partial E_L}{\partial \boldsymbol{\theta}} \right]. \quad (2.17)$$

The specific forms of $\frac{\partial E_L}{\partial \theta_k}$ are given as

$$\begin{aligned} \frac{\partial E_L}{\partial \lambda_0} &= -\sqrt{2} G\left(\frac{\lambda_1}{\sqrt{2}\sigma_{\lambda_1}}\right), \\ \frac{\partial E_L}{\partial \lambda_1} &= \left[-\frac{\lambda_1}{\sigma_{\lambda_1}^2} G\left(\frac{\lambda_1}{\sqrt{2}\sigma_{\lambda_1}}\right) + G\left(\frac{\lambda_1}{\sqrt{2}\sigma_{\lambda_1}}\right) + \frac{1}{\sqrt{2}\sigma_{\lambda_1}} \right] \mathcal{E} \\ &\quad + \frac{\sqrt{2}\rho\sigma_{\lambda_0}}{\sigma_{\lambda_1}} G\left(\frac{\lambda_1}{\sqrt{2}\sigma_{\lambda_1}}\right), \\ \frac{\partial E_L}{\partial \sigma_{\lambda_0}} &= \frac{\sqrt{2}\rho\lambda_1}{\sigma_{\lambda_1}^2} G\left(\frac{\lambda_1}{\sqrt{2}\sigma_{\lambda_1}}\right) + \rho, \\ \frac{\partial E_L}{\partial \sigma_{\lambda_1}} &= -G\left(\frac{\lambda_1}{\sqrt{2}\sigma_{\lambda_1}}\right) \frac{\sqrt{2}(\omega + \rho\sigma_{\lambda_0})}{\sigma_{\lambda_1}^2} \\ &\quad + \mathcal{E} \left[G\left(\frac{\lambda_1}{\sqrt{2}\sigma_{\lambda_1}}\right) \frac{\lambda_1^2}{\sigma_{\lambda_1}^3} + G\left(\frac{\lambda_1}{\sqrt{2}\sigma_{\lambda_1}}\right) - \frac{\lambda_1}{\sqrt{2}\sigma_{\lambda_1}^2} \right], \\ \frac{\partial E_L}{\partial \rho} &= \frac{\sqrt{2}\sigma_{\lambda_0}\lambda_1}{\sigma_{\lambda_1}} G\left(\frac{\lambda_1}{\sqrt{2}\sigma_{\lambda_1}}\right) + \sigma_{\lambda_0}, \end{aligned}$$

$$\frac{\partial E_L}{\partial \sigma_B} = 0, \frac{\partial E_L}{\partial \sigma} = 0,$$

$$\text{where } \mathcal{E} = \frac{\sqrt{2}(W + \rho\sigma_{\lambda_0}\sigma_{\lambda_1} - \lambda_0\sigma_{\lambda_1})}{\sigma_{\lambda_1}}.$$

By the above derivations, the standard error of the MLE for $\hat{E}_L$, saying $\widehat{SE}_{\hat{E}_L}$, can be evaluated as $\widehat{SE}_{\hat{E}_L} = \sqrt{\widehat{var}(\hat{E}_L)}$, where $\widehat{var}(\hat{E}_L)$ is the MLE of $Avar(\hat{E}_L)$ and can be achieved by calculating Eq. (2.17) at $\hat{\theta}$. Based on these results, we determine the degradation test planning in the following section.

2.5 Degradation Test Planning

The degradation test plan depends on the parameters of the model, which are not exactly known when we make the planning. Prior knowledge of the degradation model including previous experience (such as measurements of the degradation burn-in test, and degradation data from the field systems). In this case, expert knowledge and design specifications are used to choose a set of parameters as planning information before the planning begins, denoted by $\theta^\square$. This set of parameters is known as the planning information.

Given the planning information $\theta^\square$ and a test plan with specific number of units as well as measurement schedule of each unit, we can compute the corresponding fisher information matrix, which facilitates the evaluation of standard error for the mean of lifetime and the quantile of the degradation distribution. For simple test plans where all units are measured using the same schedule, a contour plot of the large-sample approximate standard error of the interesting statistical test-plan properties against the number of units n and the number of times each unit measured m is recommended for test selection in [1]. For test plans where units are measured using different schedules, the plots of standard errors against time are compared to choose a reasonable test plan.

The planning problem, when there is a constraint on standard error of the statistical test-plan properties and a desire to minimize the cost of degradation test, is often encountered. Generally speaking, the cost of a degradation test depends on the number of the tested units, the measurement times on each unit, and the used measurement method. Therefore, suppose that the relationship between the standard error of measurement and the cost of measurement method is formulated as

$$cost(\sigma) = C - \kappa \exp(\eta\sigma), \quad (2.18)$$

where C, κ, η are positive constant coefficients which can be determined based on experience.

From Eq. (2.18), it can be noted that a smaller σ corresponds to higher performance of the measurement method. Equation (2.18) indicates that higher the performance measure method is, the harder the performance of the measurement method can be

further improved. This phenomenon is consistent with most of the industrial cases. When the performance level of the degradation measurement is low, the cost will increase rapidly to improve the precision of equipment. However, the rate of increase slows down as the performance of the measurement equipment increases.

Based on the above discussions, the cost of a test plan can be formulated as

$$cost(n, \mathbf{m}) = c_1 + c_2 n + \sum_{k=1}^n cost(\sigma) m_k, \quad (2.19)$$

where $\mathbf{m}$ is the collection of the number of measurements of each unit, i.e., $[m_1, \dots, m_n]^T$. c_1 , and c_2 denote the fixed cost of running the test and the cost of testing each unit, respectively. However, it is noted that other models describing the relationship between the standard error of measurement and the cost of measurement method can also be used, and the proposed framework in this chapter is not limited to the model as (2.18).

Then, the problem of selecting test plan can be formulated as the following constrained optimization problem:

$$\begin{aligned} [\mathbf{m}^*, n^*] &= \arg \min_{\mathbf{m}, n} [cost(n, \mathbf{m})] \\ s.t. \quad SE &\leq \gamma, \end{aligned} \quad (2.20)$$

where γ is the maximum acceptable value for ASE and can be determined according to the specific yet practical requirement.

By optimizing (2.19), we can obtain the optimal test plan which gives the optimal number of the tested units and the testing times for each unit.

2.6 An Illustrative Example

To illustrate the procedure for planning repeated degradation test with there-source variability, we give an example of degradation test planning based on $d_{0.1}(t)$ in this section. For the purpose of comparison, the same planning information of shelf-life test design in example 3 from [1] together with a new setting of σ_B is used here. Since $d_{0.1}(t)$ depends on time t , we first give a plot of contourslice of $\widehat{SE}_{\widehat{d_{0.1}}}$ in t and the cost on the grid of m and n , where both large and small values of $\sigma^\square$ are considered. Suppose that $\sigma_B = 0.3$ for both cases. The individual costs of components in the test are set as $c_1 = 15000$, $c_2 = 1500$, $\kappa = 100$, and $\eta = 1.25$. The maximum acceptable γ is set as 0.8.

From Figs. 2.1 and 2.2, we can observe that the large-sample approximate standard error of $d_{0.1}(t)$, $\widehat{SE}_{\widehat{d_{0.1}}}$, cannot be decreased by taking more measurements when the error of measurement is small and there are only a few units, i.e., less than 5 units are measured. In addition, in the case of small measurement error, $\widehat{SE}_{\widehat{d_{0.1}}}$ cannot

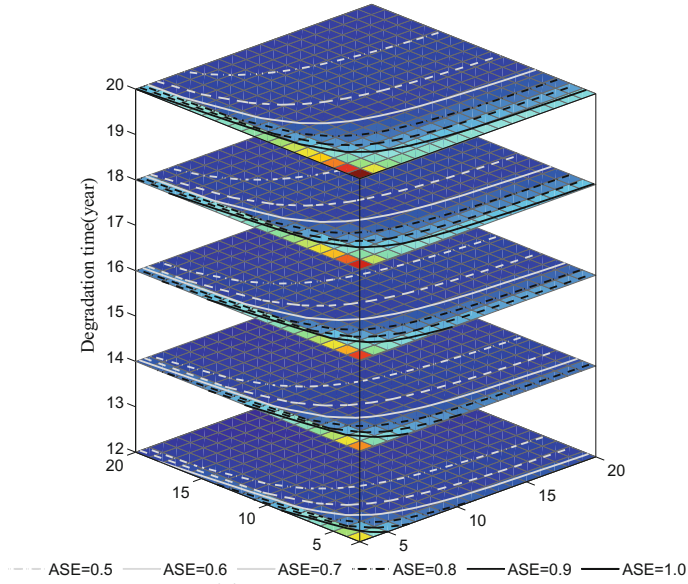


Fig. 2.1 Contourslice of the large-sample approximate standard error $\widehat{SE}_{d_{0,1}}$ when $\sigma^{\square} = 0.3$

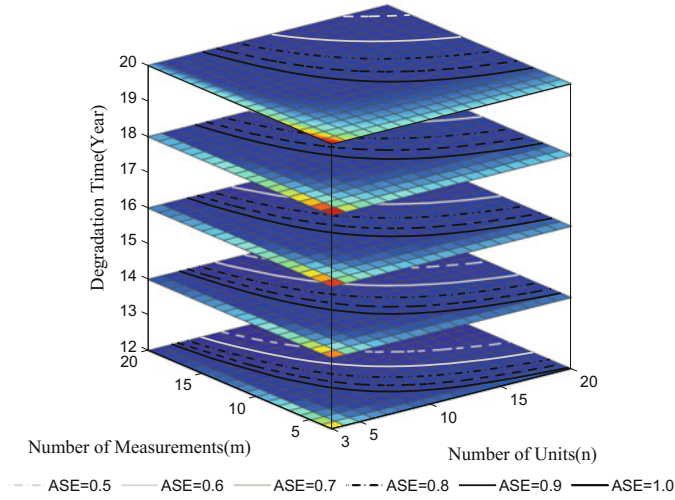


Fig. 2.2 Contourslice of the large-sample approximate standard error $\widehat{SE}_{d_{0,1}}$ when $\sigma^{\square} = 1.2$

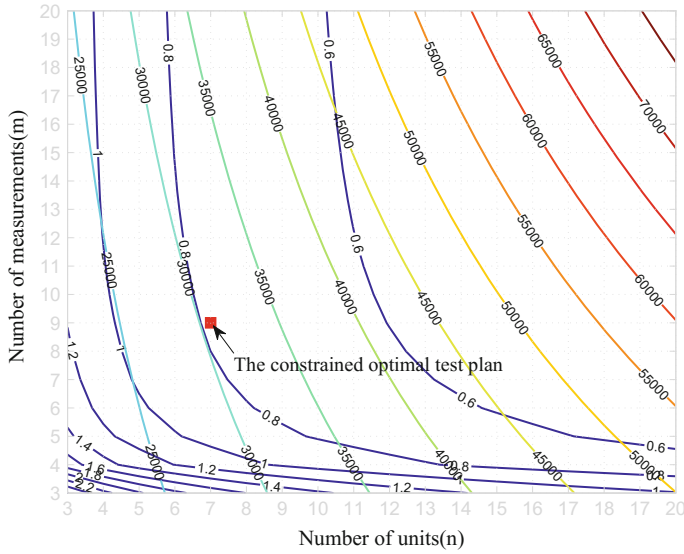


Fig. 2.3 the contour plot of cost and $\widehat{SE}_{\widehat{d}_{0,1}}$ when $\sigma^{\square} = 0.3$

be decreased efficiently by adding more units, when each unit are not measured sufficiently. In the case where the measurement error is large, $\widehat{SE}_{\widehat{d}_{0,1}}$ is more sensitive with m and n , which is indicated by Fig. 2.1. $\widehat{SE}_{\widehat{d}_{0,1}}$ is larger when the temporal variability is under consideration.

Figure 2.1 also shows that $\widehat{SE}_{\widehat{d}_{0,1}}$ increases with time t due to the temporal variability and the unit-to-unit variability in the parameter of the degradation model. Therefore, we choose the contourslice at a later time, say $t = 20$, to accomplish the test design. The top 1 slice of Fig. 2.1 is amplified in Fig. 2.2, which indicates a test plan where $n^* = 7$ items should be measured at $m^* = 9$ equally spaced times. For such a test plan, $\widehat{SE}_{\widehat{d}_p} = 0.7277$ and $cost(m^*, n^*) = 30696$.

To further investigate the influence of the measurement errors on the test planning, we check a series of contour plot of both $\widehat{SE}_{\widehat{d}_p}$ and the cost of test at different levels of measurement errors. An interesting result is found as follows. After setting $\sigma^{\square} = 0.8$, a test plan with $n = 8$, and $m = 18$ can achieve the precision level of $\widehat{SE}_{\widehat{d}_{0,1}} = 0.7863$ with a cost of 30856, which is very close to the constrained optimum design when $\sigma^{\square} = 0.3$. A comparison between Figs. 2.3 and 2.4 suggests that there are two strategies to achieve an acceptable level of $\widehat{SE}_{\widehat{d}_p}$ under the same constraint of budget. In other words, we can either spend less money on instruments with relatively worse performance but take more measurements on more items, or spend more money on instruments with better performance but cut down the number of items and measurements. These results are useful to design the degradation test plan for a practical product, particularly for vital products.

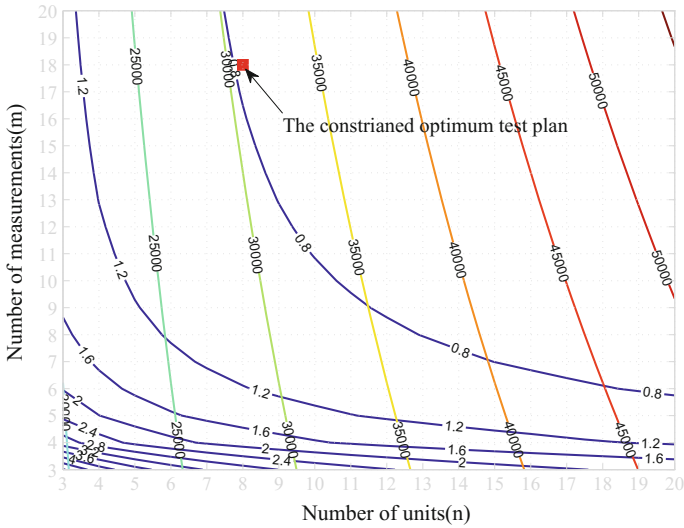


Fig. 2.4 the contour plot of cost and $\widehat{SE}_{d_{0.1}}$ when $\sigma^2 = 0.8$

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